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Conditions of Functional Null Controllability for Some Types of Singularly Perturbed Nonlinear Systems with Delays

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Abstract: Two types of singularly-perturbed nonlinear time delay controlled systems are considered. For these systems, sufficient conditions of the functional null controllability are derived. These conditions, being independent of the parameter of singular perturbation, provide the controllability of the systems for all sufficiently small values of the parameter. Illustrative examples are presented.

Keywords: singularly-perturbed nonlinear system; time delay system; functional controllability

MSC: 34K26; 93B05; 93C23

1. Introduction

Vector-valued differential equations with small positive parameters, multiplying some of the highest order derivatives, are called singularly-perturbed differential equations. Such equations serve as mathematical models in various applications where multi-time-scale dynamics appears. These parameters are called parameters of singular perturbations. In applications, they can be small masses, small time constants, small capacitances, small geotropic reactions, as well as many other small positive parameters in electrical engineering, control engineering, aerospace engineering, mechanical engineering, physics, chemistry, biology, medicine, etc. (see, e.g., [1–3] and the references therein). An important class of singularly-perturbed differential systems is the class of the systems with small time delays (of order of the parameter of singular perturbation). Such systems appear in various real-life applications, for instance in nuclear engineering [4], in botany [5], in physiology and medicine [6,7], in control engineering [8], and in communication engineering [9,10].

Various theoretical and applied issues in the area of singularly-perturbed controlled systems without delays, as well as with state and/or control delays were extensively studied in the literature (see, e.g., [1,11–13] and the references therein).

One of the basic properties of a controlled system is its controllability. The controllability means the existence of a control function transferring the system from any admissible initial position to any admissible terminal position in a finite time. This property was extensively studied in the literature (see, e.g., [14–18] and the references therein). To find out whether a singularly-perturbed system is controllable with respect to a proper definition, corresponding controllability conditions can be directly used for any pre-chosen value of the small parameter $\varepsilon > 0$ of singular perturbation. However, the stiffness and a possible high Euclidean space dimension of this system can considerably complicate this use. Moreover, such a use depends on the value of ε, and it should be repeated more and more while this parameter changes. Furthermore, it should be noted that in many applications, the current value of ε is unknown. These circumstances show how it is important to derive ε-free conditions, which provide the validity of the controllability for a singularly-perturbed system robustly with respect to ε, i.e., for all sufficiently small values of this parameter.

Controllability of singularly-perturbed systems, robust with respect to ε, was studied in the literature in a number of works. Thus, in [19–22], the complete controllability of some linear and nonlinear systems without delays was analyzed. The robust complete/output Euclidean space controllability for various linear time-dependent/time-invariant systems either with delays only in state variables or with state and control delays was analyzed in [23–32]. However, the functional controllability (approximate and exact) of singularly-perturbed time delay systems was studied only in a few papers (see [33] and the references therein). Moreover, to the best of our knowledge, the exact functional controllability of a singularly-perturbed system was studied only in [33]. Namely, in this work the functional null controllability was investigated for a singularly-perturbed linear constant coefficients system with non-small point-wise commensurate delays (of order of one) only in the slow state variable. Parameter-independent sufficient conditions for the functional null controllability of this system were derived.

In the present paper, two types of singularly-perturbed nonlinear time-dependent systems with point-wise non-commensurate delays and distributed delays are considered. The delays are small of the order of ε. For each type of these systems, ε-free functional null controllability conditions are derived. These conditions are robust with respect to ε, i.e., they guarantee the functional null controllability of a corresponding system for all sufficiently small positive values of this parameter.

The following main notations are applied in the paper:

1. E^n is the n-dimensional real Euclidean space.
2. The Euclidean norm of either a vector or a matrix is denoted by $\|\cdot\|$.
3. The upper index T denotes the transposition either of a vector x (x^T) or of a matrix A (A^T).
4. I_n denotes the identity matrix of dimension n.
5. $C[t_1, t_2; E^n]$ denotes the linear space of all vector-valued functions $x(\cdot) : [t_1, t_2] \to E^n$, continuous in the interval $[t_1, t_2]$.
6. $\mathrm{col}(x, y)$, where $x \in E^n$, $y \in E^m$, denotes the column block-vector of the dimension $n + m$ with the upper block x and the lower block y, i.e., $\mathrm{col}(x, y) = (x^T, y^T)^T$.
7. $\mathrm{Re}\lambda$ denotes the real part of a complex number λ.

2. Problem Formulation

Consider the controlled system:

$$\frac{dx(t)}{dt} = \sum_{j=0}^{N} \left[A_{1j}(t)x(t - \varepsilon h_j) + A_{2j}(t)y(t - \varepsilon h_j)\right]$$
$$+ \int_{-h}^{0} \left[G_1(t,\tau)x(t + \varepsilon\tau) + G_2(t,\tau)y(t + \varepsilon\tau)\right]d\tau + f_1\big(z(t), t\big)$$
$$+ f_2\big(z(t - \varepsilon h_1), ..., z(t - \varepsilon h_N), t\big) + B_1(t)u(t), \quad t \geq 0, \tag{1}$$

$$\varepsilon\frac{dy(t)}{dt} = \sum_{j=0}^{N} \left[A_{3j}(t)x(t - \varepsilon h_j) + A_{4j}(t)y(t - \varepsilon h_j)\right]$$
$$+ \int_{-h}^{0} \left[G_3(t,\tau)x(t + \varepsilon\tau) + G_4(t,\tau)y(t + \varepsilon\tau)\right]d\tau + f_3\big(z(t), t\big)$$
$$+ f_4\big(z(t - \varepsilon h_1), ..., z(t - \varepsilon h_N), t\big) + B_2(t)u(t), \quad t \geq 0, \tag{2}$$

where $x(t) \in E^n$, $y(t) \in E^m$, $z(\cdot) = \mathrm{col}\big(x(\cdot), y(\cdot)\big)$, $u(t) \in E^r$ ($u(t)$ is a control); $\varepsilon > 0$ is a small parameter; $N \geq 1$; $0 = h_0 < h_1 < ... < h_N = h$ are some given ε-independent constants; $A_{ij}(t)$ and $B_k(t)$, $(i = 1, ..., 4;\ j = 0, 1, ..., N;\ k = 1, 2)$ are matrix-valued functions of corresponding dimensions, continuously differentiable for $t \in [0, +\infty)$; $G_i(t, \tau)$, $(i = 1, ..., 4)$ are matrix-valued functions of corresponding dimensions, piecewise continuous in $\tau \in [-h, 0]$ for any $t \in [0, +\infty)$ and continuously differentiable with respect to $t \in [0, +\infty)$ uniformly in $\tau \in [-h, 0]$; $f_l(z, t)$, $(l = 1, 3)$ are given vector-valued functions of corresponding dimensions, continuous for $(z, t) \in E^{n+m} \times [0, +\infty)$; $f_p(q_1, ..., q_N, t)$, $(p = 2, 4)$ are given vector-valued functions of corresponding dimensions, continuous for $(q_1, ..., q_N, t) \in E^{n+m} \times ... \times E^{n+m} \times [0, +\infty)$. Moreover, the functions $f_l(z, t)$ and $f_p(q_1, ..., q_N, t)$, $(l = 1, 3;\ p = 2, 4)$ satisfy the following limit conditions uniformly in $t \in [0, \hat{t}]$:

$$\lim_{\alpha_1 \to +\infty} \frac{\|f_l(z, t)\|}{\alpha_1} = 0, \quad l = 1, 3, \tag{3}$$

$$\lim_{\alpha_2 \to +\infty} \frac{\|f_p(q_1, ..., q_N, t)\|}{\alpha_2} = 0, \quad p = 2, 4, \tag{4}$$

where $\hat{t} > 0$ is any given time instant, and:

$$\alpha_1 \stackrel{\triangle}{=} \|z\|, \ \ \alpha_2 \stackrel{\triangle}{=} \|\mathrm{col}(q_1, ..., q_N)\|, \quad z \in E^{n+m}, \ \ q_j \in E^{n+m}, \ j = 1, ..., N. \tag{5}$$

For any given $\varepsilon > 0$ and piecewise continuous control function $u(t)$, $t \in [0, +\infty)$, the system (1) and (2) is a nonlinear time-dependent functional-differential system. It is infinite-dimensional with the state variables $x(t + \varepsilon\tau)$, $y(t + \varepsilon\tau)$, $\tau \in [-h, 0]$. Moreover, for $0 < \varepsilon << 1$, this system is singularly-perturbed. Equation (1) is called a slow mode, while the Equation (2) is called a fast mode of (1) and (2). The important feature of (1) and (2) is that this system has the state delays proportional to the small multiplier ε for a part of the derivatives, namely for $dy(t)/dt$.

Let $t_c > 0$ be a given time instant independent of ε.

Definition 1. *For a given $\varepsilon \in (0, t_c/h)$, the system (1) and (2) is said to be functional null controllable at the time instant t_c if for any $\varphi_x(\cdot) \in C[-\varepsilon h, 0; E^n]$ and $\varphi_y(\cdot) \in C[-\varepsilon h, 0; E^m]$, there exists a piecewise continuous control function $u(t)$, $t \in [0, t_c]$, for which the solution $\mathrm{col}\big(x(t, \varepsilon), y(t, \varepsilon)\big)$, $t \in [0, t_c]$ of the system (1) and (2) with the initial conditions:*

$$x(\tau) = \varphi_x(\tau), \quad y(\tau) = \varphi_y(\tau), \quad \tau \in [-\varepsilon h, 0] \tag{6}$$

satisfies the terminal conditions:

$$x(t, \varepsilon) = 0, \quad y(t, \varepsilon) = 0 \ \ \forall\, t \in [t_c - \varepsilon h, t_c]. \tag{7}$$

In what follows, we study two particular types of the system (1) and (2). In the system of the first type, the slow mode is uncontrolled, and this mode does not contain the delayed state variables. In the system of the second type, the fast mode is uncontrolled, and it does not contain the delayed state variables. The objective of the paper is the following: or each of the above-mentioned types of the system (1) and (2), to derive ε-free conditions of the functional null controllability, which are valid for all sufficiently small values of ε.

3. Preliminary Results

3.1. Euclidean Space Null Controllability of a Singularly-Perturbed Linear Time Delay System

In this subsection, we consider the linear system associated with the original system (1) and (2). Namely,

$$\frac{dx(t)}{dt} = \sum_{j=0}^{N} \left[A_{1j}(t)x(t-\varepsilon h_j) + A_{2j}(t)y(t-\varepsilon h_j)\right]$$
$$+ \int_{-h}^{0} \left[G_1(t,\tau)x(t+\varepsilon\tau) + G_2(t,\tau)y(t+\varepsilon\tau)\right]d\tau + B_1(t)u(t), \quad t \geq 0, \tag{8}$$

$$\varepsilon\frac{dy(t)}{dt} = \sum_{j=0}^{N} \left[A_{3j}(t)x(t-\varepsilon h_j) + A_{4j}(t)y(t-\varepsilon h_j)\right]$$
$$+ \int_{-h}^{0} \left[G_3(t,\tau)x(t+\varepsilon\tau) + G_4(t,\tau)y(t+\varepsilon\tau)\right]d\tau + B_2(t)u(t), \quad t \geq 0. \tag{9}$$

Let $\varepsilon^0 > 0$ be a given number. Let $\tilde{t}(\varepsilon)$, $\varepsilon \in [0, \varepsilon^0]$ be a given continuously-differentiable non-increasing function of ε, and $\tilde{t}(\varepsilon) > 0$, $\varepsilon \in [0, \varepsilon^0]$. Thus,

$$0 < \tilde{t}(\varepsilon) \leq \tilde{t}(0) \quad \forall\, \varepsilon \in [0, \varepsilon^0]. \tag{10}$$

Definition 2. *For a given $\varepsilon \in (0, \varepsilon^0]$, the system (8) and (9) is said to be Euclidean space null controllable at the time instant $\tilde{t}(\varepsilon)$ if for any $\varphi_x(\cdot) \in C[-\varepsilon h, 0; E^n]$ and $\varphi_y(\cdot) \in C[-\varepsilon h, 0; E^m]$, there exists a piecewise continuous control function $u(t)$, $t \in [0, \tilde{t}(\varepsilon)]$, for which the initial-value problem (8), (9), and (6) has the solution satisfying the terminal conditions:*

$$x\big(\tilde{t}(\varepsilon)\big) = 0, \quad y\big(\tilde{t}(\varepsilon)\big) = 0. \tag{11}$$

3.2. Euclidean Space Null Controllability of the Original System (1) and (2)

Definition 3. *For a given $\varepsilon \in (0, \varepsilon^0]$, the system (1) and (2) is said to be Euclidean space null controllable at the time instant $\tilde{t}(\varepsilon)$ if for any $\varphi_x(\cdot) \in C[-\varepsilon h, 0; E^n]$ and $\varphi_y(\cdot) \in C[-\varepsilon h, 0; E^m]$, there exists a piecewise continuous control function $u(t)$, $t \in [0, \tilde{t}(\varepsilon)]$, for which the boundary-value problem (1), (2), (6), and (11) has a solution.*

Based on the limit conditions (3) and (4) and the results of [34], we directly obtain the following assertion.

Proposition 1. *Let for a given $\varepsilon \in (0, \varepsilon^0]$, the system (8) and (9) be Euclidean space null controllable at the time instant $\tilde{t}(\varepsilon)$. Then, for this ε, the singularly-perturbed nonlinear system (1) and (2) is Euclidean space null controllable at the time instant $\tilde{t}(\varepsilon)$.*

4. System of the First Type

In this section, we consider the following particular case of the system (1) and (2):

$$\frac{dx(t)}{dt} = A_{10}(t)x(t) + A_{20}(t)y(t) + f_1\big(z(t), t\big), \quad t \geq 0, \tag{12}$$

$$\varepsilon\frac{dy(t)}{dt} = \sum_{j=0}^{N} \left[A_{3j}(t)x(t-\varepsilon h_j) + A_{4j}(t)y(t-\varepsilon h_j)\right]$$
$$+ \int_{-h}^{0} \left[G_3(t,\tau)x(t+\varepsilon\tau) + G_4(t,\tau)y(t+\varepsilon\tau)\right]d\tau + f_3\big(z(t), t\big)$$
$$+ f_4\big(z(t-\varepsilon h_1), ..., z(t-\varepsilon h_N), t\big) + B_2(t)u(t), \;\; t \geq 0. \tag{13}$$

The linear system, corresponding to (12) and (13), is a particular case of the system (8) and (9), and it has the form:

$$\frac{dx(t)}{dt} = A_{10}(t)x(t) + A_{20}(t)y(t), \quad t \geq 0, \tag{14}$$

$$\varepsilon\frac{dy(t)}{dt} = \sum_{j=0}^{N}\left[A_{3j}(t)x(t-\varepsilon h_j) + A_{4j}(t)y(t-\varepsilon h_j)\right]$$
$$+ \int_{-h}^{0}\left[G_3(t,\tau)x(t+\varepsilon\tau) + G_4(t,\tau)y(t+\varepsilon\tau)\right]d\tau + B_2(t)u(t), \ \ t \geq 0. \tag{15}$$

4.1. Auxiliary Results

4.1.1. Asymptotic Decomposition of (14) and (15)

Let us decompose asymptotically the system (14) and (15) into two much simpler ε-free subsystems, the slow and fast ones.

The slow subsystem is obtained from (14) and (15) by setting there formally $\varepsilon = 0$. Thus, the slow subsystem has the form:

$$\frac{dx_s(t)}{dt} = A_{10}(t)x_s(t) + A_{20}(t)y_s(t), \quad t \geq 0, \tag{16}$$

$$0 = A_{3s}(t)x_s(t) + A_{4s}(t)y_s(t) + B_2(t)u_s(t), \quad t \geq 0, \tag{17}$$

where $x_s(t) \in E^n$, $y_s(t) \in E^m$, $u_s(t) \in E^m$, (u_s is a control);

$$A_{is}(t) = \sum_{j=0}^{N} A_{ij}(t) + \int_{-h}^{0} G_i(t,\tau)d\tau, \quad i = 3,4. \tag{18}$$

The slow subsystem (16) and (17) is a differential-algebraic system.

In what follows, we assume that:

$$\det A_{4s}(t) \neq 0, \quad t \geq 0. \tag{19}$$

Subject to this assumption, the slow subsystem (16) and (17) can be reduced to the differential equation with respect to $x_s(t)$:

$$\frac{dx_s(t)}{dt} = A_s(t)x_s(t) + B_s(t)u_s(t), \quad t \geq 0, \tag{20}$$

where:

$$\begin{aligned} A_s(t) &= A_{10}(t) - A_{20}(t)A_{4s}^{-1}(t)A_{3s}(t), \\ B_s(t) &= -A_{20}(t)A_{4s}^{-1}(t)B_2(t). \end{aligned} \tag{21}$$

The differential Equation (20) also is called the slow subsystem, associated with the system (14) and (15).

The fast subsystem is derived from the fast mode (15) of the system (14) and (15) in the following way: (a) the terms containing the state variable $x(t+\varepsilon\tau)$, $\tau \in [-h,0]$ are removed from (15); (b) the transformations of the variables $t = t_1 + \varepsilon\xi$, $y(t_1+\varepsilon\xi) \stackrel{\triangle}{=} y_f(\xi)$, $u(t_1+\varepsilon\xi) \stackrel{\triangle}{=} u_f(\xi)$ are made in the resulting system, where $t_1 \geq 0$ is any fixed time instant, and $\xi \geq 0$ is a new independent variable. Thus, we obtain the system:

$$\frac{dy_f(\xi)}{d\xi} = \sum_{j=0}^{N} A_{4j}(t_1 + \varepsilon\xi)y_f(\xi - h_j)$$
$$+ \int_{-h}^{0} G_4(t_1 + \varepsilon\xi, \tau)y_f(\xi + \tau)d\tau + B_2(t_1 + \varepsilon\xi)u_f(\xi).$$

Now, setting formally $\varepsilon = 0$ in this system and replacing t_1 with t yield the fast subsystem:

$$\frac{dy_f(\xi)}{d\xi} = \sum_{j=0}^{N} A_{4j}(t)y_f(\xi - h_j)$$
$$+ \int_{-h}^{0} G_4(t, \tau)y_f(\xi + \tau)d\tau + B_2(t)u_f(\xi), \quad \xi \geq 0, \tag{22}$$

where $t \geq 0$ is a parameter; $y_f(\xi) \in E^m$; $y_f(\xi + \tau)$, $\tau \in [-h, 0]$ is the state variable; $u_f(\xi) \in E^r$ ($u_f(\xi)$ is the control).

The new independent variable ξ is called the stretched time, and it is expressed by the original time t in the form $\xi = (t - t_1)/\varepsilon$. Thus, for any $t > t_1$, $\xi \to +\infty$ as $\varepsilon \to +0$.

The fast subsystem (22) is a differential equation with state delays. It is of a lower Euclidean dimension than the system (14) and (15).

Definition 4. *The slow subsystem (20) is said to be null controllable at the time instant $\tilde{t}(0)$ if for any $x_0 \in E^n$, there exists a piecewise continuous control function $u_s(t)$, $t \in [0, \tilde{t}(0)]$, for which (20) has a solution $x_s(t)$, $t \in [0, \tilde{t}(0)]$, satisfying the initial and terminal conditions:*

$$x_s(0) = x_0, \quad x_s\big(\tilde{t}(0)\big) = 0. \tag{23}$$

Definition 5. *For a given $t \geq 0$, the fast subsystem (22) is said to be Euclidean space null controllable if for any $\varphi_{yf}(\cdot) \in C[-h, 0; E^m]$, there exist a number $\xi_c \geq h$, independent of $\varphi_{yf}(\cdot)$, and a piecewise continuous control function $u_f(\xi)$, $\xi \in [0, \xi_c]$, for which the system (22) with the initial and terminal conditions:*

$$y_f(\tau) = \varphi_{yf}(\tau), \quad \tau \in [-h, 0], \tag{24}$$

$$y_f(\xi_c) = 0, \tag{25}$$

has a solution.

4.1.2. Null Controllability Conditions for the Slow and Fast Subsystems

Let the $n \times n$-matrix-valued function $\Psi_s(\sigma)$, $\sigma \in [0, \tilde{t}(0)]$ be the unique solution of the terminal-value problem:

$$\frac{d\Psi_s(\sigma)}{d\sigma} = -A_s^T(\sigma)\Psi_s(\sigma), \quad \sigma \in [0, \tilde{t}(0)), \quad \Psi_s\big(\tilde{t}(0)\big) = I_n. \tag{26}$$

Let, for any given $t \geq 0$, the $m \times m$-matrix-valued function $\Psi_f(\xi, t)$ be the unique solution of the initial-value problem:

$$\frac{d\Psi_f(\xi)}{d\xi} = \sum_{j=0}^{N} A_{4j}^T(t)\Psi_f(\xi - h_j) + \int_{-h}^{0} G_4^T(t, \tau)\Psi_f(\xi + \tau)d\tau, \quad \xi > 0,$$
$$\Psi_f(\xi) = 0, \quad \xi < 0, \quad \Psi_f(0) = I_m. \tag{27}$$

Consider the matrices:

$$W_s(\tilde{t}(0)) = \int_0^{\tilde{t}(0)} \Psi_s^T(\sigma)B_s(\sigma)B_s^T(\sigma)\Psi_s(\sigma)d\sigma, \tag{28}$$

$$W_f(\xi,t) = \int_0^{\xi} \Psi_f^T(\rho,t)B_2(t)B_2^T(t)\Psi_f(\rho,t)d\rho, \quad \xi \geq 0, \quad t \geq 0. \tag{29}$$

By virtue of the results of [30] (Section 3.1), we have the following two assertions.

Proposition 2. *The slow subsystem (20) is null controllable at the time instant $\tilde{t}(0)$, if and only if* $\det W_s(\tilde{t}(0)) \neq 0$.

Proposition 3. *For a given $t \geq 0$, the fast subsystem (22) is Euclidean space null controllable if and only if there exists an instant $\xi = \xi_c \geq h$ such that* $\det W_f(\xi_c, t) \neq 0$.

4.1.3. ε-Free Conditions for Euclidean Space Null Controllability of the Linear System (14) and (15)

Here, we assume:

(I_1) All roots $\lambda(t)$ of the equation:

$$\det\left[\lambda I_m - \sum_{j=0}^{N} A_{4j}(t)\exp(-\lambda h_j) - \int_{-h}^{0} G_4(t,\tau)\exp(\lambda\tau)d\tau\right] = 0 \tag{30}$$

satisfy the inequality $\mathrm{Re}\lambda(t) < -\beta$ for all $t \in [0, \tilde{t}(0)]$, where $\beta > 0$ is some constant.

Quite similarly to the results of [25], we obtain the following assertion.

Proposition 4. *Let the assumption (I_1) be valid. Let the slow subsystem (20) be null controllable at the time instant $\tilde{t}(0)$. Let, for $t = \tilde{t}(0)$, the fast subsystem (22) be Euclidean space null controllable. Then, there exists a positive number $\varepsilon_1 \leq \varepsilon^0$ such that, for all $\varepsilon \in (0, \varepsilon_1]$, the singularly-perturbed linear system (14) and (15) is Euclidean space null controllable at the time instant $\tilde{t}(\varepsilon)$.*

Based on Propositions 2 and 3, Proposition 4 can be reformulated as follows.

Proposition 5. *Let the assumption (I_1) be valid. Let* $\det W_s(\tilde{t}(0)) \neq 0$. *Let there exist a number $\xi_c \geq h$ such that* $\det W_f(\xi_c, \tilde{t}(0)) \neq 0$. *Then, there exists a positive number $\varepsilon_1 \leq \varepsilon^0$ such that, for all $\varepsilon \in (0, \varepsilon_1]$, the singularly-perturbed linear system (14) and (15) is Euclidean space null controllable at the time instant $\tilde{t}(\varepsilon)$.*

4.1.4. Euclidean Space Null Controllability of the Nonlinear System (12) and (13): ε-Free Conditions

Based on Equations (3) and (4) and the results of [34], and using Proposition 5, we directly obtain the following assertion.

Proposition 6. *Let the assumption (I_1) be valid. Let* $\det W_s(\tilde{t}(0)) \neq 0$. *Let there exist a number $\xi_c \geq h$ such that* $\det W_f(\xi_c, \tilde{t}(0)) \neq 0$. *Then, there exists a positive number $\varepsilon_1 \leq \varepsilon^0$ such that, for all $\varepsilon \in (0, \varepsilon_1]$, the singularly-perturbed nonlinear system (12) and (13) is Euclidean space null controllable at the time instant $\tilde{t}(\varepsilon)$.*

4.2. Main Result

In this subsection, we assume that $u(t) \in E^m$ and $B_2(t)$ is an $m \times m$-matrix. Furthermore, we assume:

(II_1) The matrix $B_2(t_c)$ is invertible.

(III$_1$) $f_1(0, t) = 0$ for all $t \in [0, t_c]$.

(IV$_1$) The vector-valued functions $f_1(z, t)$ and $f_3(z, t)$ satisfy the local Lipschitz condition with respect to $z \in E^{n+m}$ uniformly in $t \in [0, t_c]$.

Lemma 1. *Let the assumption (II_1) be valid. Then,* $\det W_f(\xi_c, t_c) \neq 0$ *for any* $\xi_c \geq h$.

Proof. The statement of the lemma directly follows from the definition of the matrix $W_f(\xi, t)$ (see Equation (29)). □

Theorem 1. *Let the assumptions (I_1)–(IV_1) be valid. Let* $\det W_s(t_c) \neq 0$. *Then, there exists a positive number* $\varepsilon^* < t_c/(2h)$ *such that, for any* $\varepsilon \in (0, \varepsilon^*]$, *the system (12) and (13) is functional null controllable at the time instant* t_c.

Proof. Let us choose $\tilde{t}(\varepsilon)$ in the form:

$$\tilde{t}(\varepsilon) = t_c - \varepsilon h, \quad \varepsilon \in [0, t_c/(2h)]. \tag{31}$$

Then, by virtue of Proposition 6, there exists a positive number $\varepsilon_1 < t_c/(2h)$ such that, for all $\varepsilon \in (0, \varepsilon_1]$, the system (12) and (13) is Euclidean space null controllable at the time instant $t_c - \varepsilon h$. Hence, for any $\varepsilon \in (0, \varepsilon_1]$ and any $\varphi_x(\cdot) \in C[-\varepsilon h, 0; E^n]$, $\varphi_y(\cdot) \in C[-\varepsilon h, 0; E^m]$, there exists the piecewise continuous control function $u(t) = \tilde{u}(t, \varepsilon)$, $t \in [0, t_c - \varepsilon h]$, for which the solution $\mathrm{col}\big(x(t), y(t)\big) = \mathrm{col}\big(\tilde{x}(t, \varepsilon), \tilde{y}(t, \varepsilon)\big)$ of the initial-value problem (12), (13), and (6) satisfies the zero terminal conditions, i.e.,

$$\tilde{x}(t_c - \varepsilon h, \varepsilon) = 0, \quad \tilde{y}(t_c - \varepsilon h, \varepsilon) = 0. \tag{32}$$

Note that any piecewise continuous control $u(t)$, $t \in [0, t_c - \varepsilon h]$, which differs from $\tilde{u}(t, \varepsilon)$ only at the single point $t = t_c - \varepsilon h$, generates the same solution $\mathrm{col}\big(\tilde{x}(t, \varepsilon), \tilde{y}(t, \varepsilon)\big)$ of the initial-value problem (12), (13), and (6) in the entire interval $t \in [0, t_c - \varepsilon h]$, i.e., (32) holds for any such control.

Now, for any $\varepsilon \in (0, \varepsilon_1]$, we consider the initial-value problem (12), (13), and (6) in the interval $[0, t_c]$. In the system (12) and (13), we choose the control $u(t) = \tilde{u}(t, \varepsilon)$ in the interval $[0, t_c - \varepsilon h)$, while in the interval $[t_c - \varepsilon h, t_c]$, the piecewise continuous control $u(t)$ will be chosen later. Namely, in the interval $[t_c - \varepsilon h, t_c]$, the piecewise continuous control $u(t)$ will be chosen in such a way that the solution of (12) and (13) with the initial conditions:

$$x(\rho) = \tilde{x}(\rho, \varepsilon), \quad y(\rho) = \tilde{y}(\rho, \varepsilon), \quad \rho \in [t_c - 2\varepsilon h, t_c - \varepsilon h] \tag{33}$$

will be zero for all $t \in (t_c - \varepsilon h, t_c]$. For this purpose, we introduce in the consideration the following functions:

$$X_j(t, \varepsilon) = \begin{cases} \tilde{x}(t - \varepsilon h_j, \varepsilon), & t \in \big[t_c - \varepsilon h, t_c - \varepsilon(h - h_j)\big), \\ 0, & t \in [t_c - \varepsilon(h - h_j), t_c], \end{cases} \quad j = 1, ..., N, \tag{34}$$

$$Y_j(t, \varepsilon) = \begin{cases} \tilde{y}(t - \varepsilon h_j, \varepsilon), & t \in \big[t_c - \varepsilon h, t_c - \varepsilon(h - h_j)\big), \\ 0, & t \in [t_c - \varepsilon(h - h_j), t_c], \end{cases} \quad j = 1, ..., N, \tag{35}$$

$$Z_j(t, \varepsilon) = \mathrm{col}\big(X_j(t, \varepsilon), Y_j(t, \varepsilon)\big), \quad t \in [t_c - \varepsilon h, t_c], \quad j = 1, ..., N, \tag{36}$$

$$\omega(t, \varepsilon) = \frac{t_c - t}{\varepsilon} - h, \quad t \in [t_c - \varepsilon h, t_c]. \tag{37}$$

Furthermore, we represent the integral in the right-hand side of (13) as:

$$\int_{-h}^{0}\big[G_3(t,\tau)x(t+\varepsilon\tau)+G_4(t,\tau)y(t+\varepsilon\tau)\big]d\tau =$$
$$\int_{-h}^{\omega(t,\varepsilon)}\big[G_3(t,\tau)x(t+\varepsilon\tau)+G_4(t,\tau)y(t+\varepsilon\tau)\big]d\tau$$
$$+\int_{\omega(t,\varepsilon)}^{0}\big[G_3(t,\tau)x(t+\varepsilon\tau)+G_4(t,\tau)y(t+\varepsilon\tau)\big]d\tau, \quad t\in[t_c-\varepsilon h,t_c]. \tag{38}$$

The first integral in the right-hand side of (38) can be rewritten as:

$$\int_{-h}^{\omega(t,\varepsilon)}\big[G_3(t,\tau)x(t+\varepsilon\tau)+G_4(t,\tau)y(t+\varepsilon\tau)\big]d\tau =$$
$$\int_{-h}^{\omega(t,\varepsilon)}\big[G_3(t,\tau)\tilde{x}(t+\varepsilon\tau,\varepsilon)+G_4(t,\tau)\tilde{y}(t+\varepsilon\tau,\varepsilon)\big]d\tau, \quad t\in[t_c-\varepsilon h,t_c]. \tag{39}$$

The second integral in the right-hand side of (38) can be rewritten as:

$$\int_{\omega(t,\varepsilon)}^{0}\big[G_3(t,\tau)x(t+\varepsilon\tau)+G_4(t,\tau)y(t+\varepsilon\tau)\big]d\tau =$$
$$\int_{t_c-\varepsilon h}^{t}\left[G_3\left(t,\frac{\chi-t}{\varepsilon}\right)x(\chi)+G_4\left(t,\frac{\chi-t}{\varepsilon}\right)y(\chi)\right]d\chi, \quad t\in[t_c-\varepsilon h,t_c]. \tag{40}$$

Thus, due to (38)–(40),

$$\int_{-h}^{0}\big[G_3(t,\tau)x(t+\varepsilon\tau)+G_4(t,\tau)y(t+\varepsilon\tau)\big]d\tau =$$
$$\int_{-h}^{\omega(t,\varepsilon)}\big[G_3(t,\tau)\tilde{x}(t+\varepsilon\tau)d\tau+G_4(t,\tau)\tilde{y}(t+\varepsilon\tau)\big]d\tau +$$
$$\int_{t_c-\varepsilon h}^{t}\left[G_3\left(t,\frac{\chi-t}{\varepsilon}\right)x(\chi)+G_4\left(t,\frac{\chi-t}{\varepsilon}\right)y(\chi)\right]d\chi, \quad t\in[t_c-\varepsilon h,t_c]. \tag{41}$$

Furthermore, we note that, due to the smoothness of the matrix-valued function $B_2(t)$ and the assumption (II_1), $B_2(t)$ is invertible in the entire interval $t\in[t_c-\varepsilon h,t_c]$ for all $\varepsilon\in[0,\varepsilon_2]$, where $0<\varepsilon_2\le\varepsilon_1$ is some sufficiently small number.

For any $\varepsilon\in(0,\varepsilon_2]$, we choose the above-mentioned control $u(t)$, $t\in[t_c-\varepsilon h,t_c]$ as:

$$u(t)=\bar{u}(t,\varepsilon)\stackrel{\triangle}{=}-B_2^{-1}(t)\bigg\{\int_{-h}^{\omega(t,\varepsilon)}\big[G_3(t,\tau)\tilde{x}(t+\varepsilon\tau)+G_4(t,\tau)\tilde{y}(t+\varepsilon\tau)\big]d\tau$$
$$+\sum_{j=1}^{N}\big[A_{3j}(t)X_j(t,\varepsilon)+A_{4j}(t)Y_j(t,\varepsilon)\big]+f_3(0,t)+f_4\big(Z_1(t,\varepsilon),\ldots,Z_N(t,\varepsilon),t\big)\bigg\}. \tag{42}$$

Substitution of this control into (13) and use of Equation (41) yield the following equation in the interval $[t_c-\varepsilon h,t_c]$:

$$\varepsilon\frac{dy(t)}{dt}=A_{30}(t)x(t)+A_{40}(t)y(t)$$
$$+\int_{t_c-\varepsilon h}^{t}\left[G_3\left(t,\frac{\chi-t}{\varepsilon}\right)x(\chi)+G_4\left(t,\frac{\chi-t}{\varepsilon}\right)y(\chi)\right]d\chi$$
$$+f_3\big(z(t),t\big)-f_3(0,t)+\Gamma\big(z(t-\varepsilon h_1),\ldots,z(t-\varepsilon h_N),t\big), \quad t\in[t_c-\varepsilon h,t_c], \tag{43}$$

where:

$$\Gamma\big(z(t-\varepsilon h_1),...,z(t-\varepsilon h_N),t\big) = \sum_{j=1}^{N}\big(A_{3j}(t)\ ,\ A_{4j}(t)\big)\big(z(t-\varepsilon h_j) - Z_j(t,\varepsilon)\big) \\ +f_4\big(z(t-\varepsilon h_1),...,z(t-\varepsilon h_N),t\big) - f_4\big(Z_1(t,\varepsilon),...,Z_N(t,\varepsilon),t\big). \tag{44}$$

Remember that $z(\cdot) = \mathrm{col}\big(x(\cdot),y(\cdot)\big)$.

Let us solve the system (12) and (43) in the interval $[t_c - \varepsilon h, t_c]$ subject to the initial conditions (33). To do this, we use the method of steps. Namely, let us consider this initial-value problem in the interval $\big[t_c - \varepsilon h, t_c - \varepsilon(h-h_1)\big]$. By virtue of (34)–(36), we have:

$$\Gamma\big(z(t-\varepsilon h_1),...,z(t-\varepsilon h_N),t\big) = 0, \quad t \in \big[t_c - \varepsilon h, t_c - \varepsilon(h-h_1)\big].$$

Due to the latter, Equation (43) becomes:

$$\varepsilon\frac{dy(t)}{dt} = A_{30}(t)x(t) + A_{40}(t)y(t) \\ + \int_{t_c-\varepsilon h}^{t}\left[G_3\left(t,\frac{\chi-t}{\varepsilon}\right)x(\chi) + G_4\left(t,\frac{\chi-t}{\varepsilon}\right)y(\chi)\right]d\chi \\ +f_3\big(z(t),t\big) - f_3(0,t), \qquad t \in \big[t_c - \varepsilon h, t_c - \varepsilon(h-h_1)\big]. \tag{45}$$

Moreover, due to (32) and (33), the point-wise initial conditions for the system (12) and (45) become:

$$x(t_c - \varepsilon h) = 0, \quad y(t_c - \varepsilon h) = 0. \tag{46}$$

Solving the initial-value problem (12), (45), and (46) in the interval $\big[t_c - \varepsilon h, t_c - \varepsilon(h-h_1)\big]$ and taking into account the assumptions (III_1)–(IV_1), we directly obtain that its unique solution is:

$$x(t) = x(t,\varepsilon) = 0, \quad y(t) = y(t,\varepsilon) = 0, \quad t \in \big[t_c - \varepsilon h, t_c - \varepsilon(h-h_1)\big]. \tag{47}$$

Now, based on the solution (47) of the the initial-value problem (12), (45), and (46), let us consider Equation (43) in the interval $\big[t_c - \varepsilon(h-h_1), t_c - \varepsilon(h-\bar{h}_2)\big]$, where $\bar{h}_2 = \min\{2h_1, h_2\}$. Similarly to (45), we obtain this equation in the following form:

$$\varepsilon\frac{dy(t)}{dt} = A_{30}(t)x(t) + A_{40}(t)y(t) \\ + \int_{t_c-\varepsilon(h-h_1)}^{t}\left[G_3\left(t,\frac{\chi-t}{\varepsilon}\right)x(\chi) + G_4\left(t,\frac{\chi-t}{\varepsilon}\right)y(\chi)\right]d\chi \\ +f_3\big(z(t),t\big) - f_3(0,t), \qquad t \in \big[t_c - \varepsilon(h-h_1), t_c - \varepsilon(h-\bar{h}_2)\big]. \tag{48}$$

The system (12) and (48), along with the initial conditions:

$$x\big(t_c - \varepsilon(h-h_1)\big) = 0, \quad y\big(t_c - \varepsilon(h-h_1)\big) = 0,$$

yields the unique solution:

$$x(t) = x(t,\varepsilon) = 0, \quad y(t) = y(t,\varepsilon) = 0, \quad t \in \big[t_c - \varepsilon(h-h_1), t_c - \varepsilon(h-\bar{h}_2)\big]. \tag{49}$$

Continuing to solve the system (12) and (43) in the interval $[t_c - \varepsilon h, t_c]$ subject to the initial conditions (33) by the method of steps, we obtain (similarly to (47) and (49)) the unique solution of this initial-value problem in the entire interval $[t_c - \varepsilon h, t_c]$:

$$x(t) = x(t,\varepsilon) = 0, \quad y(t) = y(t,\varepsilon) = 0, \quad t \in [t_c - \varepsilon h, t_c]. \tag{50}$$

Thus, for any $\varepsilon \in (0, \varepsilon_2]$, the control $u(t) = \bar{u}(t, \varepsilon)$, $t \in [t_c - \varepsilon h, t_c]$, given by (42), generates the unique zeroth solution of the system (12) and (13) subject to the initial conditions (33) in the entire interval $[t_c - \varepsilon h, t_c]$. Therefore, the control:

$$u(t) = \begin{cases} \tilde{u}(t, \varepsilon), & t \in [0, t_c - \varepsilon h), \\ \bar{u}(t, \varepsilon), & t \in [t_c - \varepsilon h, t_c] \end{cases} \tag{51}$$

generates the solution $x(t) = x(t, \varepsilon)$, $y(t) = y(t, \varepsilon)$, $t \in [0, t_c]$ of the initial-value problem (12), (13), and (6) satisfying the terminal conditions (7). The latter, along with Definition 1 and the notation $\varepsilon^* \stackrel{\triangle}{=} \varepsilon_2$, completes the proof of the theorem. □

Remark 1. *The idea to establish the functional null controllability of a time delay system, based on its Euclidean space null controllability and using a control variable at the final interval as a proper function of the state variable at the preceding interval, was applied in the literature in a number of works (see [15,35,36]). In these works, the simplest unperturbed linear differential-difference systems with constant coefficients were considered. The results of the present paper are an essential extension of the application of this idea to the functional null controllability analysis of singularly-perturbed nonlinear time-dependent systems with multiple point-wise and distributed delays.*

4.3. Example

Consider the following system, a particular case of (12) and (13):

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -tx_1(t) + (t+2)x_2(t) - y(t) \\ &+ \sin\big(x_1(t) - x_2(t)\big) + \cos\big(ty(t)\big) - 1, \quad t \geq 0, \end{aligned} \tag{52}$$

$$\begin{aligned} \frac{dx_2(t)}{dt} &= (t-1)x_1(t) - (t+1)x_2(t) + y(t) \\ &- \ln\big(1 + x_1^2(t) + x_2^2(t)\big) + \arctan\big(y(t)/(t+1)\big), \quad t \geq 0, \end{aligned} \tag{53}$$

$$\begin{aligned} \varepsilon\frac{dy(t)}{dt} &= x_1(t) - x_2(t) - 2y(t) + x_1(t-\varepsilon) + tx_2(t-\varepsilon) + y(t-\varepsilon) \\ &+2\int_{-1}^{0}\big[t\tau x_1(t+\varepsilon\tau) - \tau x_2(t+\varepsilon\tau)\big]d\tau + t\big(1 + x_1^2(t) + x_2^2(t) + y^2(t)\big)^{1/4} \\ &+(t+2)\cos^2\big(x_1(t-\varepsilon) + x_2(t-\varepsilon) + y(t-\varepsilon)\big) + u(t), \quad t \geq 0, \end{aligned} \tag{54}$$

where $x_1(t)$, $x_2(t)$, $y(t)$, and $u(t)$ are scalars, i.e., $n = 2$, $m = 1$, $r = 1$.

In this example, we study the functional null controllability of the system (52)–(54) at the time instant $t_c = 1$ for all sufficiently small $\varepsilon > 0$.

Comparing the systems (52)–(54) with the system (12) and (13), we obtain that in this example: $N = 1, h_1 = h = 1$, and:

$$A_{10}(t) = \begin{pmatrix} -t & t+2 \\ t-1 & -(t+1) \end{pmatrix}, \quad A_{20}(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad A_{30}(t) = (1\,,\,-1),$$
$$A_{40}(t) = -2, \quad A_{31}(t) = (1\,,\,t), \quad A_{41}(t) = 1,$$
$$G_3(t,\tau) = (2t\tau\,,\,-2\tau), \quad G_4(t,\tau) = 0, \quad B_2(t) = 1,$$
$$f_1(z(t),t) = \begin{pmatrix} \sin\big(x_1(t) - x_2(t)\big) + \cos\big(ty(t)\big) - 1 \\ -\ln\big(1 + x_1^2(t) + x_2^2(t)\big) + \arctan\big(y(t)/(t+1)\big) \end{pmatrix},$$
$$f_3(z(t),t) = t\big(1 + x_1^2(t) + x_2^2(t) + y^2(t)\big)^{1/4},$$
$$f_4(z(t-\varepsilon),t) = (t+2)\cos^2\big(x_1(t-\varepsilon) + x_2(t-\varepsilon) + y(t-\varepsilon)\big), \tag{55}$$

where $z(\cdot) = \mathrm{col}\big(x_1(\cdot), x_2(\cdot), y(\cdot)\big)$.

Thus, in this example, the limit conditions (3) and (4) are valid for $l = 1, 3$, $p = 4$. The coefficients of $x_1(\cdot)$, $x_2(\cdot)$, $y(\cdot)$, and $u(t)$ in the linear terms are smooth functions for $t \geq 0$, $\tau \in [-1, 0]$. Moreover, the assumptions (II$_1$)–(IV$_1$) also are valid. Let us show the validity of the assumption (I$_1$). In this example, Equation (30) becomes:

$$\lambda + 2 - \exp(-\lambda) = 0. \tag{56}$$

For all complex numbers λ with $\mathrm{Re}\lambda \geq -0.4$, we have the inequality:

$$\mathrm{Re}\big(\lambda + 2 - \exp(-\lambda)\big) > 0.108,$$

meaning that such λ are not roots of Equation (30). Therefore, all roots of (56) satisfy the inequality:

$$\mathrm{Re}\lambda < -0.4,$$

i.e., the assumption (I$_1$) is valid for $\beta = 0.4$.

Using (18) and (21), we obtain:

$$A_{3s}(t)(2-t\,,\,t), \quad A_{4s}(t) = -1, \quad t \in [0, t_c], \tag{57}$$

$$A_s(t) = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}, \quad B_s(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad t \in [0, t_c]. \tag{58}$$

Remember that $t_c = 1$ and, therefore, $\tilde{t}(0) = t_c = 1$.

Now, let us show that in this example, the matrix $W_s(t_c)$, given by (28), is invertible. Since the matrices $A_s(t)$ and $B_s(t)$ are constant (see Equation (58)), then, due to the results of [14], the invertibility of $W_s(t_c)$ is equivalent to the following condition:

$$\mathrm{rank}\big(B_s\,,\,A_sB_s\big) = 2. \tag{59}$$

Calculating the matrix in the left-hand side of (59), we obtain:

$$\big(B_s\,,\,A_sB_s\big) = \begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix},$$

meaning the validity of the condition (59) and, therefore, the invertibility of the matrix $W_s(t_c)$.

Thus, in this example, all the conditions of Theorem 1 are fulfilled. This yields the existence of a positive number ε^* such that, for all $\varepsilon \in (0, \varepsilon^*]$, the system (52)–(54) is functional null controllable at the time instant $t_c = 1$.

5. System of the Second Type

In this section, we consider the following particular case of the system (1) and (2):

$$\frac{dx(t)}{dt} = \sum_{j=0}^{N} \left[A_{1j}(t)x(t-\varepsilon h_j) + A_{2j}(t)y(t-\varepsilon h_j)\right]$$
$$+ \int_{-h}^{0} \left[G_1(t,\tau)x(t+\varepsilon\tau) + G_2(t,\tau)y(t+\varepsilon\tau)\right]d\tau + f_1\big(z(t),t\big)$$
$$+ f_2\big(z(t-\varepsilon h_1), \dots, z(t-\varepsilon h_N), t\big) + B_1(t)u(t), \quad t \geq 0, \tag{60}$$

$$\varepsilon\frac{dy(t)}{dt} = A_{30}(t)x(t) + A_{40}(t)y(t) + f_3\big(z(t),t\big), \quad t \geq 0. \tag{61}$$

The linear system, corresponding to (60) and (61), is a particular case of the system (8) and (9), and it has the form:

$$\frac{dx(t)}{dt} = \sum_{j=0}^{N} \left[A_{1j}(t)x(t-\varepsilon h_j) + A_{2j}(t)y(t-\varepsilon h_j)\right]$$
$$+ \int_{-h}^{0} \left[G_1(t,\tau)x(t+\varepsilon\tau) + G_2(t,\tau)y(t+\varepsilon\tau)\right]d\tau + B_1(t)u(t), \quad t \geq 0, \tag{62}$$

$$\varepsilon\frac{dy(t)}{dt} = A_{30}(t)x(t) + A_{40}(t)y(t), \quad t \geq 0. \tag{63}$$

5.1. Auxiliary Results

5.1.1. Asymptotic Decomposition of (62) and (63)

Decomposing asymptotically the system (62) and (63), we obtain its slow subsystem:

$$\frac{dx_s(t)}{dt} = A_{1s}(t)x_s(t) + A_{2s}(t)y_s(t) + B_1(t)u_s(t), \quad t \geq 0, \tag{64}$$

$$0 = A_{30}(t)x_s(t) + A_{40}(t)y_s(t), \quad t \geq 0, \tag{65}$$

where $x_s(t) \in E^n$, $y_s(t) \in E^m$, $u_s(t) \in E^n$ (u_s is a control);

$$A_{is}(t) = \sum_{j=0}^{N} A_{ij}(t) + \int_{-h}^{0} G_i(t,\tau)d\tau, \quad i = 1,2. \tag{66}$$

Like in Section 4.1.1, the slow subsystem (64) and (65) is a differential-algebraic system.

In what follows, we assume that:

$$\det A_{40}(t) \neq 0, \quad t \geq 0. \tag{67}$$

Subject to this assumption, the slow subsystem (64) and (65) can be reduced to the differential equation with respect to $x_s(t)$:

$$\frac{dx_s(t)}{dt} = A_s(t)x_s(t) + B_1(t)u_s(t), \quad t \geq 0, \tag{68}$$

where:

$$A_s(t) = A_{1s}(t) - A_{2s}(t)A_{40}^{-1}(t)A_{30}(t). \tag{69}$$

Like in Section 4.1.1, the differential Equation (68) also is called the slow subsystem, associated with the system (62) and (63).

The fast subsystem, associated with the system (62) and (63), is obtained in a way similar to that of obtaining the system (22). Thus, we have:

$$\frac{dy_f(\xi)}{d\xi} = A_{40}(t)y_f(\xi), \quad \xi \geq 0, \tag{70}$$

where $t \geq 0$ is a parameter; $\xi \geq 0$ is an independent variable; $y_f(\xi) \in E^m$ is a state variable.

The meaning of the new independent variable ξ and its connection with the original independent variable t are the same as in Section 4.1.1. However, in contrast with the results of Section 4.1.1 (see the Equation (22)), the fast subsystem (70) does not contain a control variable, i.e., it is completely uncontrollable. Therefore, the method, proposed in Sections 4.1.2–4.1.4, for the ε-free analysis of the Euclidean space null controllability of the first type system (12) and (13), is not applicable for such an analysis of the Euclidean space null controllability of the second type system (60) and (61). In this section, we apply another method for this analysis, which is based on the results of [28].

5.1.2. ε-Free Conditions for Euclidean Space Null Controllability of the Linear System (62) and (63)

Here, we assume:

(I_2) All roots $\lambda(t)$ of the equation:

$$\det\left[\lambda I_m - A_{40}(t)\right] = 0 \tag{71}$$

satisfy the inequality $\mathrm{Re}\lambda(t) < -\beta$ for all $t \in [0, \tilde{t}(0)]$, where $\beta > 0$ is some constant.

Remember that $\tilde{t}(\varepsilon)$, $\varepsilon \in [0, \varepsilon^0]$ is a given continuously-differentiable non-increasing function of ε, satisfying the inequality (10).

Let the $n \times n$-matrix-valued function $\Phi_s(\sigma)$ be the solution of the following terminal-value problems:

$$\frac{d\Phi_s(\sigma)}{d\sigma} = -A_s^T(\sigma)\Phi_s(\sigma), \quad \sigma \in \left[0, \tilde{t}(0)\right), \quad \Phi_s\left(\tilde{t}(0)\right) = I_n. \tag{72}$$

Let:

$$\Phi_f(\xi) \stackrel{\triangle}{=} \exp\left[A_{40}^T\left(\tilde{t}(0)\right)\xi\right], \quad \xi \geq 0. \tag{73}$$

From the assumption (I_2), we directly obtain:

$$\|\Phi_f(\xi)\| \leq a\exp(-\beta\xi), \quad \xi \geq 0, \tag{74}$$

where $a > 0$ is some constant.

Let:

$$\Theta_f(\xi) \stackrel{\triangle}{=} A_{30}^T\left(\tilde{t}(0)\right)\int_{\xi}^{+\infty}\Phi_f(\kappa)d\kappa, \quad \xi \geq 0. \tag{75}$$

Due to the inequality (74), the integral in the right-hand side of (75) converges and:

$$\Theta_f(\xi) = -A_{30}^T(\tilde{t}(0))\Big(A_{40}^T(\tilde{t}(0))\Big)^{-1} \exp\big[A_{40}^T(\tilde{t}(0))\xi\big], \quad \xi \geq 0.$$

Consider the matrices:

$$M_s(\tilde{t}(0)) = \int_0^{\tilde{t}(0)} \Phi_s^T(\sigma)B_1(\sigma)B_1^T(\sigma)\Phi_s(\sigma)d\sigma, \tag{76}$$

$$M_f(\tilde{t}(0)) = \int_0^{+\infty} \Theta_f^T(\xi)B_1(\tilde{t}(0))B_1^T(\tilde{t}(0))\Theta_f(\xi)d\xi. \tag{77}$$

Quite similarly to the results of [28], we obtain the following assertion.

Proposition 7. *Let the assumption (I_2) be valid. Let* $\det M_s(\tilde{t}(0)) \neq 0$, $\det M_f(\tilde{t}(0)) \neq 0$. *Then, there exists a positive number $\varepsilon_1 \leq \varepsilon^0$ such that, for all $\varepsilon \in (0, \varepsilon_1]$, the singularly-perturbed linear system (62) and (63) is Euclidean space null controllable at the time instant $\tilde{t}(\varepsilon)$.*

Remark 2. *Due to the results of [28], the matrix $M_f(\tilde{t}(0))$ is nonsingular if and only if the following system is null controllable at some $\xi = \xi_c$:*

$$\frac{d\tilde{y}(\xi)}{d\xi} = A_{40}(\tilde{t}(0))\tilde{y}(\xi) + A_{40}^{-1}(\tilde{t}(0))A_{30}(\tilde{t}(0))B_1(\tilde{t}(0))\tilde{u}(\xi), \quad \xi \geq 0, \tag{78}$$

where $\tilde{y}(\xi) \in E^m$ is a state variable; $\tilde{u}(\xi) \in E^r$ is a control variable.

5.1.3. Euclidean Space Null Controllability of the Nonlinear System (60) and (61): ε-Free Conditions

Based on Equations (3) and (4), the results of [34], and using Proposition 7, we directly obtain the following assertion.

Proposition 8. *Let the assumption (I_2) be valid. Let* $\det M_s(\tilde{t}(0)) \neq 0$, $\det M_f(\tilde{t}(0)) \neq 0$. *Then, there exists a positive number $\varepsilon_1 \leq \varepsilon^0$ such that, for all $\varepsilon \in (0, \varepsilon_1]$, the singularly-perturbed nonlinear system (60) and (61) is Euclidean space null controllable at the time instant $\tilde{t}(\varepsilon)$.*

5.2. Main Result

In this subsection, we assume that $u(t) \in E^n$ and $B_1(t)$ is an $n \times n$-matrix. Along with this, we assume:

(II_2) The matrix $B_1(t_c)$ is invertible.
(III_2) $f_3(0, t) = 0$ for all $t \in [0, t_c]$.
(IV_2) The vector-valued functions $f_1(z, t)$ and $f_3(z, t)$ satisfy the local Lipschitz condition with respect to $z \in E^{n+m}$ uniformly in $t \in [0, t_c]$.

Note that the assumption (IV_2) is the same as the assumption (IV_1) in Section 4.2. However, for the sake of the paper's readability, we repeat here this assumption.

Lemma 2. *Let the assumption (II_2) be valid. Then,* $\det M_s(t_c) \neq 0$.

Proof. Due to the smoothness of the matrix-valued function $B_1(t)$ and the assumption (II_2), there exists a number $t_1 \in (0, t_c)$ such that, for all $t \in [t_1, t_c]$, the matrix $B_1(t)$ is invertible. Using Equation (76), we can represent the matrix $M_s(t_c)$ as:

$$M_s(t_c) = \int_0^{t_1} \Phi_s^T(\sigma) B_1(\sigma) B_1^T(\sigma) \Phi_s(\sigma) d\sigma + \int_{t_1}^{t_c} \Phi_s^T(\sigma) B_1(\sigma) B_1^T(\sigma) \Phi_s(\sigma) d\sigma. \tag{79}$$

The first integral in the right-hand side of (79) is at least a positive semi-definite matrix, while the second integral is a positive definite matrix. Therefore, the sum of these integrals is a positive definite matrix, which completes the proof of the lemma. □

Theorem 2. *Let the assumptions (I_2)–(IV_2) be valid. Let* $\det M_f(t_c) \neq 0$*. Then, there exists a positive number* $\varepsilon^* < t_c/(2h)$ *such that, for any* $\varepsilon \in (0, \varepsilon^*]$*, the system (60) and (61) is functional null controllable at the time instant* t_c*.*

Proof. Using Proposition 8 and Lemma 2, the theorem is proven quite similarly to Theorem 1. □

5.3. Example

Consider the following system, a particular case of (60) and (61):

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t) + 2y_1(t) + 5t^2 y_2(t) + tx(t-\varepsilon) - y_1(t-\varepsilon) + y_2(t-\varepsilon) \\ -2t \int_{-1}^{0} \tau x(t+\varepsilon\tau) d\tau &- 10t^2 \int_{-1}^{0} \tau y_1(t+\varepsilon\tau) d\tau + \cos\left(t^2 x(t) + y_1(t) - t y_2(t)\right) \\ &- \sin^2\left(x(t-\varepsilon) + t^2 y_1(t-\varepsilon) + y_2(t-\varepsilon)\right) + tu(t), \quad t \geq 0, \end{aligned} \tag{80}$$

$$\varepsilon \frac{dy_1(t)}{dt} = 2x(t) - y_1(t) - 5t y_2(t) - \ln\left(1 + t x^2(t) + y_1^2(t) + y_2^2(t)\right), \quad t \geq 0, \tag{81}$$

$$\varepsilon \frac{dy_2(t)}{dt} = -x(t) + t y_1(t) - y_2(t) + \arctan\left(x(t) - t y_1(t) + y_2(t)\right), \quad t \geq 0, \tag{82}$$

where $x(t)$, $y_1(t)$, $y_2(t)$, and $u(t)$ are scalars, i.e., $n = 1$, $m = 2$, $r = 1$.

In this example, like in the example of Section 4.3, we study the functional null controllability of the system (80)–(82) at the time instant $t_c = 1$ for all sufficiently small $\varepsilon > 0$.

Comparing the system (80)–(82) with the system (60) and (61), we obtain that in the present example: $N = 1$, $h_1 = h = 1$, and:

$$\begin{aligned} A_{10}(t) = 1, \quad A_{20}(t) = (2\,,\,5t^2), \quad A_{30}(t) &= \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \\ A_{40}(t) = \begin{pmatrix} -1 & -5t \\ t & -1 \end{pmatrix}, \quad A_{11}(t) = t, \quad A_{21}(t) &= (-1\,,\,1), \\ G_1(t,\tau) = -2t\tau, \quad G_2(t,\tau) = (-10t^2\tau\,,\,0), \quad B_1(t) &= t, \\ f_1(z(t),t) &= \cos\left(t^2 x(t) + y_1(t) - t y_2(t)\right), \\ f_2(z(t-\varepsilon),t) &= -\sin^2\left(x(t-\varepsilon) + t^2 y_1(t-\varepsilon) + y_2(t-\varepsilon)\right), \\ f_3(z(t),t) &= \begin{pmatrix} -\ln\left(1 + t x^2(t) + y_1^2(t) + y_2^2(t)\right) \\ \arctan\left(x(t) - t y_1(t) + y_2(t)\right) \end{pmatrix}, \end{aligned} \tag{83}$$

where $z(\cdot) = \mathrm{col}\big(x(\cdot), y_1(\cdot), y_2(\cdot)\big)$.

Thus, in this example, the limit conditions (3) and (4) are valid for $l = 1, 3$, $p = 2$. The coefficients of $x(\cdot)$, $y_1(\cdot)$, $y_2(\cdot)$, and $u(t)$ in the linear terms are smooth functions for $t \geq 0$, $\tau \in [-1, 0]$. Moreover, the assumptions (II_2)–(IV_2) also are valid. Let us show the validity of the assumption (I_2). In this example, Equation (71) becomes:

$$(1 + \lambda)^2 + 5t^2 = 0, \tag{84}$$

yielding the roots:

$$\lambda_1(t) = -1 + \sqrt{5}ti, \quad \lambda_2(t) = -1 - \sqrt{5}ti, \quad t \geq 0,$$

where i is the imaginary unit.

Thus, the real parts of the roots of Equation (84) equal -1, meaning the fulfillment of the assumption (I_2) with any β larger than one.

Now, let us show that $\det M_f(t_c) \neq 0$. Due to Remark 2, this inequality is equivalent to the null controllability of the system (78), where $\tilde{t}(0) = t_c = 1$. Since the coefficients of this system are constant, then, due to the results of [14], the null controllability of this system is equivalent to the following condition:

$$\operatorname{rank}\big(A_{40}^{-1}(1)A_{30}(1)B_1(1)\,,\ A_{30}(1)B_1(1)\big) = 2. \tag{85}$$

Calculating the matrix in the left-hand side of (85), we obtain:

$$\big(A_{40}^{-1}(1)A_{30}(1)B_1(1)\,,\ A_{30}(1)B_1(1)\big) = \begin{pmatrix} -7/6 & 2 \\ -1/6 & -1 \end{pmatrix},$$

meaning the validity of the condition (85) and, therefore, the null controllability of the system (78). Thus, in this example, the matrix $M_f(t_c)$ is nonsingular. Hence, in this example, all the conditions of Theorem 2 are fulfilled. This yields the existence of a positive number ε^* such that, for all $\varepsilon \in (0, \varepsilon^*]$, the system (80)–(82) is functional null controllable at time instant $t_c = 1$.

6. Conclusions

In this paper, two types of singularly-perturbed nonlinear time-dependent-controlled differential systems with time delays (multiple point-wise and distributed) in the state variables were analyzed. The case, where the delays are small, of the order of a small positive multiplier ε for a part of the derivatives in the differential equations, was treated. In the system of the first type, the slow mode equation was uncontrolled, and this equation did not contain the delayed state variables. In the system of the second type, the fast mode equation was uncontrolled, and it did not contain the delayed state variables. The functional null controllability of the considered systems, robust with respect to the small parameter ε, was studied. For each type of system, ε-free conditions, guaranteeing the functional null controllability for all sufficiently small values of ε, were derived. These results were based on the ε-free analysis of the Euclidean space null controllability of the considered systems.

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