

## Article

# Cubic Interval-Valued Intuitionistic Fuzzy Sets and Their Application in *BCK/BCI*-Algebras

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**Abstract:** As a new extension of a cubic set, the notion of a cubic interval-valued intuitionistic fuzzy set is introduced, and its application in *BCK/BCI*-algebra is considered. The notions of  $\alpha$ -internal,  $\beta$ -internal,  $\alpha$ -external and  $\beta$ -external cubic IVIF set are introduced, and the P-union, P-intersection, R-union and R-intersection of  $\alpha$ -internal and  $\alpha$ -external cubic IVIF sets are discussed. The concepts of cubic IVIF subalgebra and ideal in *BCK/BCI*-algebra are introduced, and related properties are investigated. Relations between cubic IVIF subalgebra and cubic IVIF ideal are considered, and characterizations of cubic IVIF subalgebra and cubic IVIF ideal are discussed.

**Keywords:**  $\alpha$ -internal;  $\beta$ -internal;  $\alpha$ -external;  $\beta$ -external cubic interval-valued intuitionistic fuzzy set; cubic interval-valued intuitionistic fuzzy subalgebra; cubic interval-valued intuitionistic fuzzy ideal

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## 1. Introduction

In 2012, Jun et al. [1] introduced cubic sets, and then this notion is applied to several algebraic structures (see [2–11]).

The aim of this paper is to introduce the notion of a cubic IVIF set which is an extended concept of a cubic set. We introduce cubic interval-valued intuitionistic fuzzy set which is an extension of cubic set, and apply it to *BCK/BCI*-algebra. We investigate P-union, P-intersection, R-union and R-intersection of  $\alpha$ -internal and  $\alpha$ -external cubic IVIF sets.

We define cubic IVIF subalgebra and ideal in *BCK/BCI*-algebra, and investigate related properties. We consider relations between cubic IVIF subalgebra and cubic IVIF ideal. We discuss characterizations of cubic IVIF subalgebra and cubic IVIF ideal.

## 2. Preliminaries

A fuzzy set in a set  $X$  is defined to be a function  $\mu : X \rightarrow [0, 1]$ . For  $I = [0, 1]$ , denote by  $I^X$  the collection of all fuzzy sets in a set  $X$ . Define a relation  $\leq$  on  $I^X$  as follows:

$$(\forall \mu, \lambda \in I^X) (\mu \leq \lambda \iff (\forall x \in X) (\mu(x) \leq \lambda(x))). \quad (1)$$

The complement of  $\mu \in I^X$ , denoted by  $\mu^c$ , is defined by

$$(\forall x \in X) (\mu^c(x) = 1 - \mu(x)). \quad (2)$$

For a family  $\{\mu_i \mid i \in \Lambda\}$  of fuzzy sets in  $X$ , we define the join ( $\vee$ ) and meet ( $\wedge$ ) operations as follows:

$$\bigvee_{i \in \Lambda} \mu_i : X \rightarrow [0, 1], x \mapsto \sup\{\mu_i(x) \mid i \in \Lambda\},$$

$$\bigwedge_{i \in \Lambda} \mu_i : X \rightarrow [0, 1] x \mapsto \inf\{\mu_i(x) \mid i \in \Lambda\}.$$

An *interval-valued intuitionistic fuzzy set* (briefly, IVIF set)  $A$  over a set  $X$  (see [12]) is an object having the form

$$\tilde{A} = \{\langle x, \alpha[\tilde{A}](x), \beta[\tilde{A}](x) \rangle \mid x \in X\}$$

where  $\alpha[\tilde{A}](x) \subseteq [0, 1]$  and  $\beta[\tilde{A}](x) \subseteq [0, 1]$  are intervals and for every  $x \in X$ ,

$$\sup \alpha[\tilde{A}](x) + \sup \beta[\tilde{A}](x) \leq 1$$

Especially, if

$$\sup \alpha[\tilde{A}](x) = \inf \alpha[\tilde{A}](x) \text{ and } \sup \beta[\tilde{A}](x) = \inf \beta[\tilde{A}](x), \quad (3)$$

then the IVIF set  $\tilde{A}$  is reduced to an intuitionistic fuzzy set (see [13]).

Given two closed subintervals  $D_1 = [D_1^-, D_1^+]$  and  $D_2 = [D_2^-, D_2^+]$  of  $[0, 1]$ , we define the order “ $\ll$ ” as follows:

$$D_1 \ll D_2 \Leftrightarrow D_1^- \leq D_2^- \text{ and } D_1^+ \leq D_2^+.$$

We also define the refined minimum (briefly, rmin) and refined maximum (briefly, rmax) as follows:

$$\text{rmin}\{D_1, D_2\} = [\min\{D_1^-, D_2^-\}, \min\{D_1^+, D_2^+\}],$$

$$\text{rmax}\{D_1, D_2\} = [\max\{D_1^-, D_2^-\}, \max\{D_1^+, D_2^+\}].$$

For a family  $\{D_i = [D_i^-, D_i^+] \mid i \in \Lambda\}$  of closed subintervals of  $[0, 1]$ , we define rinf (refined infimum) and rsup (refined supremum) as follows:

$$\text{rinf} D_i = \left[ \inf_{i \in \Lambda} D_i^-, \inf_{i \in \Lambda} D_i^+ \right] \text{ and } \text{rsup} D_i = \left[ \sup_{i \in \Lambda} D_i^-, \sup_{i \in \Lambda} D_i^+ \right].$$

In this paper we use the interval-valued intuitionistic fuzzy set

$$\tilde{A} = \{\langle x, \alpha[\tilde{A}](x), \beta[\tilde{A}](x) \rangle \mid x \in X\}$$

over  $X$  in which  $\alpha[\tilde{A}](x)$  and  $\beta[\tilde{A}](x)$  are closed subintervals of  $[0, 1]$  for all  $x \in X$ , that is,  $\alpha[\tilde{A}] : X \rightarrow D[0, 1]$  and  $\beta[\tilde{A}] : X \rightarrow D[0, 1]$  are interval-valued fuzzy (briefly, IVF) sets in  $X$  where  $D[0, 1]$  is the set of all closed subintervals of  $[0, 1]$ . Also, we use the notations  $\alpha[\tilde{A}]^-(x)$  and  $\alpha[\tilde{A}]^+(x)$  to mean the left end point and the right end point of the interval  $\alpha[\tilde{A}](x)$ , respectively, and so we have  $\alpha[\tilde{A}](x) = [\alpha[\tilde{A}]^-(x), \alpha[\tilde{A}]^+(x)]$ . The interval-valued intuitionistic fuzzy set

$$\tilde{A} = \{\langle x, \alpha[\tilde{A}](x), \beta[\tilde{A}](x) \rangle \mid x \in X\}$$

over  $X$  is simply denoted by  $\tilde{A}(x) = (\alpha[\tilde{A}](x), \beta[\tilde{A}](x))$  for  $x \in X$  or  $\tilde{A} = (\alpha[\tilde{A}], \beta[\tilde{A}])$ .

An algebra  $(X; *, 0)$  of type  $(2, 0)$  is called a *BCI-algebra* if it satisfies the following axioms:

- (I)  $(\forall x, y, z \in X) ((x * y) * (x * z)) * (z * y) = 0$ ,
- (II)  $(\forall x, y \in X) ((x * (x * y)) * y = 0)$ ,
- (III)  $(\forall x \in X) (x * x = 0)$ ,
- (IV)  $(\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y)$ .

If a BCI-algebra  $X$  satisfies the following identity:

- (V)  $(\forall x \in X) (0 * x = 0)$ ,

then  $X$  is called a BCK-algebra. We can define a partial ordering  $\leq$  on  $X$  by  $x \leq y$  if and only if  $x * y = 0$ . Any BCK/BCI-algebra  $X$  satisfies the following conditions:

$$(\forall x \in X) (x * 0 = x), \quad (4)$$

$$(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x), \quad (5)$$

$$(\forall x, y, z \in X) ((x * y) * z = (x * z) * y), \quad (6)$$

$$(\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y). \quad (7)$$

A nonempty subset  $S$  of a BCK/BCI-algebra  $X$  is called a *subalgebra* of  $X$  if  $x * y \in S$  for all  $x, y \in S$ . A subset  $I$  of a BCK/BCI-algebra  $X$  is called an *ideal* of  $X$  if  $0 \in I$  and the following condition is valid.

$$(\forall x, y \in X) (x * y \in I, y \in I \Rightarrow x \in I). \quad (8)$$

We refer the reader to the books [14,15] and the paper [16] for further information regarding BCK/BCI-algebras.

### 3. Cubic Interval-Valued Intuitionistic Fuzzy Sets

**Definition 1.** Let  $X$  be a nonempty set. By a cubic interval-valued intuitionistic fuzzy set (briefly, cubic IVIF set) in  $X$  we mean a structure

$$\mathcal{A} = \{ \langle x, \mu(x), \tilde{A}(x) \rangle \mid x \in X \}$$

in which  $\mu$  is a fuzzy set in  $X$  and  $\tilde{A}$  is an interval-valued intuitionistic fuzzy set in  $X$ .

A cubic IVIF set  $\mathcal{A} = \{ \langle x, \mu(x), \tilde{A}(x) \rangle \mid x \in X \}$  is simply denoted by  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$ .

Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  be a cubic IVIF set in a nonempty set  $X$ . Given  $\varepsilon \in [0, 1]$  and  $([s_\alpha, t_\alpha], [s_\beta, t_\beta]) \in D[0, 1] \times D[0, 1]$ , we consider the sets

$$\begin{aligned} \mu[\varepsilon] &:= \{x \in X \mid \mu(x) \geq \varepsilon\}, \\ \alpha[\tilde{A}][s_\alpha, t_\alpha] &:= \{x \in X \mid \alpha[\tilde{A}](x) \ll [s_\alpha, t_\alpha]\}, \\ \beta[\tilde{A}][s_\beta, t_\beta] &:= \{x \in X \mid \beta[\tilde{A}](x) \gg [s_\beta, t_\beta]\}. \end{aligned}$$

**Definition 2.** Let  $X$  be a nonempty set. A cubic IVIF set  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is said to be

- $\alpha$ -internal if  $\mu(x) \in \alpha[\tilde{A}](x)$  for all  $x \in X$ ,
- $\beta$ -internal if  $\mu(x) \in \beta[\tilde{A}](x)$  for all  $x \in X$ ,
- internal if it is both  $\alpha$ -internal and  $\beta$ -internal.
- $\alpha$ -external if  $\mu(x) \notin (\alpha[\tilde{A}]^-(x), \alpha[\tilde{A}]^+(x))$  for all  $x \in X$ .
- $\beta$ -external if  $\mu(x) \notin (\beta[\tilde{A}]^-(x), \beta[\tilde{A}]^+(x))$  for all  $x \in X$ .
- external if it is both  $\alpha$ -external and  $\beta$ -external.

It is clear that if  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF set in  $X$  with  $\alpha[\tilde{A}](x) \cap \beta[\tilde{A}](x) = \emptyset$  for some  $x \in X$ , then  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  cannot be internal.

**Example 1.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  be a cubic IVIF set in  $I$  in which  $\tilde{A}(x) = ([0.3, 0.5], [0.1, 0.4])$  and  $\mu(x) = k$  for all  $x \in X$ .

- (1) If  $k \in (0.4, 0.5]$ , then  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is  $\alpha$ -internal which is not  $\beta$ -internal.
- (2) If  $k \in [0.1, 0.3)$ , then  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is  $\beta$ -internal which is not  $\alpha$ -internal.
- (3) If  $k \in [0.3, 0.4]$ , then  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is internal.
- (4) If either  $k \geq 0.5$  or  $k \leq 0.1$ , then  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is external.
- (5) If either  $k \geq 0.5$  or  $k \leq 0.3$ , then  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is  $\alpha$ -external but may not be  $\beta$ -external.
- (6) If either  $k \geq 0.4$  or  $k \leq 0.1$ , then  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is  $\beta$ -external but may not be  $\alpha$ -external.

**Proposition 1.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  be a cubic IVIF set in a nonempty set  $X$  in which  $\beta[\tilde{A}]^-(x) \leq \alpha[\tilde{A}]^-(x) < \alpha[\tilde{A}]^+(x) \leq \beta[\tilde{A}]^+(x)$  for all  $x \in X$ . If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is  $\alpha$ -internal, then it is  $\beta$ -internal and so internal. Also, if  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is  $\beta$ -external, then it is  $\alpha$ -external and so external.

**Proof.** Straightforward.  $\square$

**Proposition 2.** If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF set in  $X$  which is not external, then there exist  $x, y \in X$  such that  $\mu(x) \in \alpha[\tilde{A}](x)$  or  $\mu(y) \in \beta[\tilde{A}](y)$ .

**Proof.** Straightforward.  $\square$

**Theorem 1.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  be a cubic IVIF set in  $X$  in which the right end point of  $\alpha[\tilde{A}](x)$  (or  $\beta[\tilde{A}](x)$ ) is equal to the left end point of  $\beta[\tilde{A}](x)$  (or  $\alpha[\tilde{A}](x)$ ) for all  $x \in X$ . If we define  $\mu$  by

$$(\forall x \in X) (\mu(x) = \alpha[\tilde{A}]^+(x) = \beta[\tilde{A}]^-(x) \text{ or } \mu(x) = \alpha[\tilde{A}]^-(x) = \beta[\tilde{A}]^+(x)),$$

then  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is external.

**Proof.** Straightforward.  $\square$

For any cubic IVIF set  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  in  $X$ , let

$$\begin{aligned} U_\alpha(\tilde{A}) &:= \{\alpha[\tilde{A}]^+(x) \mid x \in X\}, L_\alpha(\tilde{A}) := \{\alpha[\tilde{A}]^-(x) \mid x \in X\}, \\ U_\beta(\tilde{A}) &:= \{\beta[\tilde{A}]^+(x) \mid x \in X\}, L_\beta(\tilde{A}) := \{\beta[\tilde{A}]^-(x) \mid x \in X\}. \end{aligned}$$

**Theorem 2.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  be a cubic IVIF set in  $X$ . If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is both  $\alpha$ -internal and  $\alpha$ -external (resp., both  $\beta$ -internal and  $\beta$ -external), then  $\mu(x) \in U_\alpha(\tilde{A}) \cup L_\alpha(\tilde{A})$  (resp.,  $\mu(x) \in U_\beta(\tilde{A}) \cup L_\beta(\tilde{A})$ ) for all  $x \in X$ .

**Proof.** Assume that  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is both  $\alpha$ -internal and  $\alpha$ -external. Then  $\mu(x) \in \alpha[\tilde{A}](x)$  and  $\mu(x) \notin (\alpha[\tilde{A}]^-(x), \alpha[\tilde{A}]^+(x))$  for all  $x \in X$ . It follows that  $\mu(x) = \alpha[\tilde{A}]^-(x)$  or  $\mu(x) = \alpha[\tilde{A}]^+(x)$ , that is,  $\mu(x) \in L_\alpha(\tilde{A})$  or  $\mu(x) \in U_\alpha(\tilde{A})$ . Hence  $\mu(x) \in U_\alpha(\tilde{A}) \cup L_\alpha(\tilde{A})$  for all  $x \in X$ . Similarly, if  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is both  $\beta$ -internal and  $\beta$ -external, then  $\mu(x) \in U_\beta(\tilde{A}) \cup L_\beta(\tilde{A})$  for all  $x \in X$ .  $\square$

Given an IVIF set  $\tilde{A} = (\alpha[\tilde{A}], \beta[\tilde{A}])$  in  $X$ , the complement of  $\tilde{A}$  is denoted by  $\tilde{A}^c$  and is defined as follows:

$$\tilde{A}^c : X \rightarrow D[0, 1] \times D[0, 1], x \mapsto \langle \alpha[\tilde{A}^c], \beta[\tilde{A}^c] \rangle \quad (9)$$

where

$$\alpha[\tilde{A}^c] : X \rightarrow D[0, 1], x \mapsto [\alpha[\tilde{A}^c]^-(x), \alpha[\tilde{A}^c]^+(x)] \quad (10)$$

and

$$\beta[\tilde{A}^c] : X \rightarrow D[0, 1], x \mapsto [\beta[\tilde{A}^c]^-(x), \beta[\tilde{A}^c]^+(x)] \quad (11)$$

with  $\alpha[\tilde{A}^c]^-(x) = 1 - \alpha[\tilde{A}]^+(x)$ ,  $\alpha[\tilde{A}^c]^+(x) = 1 - \alpha[\tilde{A}]^-(x)$ ,  $\beta[\tilde{A}^c]^-(x) = 1 - \beta[\tilde{A}]^+(x)$ , and  $\beta[\tilde{A}^c]^+(x) = 1 - \beta[\tilde{A}]^-(x)$ .

**Definition 3.** For any cubic IVIF sets  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  in  $X$ , we define

$$(\text{Equality}) \mathcal{A} = \mathcal{B} \Leftrightarrow \tilde{A} = \tilde{B} \text{ and } \mu = \lambda, \quad (12)$$

$$(\text{P-order}) \mathcal{A} \subseteq_P \mathcal{B} \Leftrightarrow \tilde{A} \subseteq \tilde{B} \text{ and } \mu \leq \lambda, \quad (13)$$

$$(\text{R-order}) \mathcal{A} \subseteq_R \mathcal{B} \Leftrightarrow \tilde{A} \subseteq \tilde{B} \text{ and } \mu \geq \lambda, \quad (14)$$

where  $\tilde{A} \subseteq \tilde{B}$  means that  $\alpha[\tilde{A}] \ll \alpha[\tilde{B}]$  and  $\beta[\tilde{A}] \gg \beta[\tilde{B}]$ , that is,

$$(\forall x \in X) \left( \begin{array}{l} \alpha[\tilde{A}]^-(x) \leq \alpha[\tilde{B}]^-(x), \alpha[\tilde{A}]^+(x) \leq \alpha[\tilde{B}]^+(x) \\ \beta[\tilde{A}]^-(x) \geq \beta[\tilde{B}]^-(x), \beta[\tilde{A}]^+(x) \geq \beta[\tilde{B}]^+(x) \end{array} \right).$$

It is clear that the set of all cubic IVIF sets in  $X$  forms a poset under the P-order  $\subseteq_P$  and the R-order  $\subseteq_R$ .

For any  $\tilde{A}_i = \{ \langle x, \alpha[\tilde{A}_i](x), \beta[\tilde{A}_i](x) \rangle \}$  where  $i \in \Lambda$ , we define

$$\bigcup_{i \in \Lambda} \tilde{A}_i = \left\{ \langle x, \alpha[\bigcup_{i \in \Lambda} \tilde{A}_i](x), \beta[\bigcup_{i \in \Lambda} \tilde{A}_i](x) \rangle \mid x \in X \right\} \quad (15)$$

and

$$\bigcap_{i \in \Lambda} \tilde{A}_i = \left\{ \langle x, \alpha[\bigcap_{i \in \Lambda} \tilde{A}_i](x), \beta[\bigcap_{i \in \Lambda} \tilde{A}_i](x) \rangle \mid x \in X \right\} \quad (16)$$

where  $\alpha[\bigcup_{i \in \Lambda} \tilde{A}_i](x) = \text{rsup}_{i \in \Lambda} \alpha[\tilde{A}_i](x)$ ,  $\beta[\bigcup_{i \in \Lambda} \tilde{A}_i](x) = \text{rinf}_{i \in \Lambda} \beta[\tilde{A}_i](x)$ ,  $\alpha[\bigcap_{i \in \Lambda} \tilde{A}_i](x) = \text{rinf}_{i \in \Lambda} \alpha[\tilde{A}_i](x)$ , and  $\beta[\bigcap_{i \in \Lambda} \tilde{A}_i](x) = \text{rsup}_{i \in \Lambda} \beta[\tilde{A}_i](x)$ .

**Definition 4.** Given a family  $\{\mathcal{A}_i = \langle \mu_i, \tilde{A}_i \rangle \mid i \in \Lambda\}$  of cubic IVIF sets in  $X$ , we define

$$(1) \quad \mathcal{A}_P := \left\langle \bigvee_{i \in \Lambda} \mu_i, \bigcup_{i \in \Lambda} \tilde{A}_i \right\rangle \quad (\text{P-union})$$

$$(2) \quad \mathcal{A}_R := \left\langle \bigwedge_{i \in \Lambda} \mu_i, \bigcup_{i \in \Lambda} \tilde{A}_i \right\rangle \quad (\text{R-union})$$

$$(3) \quad \mathcal{A}_P := \left\langle \bigwedge_{i \in \Lambda} \mu_i, \bigcap_{i \in \Lambda} \tilde{A}_i \right\rangle \quad (\text{P-intersection})$$

$$(4) \quad \mathcal{A}_R := \left\langle \bigvee_{i \in \Lambda} \mu_i, \bigcap_{i \in \Lambda} \tilde{A}_i \right\rangle \quad (\text{R-intersection})$$

**Proposition 3.** For any cubic IVIF sets  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$ ,  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$ ,  $\mathcal{C} = \langle \nu, \tilde{C} \rangle$  and  $\mathcal{D} = \langle \rho, \tilde{D} \rangle$  in  $X$ , we have

$$(1) \quad \mathcal{A} \subseteq_P \mathcal{B}, \mathcal{A} \subseteq_P \mathcal{C} \Rightarrow \mathcal{A} \subseteq_P \mathcal{B} \mathbin{\mathbb{M}}_P \mathcal{C}.$$

$$(2) \quad \mathcal{A} \subseteq_R \mathcal{B}, \mathcal{A} \subseteq_R \mathcal{C} \Rightarrow \mathcal{A} \subseteq_R \mathcal{B} \mathbin{\mathbb{M}}_R \mathcal{C}.$$

$$(3) \quad \mathcal{A} \subseteq_P \mathcal{C}, \mathcal{B} \subseteq_P \mathcal{C} \Rightarrow \mathcal{A} \mathbin{\mathbb{U}}_P \mathcal{B} \subseteq_P \mathcal{C}.$$

$$(4) \quad \mathcal{A} \subseteq_R \mathcal{C}, \mathcal{B} \subseteq_R \mathcal{C} \Rightarrow \mathcal{A} \mathbin{\mathbb{U}}_R \mathcal{B} \subseteq_R \mathcal{C}.$$

$$(5) \quad \mathcal{A} \subseteq_P \mathcal{B}, \mathcal{C} \subseteq_P \mathcal{D} \Rightarrow \mathcal{A} \mathbin{\mathbb{U}}_P \mathcal{C} \subseteq_P \mathcal{B} \mathbin{\mathbb{U}}_P \mathcal{D}, \mathcal{A} \mathbin{\mathbb{M}}_P \mathcal{C} \subseteq_P \mathcal{B} \mathbin{\mathbb{M}}_P \mathcal{D}.$$

$$(6) \quad \mathcal{A} \subseteq_R \mathcal{B}, \mathcal{C} \subseteq_R \mathcal{D} \Rightarrow \mathcal{A} \mathbin{\mathbb{U}}_R \mathcal{C} \subseteq_R \mathcal{B} \mathbin{\mathbb{U}}_R \mathcal{D}, \mathcal{A} \mathbin{\mathbb{M}}_R \mathcal{C} \subseteq_R \mathcal{B} \mathbin{\mathbb{M}}_R \mathcal{D}.$$

**Proof.** Straightforward.  $\square$

**Theorem 3.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  be a cubic IVIF set in  $X$ . If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is  $\alpha$ -internal (resp.,  $\alpha$ -external), then so is the complement of  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$ .

**Proof.** Assume that  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is  $\alpha$ -internal. Then  $\mu(x) \in \alpha[\tilde{A}](x)$ , that is,  $\alpha[\tilde{A}]^-(x) \leq \mu(x) \leq \alpha[\tilde{A}]^+(x)$  for all  $x \in X$ . Thus  $1 - \alpha[\tilde{A}]^+(x) \leq 1 - \mu(x) \leq 1 - \alpha[\tilde{A}]^-(x)$ , that is,  $\mu^c(x) \in \alpha[\tilde{A}^c](x)$  for all  $x \in X$ . Hence  $\mathcal{A}^c = \langle \mu^c, \tilde{A}^c \rangle$  is  $\alpha$ -internal. If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is  $\alpha$ -external, then  $\mu(x) \notin (\alpha[\tilde{A}]^-(x), \alpha[\tilde{A}]^+(x))$  for all  $x \in X$ , and so  $\mu(x) \in [0, \alpha[\tilde{A}]^-(x)]$  or  $\mu(x) \in [\alpha[\tilde{A}]^+(x), 1]$ . It follows that  $1 - \alpha[\tilde{A}]^-(x) \leq 1 - \mu(x) \leq 1$  or  $0 \leq 1 - \mu(x) \leq 1 - \alpha[\tilde{A}]^+(x)$ , and so that

$$\begin{aligned} \mu^c(x) &= 1 - \mu(x) \notin (1 - \alpha[\tilde{A}]^+(x), 1 - \alpha[\tilde{A}]^-(x)) \\ &= (\alpha[\tilde{A}^c]^-(x), \alpha[\tilde{A}^c]^+(x)) \end{aligned}$$

Therefore  $\mathcal{A}^c = \langle \mu^c, \tilde{A}^c \rangle$  is  $\alpha$ -external.  $\square$

**Theorem 4.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  be a cubic IVIF set in  $X$ . If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is  $\beta$ -internal (resp.,  $\beta$ -external), then so is the complement of  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$ .

**Proof.** It is similar to the proof of Theorem 3.  $\square$

**Corollary 1.** If a cubic IVIF set  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  in  $X$  is internal, then so is  $\mathcal{A}^c = \langle \mu^c, \tilde{A}^c \rangle$ .

**Theorem 5.** If  $\{\mathcal{A}_i = \langle \mu_i, \tilde{A}_i \rangle \mid i \in \Lambda\}$  is a family of  $\alpha$ -internal cubic IVIF sets in  $X$ , then the  $P$ -union and the  $P$ -intersection of  $\{\mathcal{A}_i = \langle \mu_i, \tilde{A}_i \rangle \mid i \in \Lambda\}$  are also  $\alpha$ -internal cubic IVIF sets in  $X$ .

**Proof.** Since  $\mathcal{A}_i$  is  $\alpha$ -internal, we have  $\mu_i(x) \in \alpha[\tilde{A}_i](x)$ , that is,

$$\alpha[\tilde{A}_i]^-(x) \leq \mu_i(x) \leq \alpha[\tilde{A}_i]^+(x)$$

for all  $x \in X$  and  $i \in \Lambda$ . It follows that

$$\alpha[\bigcup_{i \in \Lambda} \tilde{A}_i]^-(x) \leq \left( \bigvee_{i \in \Lambda} \mu_i \right)(x) \leq \alpha[\bigcup_{i \in \Lambda} \tilde{A}_i]^+(x)$$

and

$$\alpha[\bigcap_{i \in \Lambda} \tilde{A}_i]^-(x) \leq \left( \bigwedge_{i \in \Lambda} \mu_i \right)(x) \leq \alpha[\bigcap_{i \in \Lambda} \tilde{A}_i]^+(x).$$

Therefore  $\bigcup_{i \in \Lambda} \mathcal{A}_i$  and  $\bigcap_{i \in \Lambda} \mathcal{A}_i$  are  $\alpha$ -internal cubic IVIF sets in  $X$ .  $\square$

The following example shows that the  $R$ -union and  $R$ -intersection of  $\alpha$ -internal cubic IVIF sets need not be  $\alpha$ -internal.

**Example 2.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  be cubic IVIF sets in  $I = [0, 1]$  with  $\tilde{A}(x) = ([0.2, 0.45], [0.3, 0.5])$  and  $\tilde{B}(x) = ([0.6, 0.8], [0.1, 0.2])$  for all  $x \in I$ . Then  $\tilde{A} \cup \tilde{B} = (\alpha[\tilde{A} \cup \tilde{B}], \beta[\tilde{A} \cup \tilde{B}])$  and  $\tilde{A} \cap \tilde{B} = (\alpha[\tilde{A} \cap \tilde{B}], \beta[\tilde{A} \cap \tilde{B}])$ , where

$$\begin{aligned} \alpha[\tilde{A} \cup \tilde{B}](x) &= \text{rmax}\{\alpha[\tilde{A}](x), \alpha[\tilde{B}](x)\} \\ &= \text{rmax}\{[0.2, 0.45], [0.6, 0.8]\} = [0.6, 0.8] \end{aligned}$$

$$\begin{aligned}\beta[\tilde{A} \cup \tilde{B}](x) &= \text{rmin}\{\beta[\tilde{A}](x), \beta[\tilde{B}](x)\} \\ &= \text{rmin}\{[0.3, 0.5], [0.1, 0.2]\} = [0.1, 0.2]\end{aligned}$$

$$\begin{aligned}\alpha[\tilde{A} \cap \tilde{B}](x) &= \text{rmin}\{\alpha[\tilde{A}](x), \alpha[\tilde{B}](x)\} \\ &= \text{rmin}\{[0.2, 0.45], [0.6, 0.8]\} = [0.2, 0.45]\end{aligned}$$

$$\begin{aligned}\beta[\tilde{A} \cap \tilde{B}](x) &= \text{rmax}\{\beta[\tilde{A}](x), \beta[\tilde{B}](x)\} \\ &= \text{rmin}\{[0.3, 0.5], [0.1, 0.2]\} = [0.3, 0.5].\end{aligned}$$

If  $\mu(x) = 0.4$  and  $\lambda(x) = 0.7$  for all  $x \in I$ , then  $\mathcal{A}$  and  $\mathcal{B}$  are  $\alpha$ -internal. The R-union of  $\mathcal{A}$  and  $\mathcal{B}$  is

$$\mathcal{A} \uplus_R \mathcal{B} = \{\langle x, \mu(x), \tilde{B}(x) \rangle \mid x \in I\}$$

and it is not  $\alpha$ -internal. The R-intersection of  $\mathcal{A}$  and  $\mathcal{B}$  is

$$\mathcal{A} \mathbin{\mathbb{M}}_R \mathcal{B} = \{\langle x, \lambda(x), \tilde{A}(x) \rangle \mid x \in I\}$$

which is not  $\alpha$ -internal.

We provide a condition for the R-union of two  $\alpha$ -internal cubic IVIF sets to be  $\alpha$ -internal.

**Theorem 6.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  be cubic IVIF sets in  $X$  such that

$$(\forall x \in X) (\max\{\alpha[\tilde{A}]^-(x), \alpha[\tilde{B}]^-(x)\} \leq (\mu \wedge \lambda)(x)). \quad (17)$$

If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  are  $\alpha$ -internal, then so is the R-union of  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$ .

**Proof.** Assume that  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  are  $\alpha$ -internal cubic IVIF sets in  $X$  that satisfies the condition (17). Then

$$\alpha[\tilde{A}]^-(x) \leq \mu(x) \leq \alpha[\tilde{A}]^+(x)$$

and

$$\alpha[\tilde{B}]^-(x) \leq \lambda(x) \leq \alpha[\tilde{B}]^+(x)$$

for all  $x \in X$ , which imply that  $(\mu \wedge \lambda)(x) \leq \alpha[\tilde{A} \cup \tilde{B}]^+(x)$ . It follows from (17) that

$$\alpha[\tilde{A} \cup \tilde{B}]^-(x) = \max\{\alpha[\tilde{A}]^-(x), \alpha[\tilde{B}]^-(x)\} \leq (\mu \wedge \lambda)(x) \leq \alpha[\tilde{A} \cup \tilde{B}]^+(x)$$

for all  $x \in X$ . Therefore

$$\mathcal{A} \uplus_R \mathcal{B} = \{\langle x, (\mu \wedge \lambda)(x), (\tilde{A} \cup \tilde{B})(x) \rangle \mid x \in X\}$$

is an  $\alpha$ -internal cubic IVIF set in  $X$ .  $\square$

We provide a condition for the R-intersection of two  $\alpha$ -internal cubic IVIF sets to be  $\alpha$ -internal.

**Theorem 7.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  be cubic IVIF sets in  $X$  such that

$$(\forall x \in X) (\min\{\alpha[\tilde{A}]^+(x), \alpha[\tilde{B}]^+(x)\} \geq (\mu \vee \lambda)(x)). \quad (18)$$

If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  are  $\alpha$ -internal, then so is the R-intersection of  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$ .

**Proof.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  be  $\alpha$ -internal cubic IVIF sets in  $X$  which satisfy the condition (18). Then  $\alpha[\tilde{A}]^-(x) \leq \mu(x) \leq \alpha[\tilde{A}]^+(x)$  and  $\alpha[\tilde{B}]^-(x) \leq \lambda(x) \leq \alpha[\tilde{B}]^+(x)$  for all  $x \in X$ . It follows from (18) that

$$\alpha[\tilde{A} \cap \tilde{B}]^-(x) \leq (\mu \vee \lambda)(x) \leq \min\{\alpha[\tilde{A}]^+(x), \alpha[\tilde{B}]^+(x)\} = \alpha[\tilde{A} \cap \tilde{B}]^+(x)$$

for all  $x \in X$ . Therefore

$$\mathcal{A} \mathbb{M}_R \mathcal{B} = \{\langle x, (\mu \vee \lambda)(x), (\tilde{A} \cap \tilde{B})(x) \rangle \mid x \in X\}$$

is an  $\alpha$ -internal cubic IVIF set in  $X$ .  $\square$

The following example shows that the P-union and the P-intersection of  $\alpha$ -external cubic IVIF sets need not be  $\alpha$ -external.

**Example 3.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  be cubic IVIF sets in  $X$  which are given in Example 2. If  $\mu(x) = 0.7$  and  $\lambda(x) = 0.4$  for all  $x \in I$ , then  $\mathcal{A}$  and  $\mathcal{B}$  are  $\alpha$ -external. The P-union of  $\mathcal{A}$  and  $\mathcal{B}$  is

$$\mathcal{A} \mathbb{U}_P \mathcal{B} = \{\langle x, \mu(x), \tilde{B}(x) \rangle \mid x \in I\}$$

and it is not  $\alpha$ -external. The P-intersection of  $\mathcal{A}$  and  $\mathcal{B}$  is

$$\mathcal{A} \mathbb{M}_P \mathcal{B} = \{\langle x, \lambda(x), \tilde{A}(x) \rangle \mid x \in I\}$$

which is not  $\alpha$ -external.

We consider conditions for the P-union and P-intersection of two  $\alpha$ -external cubic IVIF sets to be  $\alpha$ -external.

**Theorem 8.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  be cubic IVIF sets in  $X$  such that

$$\{x \in X \mid \mu(x) \leq \alpha[\tilde{A}]^-(x), \lambda(x) \geq \alpha[\tilde{B}]^+(x)\} = \emptyset \quad (19)$$

and

$$\{x \in X \mid \mu(x) \geq \alpha[\tilde{A}]^+(x), \lambda(x) \leq \alpha[\tilde{B}]^-(x)\} = \emptyset. \quad (20)$$

If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  are  $\alpha$ -external, then so are the P-union and P-intersection of  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$ .

**Proof.** If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  are  $\alpha$ -external, then  $\mu(x) \notin (\alpha[\tilde{A}]^-(x), \alpha[\tilde{A}]^+(x))$  and  $\lambda(x) \notin (\alpha[\tilde{B}]^-(x), \alpha[\tilde{B}]^+(x))$ , that is,

$$\mu(x) \leq \alpha[\tilde{A}]^-(x) \text{ or } \mu(x) \geq \alpha[\tilde{A}]^+(x)$$

and

$$\lambda(x) \leq \alpha[\tilde{B}]^-(x) \text{ or } \lambda(x) \geq \alpha[\tilde{B}]^+(x)$$

for all  $x \in X$ . It follows from conditions (19) and (20) that

$$(\forall x \in X) (\mu(x) \leq \alpha[\tilde{A}]^-(x), \lambda(x) \leq \alpha[\tilde{B}]^-(x)) \quad (21)$$



or

$$(\forall x \in X) (\mu(x) \geq \alpha[\tilde{A}]^+(x), \lambda(x) \geq \alpha[\tilde{B}]^+(x)). \quad (22)$$

The conditions (21) and (22) induce

$$(\mu \vee \lambda)(x) \leq \max\{\alpha[\tilde{A}]^-(x), \alpha[\tilde{B}]^-(x)\}$$

and

$$(\mu \vee \lambda)(x) \geq \max\{\alpha[\tilde{A}]^+(x), \alpha[\tilde{B}]^+(x)\}$$

respectively, for all  $x \in X$ . Hence

$$(\mu \vee \lambda)(x) \notin (\alpha[\tilde{A} \cup \tilde{B}]^-(x), \alpha[\tilde{A} \cup \tilde{B}]^+(x))$$

for all  $x \in X$ , and therefore

$$\mathcal{A} \uplus_P \mathcal{B} = \{\langle x, (\mu \vee \lambda)(x), (\tilde{A} \cup \tilde{B})(x) \rangle \mid x \in I\}$$

is an  $\alpha$ -external cubic IVIF set in  $X$ . Also, (21) and (22) imply that

$$(\mu \wedge \lambda)(x) \leq \min\{\alpha[\tilde{A}]^-(x), \alpha[\tilde{B}]^-(x)\}$$

and

$$(\mu \wedge \lambda)(x) \geq \min\{\alpha[\tilde{A}]^+(x), \alpha[\tilde{B}]^+(x)\}$$

respectively, for all  $x \in X$ . Therefore

$$\mathcal{A} \oslash_P \mathcal{B} = \{\langle x, (\mu \wedge \lambda)(x), (\tilde{A} \cap \tilde{B})(x) \rangle \mid x \in I\}$$

is an  $\alpha$ -external cubic IVIF set in  $X$ .  $\square$

The following example shows that the R-union and R-intersection of  $\alpha$ -external cubic IVIF sets in  $X$  may not be  $\alpha$ -external.

**Example 4.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  be cubic IVIF sets in  $I = [0, 1]$  with  $\tilde{A}(x) = ([0.2, 0.4], [0.3, 0.5])$  and  $\tilde{B}(x) = ([0.3, 0.6], [0.2, 0.4])$  for all  $x \in I$ . Then  $\tilde{A} \cup \tilde{B} = (\alpha[\tilde{A} \cup \tilde{B}], \beta[\tilde{A} \cup \tilde{B}])$  and  $\tilde{A} \cap \tilde{B} = (\alpha[\tilde{A} \cap \tilde{B}], \beta[\tilde{A} \cap \tilde{B}])$ , where

$$\begin{aligned} \alpha[\tilde{A} \cup \tilde{B}](x) &= \text{rmax}\{\alpha[\tilde{A}](x), \alpha[\tilde{B}](x)\} \\ &= \text{rmax}\{[0.2, 0.4], [0.3, 0.6]\} = [0.3, 0.6] \end{aligned}$$

$$\begin{aligned} \beta[\tilde{A} \cup \tilde{B}](x) &= \text{rmin}\{\beta[\tilde{A}](x), \beta[\tilde{B}](x)\} \\ &= \text{rmin}\{[0.3, 0.5], [0.2, 0.4]\} = [0.2, 0.4] \end{aligned}$$

$$\begin{aligned} \alpha[\tilde{A} \cap \tilde{B}](x) &= \text{rmin}\{\alpha[\tilde{A}](x), \alpha[\tilde{B}](x)\} \\ &= \text{rmin}\{[0.2, 0.4], [0.3, 0.6]\} = [0.2, 0.4] \end{aligned}$$

$$\begin{aligned} \beta[\tilde{A} \cap \tilde{B}](x) &= \text{rmax}\{\beta[\tilde{A}](x), \beta[\tilde{B}](x)\} \\ &= \text{rmin}\{[0.3, 0.5], [0.2, 0.4]\} = [0.3, 0.5]. \end{aligned}$$

If  $\mu(x) = 0.5$  and  $\lambda(x) = 0.7$  for all  $x \in I$ , then  $\mathcal{A}$  and  $\mathcal{B}$  are both  $\alpha$ -external. The R-union of  $\mathcal{A}$  and  $\mathcal{B}$  is

$$\mathcal{A} \uplus_R \mathcal{B} = \{ \langle x, \mu(x), \tilde{B}(x) \rangle \mid x \in I \}$$

and it is not  $\alpha$ -external. If  $\mu(x) = 0.1$  and  $\lambda(x) = 0.3$  for all  $x \in I$ , then  $\mathcal{A}$  and  $\mathcal{B}$  are  $\alpha$ -external. The R-intersection of  $\mathcal{A}$  and  $\mathcal{B}$  is

$$\mathcal{A} \cap_R \mathcal{B} = \{ \langle x, \lambda(x), \tilde{A}(x) \rangle \mid x \in I \}$$

which is not  $\alpha$ -external.

We discuss conditions for the R-union and R-intersection of two cubic IVIF sets to be  $\alpha$ -external.

Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  be cubic IVIF sets in  $X$  such that

$$(\forall x \in X) (\alpha[\tilde{A}]^-(x) \leq \alpha[\tilde{B}]^-(x) \leq \alpha[\tilde{A}]^+(x) \leq \alpha[\tilde{B}]^+(x)). \quad (23)$$

If  $\mu(x) \leq \alpha[\tilde{A}]^-(x)$  for all  $x \in X$ , then

$$(\mu \wedge \lambda)(x) \leq \alpha[\tilde{A}]^-(x) \leq \max\{\alpha[\tilde{A}]^-(x), \alpha[\tilde{B}]^-(x)\}$$

and so  $(\mu \wedge \lambda)(x) \notin (\alpha[\tilde{A} \cup \tilde{B}]^-(x), \alpha[\tilde{A} \cup \tilde{B}]^+(x))$ .

If  $(\mu \wedge \lambda)(x) \geq \alpha[\tilde{B}]^+(x)$  for all  $x \in X$ , then

$$(\mu \wedge \lambda)(x) \geq \alpha[\tilde{B}]^+(x) = \max\{\alpha[\tilde{A}]^+(x), \alpha[\tilde{B}]^+(x)\}$$

and thus  $(\mu \wedge \lambda)(x) \notin (\alpha[\tilde{A} \cup \tilde{B}]^-(x), \alpha[\tilde{A} \cup \tilde{B}]^+(x))$ .

If  $(\mu \vee \lambda)(x) \leq \alpha[\tilde{A}]^-(x)$  for all  $x \in X$ , then

$$(\mu \vee \lambda)(x) \leq \alpha[\tilde{A}]^-(x) = \min\{\alpha[\tilde{A}]^-(x), \alpha[\tilde{B}]^-(x)\}$$

which implies that  $(\mu \vee \lambda)(x) \notin (\alpha[\tilde{A} \cap \tilde{B}]^-(x), \alpha[\tilde{A} \cap \tilde{B}]^+(x))$ .

If  $\mu(x) \geq \alpha[\tilde{B}]^+(x)$  for all  $x \in X$ , then

$$(\mu \vee \lambda)(x) \geq \mu(x) \geq \alpha[\tilde{B}]^+(x) \geq \min\{\alpha[\tilde{A}]^+(x), \alpha[\tilde{B}]^+(x)\}$$

which induces  $(\mu \vee \lambda)(x) \notin (\alpha[\tilde{A} \cap \tilde{B}]^-(x), \alpha[\tilde{A} \cap \tilde{B}]^+(x))$ .

Therefore we have the following theorem.

**Theorem 9.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  be cubic IVIF sets in  $X$  satisfying the condition (23). If  $\mu$  and  $\lambda$  satisfies any one of the following conditions.

$$(\forall x \in X) (\mu(x) \leq \alpha[\tilde{A}]^-(x)), \quad (24)$$

$$(\forall x \in X) ((\mu \wedge \lambda)(x) \geq \alpha[\tilde{B}]^+(x)), \quad (25)$$

then the R-union of  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  is  $\alpha$ -external. If  $\mu$  and  $\lambda$  satisfies any one of the following conditions.

$$(\forall x \in X) ((\mu \vee \lambda)(x) \leq \alpha[\tilde{A}]^-(x)), \quad (26)$$

$$(\forall x \in X) (\mu(x) \geq \alpha[\tilde{B}]^+(x)), \quad (27)$$

then the R-intersection of  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  is  $\alpha$ -external.

Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  be cubic IVIF sets in  $X$  such that

$$(\forall x \in X) (\alpha[\tilde{B}]^-(x) \leq \alpha[\tilde{A}]^-(x) \leq \alpha[\tilde{A}]^+(x) \leq \alpha[\tilde{B}]^+(x)). \quad (28)$$

If  $\mu(x) \leq \alpha[\tilde{B}]^-(x)$  for all  $x \in X$ , then

$$(\mu \wedge \lambda)(x) \leq \alpha[\tilde{B}]^-(x) \leq \max\{\alpha[\tilde{A}]^-(x), \alpha[\tilde{B}]^-(x)\}$$

and so  $(\mu \wedge \lambda)(x) \notin (\alpha[\tilde{A} \cup \tilde{B}]^-(x), \alpha[\tilde{A} \cup \tilde{B}]^+(x))$ .

If  $(\mu \wedge \lambda)(x) \geq \alpha[\tilde{B}]^+(x)$  for all  $x \in X$ , then

$$(\mu \wedge \lambda)(x) \geq \alpha[\tilde{B}]^+(x) = \max\{\alpha[\tilde{A}]^+(x), \alpha[\tilde{B}]^+(x)\}$$

and thus  $(\mu \wedge \lambda)(x) \notin (\alpha[\tilde{A} \cup \tilde{B}]^-(x), \alpha[\tilde{A} \cup \tilde{B}]^+(x))$ .

If  $\{x \in X \mid \mu(x) \geq \alpha[\tilde{B}]^+(x)\} = X$ , then

$$(\mu \vee \lambda)(x) \geq \mu(x) \geq \alpha[\tilde{B}]^+(x) \geq \min\{\alpha[\tilde{A}]^+(x), \alpha[\tilde{B}]^+(x)\}$$

which induces  $(\mu \vee \lambda)(x) \notin (\alpha[\tilde{A} \cap \tilde{B}]^-(x), \alpha[\tilde{A} \cap \tilde{B}]^+(x))$  for all  $x \in X$ .

If  $\{x \in X \mid (\mu \vee \lambda)(x) \leq \alpha[\tilde{B}]^-(x)\} = X$ , then

$$(\mu \vee \lambda)(x) \leq \alpha[\tilde{B}]^-(x) = \min\{\alpha[\tilde{A}]^-(x), \alpha[\tilde{B}]^-(x)\}$$

and so  $(\mu \vee \lambda)(x) \notin (\alpha[\tilde{A} \cap \tilde{B}]^-(x), \alpha[\tilde{A} \cap \tilde{B}]^+(x))$  for all  $x \in X$ .

Therefore we have the following theorem.

**Theorem 10.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  be cubic IVIF sets in  $X$  satisfying the condition (28). If  $\mu$  and  $\lambda$  satisfies

$$\{x \in X \mid \mu(x) \leq \alpha[\tilde{B}]^-(x)\} = X$$

or

$$\{x \in X \mid (\mu \wedge \lambda)(x) \geq \alpha[\tilde{B}]^+(x)\} = X,$$

then the R-union of  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  is  $\alpha$ -external. Also, if

$$\{x \in X \mid \mu(x) \geq \alpha[\tilde{B}]^+(x)\} = X$$

or

$$\{x \in X \mid (\mu \vee \lambda)(x) \leq \alpha[\tilde{B}]^-(x)\} = X,$$

then the R-intersection of  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  and  $\mathcal{B} = \langle \lambda, \tilde{B} \rangle$  is  $\alpha$ -external.

#### 4. Cubic IVIF Subalgebras and Ideals

In what follows, let  $X$  be a BCK/BCI-algebra unless otherwise specified.

**Definition 5.** A cubic IVIF set  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  in  $X$  is called a cubic IVIF subalgebra of  $X$  if the following conditions are valid.

$$(\forall x, y \in X) (\mu(x * y) \geq \min\{\mu(x), \mu(y)\}), \quad (29)$$

$$(\forall x, y \in X) \left( \begin{array}{l} \alpha[\tilde{A}](x * y) \leq \max\{\alpha[\tilde{A}](x), \alpha[\tilde{A}](y)\} \\ \beta[\tilde{A}](x * y) \geq \min\{\beta[\tilde{A}](x), \beta[\tilde{A}](y)\} \end{array} \right). \quad (30)$$

**Example 5.** Let  $X = \{0, a, b, c\}$  be a BCK-algebra with the Cayley table (Table 1).

**Table 1.** Tabular representation of the binary operation  $*$ .

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a cubic IVIF set  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  in  $X$  by the tabular representation in Table 2.

**Table 2.** Tabular representation of  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$ .

$X$	$\mu$	$\tilde{A} = (\alpha[\tilde{A}], \beta[\tilde{A}])$
0	0.8	$([0.1, 0.4], [0.3, 0.5])$
a	0.3	$([0.2, 0.4], [0.2, 0.4])$
b	0.6	$([0.2, 0.5], [0.1, 0.3])$
c	0.5	$([0.3, 0.5], [0.2, 0.5])$

It is routine to verify that  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF subalgebra of  $X$ .

**Proposition 4.** If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF subalgebra of  $X$ , then  $\mu(0) \geq \mu(x)$ ,  $\alpha[\tilde{A}](0) \ll \alpha[\tilde{A}](x)$  and  $\beta[\tilde{A}](0) \gg \beta[\tilde{A}](x)$  for all  $x \in X$ .

**Proof.** Since  $x * x = 0$  for all  $x \in X$ , we have

$$\begin{aligned}\mu(0) &= \mu(x * x) \geq \min\{\mu(x), \mu(x)\} = \mu(x), \\ \alpha[\tilde{A}](0) &= \alpha[\tilde{A}](x * x) \ll \text{rmax}\{\alpha[\tilde{A}](x), \alpha[\tilde{A}](x)\} = \alpha[\tilde{A}](x), \\ \beta[\tilde{A}](0) &= \beta[\tilde{A}](x * x) \gg \text{rmin}\{\beta[\tilde{A}](x), \beta[\tilde{A}](x)\} = \beta[\tilde{A}](x).\end{aligned}$$

This completes the proof.  $\square$

**Proposition 5.** If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF subalgebra of a BCI-algebra  $X$ , then  $\mu(0 * x) \geq \mu(x)$ ,  $\alpha[\tilde{A}](0 * x) \ll \alpha[\tilde{A}](x)$  and  $\beta[\tilde{A}](0 * x) \gg \beta[\tilde{A}](x)$  for all  $x \in X$ .

**Proof.** It follows from Definition 5 and Proposition 4.  $\square$

**Proposition 6.** For a cubic IVIF subalgebra  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  of  $X$ , if there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} \mu(x_n) = 1$ ,  $\lim_{n \rightarrow \infty} \alpha[\tilde{A}](x_n) = [0, 0]$  and  $\lim_{n \rightarrow \infty} \beta[\tilde{A}](x_n) = [1, 1]$ , then  $\mu(0) = 1$ ,  $\alpha[\tilde{A}](0) = [0, 0]$  and  $\beta[\tilde{A}](0) = [1, 1]$ .

**Proof.** Using Proposition 4, we have  $\mu(0) \geq \mu(x_n)$ ,  $\alpha[\tilde{A}](0) \ll \alpha[\tilde{A}](x_n)$  and  $\beta[\tilde{A}](0) \gg \beta[\tilde{A}](x_n)$ . It follows from hypothesis that

$$\begin{aligned}1 &\geq \mu(0) \geq \lim_{n \rightarrow \infty} \mu(x_n) = 1, \\ [0, 0] &\ll \alpha[\tilde{A}](0) \ll \lim_{n \rightarrow \infty} \alpha[\tilde{A}](x_n) = [0, 0], \\ [1, 1] &\gg \beta[\tilde{A}](0) \gg \lim_{n \rightarrow \infty} \beta[\tilde{A}](x_n) = [1, 1],\end{aligned}$$

Hence  $\mu(0) = 1$ ,  $\alpha[\tilde{A}](0) = [0, 0]$  and  $\beta[\tilde{A}](0) = [1, 1]$ .  $\square$

**Theorem 11.** If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF subalgebra of  $X$ , then the sets  $\mu[\varepsilon]$ ,  $\alpha[\tilde{A}][s, t]$  and  $\beta[\tilde{A}][s, t]$ , are subalgebras of  $X$  for all  $\varepsilon \in [0, 1]$  and  $[s, t] \in D[0, 1]$ .

**Proof.** For any  $\varepsilon \in [0, 1]$  and  $[s, t] \in D[0, 1]$ , let  $x, y \in X$  be such that

$$x, y \in \mu[\varepsilon] \cap \alpha[\tilde{A}][s, t] \cap \beta[\tilde{A}][s, t].$$

Then  $\mu(x) \geq \varepsilon$ ,  $\mu(y) \geq \varepsilon$ ,  $\alpha[\tilde{A}](x) \ll [s, t]$ ,  $\alpha[\tilde{A}](y) \ll [s, t]$ ,  $\beta[\tilde{A}](x) \gg [s, t]$  and  $\beta[\tilde{A}](y) \gg [s, t]$ . It follows that

$$\begin{aligned} \mu(x * y) &\geq \min\{\mu(x), \mu(y)\} \geq \min\{\varepsilon, \varepsilon\} = \varepsilon, \\ \alpha[\tilde{A}](x * y) &\ll \text{rmax}\{\alpha[\tilde{A}](x), \alpha[\tilde{A}](y)\} \ll \text{rmax}\{[s, t], [s, t]\} = [s, t], \\ \beta[\tilde{A}](x * y) &\gg \text{rmin}\{\beta[\tilde{A}](x), \beta[\tilde{A}](y)\} \gg \text{rmin}\{[s, t], [s, t]\} = [s, t], \end{aligned}$$

that is,  $x * y \in \mu[\varepsilon]$ ,  $x * y \in \alpha[\tilde{A}][s, t]$  and  $x * y \in \beta[\tilde{A}][s, t]$ . Therefore  $\mu[\varepsilon]$ ,  $\alpha[\tilde{A}][s, t]$  and  $\beta[\tilde{A}][s, t]$  are subalgebras of  $X$  for all  $\varepsilon \in [0, 1]$  and  $[s, t] \in D[0, 1]$ .  $\square$

**Corollary 2.** If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF subalgebra of  $X$ , then  $\mu[\varepsilon] \cap \alpha[\tilde{A}][s, t] \cap \beta[\tilde{A}][s, t]$  is a subalgebra of  $X$  for all  $\varepsilon \in [0, 1]$  and  $[s, t] \in D[0, 1]$ .

The following example shows that the converse of Corollary 2 is not true in general.

**Example 6.** Let  $X = \{0, 1, 2, 3, 4\}$  be a set with the Cayley table (Table 3).

**Table 3.** Tabular representation of the binary operation  $*$ .

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

Then  $X$  is a BCK-algebra (see [15]). Define a cubic IVIF set  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  in  $X$  by the tabular representation in Table 4.

**Table 4.** Tabular representation of  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$ .

$X$	$\mu$	$\tilde{A} = (\alpha[\tilde{A}], \beta[\tilde{A}])$
0	0.8	$([0.1, 0.3], [0.3, 0.6])$
1	0.3	$([0.2, 0.4], [0.2, 0.4])$
2	0.8	$([0.2, 0.5], [0.1, 0.3])$
3	0.5	$([0.3, 0.5], [0.2, 0.5])$
4	0.3	$([0.3, 0.5], [0.2, 0.5])$

It is routine to verify that  $\mu[\varepsilon] \cap \alpha[\tilde{A}][s, t] \cap \beta[\tilde{A}][s, t]$  is a subalgebra of  $X$  for all  $\varepsilon \in [0, 1]$  and  $[s, t] \in D[0, 1]$ . But,  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is not a cubic IVIF subalgebra of  $X$  since  $\mu(2 * 3) = \mu(1) = 0.3 < 0.5 = \min\{\mu(2), \mu(3)\}$ .

We provide conditions for a cubic IVIF set  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  in  $X$  to be a cubic IVIF subalgebra of  $X$ .

**Theorem 12.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  be a cubic IVIF set in  $X$  such that  $\mu[\varepsilon]$ ,  $\alpha[\tilde{A}][s_\alpha, t_\alpha]$  and  $\beta[\tilde{A}][s_\beta, t_\beta]$  are subalgebras of  $X$  for all  $\varepsilon \in [0, 1]$ , and  $([s_\alpha, t_\alpha], [s_\beta, t_\beta]) \in D[0, 1] \times D[0, 1]$ . Then  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF subalgebra of  $X$ .

**Proof.** For any  $x, y \in X$ , putting

$$\begin{aligned}\min\{\mu(x), \mu(y)\} &:= \varepsilon \\ \text{rmax}\{\alpha[\tilde{A}](x), \alpha[\tilde{A}](y)\} &:= [s_\alpha, t_\alpha], \\ \text{rmin}\{\beta[\tilde{A}](x), \beta[\tilde{A}](y)\} &:= [s_\beta, t_\beta],\end{aligned}$$

imply that  $\mu(x) \geq \varepsilon$ ,  $\mu(y) \geq \varepsilon$ ,  $\alpha[\tilde{A}](x) \ll [s_\alpha, t_\alpha]$ ,  $\alpha[\tilde{A}](y) \ll [s_\alpha, t_\alpha]$ , and  $\beta[\tilde{A}](x) \gg [s_\beta, t_\beta]$ ,  $\beta[\tilde{A}](y) \gg [s_\beta, t_\beta]$ . Hence  $x \in \mu[\varepsilon]$ ,  $y \in \mu[\varepsilon]$ ,  $x \in \alpha[\tilde{A}][s_\alpha, t_\alpha]$ ,  $y \in \alpha[\tilde{A}][s_\alpha, t_\alpha]$ ,  $x \in \beta[\tilde{A}][s_\beta, t_\beta]$ , and  $y \in \beta[\tilde{A}][s_\beta, t_\beta]$ . It follows from hypothesis that  $x * y \in \mu[\varepsilon]$ ,  $x * y \in \alpha[\tilde{A}][s_\alpha, t_\alpha]$ ,  $x * y \in \beta[\tilde{A}][s_\beta, t_\beta]$ , and so that

$$\begin{aligned}\mu(x * y) &\geq \varepsilon = \min\{\mu(x), \mu(y)\} \\ \alpha[\tilde{A}](x * y) &\ll [s_\alpha, t_\alpha] = \text{rmax}\{\alpha[\tilde{A}](x), \alpha[\tilde{A}](y)\}, \\ \beta[\tilde{A}](x * y) &\gg [s_\beta, t_\beta] = \text{rmin}\{\beta[\tilde{A}](x), \beta[\tilde{A}](y)\}.\end{aligned}$$

Therefore  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF subalgebra of  $X$ .  $\square$

The following theorem gives us a way to establish a new cubic IVIF subalgebra from old one in BCI-algebras.

**Theorem 13.** Given a cubic IVIF subalgebra  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  of a BCI-algebra  $X$ , let  $\mathcal{A}^* = \langle \mu^*, A^* \rangle$  be a cubic IVIF set in  $X$  defined by  $\mu^*(x) = \mu(0 * x)$ ,  $\alpha[\tilde{A}]^*(x) = \alpha[\tilde{A}](0 * x)$  and  $\beta[\tilde{A}]^*(x) = \beta[\tilde{A}](0 * x)$  for all  $x \in X$ . Then  $\mathcal{A}^* = \langle \mu^*, A^* \rangle$  is a cubic IVIF subalgebra of  $X$ .

**Proof.** Since  $0 * (x * y) = (0 * x) * (0 * y)$  for all  $x, y \in X$ , it follows that

$$\begin{aligned}\mu^*(x * y) &= \mu(0 * (x * y)) = \mu((0 * x) * (0 * y)) \\ &\geq \min\{\mu(0 * x), \mu(0 * y)\} \\ &= \min\{\mu^*(x), \mu^*(y)\},\end{aligned}$$

$$\begin{aligned}\alpha[\tilde{A}]^*(x * y) &= \alpha[\tilde{A}](0 * (x * y)) = \alpha[\tilde{A}]((0 * x) * (0 * y)) \\ &\ll \text{rmax}\{\alpha[\tilde{A}](0 * x), \alpha[\tilde{A}](0 * y)\} \\ &= \text{rmax}\{\alpha[\tilde{A}]^*(x), \alpha[\tilde{A}]^*(y)\},\end{aligned}$$

and

$$\begin{aligned}\beta[\tilde{A}]^*(x * y) &= \beta[\tilde{A}](0 * (x * y)) = \beta[\tilde{A}]((0 * x) * (0 * y)) \\ &\gg \text{rmin}\{\beta[\tilde{A}](0 * x), \beta[\tilde{A}](0 * y)\} \\ &= \text{rmin}\{\beta[\tilde{A}]^*(x), \beta[\tilde{A}]^*(y)\}\end{aligned}$$

for all  $x, y \in X$ . Therefore  $\mathcal{A}^* = \langle \mu^*, A^* \rangle$  is a cubic IVIF subalgebra of  $X$ .  $\square$

**Definition 6.** A cubic IVIF set  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is called a cubic IVIF ideal of  $X$  if the following conditions are valid.

$$(\forall x \in X) (\mu(0) \geq \mu(x)), \quad (31)$$

$$(\forall x \in X) (\alpha[\tilde{A}](0) \ll \alpha[\tilde{A}](x), \beta[\tilde{A}](0) \gg \beta[\tilde{A}](x)) \quad (32)$$

$$(\forall x, y \in X) (\mu(x) \geq \min\{\mu(x * y), \mu(y)\}) \quad (33)$$

$$(\forall x, y \in X) \left( \begin{array}{l} \alpha[\tilde{A}](x) \ll \text{rmax}\{\alpha[\tilde{A}](x * y), \alpha[\tilde{A}](y)\} \\ \beta[\tilde{A}](x) \gg \text{rmin}\{\beta[\tilde{A}](x * y), \beta[\tilde{A}](y)\} \end{array} \right) \quad (34)$$

**Example 7.** Let  $X = \{0, a, 1, 2, 3\}$  be a BCI-algebra with the Cayley table in Table 5.

**Table 5.** Tabular representation of the binary operation  $*$ .

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Define a cubic IVIF set  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  in  $X$  by Table 6.

**Table 6.** Tabular representation of  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$ .

$X$	$\mu$	$\tilde{A} = (\alpha[\tilde{A}], \beta[\tilde{A}])$
0	0.7	$([0.1, 0.3], [0.5, 0.6])$
a	0.7	$([0.2, 0.4], [0.4, 0.5])$
1	0.2	$([0.3, 0.6], [0.2, 0.3])$
2	0.5	$([0.3, 0.6], [0.3, 0.4])$
3	0.2	$([0.3, 0.6], [0.2, 0.3])$

Then  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF ideal of  $X$ .

**Proposition 7.** Every cubic IVIF ideal  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  of  $X$  satisfies:

$$\begin{aligned} \mu(x) &\geq \min\{\mu(y), \mu(z)\}, \\ \alpha[\tilde{A}](x) &\ll \text{rmax}\{\alpha[\tilde{A}](y), \alpha[\tilde{A}](z)\}, \\ \beta[\tilde{A}](x) &\gg \text{rmin}\{\beta[\tilde{A}](y), \beta[\tilde{A}](z)\}, \end{aligned} \quad (35)$$

for all  $x, y, z \in X$  with  $x * y \leq z$ .

**Proof.** Let  $x, y, z \in X$  be such that  $x * y \leq z$ . Then  $(x * y) * z = 0$ , and so

$$\begin{aligned} \mu(x * y) &\geq \min\{\mu((x * y) * z), \mu(z)\} = \min\{\mu(0), \mu(z)\} = \mu(z), \\ \alpha[\tilde{A}](x * y) &\ll \text{rmax}\{\alpha[\tilde{A}](x * y * z), \alpha[\tilde{A}](z)\} = \text{rmax}\{\alpha[\tilde{A}](0), \alpha[\tilde{A}](z)\} = \alpha[\tilde{A}](z), \\ \beta[\tilde{A}](x * y) &\gg \text{rmin}\{\beta[\tilde{A}](x * y * z), \beta[\tilde{A}](z)\} = \text{rmin}\{\beta[\tilde{A}](0), \beta[\tilde{A}](z)\} = \beta[\tilde{A}](z). \end{aligned}$$

It follows that

$$\begin{aligned} \mu(x) &\geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\mu(y), \mu(z)\}, \\ \alpha[\tilde{A}](x) &\ll \text{rmax}\{\alpha[\tilde{A}](x * y), \alpha[\tilde{A}](y)\} \ll \text{rmax}\{\alpha[\tilde{A}](y), \alpha[\tilde{A}](z)\}, \\ \beta[\tilde{A}](x) &\gg \text{rmin}\{\beta[\tilde{A}](x * y), \beta[\tilde{A}](y)\} \gg \text{rmin}\{\beta[\tilde{A}](y), \beta[\tilde{A}](z)\}. \end{aligned}$$

This completes the proof.  $\square$

We provide conditions for a cubic IVIF set to be a cubic IVIF ideal.

**Theorem 14.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  be a cubic IVIF set in  $X$  in which conditions (31) and (32) are valid. If  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  satisfies the condition (35) for all  $x, y, z \in X$  with  $x * y \leq z$ , then  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF ideal of  $X$ .

**Proof.** Since  $x * (x * y) \leq y$  for all  $x, y \in X$ , it follows from (35) that

$$\begin{aligned}\mu(x) &\geq \min\{\mu(x * y), \mu(y)\}, \\ \alpha[\tilde{A}](x) &\ll \text{rmax}\{\alpha[\tilde{A}](x * y), \alpha[\tilde{A}](y)\}, \\ \beta[\tilde{A}](x) &\gg \text{rmin}\{\beta[\tilde{A}](x * y), \beta[\tilde{A}](y)\},\end{aligned}$$

Therefore  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF ideal of  $X$ .  $\square$

**Lemma 1.** Every cubic IVIF ideal  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  in  $X$  satisfies the following condition.

$$(\forall x, y \in X) (x \leq y \Rightarrow \mu(x) \geq \mu(y), \alpha[\tilde{A}](x) \ll \alpha[\tilde{A}](y), \beta[\tilde{A}](x) \gg \beta[\tilde{A}](y)). \quad (36)$$

**Proof.** Assume that  $x \leq y$  for all  $x, y \in X$ . Then  $x * y = 0$ , and so

$$\begin{aligned}\mu(x) &\geq \min\{\mu(x * y), \mu(y)\} = \min\{\mu(0), \mu(y)\} = \mu(y), \\ \alpha[\tilde{A}](x) &\ll \text{rmax}\{\alpha[\tilde{A}](x * y), \alpha[\tilde{A}](y)\} = \text{rmax}\{\alpha[\tilde{A}](0), \alpha[\tilde{A}](y)\} = \alpha[\tilde{A}](y), \\ \beta[\tilde{A}](x) &\gg \text{rmin}\{\beta[\tilde{A}](x * y), \beta[\tilde{A}](y)\} = \text{rmin}\{\beta[\tilde{A}](0), \beta[\tilde{A}](y)\} = \beta[\tilde{A}](y)\end{aligned}$$

by (31)–(34).  $\square$

**Theorem 15.** In a BCK-algebra  $X$ , every cubic IVIF ideal is a cubic IVIF subalgebra.

**Proof.** Let  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  be a cubic IVIF ideal of a BCK-algebra  $X$ . Since  $x * y \leq x$  for all  $x, y \in X$ , we have  $\mu(x * y) \geq \mu(x)$ ,  $\alpha[\tilde{A}](x * y) \ll \alpha[\tilde{A}](x)$ ,  $\beta[\tilde{A}](x * y) \gg \beta[\tilde{A}](x)$  by Lemma 1. It follows from (33) and (34) that

$$\begin{aligned}\mu(x * y) &\geq \mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\mu(x), \mu(y)\}, \\ \alpha[\tilde{A}](x * y) &\ll \alpha[\tilde{A}](x) \ll \text{rmax}\{\alpha[\tilde{A}](x * y), \alpha[\tilde{A}](y)\} \ll \text{rmax}\{\alpha[\tilde{A}](x), \alpha[\tilde{A}](y)\}, \\ \beta[\tilde{A}](x * y) &\gg \beta[\tilde{A}](x) \gg \text{rmin}\{\beta[\tilde{A}](x * y), \beta[\tilde{A}](y)\} \gg \text{rmin}\{\beta[\tilde{A}](x), \beta[\tilde{A}](y)\}.\end{aligned}$$

Therefore  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF subalgebra of  $X$ .  $\square$

The converse of Theorem 15 is not true in general as seen in the following example.

**Example 8.** The cubic IVIF subalgebra  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  in Example 5 is not a cubic IVIF ideal of  $X$  since  $\mu(a) = 0.3 < 0.6 = \min\{\mu(a * b), \mu(b)\}$ .

We establish a characterization of a cubic IVIF ideal in a BCK-algebra.

**Theorem 16.** For a cubic IVIF set  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  in a BCK-algebra  $X$ , the following assertions are equivalent.

- (1)  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF ideal of  $X$ .
- (2)  $\mathcal{A} = \langle \mu, \tilde{A} \rangle$  is a cubic IVIF subalgebra of  $X$  satisfying (35) for all  $x, y, z \in X$  with  $x * y \leq z$ .



**Proof.** (1)  $\Rightarrow$  (2). It follows from Proposition 7 and Theorem 15.

(2)  $\Rightarrow$  (1). It is by Proposition 4 and Theorem 14.  $\square$

## 5. Conclusions

We have introduced a cubic interval-valued intuitionistic fuzzy set which is an extension of a cubic set, and have applied it to  $BCK/BCI$ -algebra. We have investigated P-union, P-intersection, R-union and R-intersection of  $\alpha$ -internal and  $\alpha$ -external cubic IVIF sets. We have defined cubic IVIF subalgebra and ideal in  $BCK/BCI$ -algebra, and have investigated related properties. We have considered relations between cubic IVIF subalgebra and cubic IVIF ideal. We have discussed characterizations of cubic IVIF subalgebra and cubic IVIF ideal. In the consecutive research, we will discuss P-union, P-intersection, R-union and R-intersection of  $\beta$ -internal and  $\beta$ -external cubic IVIF sets. We will apply this notion to several kind of ideals in  $BCK/BCI$ -algebras, for example, (positive) implicative ideal, commutative ideal, associative ideal,  $q$ -ideal, etc. We will also apply this notion to several algebraic substructures in various algebraic structures, for example,  $MV$ -algebra,  $BL$ -algebra,  $R_0$ -algebra, residuated lattice,  $MTL$ -algebra, semigroup, semiring, near-ring, etc. We will try to study this notion in the neutrosophic environment.

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