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On the Linear Quadratic Optimal Control for Systems Described by Singularly Perturbed Itô Differential Equations with Two Fast Time Scales

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Abstract: In this paper a stochastic optimal control problem described by a quadratic performance criterion and a linear controlled system modeled by a system of singularly perturbed Itô differential equations with two fast time scales is considered. The asymptotic structure of the stabilizing solution (satisfying a prescribed sign condition) to the corresponding stochastic algebraic Riccati equation is derived. Furthermore, a near optimal control whose gain matrices do not depend upon small parameters is discussed.

Keywords: singularly perturbed linear stochastic systems; asymptotic structure of the stabilizing solution; optimal control problem; Riccati equations of stochastic control

MSC: 93E20, 93B12, 34H15, 49N10.

1. Introduction

In the last 40 years, special attention was paid to the singular perturbation techniques applied in both analysis and synthesis of control laws with prescribed performance specifications for the regulation of systems whose mathematical models are described by a system of differential equations of high dimension, and also contain a number of small parameters multiplying derivatives of a part of the state variables of the physical phenomenon under discussion.

The large number of differential equations of the mathematical model of a physical process may be caused by the presence of some "parasitic" parameters such as small time constants, resistances, inductances, capacitances, moments of inertia, small masses, etc.

The presence of such small parameters is often a source of stiffness due to the simultaneous occurrence of slow and fast phenomena. It is known that the stiffness can produce ill-conditioning of the numerical computation involved in the process of designing the optimal control. This inconvenience leads to the idea to simplify the mathematical model by neglecting the small parameters occurring in the original model. Besides the stiffness, the necessity of the simplification of the mathematical model by neglecting the small parameters is also imposed by the fact that, in many applications, the values of these parasitic quantities are not exactly known. A fundamental problem is to check if the optimal control design based on the reduced model provides a satisfactory behavior of the full system which contains fast phenomena neglected during the designing process.

Remarkable results were obtained in the problem of the design of some near optimal controllers in the case of some deterministic systems with several time scales. Such results may be found in the monographs [1–4]. A common feature of the approaches in these works is the use of the singular perturbations techniques, initially developed in connection with the study of qualitative properties of the solutions of some classes of differential equations starting with the classical work of Tichonov [5]. The interest for studying different problems regarding the singularly perturbed controlled systems is still increasing. For the reader's convenience, we refer to the recent papers [6–10].

Lately, the interest for studying optimal control problems for stochastic systems modeled by singularly perturbed Itô differential equations also increased. Unlike the deterministic case, where the reduced model is obtained by simply removing the small parameters, in the case of stochastic optimal control problems driven by systems of singularly perturbed Itô differential equations, the definition of the reduced model is not always intuitive and it is strongly dependent upon the intensity of the white noise type perturbations affecting the diffusion part of the fast equations of the mathematical model. Hence, problems related to singularly perturbed stochastic systems could not be viewed as simple extensions of there deterministic counterparts. This makes the study of this class of systems a challenging (and relatively not fully investigated) topic. The main results obtained in this field were published in [11–14].

Very few results have been reported in the literature dealing with several fast time scales. We cite here [15] for the deterministic case and [16] for the stochastic framework. Pursuing our efforts in the study of singularly perturbed stochastic systems, we consider in this paper a stochastic optimal control problem described by a quadratic performance criterion and a linear controlled system modeled by a system of singularly perturbed Itô differential equations with two fast time scales.

Unlike [17] in the deterministic case or [14] in the stochastic case, where the fast time scales have the same order of magnitude, in the present work, we consider the case in which the two fast time scales have different order of magnitude. More precisely, if $\epsilon_j > 0, j = 1,2$ are the small parameters associated with the two fast time scales, the ratio $\frac{\epsilon_2}{\epsilon_1}$ becomes the third small parameter which needs to be considered in the asymptotic analysis performed here. The most part of our study is devoted to the analysis of the asymptotic structure of the stabilizing solution of the algebraic Riccati equation involved in the computation of the optimal control of the optimization problem under consideration. The main tool in the derivation of the asymptotic structure of the stabilizing solution of the algebraic Riccati equation under consideration around the origin $(\epsilon_1, \epsilon_2, \frac{\epsilon_2}{\epsilon_1}) = (0,0,0)$ is the implicit functions theorem. To this end, we first investigate the solvability of the system of reduced equations obtained setting $\epsilon_k = 0, k = 1,2$ and $\frac{\epsilon_2}{\epsilon_1} = 0$ in the original algebraic Riccati equation. Unlike the deterministic case, in the stochastic framework considered in this paper, the system of the reduced equations is a system of strongly interconnected Riccati type algebraic equations. For this system of interconnected Riccati type equations we introduce the concept of stabilizing solution and provide a set of necessary and sufficient conditions which guarantee the existence of such a solution. Further, employing the stabilizing solution of the system of the reduced equations and the corresponding stabilizing gain matrices we show that one may apply the implicit functions theorem to obtain the existence, as well as the asymptotic structure of, the stabilizing solution of the algebraic Riccati equation associated with the optimal control problem under consideration. Based on the dominant part independent of the small parameters of the stabilizing gain matrix, we construct a near optimal control whose gain matrices can be computed without the knowledge of the precise values of the small parameters associated with the fast time scales.

The paper is organized as follows: Section 2 provides the model description and the problem formulation. In Section 3 we show how the system of reduced Riccati equations, which are not dependent upon the small parameters, can be derived. Also, we introduce the concept of the stabilizing solution for the system of reduced algebraic Riccati equations. Then, we provide conditions which guarantee the existence of this stabilizing solution. In Section 4, we obtain the existence and the asymptotic structure of the stabilizing solution for the Riccati equation associated with the original linear quadratic control problem. Finally, we show how the asymptotic structure of the stabilizing feedback gain can be used to construct a near optimal control.

2. The Problem

Let us consider the stochastic optimal control problem asking for the minimization of the quadratic functional

$$J(\mathbf{x}_0;u)=\mathbb{E}[\int_0^\infty(\sum_{j,k=0}^{2}x_j^T(t)M_{jk}x_k(t)+2\sum_{j=0}^{2}x_j^T(t)L_ju(t)+u^T(t)Ru(t))dt \tag{1}$$

along with the trajectories of the controlled system having the state space representation described by the following system of singularly perturbed Itô differential equations

$$\begin{aligned}
dx_0(t) &= (A_{00}(\epsilon)x_0(t)+A_{01}(\epsilon)x_1(t)+A_{02}(\epsilon)x_2(t)+B_0(\epsilon)u(t))dt+\\
&+ (C_{00}(\epsilon)x_0(t)+C_{01}(\epsilon)x_1(t)+C_{02}(\epsilon)x_2(t)+D_0(\epsilon)u(t))dw(t)\\
x_0(0) &= x_0^0\\
\epsilon_k dx_k(t) &= (A_{k0}(\epsilon)x_0(t)+A_{k1}(\epsilon)x_1(t)+A_{k2}(\epsilon)x_2(t)+B_k(\epsilon)u(t))dt+\\
&+ \sqrt{\epsilon_k}(C_{k0}(\epsilon)x_0(t)+C_{k1}(\epsilon)x_1(t)+C_{k2}(\epsilon)x_2(t)+D_k(\epsilon)u(t))dw(t)\\
x_k(0) &= x_k^0, k=1,2.
\end{aligned} \tag{2}$$

In (1) and (2) $u(t)\in\mathbb{R}^m$ is the vector of the control parameters and $\mathbf{x}(t)=\begin{pmatrix} x_0^T(t) & x_1^T(t) & x_2^T(t)\end{pmatrix}^T\in\mathbb{R}^n$ is the vector of state parameters, $\mathbf{x}_0=\begin{pmatrix} x_0^{0T} & x_0^{1T} & x_0^{2T}\end{pmatrix}^T$, $n=n_0+n_1+n_2$; $x_j(t)\in\mathbb{R}^{n_j}$, $0\le j\le 2$. In (1), $M_{jk}=M_{kj}^T$, $0\le k,j\le 2$, $R=R^T$. In (2), $\epsilon_k>0$ are small parameters often not exactly known.

In order to make more intuitive the developments in this paper we assume that the small parameters $\epsilon_k, k=1,2$ satisfy the assumption:

$\mathbf{H_1}$) $\epsilon_k=\varphi_k(\eta)$, where $\varphi_k:[0,\eta^*]\to[0,\infty)$ are nondecreasing functions with the properties:
(i) $\varphi_k(\eta)=0$ if and only if $\eta=0$, $k=1,2$.
(ii) $\lim_{\eta\to 0_+}\varphi_k(\eta)=0$; $\lim_{\eta\to 0_+}\frac{\varphi_2(\eta)}{\varphi_1(\eta)}=0$.

In the sequel, the dependence of ϵ_k upon the parameter η will be suppressed.

Remark 1. *According to the terminology used in the framework of singularly perturbed differential equations, $x_0(t)$ will be called* **slow state variables** *while $x_1(t),x_2(t)$ will be named* **fast state variables**. *From the condition imposed to the values of the ratio $\frac{\epsilon_2}{\epsilon_1}$ in $\mathbf{H_1}$), it follows that the states $x_2(t)$ are faster than $x_1(t)$. That is why under the assumption $\mathbf{H_1}$) system (2) is a controlled system with two fast time scales.*

In the deterministic framework, the asymptotic structure of the solutions of some systems with several time fast scales was studied in [18] while in [19] were derived uniform upper bounds of the block components of the fundamental matrix solution of the systems of linear differential equations with several fast time scales.

In (2), $\{w(t)\}_{t\ge 0}$ is a 1-dimensional standard Wiener process defined on a given probability space $(\Omega,\mathcal{F},\mathcal{P})$. The consideration of the case with an 1-dimensional standard Wiener process is only to easy the exposition. The extension to the case of a multidimensional Wiener process can be done without difficulty.

Regarding the coefficients of system (2), we make the following assumption:

$\mathbf{H_2}$) $\epsilon=(\epsilon_1,\epsilon_2)\to(A_{jk}(\epsilon),B_j(\epsilon),C_{jk}(\epsilon),D_j(\epsilon))$ are C^1 matrix valued functions defined on a neighborhood of the origin $(\epsilon_1,\epsilon_2)=(0,0)$.

We set

$$\mathbb{A}(\epsilon) = \begin{pmatrix} A_{00}(\epsilon) & A_{01}(\epsilon) & A_{02}(\epsilon) \\ \frac{1}{\epsilon_1}A_{10}(\epsilon) & \frac{1}{\epsilon_1}A_{11}(\epsilon) & \frac{1}{\epsilon_1}A_{12}(\epsilon) \\ \frac{1}{\epsilon_2}A_{20}(\epsilon) & \frac{1}{\epsilon_2}A_{21}(\epsilon) & \frac{1}{\epsilon_2}A_{22}(\epsilon) \end{pmatrix}, \mathbb{B}(\epsilon) = \begin{pmatrix} B_0(\epsilon) \\ \frac{1}{\epsilon_1}B_1(\epsilon) \\ \frac{1}{\epsilon_2}B_2(\epsilon) \end{pmatrix},$$

$$\mathbb{C}(\epsilon) = \begin{pmatrix} C_{00}(\epsilon) & C_{01}(\epsilon) & C_{02}(\epsilon) \\ \frac{1}{\sqrt{\epsilon_1}}C_{10}(\epsilon) & \frac{1}{\sqrt{\epsilon_1}}C_{11}(\epsilon) & \frac{1}{\sqrt{\epsilon_1}}C_{12}(\epsilon) \\ \frac{1}{\sqrt{\epsilon_2}}C_{20}(\epsilon) & \frac{1}{\sqrt{\epsilon_2}}C_{21}(\epsilon) & \frac{1}{\sqrt{\epsilon_2}}C_{22}(\epsilon) \end{pmatrix}, \mathbb{D}(\epsilon) = \begin{pmatrix} D_0(\epsilon) \\ \frac{1}{\sqrt{\epsilon_1}}D_1(\epsilon) \\ \frac{1}{\sqrt{\epsilon_2}}D_2(\epsilon) \end{pmatrix} \tag{3}$$

$$\mathbb{M} = \begin{pmatrix} M_{00} & M_{01} & M_{02} \\ M_{01}^T & M_{11} & M_{12} \\ M_{02}^T & M_{12}^T & M_{22} \end{pmatrix} = \mathbb{M}^T, \mathbb{L} = \begin{pmatrix} L_0 \\ L_1 \\ L_2 \end{pmatrix}. \tag{4}$$

With these notations (1) and (2) may be written in a compact form as:

$$J(\mathbf{x}_0; u) = \mathbb{E}[\int_0^\infty (\mathbf{x}^T(t)\mathbb{M}\mathbf{x}(t) + 2\mathbf{x}^T(t)\mathbb{L}u(t) + u^T(t)Ru(t))dt] \tag{5}$$

and

$$\begin{aligned} d\mathbf{x}(t) &= (\mathbb{A}(\epsilon)\mathbf{x}(t) + \mathbb{B}(\epsilon)u(t))dt + (\mathbb{C}(\epsilon)\mathbf{x}(t) + \mathbb{D}(\epsilon)u(t))dw(t) \\ \mathbf{x}(0) &= \mathbf{x}_0. \end{aligned} \tag{6}$$

One sees that for each $\epsilon = (\epsilon_1, \epsilon_2)$ fixed, the optimal control asking for the minimization of the quadratic cost (5) in the class of controls that stabilizes system (6) is a standard stochastic linear quadratic optimal control problem, which was studied starting with [20].

In [21,22] it was shown that a stochastic linear quadratic control problem, with control dependent terms in the diffusion part of the controlled system, is still well posed even if the cost weight matrices of the states and the control are allowed to be indefinite. The optimal control is given by:

$$u_{opt}(t) = -(R + \mathbb{D}^T(\epsilon)\tilde{X}(\epsilon)\mathbb{D}(\epsilon))^{-1}(\mathbb{B}^T(\epsilon)\tilde{X}(\epsilon) + \mathbb{D}^T(\epsilon)\tilde{X}(\epsilon)\mathbb{C}(\epsilon) + \mathbb{L}^T)\mathbf{x}(t) \tag{7}$$

where $\tilde{X}(\epsilon)$ is the unique stabilizing solution of the algebraic Riccati equation of stochastic control (SARE):

$$\begin{aligned} \mathbb{A}^T(\epsilon)X + X\mathbb{A}(\epsilon) + \mathbb{C}^T(\epsilon)X\mathbb{C}(\epsilon) - (X\mathbb{B}(\epsilon) + \mathbb{C}^T(\epsilon)X\mathbb{D}(\epsilon) + \mathbb{L})\times & \\ \times(R + \mathbb{D}^T(\epsilon)X\mathbb{D}(\epsilon))^{-1}(\mathbb{B}^T(\epsilon)X + \mathbb{D}^T(\epsilon)X\mathbb{C}(\epsilon) + \mathbb{L}^T) + \mathbb{M} &= 0 \end{aligned} \tag{8}$$

satisfying the sign condition

$$R + \mathbb{D}^T(\epsilon)X\mathbb{D}(\epsilon) > 0. \tag{9}$$

The condition (9) supplies the absence of the information regarding the sign of the matrix R. In [22], necessary and sufficient conditions that guarantee the existence of the stabilizing solution of a SARE were provided as (8) satisfying the sign condition (9) and a procedure for numerical computation of this solution using the so called semidefinite programming (SDP) was proposed. Also, an iterative procedure for numerical computation of the constrained SARE of type (8) and (9) was proposed in Section 5.8 from [23]. Unfortunately, the way in which the small parameters $\epsilon_k > 0, k = 1, 2$ affect the coefficients of SARE (8) and (9) may produce ill-conditioning of the numerical computation involved in obtaining the stabilizing solution $\tilde{X}(\epsilon)$ of the SARE under consideration. In order to avoid the ill-conditioning of the numerical computations generated by the high difference between the order of magnitude of the coefficients, knowledge of the asymptotic structure of the solution $\tilde{X}(\epsilon)$ in a neighborhood of the origin $\epsilon = (\epsilon_1, \epsilon_2) = (0, 0)$ is useful. As a consequence of such a study, a system of Riccati type equations not depending upon the small parameters ϵ_k, $k = 1, 2$, often named a system of reduced algebraic Riccati equations, which allows us to compute the dominant part of the solution $\tilde{X}(\epsilon)$ can be displayed.

In the deterministic case, see for example [1–4,24], the system of reduced algebraic Riccati equations is obtained by simply removing all of the small parameters. In the stochastic framework, when the controlled systems are modeled by singularly perturbed Itô differential equations, the definition of the system of reduced algebraic Riccati equations cannot be done by a simple neglection of the small parameters. From [11] or [12,25], one sees that the definition of the system of reduced algebraic Riccati equations is strongly dependent upon the magnitude of the white noise perturbations affecting the equations of the fast variables in the controlled system.

In order to obtain the asymptotic structure with respect to the small parameters $\epsilon_k > 0, k = 1, 2$ of the stabilizing solution of SARE (8), we shall use the implicit functions theorem. To this end, we need a rigourous definition of the corresponding system of reduced algebraic Riccati equations (SRARE) and to point out a special kind of solution of this system which helps us to apply the implicit functions theorem. That is why in the next section we shall show how the system of reduced algebraic Riccati equations in the case of SARE (8) and (9) can be defined. Next, we shall introduce a concept of stabilizing solution of the obtained SRARE and we shall provide a set of conditions which guarantee the existence of this stabilizing solution of SRARE. In Section 4, using reasoning based on the implicit functions theorem, we shall obtain the asymptotic structure of the stabilizing solution of SARE (8) satisfying the sign condition (9), as well as the asymptotic structure of the corresponding stabilizing feedback gain.

3. The System of Reduced Algebraic Riccati Equations

3.1. Derivation of the System of Reduced Algebraic Riccati Equations

Setting $F = -(R + \mathbb{D}^T(\epsilon)X\mathbb{D}(\epsilon))^{-1}(\mathbb{B}^T(\epsilon)X + \mathbb{D}^T(\epsilon)X\mathbb{C}(\epsilon) + \mathbb{L}^T)$ one obtains that if X is a solution of SARE (8) satisfying the sign condition (9), then (X, F) is a solution of the system:

$$\begin{aligned} \Gamma(X,\epsilon)F + \mathbb{B}^T(\epsilon)X + \mathbb{D}^T(\epsilon)X\mathbb{C}(\epsilon) + \mathbb{L}^T &= 0 \\ \mathbb{A}^T(\epsilon)X + X\mathbb{A}(\epsilon) + \mathbb{C}^T(\epsilon)X\mathbb{C}(\epsilon) - F^T\Gamma(X,\epsilon)F + \mathbb{M} &= 0 \\ \Gamma(X,\epsilon) &= R + \mathbb{D}^T(\epsilon)X\mathbb{D}(\epsilon). \end{aligned} \tag{10}$$

Conversely, if (X, F) is a solution of system (10) satisfying $\Gamma(X, \epsilon) > 0$, then X is a solution of the constrained SARE (8) and (9). To obtain the asymptotic structure of the stabilizing solution of SARE (8) and (9), we shall analyse the asymptotic structure of the solution $(\tilde{X}(\epsilon), \tilde{F}(\epsilon))$ of system (10) with the additional property that the closed-loop system

$$d\mathbf{x}(t) = (\mathbb{A}(\epsilon) + \mathbb{B}(\epsilon)\tilde{F}(\epsilon))\mathbf{x}(t)dt + (\mathbb{C}(\epsilon) + \mathbb{D}(\epsilon)\tilde{F}(\epsilon))\mathbf{x}(t)dw(t) \tag{11}$$

is exponentially stable in mean square (ESMS).

We are looking for the solution (X, F) of (10) having the partition:

$$X = \begin{pmatrix} X_{00} & \epsilon_1 X_{01} & \epsilon_2 X_{02} \\ \epsilon_1 X_{01}^T & \epsilon_1 X_{11} & \epsilon_2 X_{12} \\ \epsilon_2 X_{02}^T & \epsilon_2 X_{12}^T & \epsilon_2 X_{22} \end{pmatrix}, \qquad F = \begin{pmatrix} F_0 & F_1 & F_2 \end{pmatrix} \tag{12}$$

where $X_{jk} \in \mathbb{R}^{n_j \times n_k}$, $F_k \in \mathbb{R}^{m \times n_k}$, $0 \le j, k \le 2$.

Employing the partitions (3) and (4) of the coefficients of SARE (8) we may obtain a partition of system (10).

To ease the exposition, let us regroup the block components from (3), (4) and (12) as:

$$\mathbb{A}(\epsilon) = \Pi^{-1}(\epsilon)\begin{pmatrix} \mathbb{A}_{11}(\epsilon) & \mathbb{A}_{12}(\epsilon) \\ \mathbb{A}_{21}(\epsilon) & A_{22}(\epsilon) \end{pmatrix}, \tag{13a}$$

$$\mathbb{B}(\epsilon) = \Pi^{-1}(\epsilon)\begin{pmatrix} \mathbb{B}_{1}(\epsilon) \\ B_{2}(\epsilon) \end{pmatrix}, \tag{13b}$$

$$\mathbb{C}(\epsilon) = \Pi^{-1}(\sqrt{\epsilon})\begin{pmatrix} \mathbb{C}_{11}(\epsilon) & \mathbb{C}_{12}(\epsilon) \\ \mathbb{C}_{21}(\epsilon) & C_{22}(\epsilon) \end{pmatrix}, \tag{13c}$$

$$\mathbb{D}(\epsilon) = \Pi^{-1}(\sqrt{\epsilon})\begin{pmatrix} \mathbb{D}_{1}(\epsilon) \\ D_{2}(\epsilon) \end{pmatrix}, \tag{13d}$$

$$\mathbb{M} = \begin{pmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ \mathbb{M}_{12}^T & M_{22} \end{pmatrix}, \tag{13e}$$

$$\mathbb{L} = \begin{pmatrix} \mathbb{L}_{1} \\ L_{2} \end{pmatrix}, \tag{13f}$$

where

$$\Pi(\epsilon) = diag(I_{n_0}, \epsilon_1 I_{n_1}, \epsilon_2 I_{n_2}), \tag{14a}$$

$$\mathbb{A}_{11}(\epsilon) = \begin{pmatrix} A_{00}(\epsilon) & A_{01}(\epsilon) \\ A_{10}(\epsilon) & A_{11}(\epsilon) \end{pmatrix}$$

$$\mathbb{A}_{12}(\epsilon) = \begin{pmatrix} A_{02}(\epsilon) \\ A_{12}(\epsilon) \end{pmatrix}, \mathbb{A}_{21}(\epsilon) = \begin{pmatrix} A_{20}(\epsilon) & A_{21}(\epsilon) \end{pmatrix} \tag{14b}$$

$$\mathbb{B}_{1}(\epsilon) = \begin{pmatrix} B_{0}(\epsilon) \\ B_{1}(\epsilon) \end{pmatrix}, \tag{14c}$$

$$\mathbb{C}_{11}(\epsilon) = \begin{pmatrix} C_{00}(\epsilon) & C_{01}(\epsilon) \\ C_{10}(\epsilon) & C_{11}(\epsilon) \end{pmatrix}$$

$$\mathbb{C}_{12}(\epsilon) = \begin{pmatrix} C_{02}(\epsilon) \\ C_{12}(\epsilon) \end{pmatrix}, \mathbb{C}_{21}(\epsilon) = \begin{pmatrix} C_{20}(\epsilon) & C_{21}(\epsilon) \end{pmatrix} \tag{14d}$$

$$\mathbb{D}_{1}(\epsilon) = \begin{pmatrix} D_{0}(\epsilon) \\ D_{1}(\epsilon) \end{pmatrix}, \tag{14e}$$

$$\mathbb{M}_{11} = \begin{pmatrix} M_{00} & M_{01} \\ M_{01}^T & M_{11} \end{pmatrix}, \mathbb{M}_{12} = \begin{pmatrix} M_{02} \\ M_{12} \end{pmatrix}, \mathbb{L}_{1} = \begin{pmatrix} L_{0} \\ L_{1} \end{pmatrix} \tag{14f}$$

From (12) one obtains the following structure of X and F:

$$X = \Pi(\epsilon)\begin{pmatrix} U_1(X,\epsilon) & \Delta(\epsilon)\mathbb{X}_{12} \\ \mathbb{X}_{12}^T & X_{22} \end{pmatrix} \tag{15}$$

$$F = \begin{pmatrix} \mathbb{F}_1 & F_2 \end{pmatrix}$$

where

$$U_1(X,\epsilon) = \begin{pmatrix} X_{00} & \epsilon_1 X_{01} \\ X_{01}^T & X_{11} \end{pmatrix} \tag{16a}$$

$$\mathbb{X}_{12} = \begin{pmatrix} X_{02} \\ X_{12} \end{pmatrix} \tag{16b}$$

$$\Delta(\epsilon) = diag(\epsilon_2 I_{n_0}, \epsilon_2/\epsilon_1 I_{n_1}), \tag{16c}$$

$$\mathbb{F}_1 = \begin{pmatrix} F_0 & F_1 \end{pmatrix}. \tag{16d}$$

We also have

$$\Pi^{-1}(\sqrt{\epsilon})X\Pi^{-1}(\sqrt{\epsilon}) = \begin{pmatrix} U_2(X,\epsilon) & \Delta(\sqrt{\epsilon})\mathbb{X}_{12} \\ \mathbb{X}_{12}^T\Delta(\sqrt{\epsilon}) & X_{22} \end{pmatrix} \tag{17}$$

where $U_2(X,\epsilon) = \begin{pmatrix} X_{00} & \sqrt{\epsilon_1}X_{01} \\ \sqrt{\epsilon_1}X_{01}^T & X_{11} \end{pmatrix}$, $\Delta(\sqrt{\epsilon}) = diag(\sqrt{\epsilon_2}I_{n_0}, \sqrt{\frac{\epsilon_2}{\epsilon_1}}I_{n_1})$.

With these notations we obtain the following partition of system (10)

$$\begin{aligned} \mathbb{B}_1^T(\epsilon)U_1(X,\epsilon) + B_2^T(\epsilon)\mathbb{X}_{12}^T + \mathbb{D}_1^T(\epsilon)U_2(X,\epsilon)\mathbb{C}_{11}(\epsilon) + D_2^T(\epsilon)\mathbb{X}_{12}^T\Delta(\sqrt{\epsilon})\mathbb{C}_{11}(\epsilon) + \\ +\mathbb{D}_1^T(\epsilon)\Delta(\sqrt{\epsilon})\mathbb{X}_{12}\mathbb{C}_{21}(\epsilon) + D_2^T(\epsilon)X_{22}\mathbb{C}_{21}(\epsilon) + \mathbb{L}_1^T + \Gamma(X,\epsilon)\mathbb{F}_1 &= 0 \end{aligned} \tag{18a}$$

$$\begin{aligned} \mathbb{B}_1^T(\epsilon)\Delta(\epsilon)\mathbb{X}_{12} + B_2^T(\epsilon)X_{22} + \mathbb{D}_1^T(\epsilon)U_2(X,\epsilon)\mathbb{C}_{12}(\epsilon) + D_2^T(\epsilon)\mathbb{X}_{12}^T\Delta(\sqrt{\epsilon})\mathbb{C}_{12}(\epsilon) + \\ +\mathbb{D}_1^T(\epsilon)\Delta(\sqrt{\epsilon})X_{12}C_{22}(\epsilon) + D_2^T(\epsilon)X_{22}C_{22}(\epsilon) + L_2^T + \Gamma(X,\epsilon)F_2 &= 0 \end{aligned} \tag{18b}$$

$$\begin{aligned} \mathbb{A}_{11}^T(\epsilon)U_1(X,\epsilon) + U_1^T(X,\epsilon)\mathbb{A}_{11}(\epsilon) + \mathbb{A}_{21}^T(\epsilon)\mathbb{X}_{12}^T + \mathbb{X}_{12}\mathbb{A}_{21}(\epsilon) + \\ \mathbb{C}_{11}^T(\epsilon)U_2(X,\epsilon)\mathbb{C}_{11}(\epsilon) + \mathbb{C}_{21}^T(\epsilon)\mathbb{X}_{12}^T\Delta(\sqrt{\epsilon})\mathbb{C}_{11}(\epsilon) + \mathbb{C}_{11}^T(\epsilon)\Delta(\sqrt{\epsilon})\mathbb{X}_{12}\mathbb{C}_{21}(\epsilon) + \\ +\mathbb{C}_{21}^T(\epsilon)X_{22}\mathbb{C}_{21}(\epsilon) - \mathbb{F}_1^T\Gamma(X,\epsilon)\mathbb{F}_1 + \mathbb{M}_{11} &= 0 \end{aligned} \tag{18c}$$

$$\begin{aligned} \mathbb{A}_{11}^T(\epsilon)\Delta(\epsilon)\mathbb{X}_{12} + \mathbb{A}_{21}^T(\epsilon)X_{22} + U_1^T(X,\epsilon)\mathbb{A}_{12}(\epsilon) + \mathbb{X}_{12}A_{22}(\epsilon) + \\ \mathbb{C}_{11}^T(\epsilon)U_2(X,\epsilon)\mathbb{C}_{12}(\epsilon) + \mathbb{C}_{21}^T(\epsilon)\mathbb{X}_{12}^T\Delta(\sqrt{\epsilon})\mathbb{C}_{12}(\epsilon) + \mathbb{C}_{11}^T(\epsilon)\Delta(\sqrt{\epsilon})\mathbb{X}_{12}C_{22}(\epsilon) + \\ \mathbb{C}_{21}^T(\epsilon)X_{22}C_{22}(\epsilon) - \mathbb{F}_1^T\Gamma(X,\epsilon)F_2 + \mathbb{M}_{12} &= 0 \end{aligned} \tag{18d}$$

$$\begin{aligned} \mathbb{A}_{12}^T(\epsilon)\Delta(\epsilon)\mathbb{X}_{12} + A_{22}^T(\epsilon)X_{22} + \mathbb{X}_{12}^T\Delta(\epsilon)\mathbb{A}_{12}(\epsilon) + X_{22}A_{22}(\epsilon) + \mathbb{C}_{12}^T(\epsilon)U_2(X,\epsilon)\mathbb{C}_{12}(\epsilon) + \\ +C_{22}^T(\epsilon)\mathbb{X}_{12}^T\Delta(\sqrt{\epsilon})\mathbb{C}_{12}(\epsilon) + \mathbb{C}_{12}^T(\epsilon)\Delta(\sqrt{\epsilon})\mathbb{X}_{12}C_{22}(\epsilon) + C_{22}^T(\epsilon)X_{22}C_{22}(\epsilon) - \\ -F_2^T\Gamma(X,\epsilon)F_2 + M_{22} &= 0 \end{aligned} \tag{18e}$$

$$\begin{aligned} \Gamma(X,\epsilon) = R + \mathbb{D}_1^T(\epsilon)U_2(X,\epsilon)\mathbb{D}_1(\epsilon) + D_2^T(\epsilon)\mathbb{X}_{12}^T\Delta(\sqrt{\epsilon})\mathbb{D}_1(\epsilon) + \\ +\mathbb{D}_1^T(\epsilon)\Delta(\sqrt{\epsilon})\mathbb{X}_{12}D_2(\epsilon) + D_2^T(\epsilon)X_{22}D_2(\epsilon) \end{aligned} \tag{18f}$$

Setting formally $\epsilon_j = 0$, $j = 1,2$ and $\frac{\epsilon_2}{\epsilon_1} = 0$, in (18) we obtain the equations:

$$\begin{aligned}
\mathbb{B}_1^T(0)U_1(X,0) + B_2^T(0)\mathbb{X}_{12}^T + \mathbb{D}_1^T(0)U_2(X,0)\mathbb{C}_{11}(0) + & \\
+D_2^T(0)X_{22}C_{22} + \mathbb{L}_1^T + \Gamma(X,0)\mathbb{F}_1 &= 0 \qquad (19a)\\
B_2^T(0)X_{22} + \mathbb{D}_1^T(0)U_2(X,0)\mathbb{C}_{12}(0) + D_2^T(0)X_{22}C_{22}(0) + L_2^T + \Gamma(X,0)F_2 &= 0 \qquad (19b)\\
\mathbb{A}_{11}^T(0)U_1(X,0) + U_1^T(X,0)\mathbb{A}_{11}(0) + \mathbb{A}_{12}^T(0)\mathbb{X}_{12}^T + \mathbb{X}_{12}\mathbb{A}_{21}(0) + & \\
+\mathbb{C}_{11}^T(0)U_2(X,0)\mathbb{C}_{11}(0) + \mathbb{C}_{21}^T(0)X_{22}\mathbb{C}_{21}(0) - \mathbb{F}_1^T\Gamma(X,0)\mathbb{F}_1 + \mathbb{M}_{11} &= 0 \qquad (19c)\\
\mathbb{A}_{21}^T(0)X_{22} + U_1^T(X,0)\mathbb{A}_{12}(0) + \mathbb{X}_{12}A_{22}(0) + \mathbb{C}_{11}^T(0)U_2(X,0)\mathbb{C}_{12}(0) + & \\
+\mathbb{C}_{21}^T(0)X_{22}C_{22}(0) - \mathbb{F}_1^T\Gamma(X,0)F_2 + \mathbb{M}_{12} &= 0 \qquad (19d)\\
A_{22}^T(0)X_{22} + X_{22}A_{22}(0) + \mathbb{C}_{12}^T(0)U_2(X,0)\mathbb{C}_{12}(0) + & \\
+C_{22}^T(0)X_{22}C_{22}(0) - F_2^T\Gamma(X,0)F_2 + M_{22} &= 0 \qquad (19e)\\
\Gamma(X,0) = R + \mathbb{D}_1^T(0)U_2(X,0)\mathbb{D}_1(0) + D_2^T(0)X_{22}D_2(0). & \qquad (19f)
\end{aligned}$$

Having in mind (15) and (16), we remark that (19) is a system of nonlinear algebraic equations with the unknowns $(X_{00}, X_{01}, X_{11}, X_{02}, X_{12}, X_{22}, F_0, F_1, F_2) \in \mathcal{S}_{n_0} \times \mathbb{R}^{n_0\times n_1} \times \mathcal{S}_{n_1} \times \mathbb{R}^{n_0\times n_2} \times \mathbb{R}^{n_1\times n_2} \times \mathcal{S}_{n_2} \times \mathbb{R}^{m\times n_0} \times \mathbb{R}^{m\times n_1} \times \mathbb{R}^{m\times n_2}$.

We recall that S_q denotes the linear space of symmetric matrices of size $q \times q$.

Assuming that $A_{22}(0)$ is invertible we obtain from (19d):

$$\begin{aligned}
\mathbb{X}_{12} = & -\mathbb{A}_{21}^T(0)X_{22}A_{22}^{-1}(0) - U_1^T(X,0)\mathbb{A}_{12}(0)A_{22}^{-1}(0) - \mathbb{C}_{11}^T(0)U_2(X,0)\mathbb{C}_{12}(0)A_{22}^{-1}(0) \qquad (20)\\
& -\mathbb{C}_{21}^T(0)X_{22}C_{22}(0)\mathbb{A}_{22}^{-1}(0) + \mathbb{F}_1^T\Gamma(X,0)F_2A_{22}^{-1}(0) - \mathbb{M}_{12}A_{22}^{-1}(0).
\end{aligned}$$

Substituting (20) in (19a) and (19c) we obtain after algebraic calculations:

$$\begin{aligned}
&(\mathbb{B}_1(0) - \mathbb{A}_{12}(0)A_{22}^{-1}(0)B_2(0))^T U_1(X,0) + (\mathbb{D}_1(0) - \mathbb{C}_{12}(0)A_{22}^{-1}(0)B_2(0))^T U_2(X,0)\\
&\times(\mathbb{C}_{11}(0) - \mathbb{C}_{12}(0)A_{22}^{-1}(0)\mathbb{C}_{21}(0)) + (D_2(0) - C_{22}(0)A_{22}^{-1}(0)B_2(0))^T X_{22}(\mathbb{C}_{21}(0) -\\
&-C_{22}(0)A_{22}^{-1}(0)\mathbb{A}_{21}(0)) + (I_m + F_2A_{22}^{-1}(0)B_2(0))^T\Gamma(X,0)(\mathbb{F}_1 - F_2A_{22}^{-1}(0)\mathbb{A}_{21}(0)) + \qquad (21a)\\
&+(\mathbb{L}_1 + \mathbb{M}_{12}A_{22}^{-1}(0)B_2(0))^T - (L_2 - M_{22}A_{22}^{-1}(0)B_2(0))^T A_{22}^{-1}(0)\mathbb{A}_{21}(0) = 0\\
&(\mathbb{A}_{11}(0) - \mathbb{A}_{12}(0)A_{22}^{-1}(0)\mathbb{A}_{21}(0))^T U_1(X,0) + U_1^T(X,0)(\mathbb{A}_{11}(0) - \mathbb{A}_{12}(0)A_{22}^{-1}\mathbb{A}_{21}(0)) +\\
&+(\mathbb{C}_{11}(0) - \mathbb{C}_{12}(0)A_{22}^{-1}(0)\mathbb{A}_{21}(0))^T U_2(X,0)(\mathbb{C}_{11}(0) - \mathbb{C}_{12}(0)A_{22}^{-1}(0)\mathbb{A}_{21}(0)) +\\
&+(\mathbb{C}_{21}(0) - C_{22}(0)A_{22}^{-1}(0)\mathbb{A}_{21}(0))^T X_{22}(\mathbb{C}_{21}(0) - C_{22}(0)A_{22}^{-1}(0)\mathbb{A}_{21}(0)) -\\
&-(\mathbb{F}_1 - F_2A_{22}^{-1}(0)\mathbb{A}_{21}(0))^T\Gamma(X,0)(\mathbb{F}_1 - F_2A_{22}^{-1}(0)\mathbb{A}_{21}(0)) + \mathbb{M}_{11} - \mathbb{M}_{12}A_{22}^{-1}(0)\mathbb{A}_{21}(0) - \qquad (21b)\\
&-\mathbb{A}_{21}^T(0)A_{22}^{-T}(0)\mathbb{M}_{12}^T + \mathbb{A}_{21}^T(0)A_{22}^{-T}(0)M_{22}A_{22}^{-1}(0)\mathbb{A}_{21}(0) = 0
\end{aligned}$$

Using (3) written for $(\epsilon_1, \epsilon_2) = (0,0)$ we introduce the notations

$$\begin{aligned}
\begin{pmatrix} A_{00}^1 & A_{01}^1 \\ A_{10}^1 & A_{11}^1 \end{pmatrix} &\triangleq \mathbb{A}_{11}(0) - \mathbb{A}_{12}(0)A_{22}^{-1}(0)\mathbb{A}_{21}(0) && \text{(22a)}\\
\begin{pmatrix} C_{00}^1 & C_{01}^1 \\ C_{10}^1 & C_{11}^1 \end{pmatrix} &\triangleq \mathbb{C}_{11}(0) - \mathbb{C}_{12}(0)A_{22}^{-1}(0)\mathbb{A}_{21}(0) && \text{(22b)}\\
\begin{pmatrix} C_{20}^1 & C_{22}^1 \end{pmatrix} &\triangleq \mathbb{C}_{21}(0) - C_{22}(0)A_{22}^{-1}(0)\mathbb{A}_{21}(0) && \text{(22c)}\\
\begin{pmatrix} B_0^1 \\ B_1^1 \end{pmatrix} &\triangleq \mathbb{B}_1(0) - \mathbb{A}_{12}(0)A_{22}^{-1}(0)B_2(0) && \text{(22d)}\\
\begin{pmatrix} D_0^1 \\ D_1^1 \end{pmatrix} &\triangleq \mathbb{D}_1(0) - \mathbb{C}_{12}(0)A_{22}^{-1}(0)B_2(0) && \text{(22e)}\\
D_2^1 &\triangleq D_2(0) - C_{22}(0)A_{22}^{-1}(0)B_2(0) && \text{(22f)}\\
\begin{pmatrix} M_{00}^1 & M_{01}^1 \\ (M_{01}^1)^T & M_{11}^1 \end{pmatrix} &\triangleq \mathbb{M}_{11} - \mathbb{M}_{12}A_{22}^{-1}(0)\mathbb{A}_{21}(0) - \mathbb{A}_{21}^T(0)A_{22}^{-T}(0)M_{12}^T + \\
&\quad +\mathbb{A}_{21}^T(0)A_{22}^{-T}(0)M_{22}A_{22}^{-1}(0)\mathbb{A}_{21}(0) && \text{(22g)}\\
\begin{pmatrix} L_0^1 \\ L_1^1 \end{pmatrix} &\triangleq \mathbb{L}_1 - \mathbb{A}_{21}^T(0)A_{22}^{-T}(0)L_2 - (\mathbb{M}_{12} - \mathbb{A}_{21}^T(0)A_{22}^{-T}(0)M_{22})A_{22}^{-1}(0)B_2(0) && \text{(22h)}\\
R^1 &= R - L_2^T A_{22}^{-1}(0)B_2(0) - B_2^T(0)A_{22}^{-T}(0)L_2 + \\
&\quad + B_2^T(0)A_{22}^{-T}(0)M_{22}A_{22}^{-1}(0)B_2(0). && \text{(22i)}
\end{aligned}$$

The next result allows us to reduce the number of equations and the number of unknowns of system (19).

Lemma 1. *Assume that $A_{22}(0)$ is invertible.*

(i) If $(X_{00}, X_{01}, X_{11}, X_{02}, X_{12}, X_{22}, F_0, F_1, F_2)$ is a solution of system (19) with the property that $A_{22}(0) + B_2(0)F_2$ is an invertible matrix, then $(X_{00}, X_{01}, X_{11}, X_{22}, F_0^1, F_1^1, F_2)$ is a solution of the following system

$$(B_0^1)^T X_{00} + (B_1^1)^T X_{01}^T + \sum_{j=0}^{2} (D_j^1)^T X_{jj} C_{j0}^1 + (L_0^1)^T + \Gamma^1(X_{00}, X_{11}, X_{22}) F_0^1 = 0 \tag{23a}$$

$$(B_1^1)^T X_{11} + \sum_{j=0}^{2} (D_j^1)^T X_{jj} C_{j1}^1 + (L_1^1)^T + \Gamma^1(X_{00}, X_{11}, X_{22}) F_1^1 = 0 \tag{23b}$$

$$B_2^T(0) X_{22} + \sum_{j=0}^{2} D_j^T(0) X_{jj} C_{j2}(0) + L_2^T + \Gamma(X_{00}, X_{11}, X_{22}) F_2 = 0 \tag{23c}$$

$$\begin{aligned}(A_{00}^1)^T X_{00} + (A_{10}^1)^T X_{01}^T + X_{00} A_{00}^1 + X_{01} A_{10}^1 + \sum_{j=0}^{2} (C_{j0}^1)^T X_{jj} C_{j0}^1 - \\ -(F_0^1)^T \Gamma^1(X_{00}, X_{11}, X_{22}) F_0^1 + M_{00}^1 = 0\end{aligned} \tag{23d}$$

$$\begin{aligned}(A_{10}^1)^T X_{11} + X_{00} A_{01}^1 + X_{01} A_{11}^1 + \sum_{j=0}^{2} (C_{j0}^1)^T X_{jj} C_{j1}^1 - \\ (F_0^1)^T \Gamma^1(X_{00}, X_{11}, X_{22}) F_1^1 + M_{01}^1 = 0\end{aligned} \tag{23e}$$

$$(A_{11}^1)^T X_{11} + X_{11} A_{11}^1 + \sum_{j=0}^{2} (C_{j1}^1)^T X_{jj} C_{j1}^1 - (F_1^1)^T \Gamma^1(X_{00}, X_{11}, X_{22}) F_1^1 + M_1^1 = 0 \tag{23f}$$

$$A_{22}^T(0) X_{22} + X_{22} A_{22}(0) + \sum_{j=0}^{2} C_{j2}^T(0) X_{jj} C_{j2}(0) - F_2^T \Gamma(X_{00}, X_{11}, X_{22}) F_2 + M_{22} = 0 \tag{23g}$$

$$\Gamma^1(X_{00}, X_{11}, X_{22}) = R^1 + \sum_{j=0}^{2} (D_j^1)^T X_{jj} D_j^1 \tag{23h}$$

$$\Gamma(X_{00}, X_{11}, X_{22}) = R + \sum_{j=0}^{2} D_j^T(0) X_{jj} D_j(0) \tag{23i}$$

where

$$F_j^1 \triangleq (I_m + F_2 A_{22}^{-1}(0) B_2(0))^{-1} (F_j - F_2 A_{22}^{-1}(0) A_{2j}(0)), \quad j = 0, 1. \tag{24}$$

(ii) If $(X_{00}, X_{01}, X_{11}, X_{22}, F_0^1, F_1^1, F_2)$ is a solution of system (23) with the property that $A_{22}(0) + B_2(0)F_2$ is an invertible matrix, then $(X_{00}, X_{01}, X_{11}, X_{02}, X_{12}, F_0, F_1, F_2)$ is a solution of system (19) where

$$F_j = (I_m + F_2 A_{22}^{-1}(0) B_2(0)) F_j^1 + F_2 A_{22}^{-1}(0) A_{2j}(0), \quad j = 0, 1 \tag{25}$$

and

$$\begin{aligned} X_{02} = & -[A_{20}^T(0) X_{22} + X_{00} A_{02}(0) + X_{01} A_{12}(0) + \sum_{j=0}^{2} C_{j0}^T(0) X_{jj} C_{j2}(0) - \\ & -(F_0)^T (R + \sum_{j=0}^{2} D_j^T(0) X_{jj} D_j(0)) + F_2 + M_{02}] A_{22}^{-1}(0) \end{aligned} \tag{26a}$$

$$\begin{aligned} X_{12} = & -[A_{21}^T(0) X_{22} + X_{11} A_{12}(0) + \sum_{j=0}^{2} C_{j1}^T(0) X_{jj} C_{j2}(0) - \\ & -F_1^T (R + \sum_{j=0}^{2} D_j^T(0) X_{jj} D_j(0)) F_2 + M_{12}] A_{22}^{-1}(0) \end{aligned} \tag{26b}$$

Proof. The result follows directly combining (21) with (19b), (19c) and taking into account (22). It is worth noticing that if $A_{22}(0)$ and $A_{22}(0) + B_2(0)F_2$ are invertible, then $I_m + F_2 A_{22}^{-1}(0)B_2(0)$ is invertible too. $\square$

Assuming that A_{11}^1 is invertible we may compute X_{01} from (23e) as:

$$\begin{aligned} X_{01} &= -[(A_{10}^1)^T X_{11} + X_{00}A_{01}^1 + \sum_{j=0}^{2}(C_{j0}^1)^T X_{jj} C_{j1}^1 - \\ &\quad -(F_0^1)^T(R^1 + \sum_{j=0}^{2}(D_j^1)^T X_{jj} D_j^1)F_1^1 + M_{01}^1](A_{11}^1)^{-1}. \end{aligned} \tag{27}$$

Substituting (27) in (23a) and (23d) we obtain after some algebraic calculation the equations:

$$\begin{aligned} &(B_{01}^1 - A_{01}^1(A_{11}^1)^{-1}B_1^1)^T X_{00} + \sum_{j=0}^{2}(D_j^1 - C_{j1}^1(A_{11}^1)^{-1}B_1^1)^T X_{jj} \times \\ &(C_{j0}^1 - C_{j1}^1(A_{11}^1)^{-1}A_{10}^1) + (I_m + F_1^1(A_{11}^1)^{-1}B_1^1)\Gamma^1(X_{00}, X_{11}, X_{22})(F_0^1 - F_1^1(A_{11}^1)^{-1}A_{10}^1) \\ &+(L_0^1 - M_{01}^1(A_{11}^1)^{-1}B_1^1)^T - (R_1^1)^T(A_{11}^1)^{-1}A_{10}^1 + (B_1^1)^T(A_{11}^1)^{-T}M_{11}^1(A_{11}^1)^{-1}A_{10}^1 = 0 \end{aligned} \tag{28a}$$

$$\begin{aligned} &(A_{00}^1 - A_{01}^1(A_{11}^1)^{-1}A_{10}^1)^T X_{00} + X_{00}(A_{00}^1 - A_{01}^1(A_{11}^1)^{-1}A_{10}^1) + \sum_{j=0}^{2}(C_{j0}^1 - C_{j1}^1(A_{11}^1)^{-1}A_{10}^1)^T \times \\ &X_{jj}(C_{j0}^1 - C_{j1}^1(A_{11}^1)^{-1}A_{10}^1) - (F_0^1 - F_1^1(A_{11}^1)^{-1}A_{10}^1)^T\Gamma^1(X_{00}, X_{11}, X_{22})(F_0^1 - F_1^1(A_{11}^1)^{-1}A_{10}^1) \\ &+M_{00}^1 - (A_{10}^1)^T(A_{11}^1)^{-T}(M_{01}^1)^T - M_{01}^1(A_{11}^1)^{-1}A_{10}^1 + (A_{10}^1)^T(A_{11}^1)^{-T}M_{11}^1(A_{11}^1)^{-1}A_{10}^1 = 0 \end{aligned} \tag{28b}$$

We introduce the notations:

$$A_{00}^0 = A_{00}^1 - A_{01}^1(A_{11}^1)^{-1}A_{10}^1 \tag{29a}$$
$$B_0^0 = B_0^1 - A_{01}^1(A_{11}^1)^{-1}B_1^1 \tag{29b}$$
$$C_{j0}^0 = C_{j0}^1 - C_{j1}^1(A_{11}^1)^{-1}A_{10}^1 \tag{29c}$$
$$D_j^0 = D_j^1 - C_j^1(A_{11}^1)^{-1}B_1^1, 0 \le j \le 2 \tag{29d}$$
$$\begin{aligned} M_{00}^0 &= M_{00}^1 - (A_{10}^1)^T(A_{11}^1)^{-T}(M_{01}^1)^T - M_{01}^1(A_{11}^1)^{-1}A_{10}^1 + \\ &\quad +(A_{10}^1)^T(A_{11}^1)^{-T}M_{11}^1(A_{11}^1)^{-1}A_{10}^1 \end{aligned} \tag{29e}$$
$$L_0^0 = L_0^1 - (A_{10}^1)^T(A_{11}^1)^{-T}L_1^1 - (M_{01}^1 - (A_{10}^1)^T(A_{11}^1)^{-T}M_{11}^1)(A_{11}^1)^{-1}B_1^1 \tag{29f}$$
$$R^0 = R^1 - (B_1^1)^T(A_{11}^1)^{-T}L_1^1 - (L_1^1)^T(A_{11}^1)^{-1}B_1^1 + (B_1^1)^T(A_{11}^1)^{-T}M_{11}^1(A_{11}^1)^{-1}B_1^1. \tag{29g}$$

The next result allows us to reduce the number of unknowns and the number of the equations in system (23).

Lemma 2. *Assume that the matrices $A_{22}(0)$ and A_{11}^1 are invertible.*

(i) If $(X_{00}, X_{01}, X_{11}, X_{22}, F_0^1, F_1^1, F_2)$ is a solution of system (23) such that $A_{11}^1 + B_1^1 F_1^1$ is an invertible matrix, then $(X_{00}, X_{11}, X_{22}, F_0^1, F_1^1, F_2^2)$ is a solution of the following system:

$$(B_{00}^0)^T X_{00} + \sum_{j=0}^{2} (D_j^0)^T X_{jj} C_{j0}^0 + (L_0^0)^T + \Gamma^0(X_{00}, X_{11}, X_{22}) F_0^0 = 0 \tag{30a}$$

$$(B_{11}^1)^T X_{11} + \sum_{j=0}^{2} (D_j^1)^T X_{jj} C_{j1}^1 + (L_1^1)^T + \Gamma^1(X_{00}, X_{11}, X_{22}) F_1^1 = 0 \tag{30b}$$

$$(B_2^2)^T X_{22} + \sum_{j=0}^{2} (D_j^2)^T X_{jj} C_{j2}^2 + (L_2^2)^T + \Gamma^2(X_{00}, X_{11}, X_{22}) F_2^2 = 0 \tag{30c}$$

$$(A_{00}^0)^T X_{00} + X_{00} A_{00}^0 + \sum_{j=0}^{2} (C_{j0}^0)^T X_{jj} C_{j0}^0 - (F_0^0)^T \Gamma^0(X_{00}, X_{11}, X_{22}) F_0^0 + M_{00}^0 = 0 \tag{30d}$$

$$(A_{11}^1)^T X_{11} + X_{11} A_{11}^1 + \sum_{j=0}^{2} (C_{j1}^1)^T X_{jj} C_{j1}^1 - (F_1^1)^T \Gamma^1(X_{00}, X_{11}, X_{22}) F_1^1 + M_{11}^1 = 0 \tag{30e}$$

$$(A_{22}^2)^T X_{22} + X_{22} A_{22}^2 + \sum_{j=0}^{2} (C_{j2}^2)^T X_{jj} C_{j2}^2 - (F_2^2)^T \Gamma^2(X_{00}, X_{11}, X_{22}) F_2^2 + M_{22}^2 = 0 \tag{30f}$$

$$\Gamma^k(X_{00}, X_{11}, X_{22}) \triangleq R^k + \sum_{j=0}^{2} (D_j^k)^T X_{jj} D_j^k, k = 0, 1, 2 \tag{30g}$$

where

$$F_0^0 \triangleq (I_m + F_1^1 (A_{11}^1)^{-1} B_1^1)^{-1} (F_0^1 - F_1^1 (A_{11}^1)^{-1} A_{10}^1) \tag{31a}$$

$$F_2^2 \triangleq F_2 \tag{31b}$$

and

$$\begin{aligned} A_{22}^2 &\triangleq A_{22}(0),\ B_2^2 \triangleq B_2(0),\ C_{j2}^2 \triangleq C_{j2}(0),\ D_j^2 \triangleq D_j(0), \quad 0 \le j \le 2, \\ L_2^2 &= L_2, \quad M_{22}^2 = M_{22}, \quad R^2 \triangleq R \end{aligned} \tag{32}$$

(ii) If $(X_{00}, X_{11}, X_{22}, F_0^0, F_1^1, F_2^2)$ is a solution of system (30) with the property that $A_{11}^1 + B_{11}^1 F_1^1$ is an invertible matrix, then $(X_{00}, X_{01}, X_{11}, X_{22}, F_0^1, F_1^1, F_2)$ is a solution of system (23), where

$$\begin{aligned} F_0^1 &= (I_m + F_1^1 (A_{11}^1)^{-1} B_1^1) F_0^0 + F_1^1 (A_{11}^1)^{-1} A_{10}^1 \\ F_2 &= F_2^2 \end{aligned} \tag{33}$$

and X_{01} is computed via (27).

Proof. The proof may be done by direct calculation implying (23), (27), (33). The notations (32) were adopted only for the sake of symmetry of the equations (30). □

For the values of X_{jj} for which the matrices $\Gamma^k(X_{00}, X_{11}, X_{22})$ are invertible, we may eliminate the unknowns F^k_{kk} from (30) obtaining the following system of nonlinear equations with the unknown $(X_0, X_1, X_2) := (X_{00}, X_{11}, X_{22})$:

$$(A^k_{kk})^T X_k + X_k A^k_{kk} + \sum_{j=0}^{2}(C^k_{jk})^T X_j C^k_{jk} - (X_k B^k_k + \sum_{j=0}^{2}(C^k_{jk})^T X_j D^k_j + L^k_k) \times$$

$$(\Gamma^k(X_0, X_1, X_2))^{-1}((B^k_k)^T X_k + \sum_{j=0}^{2}(D^k_j)^T X_j C^k_{jk} + (L^k_k)^T) + M^k_{kk} = 0, \tag{34a}$$

$$\Gamma^k(X_0, X_1, X_2) = R^k + \sum_{j=0}^{2}(D^k_j)^T X_j D^k_j, \quad k = 0, 1, 2. \tag{34b}$$

Remark 2. *(a) In the deterministic case, i.e., the special case of (2) when $C_{jk}(\epsilon) = 0$, $D_j(\epsilon) = 0$, $j, k = 0, 1, 2$, system (34) reduces to*

$$(A^k_{kk})^T X_k + X_k A^k_{kk} - (X_k B^k_k + L^k_k)(R^k)^{-1}((B^k_k)^T X_k + (L^k_k)^T) + M^k_k = 0, k = 0, 1, 2. \tag{35}$$

System (35) is a system of three uncoupled algebraic Riccati equations of lower dimensions named the system of reduced algebraic Riccati equations (for details see e.g., [24]). That is why, in the sequel, system (34) will be named **system of reduced algebraic Riccati equations** *(SRARE), associated with SARE (8). We shall see that in this stochastic framework, system (34), plays a similar role as system (35) in the deterministic case. Unlike the deterministic case, where the system of reduced algebraic Riccati Equation (35) is obtained by simply removing the small parameters ϵ_k, $k = 1, 2$ in the controlled system, in the stochastic framework SRARE (34) cannot be obtained directly by such a procedure.*

(b) When $C_{jk}(0) = 0$, $D_j(0) = 0$, $j = 1, 2$, $k = 0, 1, 2$, system (34) becomes the system of reduced algebraic Riccati equations derived in [16]. In this special case (34) is:

$$\begin{aligned}(A^0_{00})^T X_0 + X_0 A^0_{00} + (C^0_{00})^T X_0 C^0_{00} - (X_0 B^0_0 + (C^0_{00})^T X_0 D^0_0 + L^0_0 \times \quad) \\ (R^0 + (D^0_0)^T X_0 D^0_0)^{-1}((B^0_0)^T X_0 + (D^0_0)^T X_0 C^0_{00} + (L^0_0)^T) + M^0_{00} = 0\end{aligned} \tag{36a}$$

$$\begin{aligned}(A^1_{11})^T X_1 + X_1 A^1_{11} + (C^1_{01})^T X_0 C^1_{01} - (X_1 B^1_1 + (C^1_{01})^T X_0 D^1_0 + L^1_1) \times \\ (R^1 + (D^1_0)^T X_0 D^1_0)^{-1}((B^1_1)^T X_1 + (D^1_0)^T X_0 C^1_{01} + (L^1_1)^T) + M^1_{11} = 0\end{aligned} \tag{36b}$$

$$\begin{aligned}(A^2_{22})^T X_2 + X_2 A^2_{22} + (C^2_{02})^T X_0 C^2_{02} - (X_2 B^2_2 + (C^2_{02})^T X_0 D^2_0 + L^2_2) \times \\ (R^2 + (D^2_0)^T X_0 D^2_0)^{-1}((B^2_2)^T X_2 + (D^2_0)^T X_0 C^2_{02} + (L^2_2)^T) + M^2_{22} = 0.\end{aligned} \tag{36c}$$

One sees that (36a) is the SARE of type (8) associated with the stochastic reduced linear quadratic optimal control problem described by

$$dx_0(t) = (A^0_{00} x_0(t) + B^0_0 u(t))dt + (C^0_{00} x_0(t) + D^0_0 u(t))dw(t), \quad x_0(0) = x^0_0$$

and

$$J_0(x^0_0; u) = \mathbb{E}[\int_0^\infty (x^T_0(t) M^0_{00} x_0(t) + 2x^T_0(t) L^0_0 u(t) + u^T(t) R^0 u(t))dt].$$

The equation (36b) and (36c) can be interpreted as algebraic Riccati equations associated with some deterministic reduced linear quadratic control problems described by:

$$\dot{x}_k(t) = A^k_{kk} x_k(t) + B^k_k u(t)$$

$$x_k(0) = x^0_k$$

and

$$J_k(x_k^0, u) = \int_0^\infty (x_k^T(t)\tilde{M}_k x_k(t) + 2x_k^T(t)\tilde{L}_k u(t) + u^T(t)\tilde{R}_k u(t)dt$$

where

$$\tilde{M}_k = M_k^k + (C_{0k}^k)^T x_0 C_{0k}^k$$
$$\tilde{L}_k = L_k^k + (C_{0k}^k)^T x_0 D_0^k$$
$$\tilde{R}_k = R^k + (D_0^k)^T x_0 D_0^k, \quad k = 1, 2.$$

The solution X_0 of SARE (36a) is involved as a parameter that affects the weights matrices from the performance criteria $J_k(x_k^0; u)$.

(c) A complete decoupling of the equations from SRARE (34) may be possible in the special case when the following conditions are simultaneously satisfied:

$$D_j(0) = 0,\ j = 0, 1, 2, C_{jk}(0) = 0,\ k = 1, 2, C_{il}(0) = 0,\ i = 1, 2,\ l = 0, 1, 2.$$

In the next subsection we introduce the concept of stabilizing solution of SRARE (34) and we shall provide a set of conditions equivalent to the existence of that solution.

3.2. The Stabilizing Solution of the SRARE

Let $\mathfrak{X}$ be the linear space defined by $\mathfrak{X} = \mathcal{S}_{n_0} \times \mathcal{S}_{n_1} \times \mathcal{S}_{n_2}$. An element $\mathbf{X}$ lies in $\mathfrak{X}$ if and only if $\mathbf{X} = (X_0, X_1, X_2)$, X_k being symmetric matrices of size $n_k \times n_k$.
On $\mathfrak{X}$ we introduce the inner product

$$< \mathbf{X}, \mathbf{Y} >= \sum_{j=0}^{2} Tr[X_j, Y_j] \tag{37}$$

for all $\mathbf{X} = (X_0, X_1, X_2)$, $\mathbf{Y} = (Y_0, Y_1, Y_2) \in \mathfrak{X}$. In (37) $Tr[\cdot]$ is the trace operator. Equipped with the inner product (37), $\mathfrak{X}$ becomes a finite dimensional real Hilbert space.

On $\mathfrak{X}$ we consider the order relation $\succcurlyeq$ induced by the closed, solid, convex cone

$$\mathfrak{X} = \{\mathbf{X} \in \mathfrak{X} | \mathbf{X} = (X_0, X_1, X_2), X_j \geq 0, j = 0, 1, 2\}.$$

Here, $X_j \geq 0$ means that X_j is a positive semidefinite matrix. In the sequel, we rewrite SRARE (34) as a generalized Riccati equation on $\mathfrak{X}$ as

$$\begin{aligned} \mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} + \mathbf{\Pi}_1[\mathbf{X}] - (\mathbf{X}\mathbf{B} + \mathbf{\Pi}_2[\mathbf{X}] + \mathbf{L}) \times \\ (\mathbf{R} + \mathbf{\Pi}_3[\mathbf{X}])^{-1}(\mathbf{X}\mathbf{B} + \mathbf{\Pi}_2[\mathbf{X}] + \mathbf{L})^T + \mathbf{M} \quad &= \quad 0 \end{aligned} \tag{38}$$

where $\mathbf{A} = (A_{00}^0, A_{11}^1, A_{22}^2) \in \mathbb{R}^{n_0 \times n_0} \times \mathbb{R}^{n_1 \times n_1} \times \mathbb{R}^{n_2 \times n_2}$, $\mathbf{B} = (B_0^0, B_1^1, B_2^2) \in \mathbb{R}^{n_0 \times m} \times \mathbb{R}^{n_1 \times m} \times \mathbb{R}^{n_2 \times m}$, $\mathbf{L} = (L_0^0, L_1^1, L_2^2) \in \mathbb{R}^{n_0 \times m} \times \mathbb{R}^{n_1 \times m} \times \mathbb{R}^{n_2 \times m}$, $\mathbf{M} = (M_0^0, M_1^1, M_2^2) \in \mathfrak{X}$, $\mathbf{R} = (R^0, R^1, R^2) \in \mathcal{S}_m \times \mathcal{S}_m \times$

$\mathcal{S}_m, \mathbf{X} \to \mathbf{\Pi}_1[\mathbf{X}] : \mathfrak{X} \to \mathfrak{X}, \mathbf{X} \to \mathbf{\Pi}_2[\mathbf{X}] : \mathfrak{X} \to \mathbb{R}^{n_0 \times m} \times \mathbb{R}^{n_1 \times m} \times \mathbb{R}^{n_2 \times n}, \mathbf{X} \to \mathbf{\Pi}_3[\mathbf{X}] : \mathfrak{X} \to \mathcal{S}_m \times \mathcal{S}_m \times \mathcal{S}_m$ are defined by

$$\begin{aligned}
\mathbf{\Pi}_1[\mathbf{X}] &= \sum_{j=0}^{2}((C_{j0}^0)^T X_j C_{j0}^0, (C_{j1}^1)^T X_j C_{j1}^1, (C_{j2}^2)^T X_j X_j C_{j2}^2) \\
\mathbf{\Pi}_2[\mathbf{X}] &= \sum_{j=0}^{2}((C_{j0}^0)^T X_j D_j^0, (C_{j1}^1)^T X_j D_j^1, (C_{j2}^2)^T X_j D_j^2) \\
\mathbf{\Pi}_3[\mathbf{X}] &= \sum_{j=0}^{2}((D_j^0)^T X_j D_j^0, (D_j^1)^T X_j D_j^1, (D_j^2)^T X_j D_j^2).
\end{aligned} \tag{39}$$

Based on the operators $\mathbf{\Pi}_k$, $k = 1,2,3$ we may define the following operator $\mathbf{X} \to \mathbf{\Pi}[\mathbf{X}] \triangleq (\mathbf{\Pi}_1[\mathbf{X}], \mathbf{\Pi}_2[\mathbf{X}], \mathbf{\Pi}_3[\mathbf{X}])$.

A feedback gain is a triple of the form $\mathbf{F} = (F_0, F_1, F_2)$ where $F_k \in \mathbb{R}^{m \times n_k}$, $k = 0,1,2$. For any feedback gain $\mathbf{F}$, we associate the following linear operator: $\mathbf{X} \to \mathcal{L}_{\mathbf{F}}[\mathbf{X}] : \mathfrak{X} \to \mathfrak{X}$ by $\mathcal{L}_{\mathbf{F}}[\mathbf{X}] = (\mathcal{L}_{\mathbf{F}0}[\mathbf{X}], \mathcal{L}_{\mathbf{F}1}[\mathbf{X}], \mathcal{L}_{\mathbf{F}2}[\mathbf{X}])$, where for each $k = 0,1,2$ we have:

$$\mathcal{L}_{\mathbf{F}k}[\mathbf{X}] = (A_{kk}^k + B_k^k F_k) X_k + X_k (A_{kk}^k + B_k^k F_k)^T + \sum_{j=0}^{2} (C_{jk}^k + D_j^k F_k) X_j (C_{jk}^k + D_j^k F_k)^T. \tag{40}$$

The next result summarizes some useful properties of the operator $\mathcal{L}_{\mathbf{F}}$.

Proposition 1. *(i) The adjoint operator $\mathcal{L}_{\mathbf{F}}^*$ of the operator $\mathcal{L}_{\mathbf{F}}$ (with respect to the inner product (37)) is given by $\mathcal{L}_{\mathbf{F}}^*[\mathbf{X}] = (\mathcal{L}_{\mathbf{F}0}^*[\mathbf{X}], \mathcal{L}_{\mathbf{F}1}^*[\mathbf{X}], \mathcal{L}_{\mathbf{F}2}^*[\mathbf{X}])$, where for each $k = 0,1,2$:*

$$\mathcal{L}_{\mathbf{F}k}^*[\mathbf{X}] = (A_{kk}^k + B_k^k F_k)^T X_k + X_k (A_{kk}^k + B_k^k F_k) + \sum_{j=0}^{2} (C_{jk}^k + D_j^k F_k)^T X_j (C_{jk}^k + D_j^k F_k). \tag{41}$$

(ii) The operator $\mathcal{L}_{\mathbf{F}}$ generates positive evolution on the space $\mathfrak{X}$ i.e., $e^{\mathcal{L}_{\mathbf{F}} t} \mathfrak{X}_+ \subset \mathfrak{X}_+$ for all $t \geq 0$.

(iii) The spectrum of the linear operator $\mathcal{L}_{\mathbf{F}}$ is located in the half plane $\mathbb{C}_- = \{\lambda \in \mathbb{C}, Re\lambda < 0\}$ if and only if there exists $\mathbf{Y} = (Y_0, Y_1, Y_2) \succ 0$ such that $\mathcal{L}_{\mathbf{F}}[\mathbf{Y}] \prec 0$.

Proof. (i) follows by direct calculation specializing the definition of the adjoint operator to the case of the operator defined in (40) and the inner product (37).

(ii) follows applying Corollary 2.2.6 from [23].

(iii) follows from the equivalence $(iv) \leftrightarrow (v)$ in the Corollary 2.3.9 from [23]. □

Now we are in the position to introduce the concept of stabilizing solution of SRARE (34).

Definition 1. *A solution $\tilde{\mathbf{X}} = (\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ of SRARE (34) is named* **stabilizing solution** *if the spectrum of the linear operator $\mathcal{L}_{\tilde{\mathbf{F}}}$ is located in the half plane $\mathbb{C}_-$, $\mathcal{L}_{\tilde{\mathbf{F}}}$ being the linear operator of type (40) associated with the feedback gain $\tilde{\mathbf{F}} = (\tilde{F}_0, \tilde{F}_1, \tilde{F}_2)$, where for each $k = 0,1,2$*

$$\tilde{F}_k \triangleq -(R^k + \sum_{j=0}^{2} (D_j^k)^T \tilde{X}_j D_j^k)^{-1} ((B_k^k)^T \tilde{X}_k + \sum_{j=0}^{2} (D_j^k)^T \tilde{X}_j C_{jk}^k + (L_k^k)^T). \tag{42}$$

Before stating the result providing the conditions which guarantee the existence of the stabilizing solution of SRARE (34), we introduce the concept of stabilizability of the triple $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$.

Definition 2. *We say that the triple* $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$ *is* **stabilizable** *if there exists a feedback gain* $\mathbf{F} = (F_0, F_1, F_2)$ *with the property that the spectrum of the corresponding linear operator* $\mathcal{L}_{\mathbf{F}}$ *of type (40) is inclosed in the half-plane* $\mathbb{C}_-$.

The next result provides a set of conditions equivalent to the stabilizability of the triple $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$.

Proposition 2. *The following are equivalent:*

(i) the triple $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$ *is stabilizable,*

(ii) there exist $\mathbf{Y} = (Y_0, Y_1, Y_2)$, $\mathbf{Z} = (Z_0, Z_1, Z_2)$, $Y_k \in \mathcal{S}_{n_k}$, $Y_k > 0$, $Z_k \in \mathbb{R}^{m \times n_k}$, $k = 0, 1, 2$, *satisfying the following system of LMIs:*

$$\begin{pmatrix} \Xi_1^k(\mathbf{Y}, \mathbf{Z}) & \Xi_2^k(\mathbf{Y}, \mathbf{Z}) \\ (\Xi_2^k(\mathbf{Y}, \mathbf{Z}))^T & \Xi_3^k(\mathbf{Y}) \end{pmatrix} < 0 \tag{43}$$

where

$$\Xi_1^k(\mathbf{Y}, \mathbf{Z}) = A_{kk}^k Y_k + Y_k (A_{kk}^k)^T + B_k^k Z_k + Z_k^T (B_k^k)^T$$

$$\Xi_2^k(\mathbf{Y}, \mathbf{Z}) = \begin{pmatrix} C_{0k}^k Y_0 + D_0^k Z_0 & C_{1k}^k Y_1 + D_1^k Z_1 & C_{2k}^k Y_2 + D_2^k Z_2 \end{pmatrix}, k = 0, 1, 2$$

$$\Xi_3^k(\mathbf{Y}) = diag(-Y_0, -Y_1, -Y_2).$$

Furthermore, if $(\mathbf{Y}, \mathbf{Z})$ *is a solution of the system of LMIs (43), then* $\mathbf{F} = (Z_0 Y_0^{-1}, Z_1 Y_1^{-1}, Z_2 Y_2^{-1})$ *is a stabilizing feedback gain.*

Proof. Following from (iii) of Proposition 3 combined with Schur complement technique. □

To obtain the asymptotic structure of the stabilizing solution of SARE (8) satisfying the sign condition (9), we shall look for conditions under which SRARE (34) has a stabilizing solution $\tilde{\mathbf{X}} = (\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ satisfying the sign conditions

$$R^k + \sum_{j=0}^{2} (D_j^k)^T \tilde{X}_j D_j^k > 0, \quad k = 0, 1, 2. \tag{44}$$

Theorem 1. *Assume that the matrices* $A_{22}^2 \triangleq A_{22}(0)$ *and* $A_{11}^1 \triangleq A_{11}(0) - A_{12}(0)A_{22}^{-1}(0)A_{21}(0)$ *are invertible. Under these conditions the following are equivalent:*

(i) (a) the triple $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$ *is stabilizable,*

(b) there exists $\mathbf{Y} = (Y_0, Y_1, Y_2) \in \mathfrak{X}$ *satisfying the following system of LMIs*

$$\begin{pmatrix} \Theta_{1k}(\mathbf{Y}) + M_{kk}^k & \Theta_{2k}(\mathbf{Y}) + L_k^k \\ (\Theta_{2k}(\mathbf{Y}) + L_k^k)^T & \Theta_{3k}(\mathbf{Y}) + R^k \end{pmatrix} > 0 \tag{45}$$

where

$$\Theta_{1k}(\mathbf{Y}) = (A_{kk}^k)^T Y_k + Y_k A_{kk}^k + \sum_{j=0}^{2} (C_{jk}^k)^T Y_j C_{jk}^k$$

$$\Theta_{2k}(\mathbf{Y}) = Y_k B_k^k + \sum_{j=0}^{2} (C_{jk}^k)^T Y_j D_j^k$$

$$\Theta_{3k}(\mathbf{Y}) = \sum_{j=0}^{2} (D_j^k)^T Y_j D_j^k, k = 0, 1, 2,$$

(ii) the SRARE (34) has a unique stabilizing solution $\tilde{\mathbf{X}} = (\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ *satisfying the sign conditions (44).*

Proof. (hint) $(i) \Rightarrow (ii)$. For each $p = 0, 1, ...$ one computes $\mathbf{X}^{p+1} = (X_0^{p+1}, X_1^{p+1}, X_2^{p+1})$ as the unique solution of the linear equation on $\mathfrak{X}$

$$\mathcal{L}^*_{\mathbf{F}^p}[\mathbf{X}^{p+1}] + \mathbf{M} + \mathbf{L}\mathbf{F}^p + (\mathbf{F}^p)^T\mathbf{L}^T + (\mathbf{F}^p)^T\mathbf{R}\mathbf{F}^p + \frac{\gamma^2}{p+1}\mathbb{I} = 0 \tag{46}$$

where $\mathcal{L}^*_{\mathbf{F}^p}$ is the adjoint of the linear operator $\mathcal{L}_{\mathbf{F}^p}$ described by (41) with $\mathbf{F}$ replaced by $\mathbf{F}^p$, $\mathbb{I} = (I_{n_0}, I_{n_1}, I_{n_2}) \in \mathfrak{X}$. In (46), $\mathbf{F}^p = (F_0^p, F_1^p, F_2^p)$ are given by

$$F_k^p = -(R^k + \sum_{j=0}^{2}(D_j^k)^T X_j^p D_j^k)^{-1}((B_k^k)^T X_k^p + \sum_{j=0}^{2}(D_j^k)^T X_j^p C_{jk}^k + (L_k^k)^T), \quad p \geq 1. \tag{47}$$

When $p = 0$ the feedback gain $\mathbf{F}^0 = (F_0^0, F_1^0, F_2^0)$ is obtained based on the assumption of stabilizability of the triple $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$. It has the property that the spectrum of the corresponding linear operator $\mathcal{L}_{\mathbf{F}^0}$ is located in the half place $\mathbb{C}_-$. One shows inductively for $p = 1, 2, ...$ that the spectrum of each operator $\mathcal{L}_{\mathbf{F}^p}$ is located in the half plane $\mathbb{C}_-$. Hence, $\mathbf{X}^{p+1}$ is well defined as the unique solution of the linear equation (46). Moreover, based on the assumption (i) b) from the statement, one gets that $\mathbf{X}^p \succeq \mathbf{X}^{p+1} \succeq \mathbf{Y}$, $\forall p \geq 0$, where $\mathbf{Y}$ is a solution of the LMIs (45). So, we have obtained that the sequence $\{\mathbf{X}^p\}_{p\geq 0}$ is convergent.

We set $\tilde{\mathbf{X}} = \lim_{p\to\infty} \mathbf{X}^p$. One proves that under the considered assumptions, $\tilde{\mathbf{X}}$ obtained in this way, is just the stabilizing solution of SRARE (34). Since, $\tilde{\mathbf{Y}} \succeq \mathbf{Y}$, where $\mathbf{Y}$ is a solution of (45), it follows that $\tilde{\mathbf{X}}$ satisfies the sign conditions (44). The uniqueness of the stabilizing solution of SRARE (34) that satisfies the sign condition (44) is a direct consequence of its maximality property.

The proof of the implication $(ii) \to (i)$ is based on the fact that if the Riccati equation of type (38) has a stabilizing solution $\tilde{\mathbf{X}}$, satisfying the sign condition (44), then the algebraic Riccati type equation obtained replacing in (38) the term $\mathbf{M}$ by $\mathbf{M} + \delta\mathbb{I}$ has a small enough solution for $\delta < 0$. The details are omitted. □

Remark 3. *The iterations described by (46) and (47) can be used for numerical computation of the stabilizing solution* $(\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ *of SRARE (34) satisfying the sign conditions (44).*

4. The Main Results

4.1. The Asymptotic Structure of the Stabilizing Solution of SARE

In this section we shall use the stabilizing solution of SRARE (34) to derive the asymptotic structure of the stabilizing solution of SARE (8) satisfying the sign condition (9).
Let $\tilde{\mathbf{X}} = (\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ be the stabilizing solution of SRARE (34) satisfying the sign conditions (44). Let $\tilde{\mathbf{F}} = (\tilde{F}_0, \tilde{F}_1, \tilde{F}_2)$ be the corresponding stabilizing feedback gain associated via (42). We set

$$\tilde{F}_0^1 \triangleq (I_m + \tilde{F}_1(A_{11}^1)^{-1}B_1^1)\tilde{F}_0 + \tilde{F}_1(A_{11}^1)^{-1}A_{10}^1 \tag{48a}$$

$$\tilde{F}_1^1 \triangleq \tilde{F}_1 \tag{48b}$$

$$\begin{aligned} \tilde{X}_{01} \triangleq\; & -[(A_{10}^1)^T\tilde{X}_1 + \tilde{X}_0 A_{01}^1 + \sum_{j=0}^{2}(C_{j0}^1)^T\tilde{X}_j C_{j1}^1 - \\ & -(\tilde{F}_0^1)^T(R^1 + \sum_{j=0}^{2}(D_j^1)^T\tilde{X}_j D_j^1)\tilde{F}_1^1 + M_{01}^1](A_{11}^1)^{-1} \end{aligned} \tag{49}$$

From (42), (48) and (49) we obtain via Lemma 2 (ii) that $(\tilde{X}_0, \tilde{X}_{01}, \tilde{X}_1, \tilde{X}_2, \tilde{F}_0^1, \tilde{F}_1^1, \tilde{F}_2)$ is a solution of system (23). To this end, we took into account that if the eigenvalues of the linear operator $\mathcal{L}_{\tilde{\mathbf{F}}}$ are inclosed in the half plane $\mathbb{C}_-$ then the matrix $A_{11}^1 + B_1^1\tilde{F}_1^1$ is a Hurwitz matrix. Hence, it is invertible.

Further, we define

$$\tilde{\tilde{F}}_j \triangleq (I_m + \tilde{F}_2 A_{22}^{-1}(0)B_2(0))\tilde{F}_j^1 + \tilde{F}_2 A_{22}^{-1}(0)A_{2j}(0), j = 0,1 \tag{50a}$$

$$\tilde{\tilde{F}}_2 \triangleq \tilde{F}_2 \tag{50b}$$

$$\begin{aligned}\tilde{X}_{02} \triangleq\; & -[A_{20}^T(0)\tilde{X}_2 + \tilde{X}_0 A_{02}(0) + \tilde{X}_{01}A_{12}(0) + \sum_{j=0}^{2} C_{j0}^T(0)\tilde{X}_j C_{j2}(0) - \\ & -\tilde{\tilde{F}}_0^T (R + \sum_{j=0}^{2} D_j^T(0)\tilde{X}_j D_j(0))\tilde{\tilde{F}}_2 + M_{02}]A_{22}^{-1}(0)\end{aligned} \tag{51a}$$

$$\begin{aligned}\tilde{X}_{12} \triangleq\; & -[A_{21}^T(0)\tilde{X}_2 + \tilde{X}_1 A_{12}(0) + \sum_{j=0}^{2} C_{j1}^T(0)\tilde{X}_j C_{j2}(0) - \\ & -\tilde{\tilde{F}}_1^T (R + \sum_{j=0}^{2} D_j^T(0)\tilde{X}_j D_j(0))\tilde{\tilde{F}}_2 + M_{12}]A_{22}^{-1}(0).\end{aligned} \tag{51b}$$

Since the eigenvalues of the linear operator $\mathcal{L}_{\tilde{\mathbf{F}}}$ are in the half plane $\mathbb{C}_-$, we deduce via (50b) that the matrix $A_{22}(0) + B_2(0)\tilde{\tilde{F}}_2$ is a Hurwitz matrix. Hence, it is invertible.

Applying Lemma 1 (ii), we deduce that $(\tilde{X}_0, \tilde{X}_{01}, \tilde{X}_1, \tilde{X}_{02}, \tilde{X}_{12}, \tilde{X}_2, \tilde{\tilde{F}}_0, \tilde{\tilde{F}}_1, \tilde{\tilde{F}}_2)$ is a solution of system (19) constructed starting from the stabilizing solution $(\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ of SRARE (34).

Now, we are in the position to state the first main result of this paper:

Theorem 2. *Assume: (a) the assumptions* $\mathbf{H}_1$) *and* $\mathbf{H}_2$) *are fulfilled;*

(b) the matrices $A_{22}(0)$ *and* $A_{11}(0) - A_{12}(0)A_{22}^{-1}(0)A_{21}(0)$ *are invertible;*

(c) conditions from (i) of Theorem 1 are fulfilled.

Under these conditions there exists $\mu^* > 0$ *with the property that for any* $\epsilon_k > 0$, $k = 1,2$, *such that* $0 < \epsilon_1 + \epsilon_2 + \frac{\epsilon_2}{\epsilon_1} \leq (\mu^*)^2$, *the SARE (8) has a stabilizing solution* $\tilde{X}(\epsilon_1, \epsilon_2)$ *satisfying the sign condition (9). Furthermore* $\tilde{X}(\epsilon_1, \epsilon_2)$ *and the corresponding stabilizing feedback gain* $\tilde{F}(\epsilon_1, \epsilon_2)$ *have the asymptotic structure:*

$$\tilde{X}(\epsilon_1, \epsilon_2) = \begin{pmatrix} \tilde{X}_1 + O(\mu) & \epsilon_1(\tilde{X}_{01} + O(\mu)) & \epsilon_2(\tilde{X}_{02} + O(\mu)) \\ \epsilon_1(\tilde{X}_{01} + O(\mu))^T & \epsilon_1(\tilde{X}_1 + O(\mu)) & \epsilon_2(\tilde{X}_{12} + O(\mu)) \\ \epsilon_2(\tilde{X}_{02} + O(\mu))^T & \epsilon_2(\tilde{X}_{12} + O(\mu))^T & \epsilon_2(\tilde{X}_2 + O(\mu)) \end{pmatrix} \tag{52}$$

$$\tilde{F}(\epsilon_1, \epsilon_2) = \begin{pmatrix} \tilde{\tilde{F}}_0 + O(\mu) & \tilde{\tilde{F}}_1 + O(\mu) & \tilde{\tilde{F}}_2 + O(\mu) \end{pmatrix} \tag{53}$$

where $\mu = (\epsilon_1 + \epsilon_2 + \frac{\epsilon_2}{\epsilon_1})^{\frac{1}{2}}$, $(\tilde{X}_{01}, \tilde{X}_{02}, \tilde{X}_{12})$ *being computed by (49) and (51) based on the stabilizing solution* $(\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ *of SRARE (34) satisfying the sign conditions (44) and* $\tilde{\tilde{F}}_k$ *are computed by (48) and (50) starting from the stabilizing feedback gains* $\tilde{F}_j$, $j = 0,1,2$, *associated with the stabilizing solution of SRARE (34).*

Proof. The existence of the stabilizing solution $\tilde{X}(\epsilon_1, \epsilon_2)$, as well as the asymptotic structure from (52) and (53), are obtained applying the implicit functions theorem in the case of system (18). To this end, we regard system (18) as an equation of the form:

$$\Phi(\mathbb{W}, \xi) = 0 \tag{54}$$

on the finite dimensional Banach space $\mathfrak{W} \triangleq \mathcal{S}_{n_0} \times \mathbb{R}^{n_0 \times n_1} \times \mathcal{S}_{n_1} \times \mathbb{R}^{n_0 \times n_2} \times \mathbb{R}^{n_1 \times n_2} \times \mathcal{S}_{n_2} \times \mathbb{R}^{m \times n_0} \times \mathbb{R}^{m \times n_1} \times \mathbb{R}^{m \times n_2}$. In (54), $\mathbb{W} = (X_0, X_{01}, X_1, X_{02}, X_{12}, X_2, F_0, F_1, F_2)$ and $\xi = (\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\frac{\epsilon_2}{\epsilon_1}})$. From (18) one sees that $\mathbb{W} \to \Phi(\mathbb{W}, \xi)$ is a C^∞-function and from the assumption $\mathbf{H}_2$) we have that $\xi \to \Phi(\mathbb{W}, \xi)$ is a C^1-function in a neighborhood of the origin $\mathbf{0} = (0,0,0)$. We also remark that the reduced equation $\Phi(\mathbb{W}, \mathbf{0}) = 0$ coincides with system (19). So, from the developments in the first part of this section we deduce that $(\tilde{\mathbb{W}}, \mathbf{0})$ is a solution of the equation (54) when $\tilde{\mathbb{W}} = (\tilde{X}_0, \tilde{X}_{01}, \tilde{X}_1, \tilde{X}_{02}, \tilde{X}_{12}, \tilde{X}_2, \tilde{\tilde{F}}_0, \tilde{\tilde{F}}_1, \tilde{\tilde{F}}_2)$. Let $\mathbf{\Phi}_{\mathbb{W}}(\tilde{\mathbb{W}}, \mathbf{0})$ be the partial derivative of $\mathbf{\Phi}(\mathbb{W}, \xi)$ evaluated in $(\mathbb{W}, \xi) = (\tilde{\mathbb{W}}, \mathbf{0})$.

First we show that the operator $\hat{\mathbb{W}} \to \gneqq_{\mathbb{W}}(\tilde{\mathbb{W}}, \mathbf{0})[\hat{\mathbb{W}}] : \mathfrak{W} \to \mathfrak{W}$ is injective.

To this end we consider the linear equation

$$\Phi_{\mathbb{W}}(\tilde{\mathbb{W}}, \mathbf{0})[\hat{\mathbb{W}}] = 0 \tag{55}$$

with the unknowns $\hat{\mathbb{W}} = (\hat{X}_0, \hat{X}_{01}, \hat{X}_1, \hat{X}_{02}, \hat{X}_{12}, \hat{X}_2, \hat{F}_0, \hat{F}_1, \hat{F}_2) \in \mathfrak{W}$. After some algebraic manipulations one obtains that (55) reduces to the linear equation

$$\mathcal{L}^*_{\tilde{\mathbf{F}}}[\hat{\mathbf{Y}}] = 0 \tag{56}$$

with the unknowns $\hat{\mathbf{Y}} = (\hat{X}_0, \hat{X}_1, \hat{X}_2) \in \mathfrak{X}$ and $\mathcal{L}_{\tilde{\mathbf{F}}}$ is the operator of type (40) associated with the stabilizing feedback gain $\tilde{\mathbf{F}} = (\tilde{F}_0, \tilde{F}_1, \tilde{F}_2)$. Equation (56) only has the solution $\hat{\mathbf{Y}} = (0,0,0)$ because the spectrum of the linear operator $\mathcal{L}_{\tilde{\mathbf{F}}}$ lies in the half plane $\mathbb{C}_-$. Finally, one obtains that Equation (55) has only the zero solution. This means that the kernel of the linear operator $\hat{\mathbb{W}} \to \mathbf{\Phi}_{\mathbb{W}}(\tilde{\mathbb{W}}, \mathbf{0})[\hat{\mathbb{W}}]$ is the null subspace. Since $\mathfrak{W}$ is a finite dimensional vector space, we may conclude that $\hat{\mathbb{W}} \to \Phi_{\mathbb{W}}(\tilde{\mathbb{W}}, \mathbf{0})[\hat{\mathbb{W}}]$ is invertible. Hence, we may apply the implicit functions theorem (see [26]) in the case of Equation (54). This allows us to deduce that there exist $\mu_0 > 0$ and a C^1-function $\xi \to \mathbb{W}(\xi) : \mathcal{B}(0, \mu_0) \to \mathfrak{W}$, which satisfy $\Phi(\mathbb{W}(\xi), \xi) = 0$, for all $\xi \in \mathcal{B}(0, \mu_0) \triangleq \{\xi \in \mathbb{R}^3 || \xi| < \mu_0\}$. Further, $\mathbb{W}(\xi) = \tilde{\mathbb{W}} + O(|\xi|)$, which yields

$$\begin{aligned}
X_j(\epsilon_1, \epsilon_2) &= \tilde{X}_j + O(|\xi|), 0 \le j \le 2 \\
X_{01}(\epsilon_1, \epsilon_2) &= \tilde{X}_{01} + O(|\xi|), \\
X_{k2}(\epsilon_1, \epsilon_2) &= \tilde{X}_{k2} + O(|\xi|), k = 0, 1 \\
F_l(\epsilon_1, \epsilon_2) &= \tilde{\tilde{F}}_l + O(|\xi|), 0 \le l \le 2.
\end{aligned} \tag{57}$$

Plugging (57) into (12) we obtain (52), (53). We also obtain that $(\tilde{X}(\epsilon_1, \epsilon_2), \tilde{F}(\epsilon_1, \epsilon_2))$, constructed as above, satisfies (10). On the other hand, from (18f) and (52) we deduce that there exists $0 \le \mu_1 \le \mu_0$ with the property that $\tilde{X}(\epsilon_1, \epsilon_2)$ satisfies (9) for any $\epsilon_1 > 0, \epsilon_2 > 0$, such that $\epsilon_1 + \epsilon_2 + \frac{\epsilon_2}{\epsilon_1} < \mu_1^2$. Thus, we have obtained that $\tilde{X}(\epsilon_1, \epsilon_2)$ with the asymptotic structure given in (52) is a solution of SARE (8) which satisfies (9).

By a standard argument, based on singular perturbations technique, one shows that there exists $0 < \mu^* \le \mu_1$ such that the closed-loop system (11), where $\tilde{F}_1(\epsilon_1, \epsilon_2)$ has the asymptotic structure (53), is ESMS. Therefore, $\tilde{X}(\epsilon_1, \epsilon_2)$ defined by (52) is just the stabilizing solution of (8) for any $\epsilon_1 > 0, \epsilon_2 > 0$ such that $\epsilon_1 + \epsilon_2 + \frac{\epsilon_2}{\epsilon_1} \le (\mu^*)^2$. Thus the proof is complete. □

In the sequel, $(\tilde{\tilde{F}}_0 \quad \tilde{\tilde{F}}_1 \quad \tilde{\tilde{F}}_2)$ will be named dominant part of the stabilizing feedback gain.

4.2. A Near Optimal Control

In this subsection, we show that the dominant part of the optimal gain matrix $\tilde{F}(\epsilon_1, \epsilon_2)$ can be used to obtain a near optimal stabilizing feedback gain for the optimal control problem described by the quadratic functional (1) and the stochastic controlled system (2).

We consider the control

$$u_{app}(t) = \tilde{\tilde{F}}_0 x_0(t) + \tilde{\tilde{F}}_1 x_1(t) + \tilde{\tilde{F}}_2 x_2(t) \tag{58}$$

$\tilde{\tilde{F}}$ being constructed by (48) and (50) based on the stabilizing feedback gains $\tilde{F}_j$ associated with the stabilizing solution $(\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ of SRARE (34).

Setting, $\mathbf{F}_{app} = \left(\tilde{\tilde{F}}_0, \tilde{\tilde{F}}_1, \tilde{\tilde{F}}_2\right)$ we may rewrite (58) in the following compact form:

$$u_{app}(t) = \mathbf{F}_{app}\mathbf{x}(t). \tag{59}$$

Substituting (59) in (6) we obtain the closed-loop system

$$d\mathbf{x}(t) = (\mathbb{A}(\epsilon) + \mathbb{B}(\epsilon)\mathbf{F}_{app})\mathbf{x}(t)dt + (\mathbb{C}(\epsilon) + \mathbb{D}(\epsilon)\mathbf{F}_{app})\mathbf{x}(t)dw(t). \tag{60}$$

The next result provides an upper bound of the deviation of the value $J(\mathbf{x}_0; u_{app})$ from the minimal value $J(\mathbf{x}_0, u_{opt})$.

Theorem 3. *Assume that the assumptions of Theorem 2 are fulfilled. Then there exist $\tilde{\mu} > 0$ such that the closed-loop system (60) is ESMS for any $\epsilon_k > 0$, $k = 1, 2$ which satisfy $\epsilon_1 + \epsilon_2 + \frac{\epsilon_2}{\epsilon_1} < \tilde{\mu}^2$. Moreover, the loss of the performance produced by the use of the control (59) instead of the optimal control (7) is given by*

$$0 \le J(\mathbf{x}_0, u_{app}) - J(\mathbf{x}_0, u_{opt}) \le \gamma(\epsilon_1 + \epsilon_2 + \frac{\epsilon_2}{\epsilon_1})|\mathbf{x}_0|^2.$$

Proof. This may be done following a similar technique as the one used in [12] in the case of a single fast time scale. The details are omitted. □

5. Conclusions

The goal of the work has been the derivation of the asymptotic structure of the stabilizing solution of an algebraic Riccati equation arising in connection with a stochastic linear quadratic optimal control problem for a controlled system described by singularly perturbed Itô differential equations with two fast time scales.

The main conclusion of our study is that, in the stochastic case when the controlled system contains state multiplicative and/or control multiplicative white noise perturbations, the reduced system of algebraic Riccati equations cannot be directly obtained by neglecting the small parameters associated with the fast time scales of the controlled system as in the deterministic framework.

In Section 3 we have shown in detail how the system of reduced algebraic Riccati equations can be defined in the considered stochastic framework. In the second part of Section 3, we have introduced the concept of a stabilizing solution of SRARE, and we have provided a set of conditions equivalent to the existence of this kind of solution of SRARE which satisfy a prescribed sign condition of type (44). Employing the stabilizing solution of SRARE, as well as the corresponding stabilizing feedback gains, we have obtained the asymptotic structure of the stabilizing solution of SARE and of the corresponding stabilizing feedback gain. The dominant part of the stabilizing feedback gain was used to construct a near optimal control whose gain matrices do not depend upon the small parameters associated with the fast time scales. The extension of the study to the case of singularly perturbed linear stochastic systems with N fast time scales, also including more complex systems such as jump Markov perturbations [27], Levy noise perturbations [28] and semi-Markov switched systems [29] remains a challenge for future research.

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