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High Perturbations of a Fractional Kirchhoff Equation with Critical Nonlinearities

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Abstract: This paper concerns a fractional Kirchhoff equation with critical nonlinearities and a negative nonlocal term. In the case of high perturbations (large values of α, i.e., the parameter of a subcritical nonlinearity), existence results are obtained by the concentration compactness principle together with the mountain pass theorem and cut-off technique. The multiplicity of solutions are further considered with the help of the symmetric mountain pass theorem. Moreover, the nonexistence and asymptotic behavior of positive solutions are also investigated.

Keywords: fractional Kirchhoff equation; multiplicity; critical exponent; concentration compactness principle; mountain pass theorem

MSC: 35J60; 35A15; 35B33

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1. Introduction and Main Results

In this work, we study the existence of solutions for the following fractional Kirchhoff equation:

$$\begin{cases} \left(a - b\displaystyle\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)^s u = \lambda u + \alpha |u|^{q-2}u + |u|^{2_s^*-2}u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded set containing 0 with a Lipschitz boundary, dimension $n > 2s$ with $s \in (0,1)$, $a, b, \lambda, \alpha > 0$, $1 < q < 2_s^*$, where $2_s^* = \frac{2n}{n-2s}$ is the fractional critical Sobolev exponent. The fractional Laplacian operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) = C(s) P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\mathrm{d}y, \ u \in \mathbb{S}(\mathbb{R}^n),$$

where $P.V.$ stands for the Cauchy principal value, $C(s)$ is a normalized constant, and $\mathbb{S}(\mathbb{R}^n)$ is the Schwartz space of the rapidly decaying function. According to [1,2],the following problem

$$\begin{cases} (-\Delta)^s u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2}$$

has a sequence of eigenvalues satisfying $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty$.

Due to its interesting theoretical structure and concrete applications in many fields, such as phase transitions, Markov processes and fractional quantum mechanics, minimal surfaces, and so on [3], more and more papers have focused on fractional and nonlocal operators of the elliptic type. For example, based on the classical Brezis–Nirenberg problem [4], Servadei and Valdinoci [5] studied the following nonlocal fractional counterpart of the Brezis–Nirenberg problem:

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{2_s^*-2}u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3}$$

and obtained a nontrivial solution when $0 < \lambda < \lambda_1$ and $N \geq 4s$. Refs. [6,7] considered problem (3) in the lower dimension. Servadei [8] further investigated problem (3) in the resonant case. Figueiredo, Bisci, and Servadei [9] studied the number of nontrivial solutions of problem (3) under consideration with the topology of Ω when λu is replaced by $\lambda|u|^{q-2}u$ with $2 \leq q < 2_s^*$. Mukherjee and Sreenadh [10] considered the existence, nonexistence, and regularity results for a weak solution of problem (3) with Hardy–Littlewood–Sobolev critical nonlinearity. Fu and Xia [11] investigated the multiplicity results of problem (3) with a nonhomogeneous term $f(x)$. When λu is replaced by a Carathéodory function, satisfying a different subcritical condition, Fiscella, Bisci, and Servadei [12] obtained different results of multiple solutions for problem (3). Fiscella, Bisci, and Servadei [1], Servadei et al. [5,13,14], and Li and Sun [15] considered the generalization of problem (3) when $(-\Delta)^s$ is replaced by the integrodifferential operator defined as

$$L_K u(x) = \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)\mathrm{d}y, \ x \in \mathbb{R}^n,$$

where K is a measurable function satisfying some suitable conditions.

In the local setting ($s = 1$), problem (1) can be viewed as a deformation of a stationary analogue of the following Kirchhoff problem:

$$\rho \frac{\partial^2 u}{\partial^2 t} - \left(\frac{P_0}{h} + \frac{F}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \mathrm{d}x \right) \frac{\partial^2 u}{\partial^2 x} = f(x,u), \tag{4}$$

proposed by Kirchhoff [16] in 1883. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (4) have the following meanings: L is the length of the string. h is the area of the cross section. F is the Young's modulus of the material. ρ is the mass density, and P_0 is the initial tension. Problem (4) was proposed and studied as the fundamental equation for understanding several physical systems, where u describes a process which depends on its average. We should point out that a fractional Kirchhoff equation was first introduced and studied by Fiscella and Valdinoci [17]. After that, many researchers paid attention to fractional Kirchhoff equations with critical nonlinearities under fractional Laplacian operator $(-\Delta)^s$, fractional p-Laplacian operator $(-\Delta)^s_p$, integrodifferential operator L_K, or fractional n/s-Laplacian operator $(-\Delta)^s_{n/s}$, see, e.g., [3,18–21] and the references therein.

In this paper, we considered a new Kirchhoff problem with a fractional Laplacian operator and a negative nonlocal term, that is, this Kirchhoff problem involves a nonlocal coefficient $a - b\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y$. It stems from the Young's modulus, which can also be used in computing tension; when the atoms are pulled apart instead of squeezed together, the strain is negative because the atoms are stretched instead of compressed, and this leads to a negative Young's modulus [22]. Yin and Liu [23] first proposed and investigated this new nonlocal problem,

$$\begin{cases} -\left(a - b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = |u|^{q-2}u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{5}$$

when $2 < q < 2^*$. From then on, these new Kirchhoff problems with a negative nonlocal term have attracted a lot of attention. Refs. [24–27] obtained some important results with a Laplacian operator and a p-Laplace operator. As for a $p(x)$-Laplacian operator, refs. [28–30] gained certain wonderful conclusions. Some useful results were also obtained in [31–33] for the Kohn–Laplacian on the first Heisenberg group.

However, there are few papers considering critical Kirchhoff problems with negative nonlocal terms like (1), since the fractional Laplacian operator $(-\Delta)^s$ is also a nonlocal term and then problem (1) is the bi-nonlocality. This, together with the lack of compactness caused by critical nonlinearities, produces several difficulties in its study. On the other hand, as pointed out by Qian [25], the sign of the nonlocal term plays an important role in nonlocal problems, so it is necessary to investigate problem (1). Firstly, we give the following nonexistence of a positive solution for problem (1).

Theorem 1. *Suppose $\lambda \geq a\lambda_1$ and $1 < q < 2_s^*$; then problem (1) has no positive solution.*

To show the existence results for problem (1), define $T = \min\{T_1, T_2\}$, where

$$T_1 = \left[\frac{a(q-2)}{64b}\right]^{\frac{1}{2}}, \; T_2 = \left(\frac{s}{8nb}\right)^{\frac{1}{4}} \left(\frac{aS_s}{2}\right)^{\frac{n}{8s}}$$

and S_s can be found in (7). We also define $\Lambda = \min\left\{\Lambda_0, \left[\frac{aT^2 n(q-2)}{4qs|\Omega|}\right]^{\frac{2s}{n}}\right\}$ with $\Lambda_0 = \dfrac{aS_s}{2}\left(\dfrac{1}{2|\Omega|}\right)^{\frac{2s}{n}}$. By virtue of the concentration compactness principle together with the mountain pass theorem and cut-off technique, we obtain the existence and asymptotic behavior of solutions for problem (1) as follows.

Theorem 2. *Suppose $2 < q \leq \min\{4, 2_s^*\}$ and $a\lambda_1 - \Lambda < \lambda < a\lambda_1$; then there exists $\alpha^* > 0$ such that if $\alpha > \alpha^*$, then problem (1) has at least one positive solution u_λ.*

Theorem 3. *For every sequence $\{\lambda_n\}$ with $\lambda_n \nearrow a\lambda_1$, let u_{λ_n} be the positive solution to problem (1) provided by Theorem 2, then $\lim_{n\to\infty} \|u_{\lambda_n}\| = 0$.*

As for the multiplicity of solutions, we further obtain the following result by using the symmetric mountain pass theorem.

Theorem 4. *Suppose $2 < q \leq \min\{4, 2_s^*\}$ and $a\lambda_1 - \Lambda < \lambda < a\lambda_1$; then for any $k \in \mathbb{N}$, there exists $\alpha_k^* > 0$ such that problem (1) admits k pairs of positive solutions for $\alpha > \alpha_k^*$.*

Remark 1. *In comparing our results with those obtained by Jin, Liu, and Zhang [3], who considered problem (1) with $a = 1$ and $b < 0$, and Yang, Liu, and Ouyang [34] in the local setting, we find that the sign of the nonlocal term also plays an important role in the nonlocal fractional setting. Furthermore, we also obtain asymptotic behavior of positive solutions for problem (1), which [3,34] did not discuss.*

Remark 2. *In Ref. [25], Qian considered problem (1) in the local setting with $\alpha = 0$ and $n \geq 5$. Hence, our results extend those obtained by [25] to the nonlocal fractional setting with $n > 2s$, and we also obtain multiple results which were not considered in [25].*

This article contains four more sections. In Section 2, we present some preliminaries and introduce the variational framework and truncated functional of problem (1). In Section 3, we show a compactness result for the truncated functional. In Section 4, we prove the existence, nonexistence, multiplicity, and asymptotic behavior of solutions for problem (1). Finally, we conclude in Section 5.

2. Preliminaries

In this section, we collect some basic definitions and results of fractional Sobolev spaces and then introduce the variational framework and truncated functional of problem (1). For convenience, $L^r(\Omega)$ with $r \geq 1$ denotes a Lebesgue space. $C(A,B)$ and $C^1(A,B)$ denote, respectively, the spaces of continuous and continuously Fréchet differentiable maps from A into B. The Fréchet derivative of I at the point u will be denoted by $I'(u)$. For $0 < s < 1$, the fractional Sobolev space $H^s(\mathbb{R}^n)$ is defined as

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n+2s}{2}}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\}.$$

Following Fiscella [35,36], in view of the boundary condition, we consider its subspace

$$W_0 = \{u \in H^s(\mathbb{R}^n) : u = 0 \,\text{a.e. in}\, \mathbb{R}^n \setminus \Omega\},$$

equipped with a scalar product and the norm defined as

$$\begin{aligned} (u,v) &= \int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} \mathrm{d}x\mathrm{d}y, \\ \|u\| &= \left(\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}. \end{aligned} \tag{6}$$

Then W_0 is a Hilbert space. By the results of [5], the embedding of $W_0 \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [1, 2_s^*]$ and compact whenever $r \in [1, 2_s^*)$. Let S_s be the fractional Sobolev constant as

$$S_s \doteq \inf_{u \in H^s(\mathbb{R}^n), u \neq 0} \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\mathrm{d}y}{\left(\displaystyle\int_{\mathbb{R}^n} |u(x)|^{2_s^*} \,\mathrm{d}x \right)^{\frac{2}{2_s^*}}} > 0. \tag{7}$$

The energy functional corresponding to problem (1) given by

$$I(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\int_\Omega |u|^2 \,\mathrm{d}x - \frac{\alpha}{q}\int_\Omega |u|^q \,\mathrm{d}x - \frac{1}{2_s^*}\int_\Omega |u|^{2_s^*} \,\mathrm{d}x, \tag{8}$$

and a function $u \in W_0$ is called a solution of problem (1) if for any $v \in W_0$,

$$(a - b\|u\|^2)\langle u, v\rangle = \lambda \int_\Omega uv \,\mathrm{d}x + \alpha \int_\Omega |u|^{q-2}uv \,\mathrm{d}x + \int_\Omega |u|^{2_s^*-2}uv \,\mathrm{d}x. \tag{9}$$

Obviously, $I \in C^1(W_0, \mathbb{R})$ and the weak solutions of problem (1) are exactly the critical points of I.

Define a smooth cut-off function ψ satisfying

$$\begin{cases} \psi(t) = 1, & t \in [0,1), \\ \psi(t) = 0, & t \in (2,\infty), \\ 0 \leq \psi(t) \leq 1, & t \in [1,2], \\ -2 \leq \psi'(t) \leq 0, & t \in [0,\infty), \end{cases}$$

then further define the following truncated functional I_T on W_0 relevant to I as

$$I_T(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\Psi_T(u)\|u\|^4 - \frac{\lambda}{2}\int_\Omega |u|^2 \,\mathrm{d}x - \frac{\alpha}{q}\int_\Omega |u|^q \,\mathrm{d}x - \frac{1}{2_s^*}\int_\Omega |u|^{2_s^*} \,\mathrm{d}x, \tag{10}$$

where $\Psi_T(u) = \psi(\frac{\|u\|^2}{T^2})$ for any $T > 0$. Obviously, $I_T \in C^1(W_0, \mathbb{R})$ is well defined and

$$\langle I_T'(u), v\rangle = \left[a - \frac{b}{2T^2}\psi'(\frac{\|u\|^2}{T^2})\|u\|^4 - b\Psi_T(u)\|u\|^2\right]\langle u, v\rangle,$$
$$- \lambda \int_\Omega uv\,\mathrm{d}x - \alpha\int_\Omega |u|^{q-2}uv\,\mathrm{d}x - \int_\Omega |u|^{2_s^*-2}uv\,\mathrm{d}x.$$

By the definition of T in Section 1, we also have

$$\begin{cases} 0 \le \Psi_T(u)\|u\|^2 \le 2T^2, \\ 0 \le \Psi_T(u)\|u\|^4 \le 4T^4, \\ -16T^6 \le \psi'(\frac{\|u\|^2}{T^2})\|u\|^6 \le 0 \end{cases} \tag{11}$$

with

$$\begin{cases} a - 2bT^2 \ge \frac{a}{2}, \\ 4bT^4 \le \frac{s}{2n}(\frac{aS_s}{2})^{\frac{n}{2s}}, \\ \frac{a(q-2)}{4q}T^2 > \frac{8b}{q}T^4. \end{cases} \tag{12}$$

Moreover, if u is a critical point of I_T with $\|u\| \le T$, then u is also a critical point of I.

3. A Compactness Result for I_T

In this section, we show that the functional I_T satisfies $(PS)_d$ conditions. We say that $\{u_k\}$ is a $(PS)_d$ sequence in W_0 for I_T if $I_T(u_k) \to d$ and $I_T'(u_k) \to 0$ in W_0' as $k \to \infty$, and I_T satisfies $(PS)_d$ conditions if any $(PS)_d$ sequence $\{u_k\}$ in W_0 has a strongly convergent subsequence.

Lemma 1. *If $\{u_k\} \subset W_0$ is a $(PS)_d$ sequence for I_T with $2 < q < 2_s^*$, then $\{u_k\}$ is bounded in W_0.*

Proof. Since $\{u_k\} \subset W_0$ is a $(PS)_d$ sequence for I_T, we have

$$I_T(u_k) \to d \text{ and } I_T'(u_k) \to 0 \text{ in } W_0' \text{ as } k \to \infty.$$

Since $2 < q < 2_s^*$, we can obtain from (7) and (11) that

$$\begin{aligned} d + o(\|u_k\|) =& I_T(u_k) - \frac{1}{2}\langle I_T'(u_k), u_k\rangle \\ =& \frac{b}{4}\Psi_T(u_k)\|u_k\|^4 + \frac{b}{4T^2}\psi'(\frac{\|u_k\|^2}{T^2})\|u_k\|^6 \\ &+ \alpha(\frac{1}{2} - \frac{1}{q})\int_\Omega |u_k|^q\,\mathrm{d}x + (\frac{1}{2} - \frac{1}{2_s^*})\int_\Omega |u_k|^{2_s^*}\,\mathrm{d}x \\ \ge& -4bT^4 + (\frac{1}{2} - \frac{1}{2_s^*})\int_\Omega |u_k|^{2_s^*}\,\mathrm{d}x, \end{aligned} \tag{13}$$

and

$$\begin{aligned} d + o(\|u_k\|) =& I_T(u_k) - \frac{1}{p}\langle I_T'(u_k), u_k\rangle \\ \ge& -4bT^4 + \alpha(\frac{1}{2} - \frac{1}{q})\int_\Omega |u_k|^q\,\mathrm{d}x. \end{aligned} \tag{14}$$

Thus, $\{u_k\}$ is bounded in $L^{2_s^*}(\Omega)$ and $L^q(\Omega)$. By Hölder's inequality, we can further obtain that $\{u_k\}$ is bounded in $L^2(\Omega)$ due to $2 < q$. Hence, there exists a constant $M > 0$ such that

$$\begin{aligned} d+o(1) \geq & I_T(u_k) \\ \geq & \frac{a}{2}\|u_k\|^2 - bT^4 - \frac{\lambda}{2}\int_\Omega |u_k|^2\,\mathrm{d}x - \frac{\alpha}{q}\int_\Omega |u_k|^q\,\mathrm{d}x - \frac{1}{2_s^*}\int_\Omega |u_k|^{2_s^*}\,\mathrm{d}x \\ \geq & \frac{a}{2}\|u_k\|^2 - M, \end{aligned}$$

which implies that $\{u_k\}$ is bounded in W_0. □

Lemma 2. *I_T satisfies the $(PS)_d$ conditions for all $d < d^* = \frac{s}{2n}(\frac{aS_s}{2})^{\frac{n}{2s}}$ and $2 < q < 2_s^*$.*

Proof. Let $\{u_k\}$ be a $(PS)_d$ sequence; by Lemma 1, $\{u_k\}$ is bounded in W_0 and then there exists a subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$) and a function $u \in W_0$ such that $u_k \rightharpoonup u$ in W_0 and $u_k \to u$ in $L^\beta(\Omega)$ with $\beta \in [1, 2_s^*)$. Thus, from the fractional concentration compactness lemma [37], there exist a countable sequence of points $\{x_j\}_{j\in\mathcal{J}} \subseteq \overline{\Omega}$ and the families of positive numbers $\{\mu_j\}_{j\in\mathcal{J}}$, $\{\nu_j\}_{j\in\mathcal{J}}$ such that

$$\int_{\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y \rightharpoonup \mu \geq \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j}, \tag{15}$$

$$|u_k|^{2_s^*} \rightharpoonup \nu = |u|^{2_s^*} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j}, \tag{16}$$

$$\mu_j \geq S_s \nu_j^{2/2_s^*},\ \forall j \in \mathcal{J}, \tag{17}$$

in the sense of measure, where δ_{x_j} is the Dirac measure concentrated at x_j. We claim that $\mathcal{J} = \emptyset$. If not, there exists a $j_0 \in \mathcal{J}$. For this x_{j_0} and any $\epsilon > 0$ small, define $\chi_{j_0}^\epsilon = \chi(\frac{x-x_{j_0}}{\epsilon})$, where $\chi \in C_0^\infty(\mathbb{R}^n)$, satisfying $0 \leq \chi(x) \leq 1$, $\chi(x) = 1$ in $B_1(0)$, $\chi(x) = 0$ in $\mathbb{R}^n \setminus B_2(0)$, and $|\nabla\chi(x)| \leq \frac{2}{\epsilon}$. Therefore, $\{\chi_{j_0}^\epsilon u_k\}$ is bounded in W_0. One can obtain from $\langle I_T'(u_k), \chi_{j_0}^\epsilon u_k\rangle \to 0$ that

$$\begin{aligned} &\left[a - \frac{b}{2T^2}\psi'(\frac{\|u_k\|^2}{T^2})\|u_k\|^4 - b\Psi_T(u_k)\|u_k|^2\right]\langle u_k, \chi_{j_0}^\epsilon u_k\rangle, \\ &= \lambda\int_\Omega u_k\chi_{j_0}^\epsilon\,\mathrm{d}x + \alpha\int_\Omega |u_k|^q\chi_{j_0}^\epsilon\,\mathrm{d}x + \int_\Omega |u_k|^{2_s^*}\chi_{j_0}^\epsilon\,\mathrm{d}x. \end{aligned} \tag{18}$$

On the one hand,

$$\begin{aligned} &\langle u_k, \chi_{j_0}^\epsilon u_k\rangle \\ =& \int_{\mathbb{R}^{2n}} \frac{(u_k(x)-u_k(y))(\chi_{j_0}^\epsilon(x)u_k(x) - \chi_{j_0}^\epsilon(y)u_k(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ =& \int_{\mathbb{R}^{2n}} \frac{u_k(x)(u_k(x)-u_k(y))(\chi_{j_0}^\epsilon(x) - \chi_{j_0}^\epsilon(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ &+ \int_{\mathbb{R}^{2n}} \frac{\chi_{j_0}^\epsilon(y)|u_k(x)-u_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{19}$$

We can obtain from the fact that $\{u_k\}$ is bounded in W_0 and Hölder's inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \frac{u_k(x)(u_k(x)-u_k(y))(\chi_{j_0}^{\epsilon}(x)-\chi_{j_0}^{\epsilon}(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\
\leq &\left(\int_{\mathbb{R}^{2n}} \frac{|u_k(x)(\chi_{j_0}^{\epsilon}(x)-\chi_{j_0}^{\epsilon}(y))|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{2n}} \frac{|u_k(x)-u_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \\
\leq &C\left(\int_{\mathbb{R}^{2n}} \frac{|u_k(x)(\chi_{j_0}^{\epsilon}(x)-\chi_{j_0}^{\epsilon}(y))|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \to 0, \text{ as } \epsilon\to 0,\ k\to\infty,
\end{aligned}$$

where the last result is from Lemma 3.4 in [38]. This, together with (15) and (19), bi-nonlocality leads to

$$\begin{aligned}
\lim_{\epsilon\to 0}\lim_{k\to\infty}\langle u_k, \chi_{j_0}^{\epsilon} u_k\rangle &= \lim_{\epsilon\to 0}\lim_{k\to\infty}\int_{\mathbb{R}^{2n}} \frac{\chi_{j_0}^{\epsilon}(y)|u_k(x)-u_k(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\
&\geq \lim_{\epsilon\to 0}\int_{\mathbb{R}^{n}} \frac{\chi_{j_0}^{\epsilon}(y)|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y + \mu_{j_0} = \mu_{j_0}.
\end{aligned} \tag{20}$$

Moreover, one can obtain from (17) that

$$\lim_{\epsilon\to 0}\lim_{k\to\infty}\int_{\Omega}|u_k|^{2_s^*}\chi_{j_0}^{\epsilon}\,\mathrm{d}x = \lim_{\epsilon\to 0}\int_{\Omega}|u|^{2_s^*}\chi_{j_0}^{\epsilon}\,\mathrm{d}x + \nu_{j_0} = \nu_{j_0}, \tag{21}$$

and

$$\lim_{\epsilon\to 0}\lim_{k\to\infty}\int_{\Omega}u_k\chi_{j_0}^{\epsilon}\,\mathrm{d}x = \lim_{\epsilon\to 0}\lim_{k\to\infty}\int_{\Omega}|u_k|^{q}\chi_{j_0}^{\epsilon}\,\mathrm{d}x = 0. \tag{22}$$

On the other hand, it follows from (11) and (12) that

$$a - \frac{b}{2T^2}\psi'(\frac{\|u_k\|^2}{T^2})\|u_k\|^4 - b\Psi_T(u_k)\|u_k\|^2 \geq a - 2bT^2 \geq \frac{a}{2}. \tag{23}$$

Combining (18) with (20)–(23) leads to

$$\nu_{j_0} \geq \frac{a}{2}\mu_{j_0}.$$

Using (17), we further obtain that

$$\nu_{j_0} \geq (\frac{aS_s}{2})^{\frac{n}{2s}}.$$

This, together with (12) and (13), leads to

$$\begin{aligned}
d + o(\|u_k\|) =&I_T(u_k) - \frac{1}{2}\langle I_T'(u_k), u_k\rangle \\
\geq &-4bT^4 + (\frac{1}{2}-\frac{1}{2_s^*})\int_{\Omega}|u_k|^{2_s^*}\,\mathrm{d}x \\
\geq &-4bT^4 + \frac{s}{n}\nu_{j_0} \\
\geq &-4bT^4 + \frac{s}{n}(\frac{aS_s}{2})^{\frac{n}{2s}} \\
\geq &\frac{s}{2n}(\frac{aS_s}{2})^{\frac{n}{2s}},
\end{aligned}$$

which contradicts $d < d_*$. So $\mathcal{J} = \emptyset$ and then $\lim_{k\to\infty}\int_{\Omega}|u_k|^{2_s^*}\,\mathrm{d}x = \int_{\Omega}|u|^{2_s^*}\,\mathrm{d}x$. By the Brézis–Lieb lemma [39], we have

$$\lim_{k\to\infty}\int_{\Omega}|u_k-u|^{2_s^*}\,\mathrm{d}x = 0.$$

By Hölder's inequality, we can further obtain

$$\left|\int_\Omega |u_k|^{2_s^*-1}(u_k-u)\,\mathrm{d}x\right| \le \left(\int_\Omega |u_k|^{2_s^*}\,\mathrm{d}x\right)^{\frac{2_s^*-1}{2_s^*}} \left(\int_\Omega |u_k-u|^{2_s^*}\,\mathrm{d}x\right)^{\frac{1}{2_s^*}} \to 0, \tag{24}$$

$$\left|\int_\Omega |u_k|^{q-1}(u_k-u)\,\mathrm{d}x\right| \le \left(\int_\Omega |u_k|^{q}\,\mathrm{d}x\right)^{\frac{q-1}{q}} \left(\int_\Omega |u_k-u|^{q}\,\mathrm{d}x\right)^{\frac{1}{q}} \to 0, \tag{25}$$

$$\left|\int_\Omega u_k(u_k-u)\,\mathrm{d}x\right| \le \left(\int_\Omega |u_k|^{2}\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega |u_k-u|^{2}\,\mathrm{d}x\right)^{\frac{1}{2}} \to 0, \tag{26}$$

as $k \to \infty$. Since $\langle I_T'(u_k), u_k - u\rangle \to 0$, one can deduce from (23) and (24)–(26)

$$\lim_{k\to\infty}\left[a - \frac{b}{2T^2}\psi'\left(\frac{\|u_k\|^2}{T^2}\right)\|u_k\|^4 - b\Psi_T(u_k)\|u_k\|^2\right]\langle u_k, u_k-u\rangle = 0,$$

which results in $\|u_k\| \to \|u\|$. This, together with the weak convergence of $\{u_k\}$ in W_0, leads to $u_k \to u$ in W_0. This ends the proof. □

4. Existence and Nonexistence Results

In this part, we firstly prove the nonexistence result for problem (1).

Proof of Theorem 1. Suppose $u \in W_0$ is a positive solution of problem (1) and e_1 is a positive eigenfunction associated with λ_1, then we have

$$\begin{aligned}\lambda\int_\Omega ue_1\,\mathrm{d}x =& (a-b\|u\|^2)\langle u, e_1\rangle - \alpha\int_\Omega u^{q-1}e_1\,\mathrm{d}x - \int_\Omega u^{2_s^*-1}e_1\,\mathrm{d}x \\ <& (a-b\|u\|^2)\langle u, e_1\rangle \\ =& \lambda_1(a-b\|u\|^2)\int_\Omega ue_1\,\mathrm{d}x \\ <& a\lambda_1\int_\Omega ue_1\,\mathrm{d}x.\end{aligned}$$

So problem (1) has no positive solution when $\lambda \ge a\lambda_1$. This ends the proof. □

For the existence result of problem (1), we need the following general mountain pass theorem [40], which can help to find a $(PS)_d$ sequence for I_T.

Theorem 5. *Let H be a Banach space and $I \in C^1(H, \mathbb{R})$ with $I(0) = 0$. Suppose the following:*

(1) There exist $\tau, \delta > 0$ such that $I(u) \ge \delta$ for all $u \in H$ with $\|u\| = \tau$.
(2) There exists $v_0 \in H$ such that $\|v_0\| > \tau$ and $I(v_0) < 0$.

Define

$$\Gamma = \{\gamma \in C([0,1], H) : \gamma(0) = 0,\ \gamma(1) = v_0\}$$

and

$$d = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

then there exists a $(PS)_d$ sequence $\{u_n\} \subset H$ and $d \ge \delta$.

Now, we begin to prove that I_T satisfies the assumptions of the mountain pass geometry.

Lemma 3. *Assume $0 < \lambda < a\lambda_1$ and $2 < q < 2_s^*$, then there exist positive constants τ, δ such that the following are obtained:*

1. *$I_T(u) \ge \delta$ for all $\|u\| = \tau$.*
2. *There exists $u_* \in W_0$ such that $\|u_*\| > \tau$ and $I_T(u_*) < 0$.*

Proof. For any $u \in W_0$, we can obtain from (7) and (10) that

$$I_T(u) \geq \frac{a\lambda_1 - \lambda}{2\lambda_1}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{\alpha C}{q}\|u\|^q - \frac{1}{2_s^*}S_s^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}.$$

Since $2 < q < 2_s^*$, let $u \in W_0$ with $\|u\| = \tau$ where τ is sufficiently small, satisfying

$$0 < \frac{a\lambda_1 - \lambda}{2\lambda_1}\tau^2 - \frac{b}{4}\tau^4 - \frac{\alpha C}{q}\tau^q - \frac{1}{2_s^*}S_s^{-\frac{2_s^*}{2}}\tau^{2_s^*} \doteq \delta,$$

then we obtain

$$\inf_{u \in W_0, \|u\| = \tau} I_T(u) \geq \delta > 0.$$

On the other hand, suppose e_1 is a positive eigenfunction associated with λ_1, then for $t > 0$, we obtain

$$I_T(te_1) \leq \frac{at^2}{2}\|e_1\|^2 - \frac{\lambda t^2}{2}\int_\Omega |e_1|^2\,\mathrm{d}x - \frac{\alpha t^q}{q}\int_\Omega |e_1|^q\,\mathrm{d}x - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |e_1|^{2_s^*}\,\mathrm{d}x \to -\infty,$$

as $t \to +\infty$. Hence, set $u_* = t_0 e_1$ with $t_0 > 0$ sufficiently large such that $\|u_*\| > \tau$ and $I_T(u_*) < 0$. The proof is completed. □

Lemma 4. *For $0 < \lambda < a\lambda_1$, one has such that*

$$\sup_{t \geq 0} I_T(te_1) \leq \frac{s}{n}(a\lambda_1 - \lambda)^{\frac{n}{2s}}|\Omega| \doteq \rho,$$

where e_1 is a positive eigenfunction associated with λ_1.

Proof. For any $u \in W_0$ and $0 < \lambda < a\lambda_1$, by Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
\frac{a\lambda_1 - \lambda}{2}\int_\Omega |u|^2\,\mathrm{d}x \leq& \frac{a\lambda_1 - \lambda}{2}\left(\int_\Omega |u|^{2_s^*}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}}|\Omega|^{\frac{2_s^*-2}{2_s^*}} \\
=&\left(\frac{1}{2}\int_\Omega |u|^{2_s^*}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}}\left[(a\lambda_1 - \lambda)2^{\frac{2-2_s^*}{2_s^*}}|\Omega|^{\frac{2_s^*-2}{2_s^*}}\right] \\
\leq&\frac{1}{2_s^*}\int_\Omega |u|^{2_s^*}\,\mathrm{d}x + \frac{s}{n}(a\lambda_1 - \lambda)^{\frac{n}{2s}}|\Omega| \\
=&\frac{1}{2_s^*}\int_\Omega |u|^{2_s^*}\,\mathrm{d}x + \rho.
\end{aligned} \tag{27}$$

For every $t \geq 0$, choosing $u = te_1$ in (27) and using the fact $\|e_1\|^2 = \lambda_1 \int_\Omega |e_1|^2$, we have

$$\begin{aligned}
I_T(te_1) &\leq \frac{a}{2}\|te_1\|^2 - \frac{\lambda}{2}\int_\Omega |te_1|^2\,\mathrm{d}x - \frac{1}{2_s^*}\int_\Omega |te_1|^{2_s^*}\,\mathrm{d}x \\
&= \frac{a\lambda_1 - \lambda}{2}\int_\Omega |te_1|^2\,\mathrm{d}x - \frac{1}{2_s^*}\int_\Omega |te_1|^{2_s^*}\,\mathrm{d}x \leq \rho.
\end{aligned}$$

This ends the proof. □

Next, we want to prove Theorem 2 by showing the existence of positive critical points for I_T and further prove that these critical points are also positive solutions of problem (1).

Proof of Theorem 2. Define

$$\Gamma \doteq \{\phi \in C([0,1], W_0) : \phi(0) = 0,\ \phi(1) = u_*\}$$
$$d_\lambda \doteq \inf_{\phi\in\Gamma} \max_{t\in[0,1]} I_T(\phi(t)),$$

where u_* is defined in Lemma 3. By Lemma 3 and Theorem 5, there exists a sequence $\{u_k\} \in W_0$ such that

$$I_T(u_k) \to d_\lambda \geq \delta > 0 \text{ and } I'_T(u_k) \to 0.$$

For $a\lambda_1 - \Lambda_0 < \lambda < a\lambda_1$ with $\Lambda_0 = \frac{aS_s}{2}\left(\frac{1}{2|\Omega|}\right)^{\frac{2s}{n}}$, it further follows from Lemma 4 that

$$d_\lambda \leq \rho < d_*.$$

Therefore, one can obtain from Lemma 2 that there exists a subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$) and a function $u_\lambda \in W_0$ such that $u_k \to u_\lambda$ in W_0. Subsequently, for $a\lambda_1 - \Lambda_0 < \lambda < a\lambda_1$, we have

$$\rho \geq I_T(u_\lambda) = \lim_{k\to\infty} I_T(u_k) = d_\lambda \geq \delta > 0 \text{ and } I'_T(u_\lambda) = 0, \tag{28}$$

which means that u_λ is a nonzero and non-negative critical point of I_T. Now, we come to show that $\|u_\lambda\| \leq T$.

It follows from (11) and (28) and $2 < q < 2_s^*$ that

$$\begin{aligned}
\rho >& I_T(u_\lambda) - \frac{1}{2}\langle I'_T(u_\lambda), u_\lambda\rangle \\
=& \frac{b}{4}\Psi_T(u_\lambda)\|u_\lambda\|^4 + \frac{b}{4T^2}\psi'(\frac{\|u_\lambda\|^2}{T^2})\|u_\lambda\|^6 \\
&+ \alpha(\frac{1}{2} - \frac{1}{q})\int_\Omega |u_\lambda|^q\,\mathrm{d}x + (\frac{1}{2} - \frac{1}{2_s^*})\int_\Omega |u_\lambda|^{2_s^*}\,\mathrm{d}x \\
\geq& -4bT^4 + \alpha(\frac{1}{2} - \frac{1}{q})\int_\Omega |u_\lambda|^q\,\mathrm{d}x,
\end{aligned} \tag{29}$$

which results in $\int_\Omega |u_\lambda|^q\,\mathrm{d}x \to 0$ as $\alpha \to \infty$. For $\alpha > 0$ big enough, we can further obtain from Hölder's inequality and (12) that

$$\frac{a\lambda_1(q-2)}{2q}\int_\Omega |u_\lambda|^2\,\mathrm{d}x \leq \frac{a(q-2)}{4q}T^2 - \frac{8b}{q}T^4. \tag{30}$$

For $a\lambda_1 - \Lambda < \lambda < a\lambda_1$, Set $\Lambda = \min\left\{\Lambda_0, \left[\frac{aT^2n(q-2)}{4qs|\Omega|}\right]^{\frac{2s}{n}}\right\}$, then (28) shows that $I_T(u_\lambda) \leq \rho \leq \frac{a(q-2)}{4q}T^2$. Moreover, one can obtain from from (11) and (30) and $2 < q \leq \min\{4, 2_s^*\}$ that

$$\begin{aligned}
\frac{a(q-2)}{4q}T^2 \geq& I_T(u_\lambda) - \frac{1}{q}\langle I'_T(u_\lambda), u_\lambda\rangle \\
=& a(\frac{1}{2} - \frac{1}{q})\|u_\lambda\|^2 + \frac{b}{2qT^2}\psi'(\frac{\|u_\lambda\|^2}{T^2})\|u_\lambda\|^6 + b(\frac{1}{q} - \frac{1}{4})\Psi_T(u_\lambda)\|u_\lambda\|^4 \\
&- \lambda(\frac{1}{2} - \frac{1}{q})\int_\Omega |u_\lambda|^2\,\mathrm{d}x + (\frac{1}{q} - \frac{1}{2_s^*})\int_\Omega |u_\lambda|^{2_s^*}\,\mathrm{d}x \\
\geq& \frac{a(q-2)}{2q}\|u_\lambda\|^2 - \frac{8b}{q}T^4 - \frac{a\lambda_1(q-2)}{2q}\int_\Omega |u_\lambda|^2\,\mathrm{d}x \\
\geq& \frac{a(q-2)}{2q}\|u_\lambda\|^2 - \frac{a(q-2)}{4q}T^2,
\end{aligned} \tag{31}$$

which also results in $\|u_\lambda\| < T$. Then u_λ is also a nontrivial and non-negative solution of problem (1). Similar to the proof of Theorem 1.1 in [41], one can further obtain that u_λ is positive. $\square$

Proof of Theorem 3. For every sequence $\{\lambda_n\}$ with $\lambda_n \nearrow a\lambda_1$, let u_{λ_n} be the positive solution to problem (1) provided by Theorem 2. Since $2 < q \leq \min\{4, 2_s^*\}$, we have

$$\begin{aligned} d_{\lambda_n} =& I(u_{\lambda_n}) - \frac{1}{2}\langle I'(u_{\lambda_n}), u_{\lambda_n}\rangle \\ =& \frac{b}{4}\|u_{\lambda_n}\|^4 + \alpha(\frac{1}{2} - \frac{1}{q}) \int_\Omega |u_{\lambda_n}|^q \, dx + (\frac{1}{2} - \frac{1}{2_s^*}) \int_\Omega |u_{\lambda_n}|^{2_s^*} \, dx \\ \geq& \frac{b}{4}\|u_{\lambda_n}\|^4. \end{aligned}$$

This, together with $0 < d_{\lambda_n} \leq \frac{s}{n}(a\lambda_1 - \lambda_n)^{\frac{n}{2s}}|\Omega|$, leads to $\lim_{n\to\infty}\|u_{\lambda_n}\| = 0$. $\square$

In order to obtain multiple solutions for problem (1), we use the following version of the symmetric mountain pass theorem of [40,42].

Theorem 6. *Let $H = V \oplus E$ be a real infinite dimensional Banach space with $dimV < \infty$, and suppose that $I \in C^1(H, \mathbb{R})$ is a functional satisfying the following conditions:*

(1) $I(u) = I(-u)$, $I(0) = 0$;
(2) There exist positive constants τ, δ such that $I(u) \geq \delta$ for all $u \in E$ with $\|u\| = \tau$;
(3) There exist a subspace $\hat{V} \subset H$ with $dimV < dim\hat{V} < \infty$ such that $\max_{u\in\hat{V}} I(u) \leq \rho$ for some $\rho > 0$;
(4) I satisfies $(PS)_d$ for any $d \in (0, \rho)$.

Then I possesses at least $dim\hat{V} - dimV$ pairs of nontrivial critical points.

For any $m \in \mathbb{N}$, set $W_0 = V_m \oplus V_m^\perp$, where $V_m = span\{e_1, e_2, \cdots, e_m\}$ denotes the linear subspace generated by the first m eigenfunctions of $(-\Delta)^s$. For any $j \in \mathbb{N}$ with $j > m$, we further define the j-dimensional subspace $\hat{V}_j = span\{e_1, e_2, \cdots, e_j\}$. Since $\hat{V}_j$ is a finite dimensional space, there exists a constant $\beta > 0$ such that $\int_\Omega |u|^q \, dx \geq \beta\|u\|^q$ for any $u \in \hat{V}_j$. Hence, we have

$$\begin{aligned} I_T(u) \leq& \frac{a}{2}\|u\|^2 - \frac{\alpha}{q}\int_\Omega |u|^q \, dx \\ \leq& \frac{a}{2}\|u\|^2 - \frac{\alpha\beta}{q}\|u\|^q \\ \leq& a(\frac{1}{2} - \frac{1}{q})\left(\frac{a}{\alpha\beta}\right)^{\frac{2}{q-2}} \doteq d_j, \ u \in \hat{V}_j. \end{aligned} \tag{32}$$

Right now, we are in a position to prove Theorem 4 by applying Theorem 6 to I_T.

Proof of Theorem 4. It is obvious that $I_T(u) = I_T(-u)$, $I_T(0) = 0$. Set $V = V_m$ and $E = V_m^\perp$, then one can obtain from Lemma 3 that (2) in Theorem 6 holds. For any $k \in \mathbb{N}$, let $j = k + m$, that is, $\hat{V}_{k+m} = span\{e_1, e_2, \cdots, e_{k+m}\}$, then $dim\hat{V} = k + m$ and $\max_{u\in\hat{V}} I_T(u) \leq d_*$ follows from (32). For big enough α such that $d_{k+m} < d^*$, then I_T satisfies (4) in Theorem 6 from Lemma 2. Therefore, Theorem 4 assures that I_T has at least k pairs of nontrivial critical points for $\alpha > 0$ sufficiently big. Arguing exactly as in the proof of Theorem 2, we can also obtain that these critical points of I_T are also positive solutions of problem (1). So, Theorem 4 is proved. $\square$

5. Conclusions

In this article, we explored the existence, multiplicity, and asymptotic behavior of solutions for a fractional Kirchhoff equation. This problem includes a fractional Laplacian operator, a fractional critical Sobolev exponent, and a negative nonlocal term. Based on the cut-off technique, we utilized the concentration compactness principle to overcome the lack of compactness due to the critical nonlinearities. The mountain pass theorem and symmetric mountain pass theorem were used to prove the existence and multiplicity of solutions by showing that the positive critical points of the truncated functional are really positive solutions of problem (1). Moreover, the nonexistence and asymptotic behavior of positive solutions were also investigated. Our results supplement and extend the results obtained in [3,25,34]. In further studies, we shall investigate a system and variable exponents cases of this kind of problem even with logarithmic perturbation. These will make this problem more difficult and interesting.

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