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Modified Viscosity Subgradient Extragradient-Like Algorithms for Solving Monotone Variational Inequalities Problems

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Abstract: Variational inequality theory is an effective tool for engineering, economics, transport and mathematical optimization. Some of the approaches used to resolve variational inequalities usually involve iterative techniques. In this article, we introduce a new modified viscosity-type extragradient method to solve monotone variational inequalities problems in real Hilbert space. The result of the strong convergence of the method is well established without the information of the operator's Lipschitz constant. There are proper mathematical studies relating our newly designed method to the currently state of the art on several practical test problems.

Keywords: projection methods; strong convergence; extragradient method; monotone mapping; variational inequalities

1. Introduction

Assume that $\mathcal{C}$ is a nonempty, closed and convex subset of a real Hilbert space $\mathbb{H}$, and $\mathbb{R}$ and $\mathbb{N}$ are the sets of real numbers and natural numbers, respectively. In this paper, we consider the classical variational inequalities problems [1,2] (in short, $VI(F,\mathcal{C})$) and the solution set of variational inequalities problem represent by $SVI(F,\mathcal{C})$. Assume that F is an operator $F : \mathbb{H} \to \mathbb{H}$ and the variational inequalities problem for an operator $F : \mathbb{H} \to \mathbb{H}$ is defined in the following way:

$$\text{Find } u^* \in \mathcal{C} \text{ such that } \langle F(u^*), y - u^* \rangle \geq 0, \ \forall y \in \mathcal{C}. \tag{1}$$

The problem (1) is well defined and equivalent to solve the following fixed point problem:

$$\text{Find a point } u^* \in \mathcal{C} \text{ such that } u^* = P_{\mathcal{C}}[u^* - \zeta F(u^*)],$$

for some $0 < \zeta < \frac{1}{L}$ where L is the Lipschitz constant of the operator F. We assume that the followings conditions have been satisfied:

(b1) The solution set is represented by $SVI(F,\mathcal{C})$ and it is nonempty;

(b2) An operator $F : \mathbb{H} \to \mathbb{H}$ is monotone—i.e.,

$$\langle F(u_1) - F(u_2), u_1 - u_2 \rangle \geq 0, \ \forall u_1, u_2 \in \mathcal{C};$$

(b3) F is Lipschitz continuous if there exists $L > 0$, such that

$$\|F(u_1) - F(u_2)\| \leq L\|u_1 - u_2\|, \ \forall u_1, u_2 \in \mathcal{C}.$$

The variational inequalities theory is a useful technique for investigating a large number of problems in physics, economics, engineering and optimization theory. It was firstly introduced by Stampacchia [1] in 1964 and also well established that the problem (1) is an important problem in nonlinear analysis. It is an advantageous mathematical model that puts together several topics of applied mathematics, such as the network equilibrium problems, the necessary optimality conditions, the systems of non-linear equations and the complementarity problems [3–7].

The projection method and its modified version methods are crucial for finding the numerical solutions of variational inequality problems. Many studies have been suggested and researched different types of projection methods to solve the variational inequalities problem (see for more details [8–18]) and others, as in [19–28]. The simplistic methodology is the gradient method for which only one projection on a feasible set is required. A convergence of the method, however, requires strong monotonicity on F. To prevent the strong monotonicity hypothesis, Korpelevich [8] and Antipin [29] introduced the following extragradient method.

$$\begin{cases} u_n \in \mathcal{C}, \\ v_n = P_{\mathcal{C}}[u_n - \zeta F(u_n)], \\ u_{n+1} = P_{\mathcal{C}}[u_n - \zeta F(v_n)], \end{cases}$$

for some $0 < \zeta < \frac{1}{L}$. The subgradient extragradient algorithm was recently developed by Censor et al. [10] to resolve problem (1) in real Hilbert space. Their method has the form of

$$\begin{cases} u_n \in \mathcal{C}, \\ v_n = P_{\mathcal{C}}[u_n - \zeta F(u_n)], \\ u_{n+1} = P_{\mathbb{H}_n}[u_n - \zeta F(v_n)], \end{cases} \tag{2}$$

where $0 < \zeta < \frac{1}{L}$ and $\mathbb{H}_n = \{z \in \mathbb{H} : \langle u_n - \zeta F(u_n) - v_n, z - v_n \rangle \leq 0\}$.

In this article, motivated by the methods in [10,30,31] and the viscosity method [14] we introduce a new viscosity subgradient–extragradient algorithm to solve variational inequality problems involving monotone operators in Hilbert space. It is important to note that, our proposed algorithm operates more effectively than the existing ones. Particularly in comparison to the results of Yang et al. [30], our algorithm operates efficiently in most situations. Analogously to the results of Yang et al. [30], proof of the convergence of Algorithm 1, it is not compulsory to have the information of the Lipschitz constant of the operator F. The proposed algorithm could be seen as a modification of the methods that are found in [8,10,30,31]. Under mild conditions, a strong convergence theorem was proven to be associated with the proposed method. Numerical experimental studies have been shown that the new method considers being more effective than the current ones in [30].

The rest of the article is arranged in the following way: Section 2 provides a few definitions and basic results that are used throughout the paper. Section 3 contains the main algorithm and convergence theorem. Section 4 includes the numerical results that illustrate the algorithmic efficacy of the introduced method.

Algorithm 1 An Explicit Method for Monotone Variational Inequality Problems

Step 0: Let $u_0 \in \mathcal{C}$, $\mu \in (0,1)$, $\zeta_0 > 0$ and a sequence $\beta_n \subset (0,1)$ with $\beta_n \to 0$ and $\sum_n^{\infty} \beta_n = +\infty$.

Step 1: Assume that $\{u_n\}$ is given and compute

$$v_n = P_{\mathcal{C}}[u_n - \zeta_n F(u_n)].$$

If $u_n = v_n$; STOP. Else, move to **Step 2**.

Step 2: Create a half-space

$$\mathbb{H}_n = \{z \in \mathbb{H} : \langle u_n - \zeta_n F(u_n) - v_n, z - v_n \rangle \leq 0\}.$$

Step 3:

$$u_{n+1} = \beta_n f(u_n) + (1 - \beta_n) z_n,$$

while $z_n = P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)]$.

Step 4: Compute

$$\zeta_{n+1} = \begin{cases} \min\left\{\zeta_n, \frac{\mu\|u_n - v_n\|^2 + \mu\|z_n - v_n\|^2}{2\left\langle F(u_n) - F(v_n), z_n - v_n \right\rangle}\right\} & \text{if} \quad \left\langle F(u_n) - F(v_n), z_n - v_n \right\rangle > 0, \\ \zeta_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and return to **Step 1**.

2. Background

A metric projection $P_{\mathcal{C}}(u_1)$ for $u_1 \in \mathbb{H}$ onto a closed and convex subset $\mathcal{C}$ of $\mathbb{H}$ is defined by

$$P_{\mathcal{C}}(u_1) = \arg\min\{\|u_2 - u_1\| : u_2 \in \mathcal{C}\}.$$

Lemma 1 ([32]; Page 31)**.** *For $u, v \in \mathbb{H}$ and $a \in \mathbb{R}$, then the following relationship holds.*

(i). $\|au + (1-a)v\|^2 = a\|u\|^2 + (1-a)\|v\|^2 - a(1-a)\|u - v\|^2.$

(ii). $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle.$

Lemma 2 ([32,33])**.** *Assume $\mathcal{C}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathbb{H}$ and let $P_{\mathcal{C}} : \mathbb{H} \to \mathcal{C}$ be a metric projection from $\mathbb{H}$ onto $\mathcal{C}$. Then:*

(i). Let $u_1 \in \mathcal{C}$ and $u_2 \in \mathbb{H}$

$$\|u_1 - P_{\mathcal{C}}(u_2)\|^2 + \|P_{\mathcal{C}}(u_2) - u_2\|^2 \leq \|u_1 - u_2\|^2.$$

(ii). $u_3 = P_{\mathcal{C}}(u_1)$ if and only if

$$\langle u_1 - u_3, u_2 - u_3 \rangle \leq 0, \ \forall\, u_2 \in \mathcal{C}.$$

(iii). For $u_2 \in \mathcal{C}$ and $u_1 \in \mathbb{H}$

$$\|u_1 - P_{\mathcal{C}}(u_1)\| \leq \|u_1 - u_2\|.$$

Lemma 3 ([34])**.** *Assume that $\{\chi_n\}$ is a sequence of non-negative real numbers such that*

$$\chi_{n+1} \leq (1 - \alpha_n)\chi_n + \alpha_n \delta_n, \ \forall\, n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ meet with the following criteria:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \textit{and} \ \limsup_{n\to\infty} \delta_n \leq 0.$$

Then, $\lim_{n\to\infty} \chi_n = 0$.

Lemma 4 ([35]). *Assume that* $\{\chi_n\}$ *is a sequence of real numbers such that there is a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $\chi_{n_i} < \chi_{n_{i+1}}$ *for all* $i \in \mathbb{N}$*. Then, there is a non decreasing sequence* $m_k \subset \mathbb{N}$ *such that* $m_k \to \infty$ *as* $k \to \infty$*, and the following conditions are fullfilled by all (sufficiently large) numbers* $k \in \mathbb{N}$*:*

$$\chi_{m_k} \leq \chi_{m_{k+1}} \text{ and } \chi_k \leq \chi_{m_{k+1}}.$$

In fact, $m_k = \max\{j \leq k : \chi_j \leq \chi_{j+1}\}$.

Lemma 5 ([36]). *Assume that* $\mathcal{C}$ *is a nonempty closed convex set in* $\mathbb{H}$ *and an operator* $F : \mathcal{C} \to \mathbb{H}$ *is monotone and continuous. Then,* u^* *is a solution of the problem* (1) *if and only if* u^* *is a solution of the following problem:*

$$\text{Find } x \in \mathcal{C} \text{ such that } \langle F(y), y - x\rangle \geq 0, \ \forall y \in \mathcal{C}.$$

3. Algorithm and Corresponding Strong Convergence Theorem

We provide a method consisting of two convex minimization problems through a viscosity and an explicit stepsize formula which are being used to enhance the rate of convergence the iterative sequence and to make the method independent of the Lipschitz constant L. The detailed method is given below:

Remark 1. $\mathbb{H}_n$ *is a half-space and so* $\mathbb{H}_n$ *is a closed and convex set in* $\mathbb{H}$.

Lemma 6. *The sequence* $\{\zeta_n\}$ *is decreasing monotonically with a lower bound* $\min\left\{\frac{\mu}{L}, \zeta_0\right\}$ *and converges to* $\zeta > 0$.

Proof. From the sequence $\{\zeta_n\}$, we see that this sequence is monotone and nonincreasing. It is given that F is Lipschitz-continuous with $L > 0$. Let $\langle F(u_n) - F(v_n), z_n - v_n\rangle > 0$, such that

$$\begin{aligned}\frac{\mu(\|u_n - v_n\|^2 + \|z_n - v_n\|^2)}{2\langle F(u_n) - F(v_n), z_n - v_n\rangle} &\geq \frac{2\mu\|u_n - v_n\|\|z_n - v_n\|}{2\|F(u_n) - F(v_n)\|\|z_n - v_n\|} \\ &\geq \frac{2\mu\|u_n - v_n\|\|z_n - v_n\|}{2\|u_n - v_n\|\|z_n - v_n\|} \\ &\geq \frac{\mu}{L}. \end{aligned} \tag{3}$$

The above discussion implies that the sequence $\{\zeta_n\}$ has a lower bound $\min\left\{\frac{\mu}{L}, \zeta_0\right\}$. Moreover, there exists number $\zeta > 0$, such that $\lim_{n\to\infty} \zeta_n = \zeta$. □

Lemma 7. *Assume that an operator* $F : \mathcal{C} \to \mathbb{H}$ *satisfies the conditions* (b1)-(b3)*. For each* $u^* \in SVI(F, \mathcal{C}) \neq \emptyset$*, we have*

$$\|z_n - u^*\|^2 \leq \|u_n - u^*\|^2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|u_n - v_n\|^2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|z_n - v_n\|^2.$$

Proof. Let consider the following

$$\begin{aligned}\|z_n - u^*\|^2 &= \|P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)] - u^*\|^2 \\ &= \|P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)] + [u_n - \zeta_n F(v_n)] - [u_n - \zeta_n F(v_n)] - u^*\|^2 \\ &= \|[u_n - \zeta_n F(v_n)] - u^*\|^2 + \|P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)] - [u_n - \zeta_n F(v_n)]\|^2 \\ &\quad + 2\langle P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)] - [u_n - \zeta_n F(v_n)], [u_n - \zeta_n F(v_n)] - u^*\rangle. \end{aligned} \tag{4}$$

From the assumption that $u^* \in SVI(F,\mathcal{C}) \subset \mathcal{C} \subset \mathbb{H}_n$, we have

$$\begin{aligned}&\big\|P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)] - [u_n - \zeta_n F(v_n)]\big\|^2 \\ &\quad + \big\langle P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)] - [u_n - \zeta_n F(v_n)], [u_n - \zeta_n F(v_n)] - u^*\big\rangle \\ &= \big\langle [u_n - \zeta_n F(v_n)] - P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)], u^* - P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)]\big\rangle \le 0,\end{aligned} \tag{5}$$

implies that

$$\begin{aligned}&\big\langle P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)] - [u_n - \zeta_n F(v_n)], [u_n - \zeta_n F(v_n)] - u^*\big\rangle \\ &\le -\big\|P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)] - [u_n - \zeta_n F(v_n)]\big\|^2.\end{aligned} \tag{6}$$

Now, using the Equation (4) implies that

$$\begin{aligned}\|z_n - u^*\|^2 &\le \big\|u_n - \zeta_n F(v_n) - u^*\big\|^2 - \big\|P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)] - [u_n - \zeta_n F(v_n)]\big\|^2 \\ &\le \|u_n - u^*\|^2 - \|u_n - z_n\|^2 + 2\zeta_n\big\langle F(v_n), u^* - z_n\big\rangle.\end{aligned} \tag{7}$$

Given that u^* is a solution of $VI(F,\mathcal{C})$, we get

$$\langle F(u^*), y - u^*\rangle \ge 0, \ \forall y \in \mathcal{C}. \tag{8}$$

Due to the monotonicity of F on $\mathcal{C}$, we can obtain

$$\langle F(v_n) - F(u^*), v_n - u^*\rangle \ge 0, \ \forall y \in \mathcal{C}. \tag{9}$$

Since $v_n \in \mathcal{C}$, it follows that

$$\langle F(v_n), v_n - u^*\rangle \ge 0. \tag{10}$$

Thus, we have

$$\big\langle F(v_n), u^* - z_n\big\rangle = \big\langle F(v_n), u^* - v_n\big\rangle + \big\langle F(v_n), v_n - z_n\big\rangle \le \big\langle F(v_n), v_n - z_n\big\rangle. \tag{11}$$

From (7) and (11), we get

$$\begin{aligned}\|z_n - u^*\|^2 &\le \|u_n - u^*\|^2 - \|u_n - z_n\|^2 + 2\zeta_n\big\langle F(v_n), v_n - z_n\big\rangle \\ &= \|u_n - u^*\|^2 - \|u_n - v_n + v_n - z_n\|^2 + 2\zeta_n\big\langle F(v_n), v_n - z_n\big\rangle \\ &\le \|u_n - u^*\|^2 - \|u_n - v_n\|^2 - \|v_n - z_n\|^2 + 2\big\langle u_n - \zeta_n F(v_n) - v_n, z_n - v_n\big\rangle.\end{aligned} \tag{12}$$

Note that $z_n = P_{\mathbb{H}_n}[u_n - \zeta_n F(v_n)]$ and by the definition of ζ_{n+1}, we have

$$\begin{aligned}&2\big\langle u_n - \zeta_n F(v_n) - v_n, z_n - v_n\big\rangle \\ &= 2\big\langle u_n - \zeta_n F(u_n) - v_n, z_n - v_n\big\rangle + 2\zeta_n\big\langle F(u_n) - F(v_n), z_n - v_n\big\rangle \\ &\le \frac{2\zeta_n}{\zeta_{n+1}}\zeta_{n+1}\big\langle F(u_n) - F(v_n), z_n - v_n\big\rangle \le \frac{\zeta_n}{\zeta_{n+1}}\big[\mu\|u_n - v_n\|^2 + \mu\|z_n - v_n\|^2\big].\end{aligned} \tag{13}$$

From expression (12) and (13), we obtain

$$\begin{aligned}&\|z_n - u^*\|^2 \\ &\le \|u_n - u^*\|^2 - \|u_n - v_n\|^2 - \|v_n - z_n\|^2 + \frac{\zeta_n}{\zeta_{n+1}}\big[\mu\|u_n - v_n\|^2 + \mu\|z_n - v_n\|^2\big] \\ &\le \|u_n - u^*\|^2 - \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\|u_n - v_n\|^2 - \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\|z_n - v_n\|^2.\end{aligned} \tag{14}$$

□

Theorem 1. *Assume that an operator $F : \mathcal{C} \to \mathbb{H}$ satisfies the conditions* (b1)-(b3) *and u^* belongs to solution set $SVI(F, \mathcal{C})$. Then, the sequences $\{u_n\}$, $\{v_n\}$ and $\{z_n\}$ generated by Algorithm 1 strongly converge to u^*.*

Proof. **Claim 1:** The sequence $\{u_n\}$ is bounded in $\mathbb{H}$.

From Lemma 7, we have

$$\|z_n - u^*\|^2 \leq \|u_n - u^*\|^2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|u_n - v_n\|^2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|z_n - v_n\|^2. \tag{15}$$

Since $\zeta_n \to \zeta$, then exits a fixed number $\epsilon \in (0, 1 - \mu)$ such that

$$\lim_{n\to\infty}\left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right) = 1 - \mu > \epsilon > 0.$$

Thus, there is a finite number $N_1 \in \mathbb{N}$ such that

$$\left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right) > \epsilon > 0, \ \forall n \geq N_1. \tag{16}$$

Thus, from (15), we obtain

$$\|z_n - u^*\|^2 \leq \|u_n - u^*\|^2, \ \forall n \geq N_1. \tag{17}$$

Let $u^* \in SVI(F, \mathcal{C})$. By definition of the sequence $\{u_{n+1}\}$ and due to contraction f with constant $\rho \in [0, 1)$ and $n \geq N_1$, we obtain

$$\begin{aligned}
\|u_{n+1} - u^*\| &= \|\beta_n f(u_n) + (1-\beta_n)z_n - u^*\| \\
&= \|\beta_n[f(u_n) - u^*] + (1-\beta_n)[z_n - u^*]\| \\
&= \|\beta_n[f(u_n) + f(u^*) - f(u^*) - u^*] + (1-\beta_n)[z_n - u^*]\| \\
&\leq \beta_n\|f(u_n) - f(u^*)\| + \beta_n\|f(u^*) - u^*\| + (1-\beta_n)\|z_n - u^*\| \\
&\leq \beta_n\rho\|u_n - u^*\| + \beta_n\|f(u^*) - u^*\| + (1-\beta_n)\|z_n - u^*\|.
\end{aligned} \tag{18}$$

Consider the expressions (17) and (18) and $\beta_n \subset (0, 1)$, we have

$$\begin{aligned}
\|u_{n+1} - u^*\| &\leq \beta_n\rho\|u_n - u^*\| + \beta_n\|f(u^*) - u^*\| + (1-\beta_n)\|u_n - u^*\| \\
&= [1 - \beta_n + \rho\beta_n]\|u_n - u^*\| + \beta_n(1-\rho)\frac{\|f(u^*) - u^*\|}{(1-\rho)} \\
&\leq \max\left\{\|u_n - u^*\|, \frac{\|f(u^*) - u^*\|}{(1-\rho)}\right\} \\
&\leq \max\left\{\|u_{N_1} - u^*\|, \frac{\|f(u^*) - u^*\|}{(1-\rho)}\right\}.
\end{aligned} \tag{19}$$

Finally, we deduce that the sequence $\{u_n\}$ is bounded.

Claim 2: If $\lim_{n\to\infty}\|u_n - v_n\| = 0$, then, as a subsequence, $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\} \rightharpoonup u^* \in SVI(F, \mathcal{C})$ as $k \to \infty$.

The reflexivity of $\mathbb{H}$ and the boundedness of $\{u_n\}$ imply that there exists a subsequence $\{u_{n_k}\}$ such that $\{u_{n_k}\} \rightharpoonup u^* \in \mathbb{H}$ as $k \to \infty$. It is sufficient to prove that $u^* \in SVI(F, \mathcal{C})$. Due to $\lim_{n\to\infty} \|u_n - v_n\| = 0$, we also have $\{v_{n_k}\} \rightharpoonup u^*$ as $k \to \infty$. In addition, the fact that

$$v_{n_k} = P_{\mathcal{C}}[u_{n_k} - \zeta_{n_k} F(u_{n_k})],$$

that is equivalent to

$$\langle u_{n_k} - \zeta_{n_k} F(u_{n_k}) - v_{n_k}, y - v_{n_k} \rangle \leq 0, \ \forall\, y \in \mathcal{C}.$$

That is, we have

$$\langle u_{n_k} - v_{n_k}, y - v_{n_k} \rangle \leq \zeta_{n_k} \langle F(u_{n_k}), y - v_{n_k} \rangle, \ \forall\, y \in \mathcal{C}. \tag{20}$$

From the monotonicity condition on F, we have

$$\langle F(u_{n_k}) - F(y), u_{n_k} - y \rangle \geq 0, \ \forall\, y \in \mathcal{C},$$

that is

$$\langle F(y), y - u_{n_k} \rangle \geq \langle F(u_{n_k}), y - u_{n_k} \rangle, \ \forall\, y \in \mathcal{C}. \tag{21}$$

Combining expressions (20) and (21), we obtain

$$\begin{aligned} 0 &\leq \langle v_{n_k} - u_{n_k}, y - v_{n_k} \rangle + \zeta_{n_k} \langle F(u_{n_k}), y - v_{n_k} \rangle \\ &= \langle v_{n_k} - u_{n_k}, y - v_{n_k} \rangle + \zeta_{n_k} \langle F(u_{n_k}), y - u_{n_k} \rangle + \zeta_{n_k} \langle F(u_{n_k}), u_{n_k} - v_{n_k} \rangle \\ &\leq \langle v_{n_k} - u_{n_k}, y - v_{n_k} \rangle + \zeta_{n_k} \langle F(y), y - u_{n_k} \rangle + \zeta_{n_k} \langle F(u_{n_k}), u_{n_k} - v_{n_k} \rangle, \end{aligned} \tag{22}$$

for all $y \in \mathcal{C}$, since $\lim_{k\to\infty} \zeta_{n_k} = \zeta > 0$ (see Lemma 6) and the sequence $\{u_n\}$ is bounded in $\mathbb{H}$. As $\lim_{n\to\infty} \|u_n - v_n\| = 0$, and pass the limit in (22) as $k \to \infty$, we obtain

$$\langle F(y), y - u^* \rangle \geq 0, \ \forall y \in \mathcal{C}. \tag{23}$$

Apply the well-known Minty Lemma 5, this is what we infer: $u^* \in SVI(F, \mathcal{C})$.

Claim 3: The sequence $\{u_n\}$ is strong convergent in $\mathbb{H}$.

The strong convergence of the sequence $\{u_n\}$ is as follows. The continuity and monotonicity of the operator F and the Minty lemma gives that $SVI(F, \mathcal{C})$ is a closed and convex set (see [37,38] for more details). As mapping f is a contraction, so is $P_{SVI(F,\mathcal{C})} \circ f$. By using the Banach contraction principle to guarantee that an unique element exists, $u^* \in SVI(F, \mathcal{C})$, such that

$$u^* = P_{SVI(F,\mathcal{C})}(f(u^*)).$$

Hence, we have

$$\langle f(u^*) - u^*, y - u^* \rangle \geq 0, \ \forall\, y \in SVI(F, \mathcal{C}). \tag{24}$$

Now, considering $u_{n+1} = \beta_n f(u_n) + (1-\beta_n)z_n$, and using Lemma 1 (i) and Lemma 7, we have

$$\begin{aligned}
\|u_{n+1} - u^*\|^2 &= \|\beta_n f(u_n) + (1-\beta_n)z_n - u^*\|^2 \\
&= \|\beta_n[f(u_n) - u^*] + (1-\beta_n)[z_n - u^*]\|^2 \\
&= \beta_n\|f(u_n) - u^*\|^2 + (1-\beta_n)\|z_n - u^*\|^2 - \beta_n(1-\beta_n)\|f(u_n) - z_n\|^2 \\
&\le \beta_n\|f(u_n) - u^*\|^2 + (1-\beta_n)\Big[\|u_n - u^*\|^2 - \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\|u_n - v_n\|^2 \\
&\quad - \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\|z_n - v_n\|^2\Big] - \beta_n(1-\beta_n)\|f(u_n) - z_n\|^2 \\
&\le \beta_n\|f(u_n) - u^*\|^2 + \|u_n - u^*\|^2 - (1-\beta_n)\Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\big[\|z_n - v_n\|^2 + \|u_n - v_n\|^2\big].
\end{aligned} \tag{25}$$

The remainder of the proof can be divided into two cases:

Case 1: Assume that there is a fixed number $N_2 \in \mathbb{N}$ $(N_2 \ge N_1)$ such that

$$\|u_{n+1} - u^*\| \le \|u_n - u^*\|, \ \forall\, n \ge N_2. \tag{26}$$

Thus, $\lim_{n\to\infty}\|u_n - u^*\|$ exists and let $\lim_{n\to\infty}\|u_n - u^*\| = l$. By using expression (25), we have

$$\begin{aligned}
&(1-\beta_n)\Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\big[\|z_n - v_n\|^2 + \|u_n - v_n\|^2\big] \\
&\le \beta_n\|f(u_n) - u^*\|^2 + \|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2.
\end{aligned} \tag{27}$$

Due to the existence of $\lim_{n\to\infty}\|u_n - u^*\| = l$, and $\beta_n \to 0$, we obtain

$$\lim_{n\to\infty}\|u_n - v_n\| = \lim_{n\to\infty}\|z_n - v_n\| = 0. \tag{28}$$

It follows that

$$\lim_{n\to\infty}\|u_n - z_n\| \le \lim_{n\to\infty}\|u_n - v_n\| + \lim_{n\to\infty}\|v_n - z_n\| = 0. \tag{29}$$

Hence, we obtain

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|\beta_n f(u_n) + (1-\beta_n)z_n - u_n\| \\
&= \|\beta_n[f(u_n) - u_n] + (1-\beta_n)[z_n - u_n]\| \\
&\le \beta_n\|f(u_n) - u_n\| + (1-\beta_n)\|z_n - u_n\| \to 0.
\end{aligned} \tag{30}$$

The sequence $\{u_n\}$ is bounded and implies that the sequences $\{v_n\}$ and $\{z_n\}$ are also bounded. Thus, we can take a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to some $\hat{u} \in \mathcal{C}$ and

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(u^*) - u^*, u_n - u^*\rangle \\
&= \limsup_{k\to\infty}\langle f(u^*) - u^*, u_{n_k} - u^*\rangle = \langle f(u^*) - u^*, \hat{u} - u^*\rangle \le 0.
\end{aligned} \tag{31}$$

We have $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$. It means that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(u^*) - u^*, u_{n+1} - u^*\rangle \\
&\le \limsup_{k\to\infty}\langle f(u^*) - u^*, u_{n+1} - u_n\rangle + \limsup_{k\to\infty}\langle f(u^*) - u^*, u_n - u^*\rangle \le 0.
\end{aligned} \tag{32}$$

From Lemma 7 and Lemma 1 (ii) ($\forall\, n \geq N_2$), we obtain

$$\begin{aligned}
&\|u_{n+1} - u^*\|^2 \\
&= \|\beta_n f(u_n) + (1-\beta_n) z_n - u^*\|^2 \\
&= \|\beta_n [f(u_n) - u^*] + (1-\beta_n)[z_n - u^*]\|^2 \\
&\leq (1-\beta_n)^2 \|z_n - u^*\|^2 + 2\beta_n \langle f(u_n) - u^*, (1-\beta_n)[z_n - u^*] + \beta_n [f(u_n) - u^*] \rangle \\
&= (1-\beta_n)^2 \|z_n - u^*\|^2 + 2\beta_n \langle f(u_n) - f(u^*) + f(u^*) - u^*, u_{n+1} - u^* \rangle \\
&= (1-\beta_n)^2 \|z_n - u^*\|^2 + 2\beta_n \langle f(u_n) - f(u^*), u_{n+1} - u^* \rangle + 2\beta_n \langle f(u^*) - u^*, u_{n+1} - u^* \rangle \\
&\leq (1-\beta_n)^2 \|z_n - u^*\|^2 + 2\beta_n \rho \|u_n - u^*\| \|u_{n+1} - u^*\| + 2\beta_n \langle f(u^*) - u^*, u_{n+1} - u^* \rangle \\
&\leq (1 + \beta_n^2 - 2\beta_n) \|u_n - u^*\|^2 + 2\beta_n \rho \|u_n - u^*\|^2 + 2\beta_n \langle f(u^*) - u^*, u_{n+1} - u^* \rangle \\
&= (1 - 2\beta_n) \|u_n - u^*\|^2 + \beta_n^2 \|u_n - u^*\|^2 + 2\beta_n \rho \|u_n - u^*\|^2 + 2\beta_n \langle f(u^*) - u^*, u_{n+1} - u^* \rangle \\
&= [1 - 2\beta_n (1-\rho)] \|u_n - u^*\|^2 + 2\beta_n (1-\rho) \left[\frac{\beta_n \|u_n - u^*\|^2}{2(1-\rho)} + \frac{\langle f(u^*) - u^*, u_{n+1} - u^* \rangle}{1-\rho} \right].
\end{aligned} \tag{33}$$

It follows (32) that

$$\limsup_{n \to \infty} \left[\frac{\beta_n \|u_n - u^*\|^2}{2(1-\rho)} + \frac{\langle f(u^*) - u^*, u_{n+1} - u^* \rangle}{1-\rho} \right] \leq 0. \tag{34}$$

Choose $n \geq N_3 \in \mathbb{N}$ ($N_3 \geq N_2$) large enough such that $2\beta_n(1-\rho) < 1$. Now, by using expressions (33) and (34) and applying Lemma 3, conclude that $\|u_n - u^*\| \to 0$, as $n \to \infty$.

Case 2: Assume that there is a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i} - u^*\| \leq \|u_{n_{i+1}} - u^*\|, \ \forall\, i \in \mathbb{N}.$$

Thus, by Lemma 4 there is a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \to \infty$, such that

$$\|u_{m_k} - u^*\| \leq \|u_{m_{k+1}} - u^*\| \ \text{ and } \ \|u_k - u^*\| \leq \|u_{m_{k+1}} - u^*\|, \ \forall\, k \in \mathbb{N}. \tag{35}$$

Similar to case 1 and from (25), we obtain

$$\begin{aligned}
&(1 - \beta_{m_k}) \left(1 - \frac{\mu \zeta_{m_k}}{\zeta_{m_k+1}}\right) [\|z_{m_k} - v_{m_k}\|^2 + \|u_{m_k} - v_{m_k}\|^2] \\
&\leq \beta_{m_k} \|f(u_{m_k}) - u^*\|^2 + \|u_{m_k} - u^*\|^2 - \|u_{m_k+1} - u^*\|^2.
\end{aligned} \tag{36}$$

Due to $\beta_{m_k} \to 0$, and $\left(1 - \frac{\mu \zeta_{m_k}}{\zeta_{m_k+1}}\right) \to 1 - \mu$, we deduce the following:

$$\lim_{n \to \infty} \|u_{m_k} - v_{m_k}\| = \lim_{k \to \infty} \|z_{m_k} - v_{m_k}\| = 0. \tag{37}$$

It follows that

$$\lim_{k \to \infty} \|u_{m_k} - z_{m_k}\| \leq \lim_{k \to \infty} \|u_{m_k} - v_{m_k}\| + \lim_{k \to \infty} \|v_{m_k} - z_{m_k}\| = 0. \tag{38}$$

Similar to case 1, we can easily obtain that

$$\lim_{k \to \infty} \|u_{m_{k+1}} - u_{m_k}\| = 0, \quad \text{and} \quad \limsup_{k \to \infty} \langle f(u^*) - u^*, u_{m_k+1} - u^* \rangle \leq 0. \tag{39}$$

By using (35) and the same argument as in (33), we have

$$
\begin{aligned}
&\|u_{m_k+1}-u^*\|^2 \\
&= [1-2\beta_{m_k}(1-\rho)]\|u_{m_k}-u^*\|^2 + 2\beta_{m_k}(1-\rho)\left[\frac{\beta_{m_k}\|u_{m_k}-u^*\|^2}{2(1-\rho)} + \frac{\langle f(u^*)-u^*, u_{m_k+1}-u^*\rangle}{1-\rho}\right] \\
&\leq [1-2\beta_{m_k}(1-\rho)]\|u_{m_k+1}-u^*\|^2 + 2\beta_{m_k}(1-\rho)\left[\frac{\beta_{m_k}\|u_{m_k}-u^*\|^2}{2(1-\rho)} + \frac{\langle f(u^*)-u^*, u_{m_k+1}-u^*\rangle}{1-\rho}\right].
\end{aligned} \tag{40}
$$

It follows that

$$
\|u_{m_k+1}-u^*\|^2 \leq \frac{\beta_{m_k}\|u_{m_k}-u^*\|^2}{2(1-\rho)} + \frac{\langle f(u^*)-u^*, u_{m_k+1}-u^*\rangle}{1-\rho}. \tag{41}
$$

Due to $\beta_{m_k} \to 0$ as $k \to \infty$, and $\limsup_{k\to\infty}\langle f(u^*)-u^*, u_{m_k+1}-u^*\rangle \leq 0$, we obtain

$$
\|u_{m_k+1}-u^*\|^2 \to 0, \text{ as } k \to \infty. \tag{42}
$$

Finally, the inequality

$$
\lim_{n\to\infty}\|u_k-u^*\|^2 \leq \lim_{n\to\infty}\|u_{m_k+1}-u^*\|^2 \leq 0. \tag{43}
$$

Consequently, $u_n \to u^*$. This completes the proof of the theorem. □

4. Numerical Illustrations

The experimental results are discussed in this section to illustrate the efficacy of our proposed Algorithm 1 (m-EgA3) compared to Algorithm 1 (m-EgA1) in [30] and Algorithm 2 (m-EgA2) in [30].

Example 1. *Consider the HpHard problem which is taken from [39] and considered by many authors for numerical tests (see [40–42]), where $F : \mathbb{R}^m \to \mathbb{R}^m$ is an operator defined by $F(u) = Mu + q$ with $q \in \mathbb{R}^m$ and*

$$
M = NN^T + B + D,
$$

where N is an $m \times m$ matrix, B is an $m \times m$ skew–symmetric matrix and D is an $m \times m$ positive definite diagonal matrix. The feasible set is defined by

$$
\mathcal{C} = \{u \in \mathbb{R}^m : Qu \leq b\},
$$

where Q is an $100 \times m$ matrix and b is a nonnegative vector in $\mathbb{R}^m$. It is clear that F is monotone and Lipschitz continuous with $L = \|M\|$. For $q = 0$, the solution set of the corresponding variational inequality is $VI(\mathcal{C}, F) = \{0\}$. In this experiment, we take the initial point $u_0 = (1, 1, \cdots, 1)$ and $D_n = \|u_n - v_n\| \leq TOL = 10^{-3}$. Moreover, the control parameters $\zeta_0 = \frac{0.7}{L}$ and $\mu = 0.9$ for Algorithm 1 (m-EgA1) in [30]; $\zeta_0 = \frac{0.7}{L}$, $\mu = 0.9$ and $\beta_n = \frac{1}{30(k+2)}$ for Algorithm 2 (m-EgA2) in [30]; $\zeta_0 = \frac{0.7}{L}$, $\mu = 0.9$, $\beta_n = \frac{1}{n+4}$ and $f(u) = \frac{u}{2}$ for Algorithm 1 (m-EgA3). The numerical results of all methods have been reported in Figures 1–8 and Table 1.

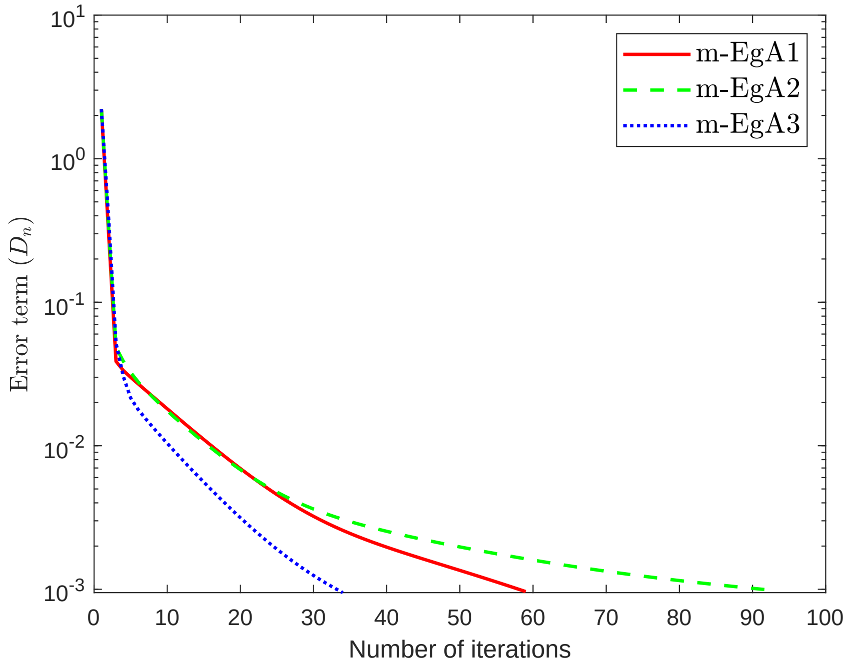

Figure 1. Numerical behaviour of Algorithm 1 compared to Algorithm 1 in [30] and Algorithm 2 in [30] for Example 1, when $m = 5$.

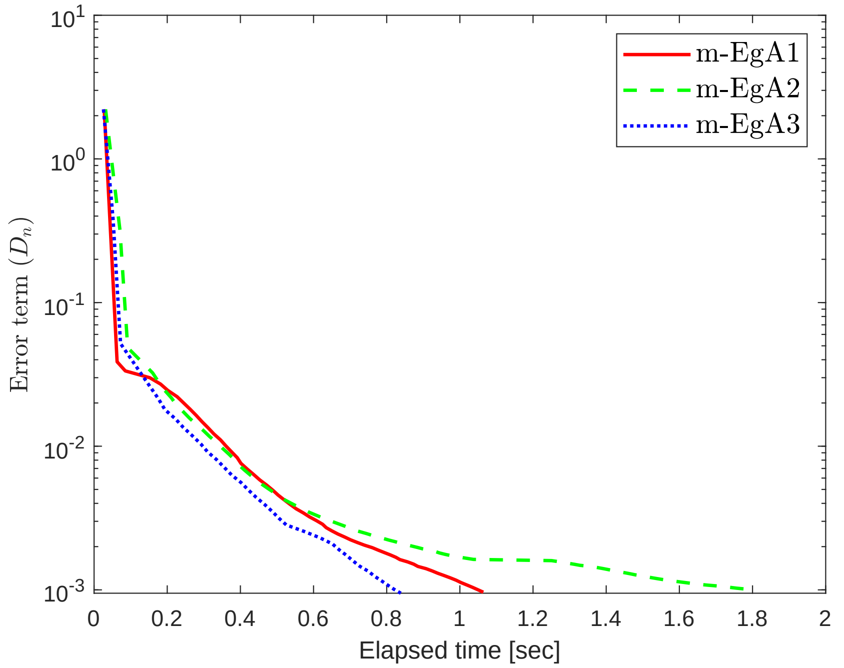

Figure 2. Numerical behaviour of Algorithm 1 compared to Algorithm 1 in [30] and Algorithm 2 in [30] for Example 1, when $m = 5$.

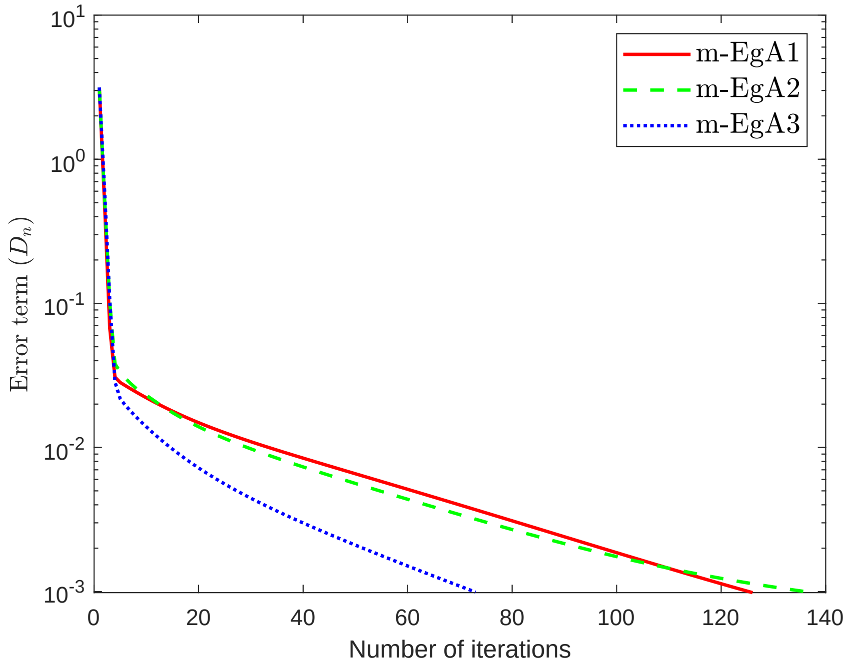

Figure 3. Numerical behaviour of Algorithm 1 compared to Algorithm 1 in [30] and Algorithm 2 in [30] for Example 1, when $m = 10$.

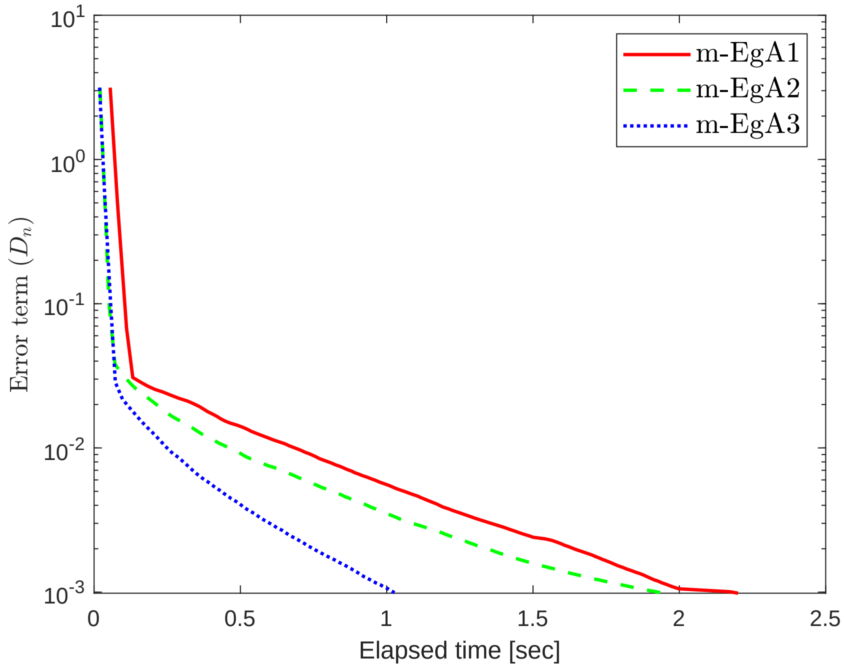

Figure 4. Numerical behaviour of Algorithm 1 compared to Algorithm 1 in [30] and Algorithm 2 in [30] for Example 1, when $m = 10$.

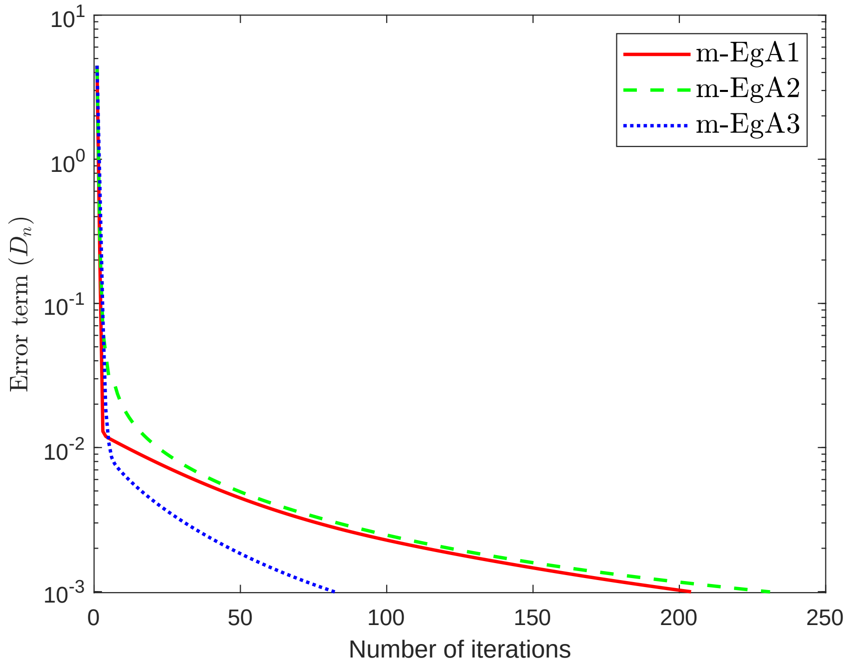

Figure 5. Numerical behaviour of Algorithm 1 compared to Algorithm 1 in [30] and Algorithm 2 in [30] for Example 1, when $m = 20$.

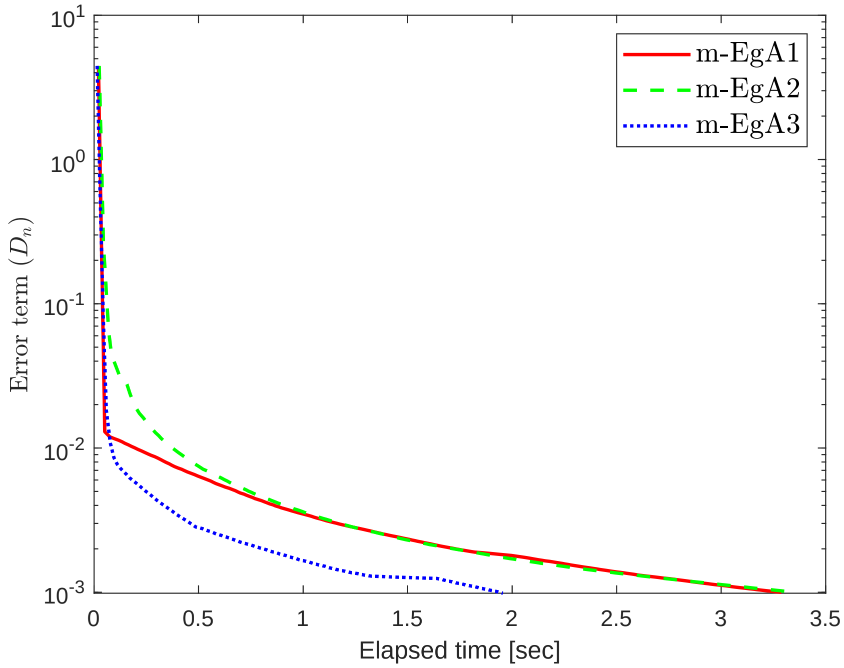

Figure 6. Numerical behaviour of Algorithm 1 compared to Algorithm 1 in [30] and Algorithm 2 in [30] for Example 1, when $m = 20$.

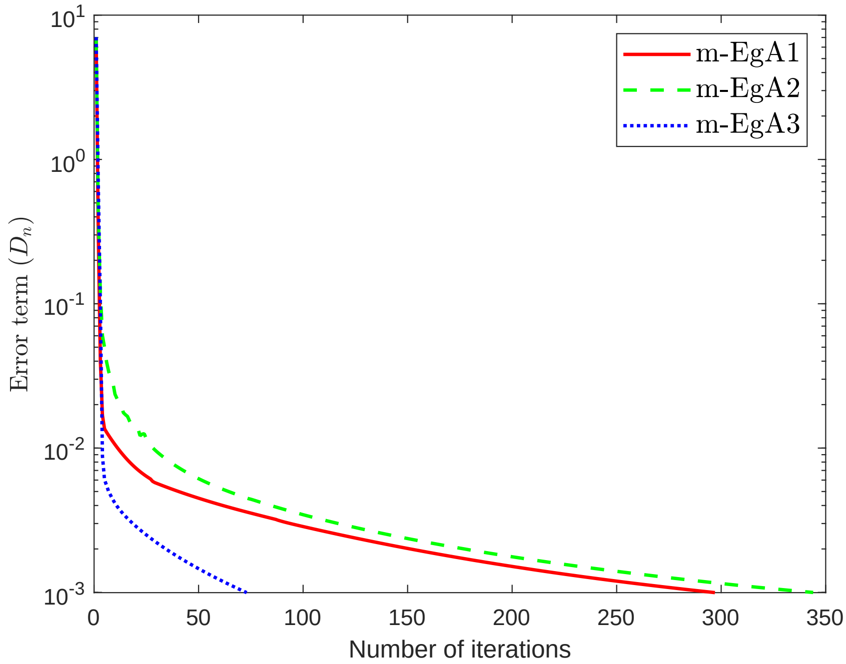

Figure 7. Numerical behaviour of Algorithm 1 compared to Algorithm 1 in [30] and Algorithm 2 in [30] for Example 1, when $m = 50$.

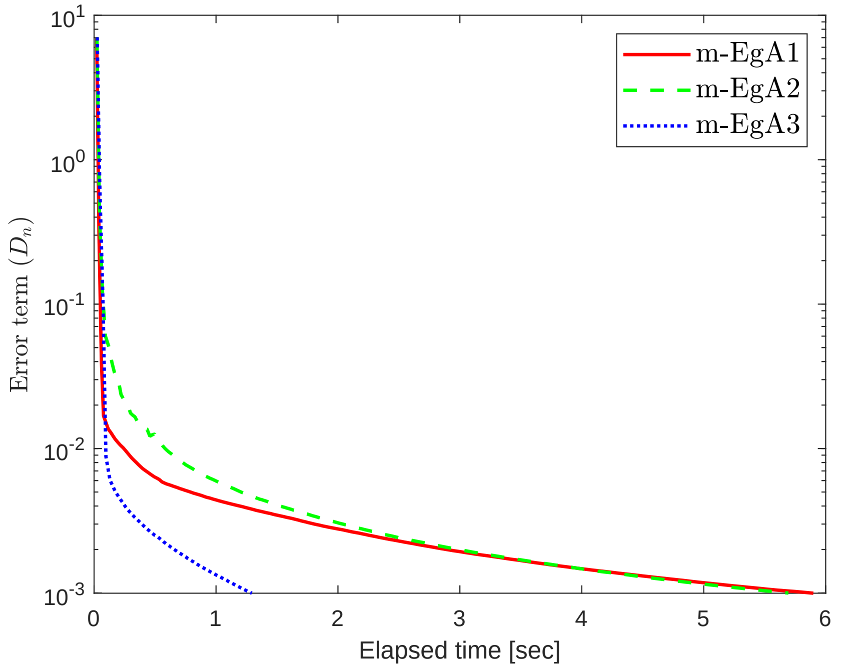

Figure 8. Numerical behaviour of Algorithm 1 compared to Algorithm 1 in [30] and Algorithm 2 in [30] for Example 1, when $m = 50$.

Table 1. Numerical results numeric values for Figures 1–8.

	m-EgA1 [30]		m-EgA2 [30]		m-EgA3	
m	**Iter.**	**Time**	**Iter.**	**Time**	**Iter.**	**Time**
5	59	1.0641	92	1.8107	34	0.8386
10	126	2.2007	137	1.9408	73	1.0267
20	204	3.2879	231	3.3654	83	11.9559
50	297	5.8990	344	5.6944	73	1.2942

Example 2. *Assume that* $\mathbb{H} = L^2([0,1])$ *is a Hilbert space with an inner product*

$$\langle u, v\rangle = \int_0^1 u(t)v(t)dt, \ \forall u, v \in \mathbb{H},$$

and the induced norm is

$$\|u\| = \sqrt{\int_0^1 |u(t)|^2 dt}.$$

Let $\mathcal{C} := \{u \in L^2([0,1]) : \|u\| \leq 1\}$ *be the unit ball and* $F : \mathcal{C} \to \mathbb{H}$ *is defined by*

$$F(u)(t) = \int_0^1 \big(u(t) - H(t,s)f(u(s))\big)ds + g(t),$$

where

$$H(t,s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}, \quad f(u) = \cos(u), \quad g(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

We can see in [41], that F is Lipschitz-continuous with Lipschitz constant $L = 2$ *and monotone. Figures 9–11 and Table 2 show the numerical results by taking different initial values* u_0 *and* $\epsilon = 10^{-3}$. *In this experiment, we take the different initial points* u_0 *and* $D_n = \|u_n - v_n\| \leq TOL = 10^{-3}$. *Moreover, the control parameters* $\zeta_0 = \frac{0.6}{L}$ *and* $\mu = 0.45$ *for Algorithm 1 (m-EgA1) in [30];* $\zeta_0 = \frac{0.6}{L}$, $\mu = 0.45$ *and* $\beta_n = \frac{1}{100(k+2)}$ *for Algorithm 2 (m-EgA2) in [30];* $\zeta_0 = \frac{0.6}{L}$, $\mu = 0.45$, $\beta_n = \frac{1}{n+2}$ *and* $f(u) = \frac{u}{3}$ *for Algorithm 1 (m-EgA3).*

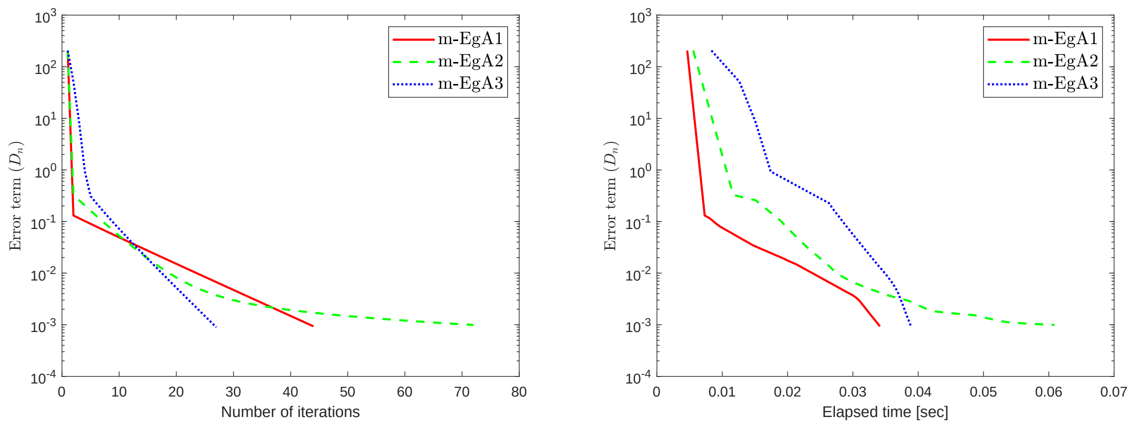

Figure 9. Numerical behaviour of Algorithm 1 compared to Algorithm 1 in [30] and Algorithm 2 in [30] for Example 1, when $u_0 = t$.

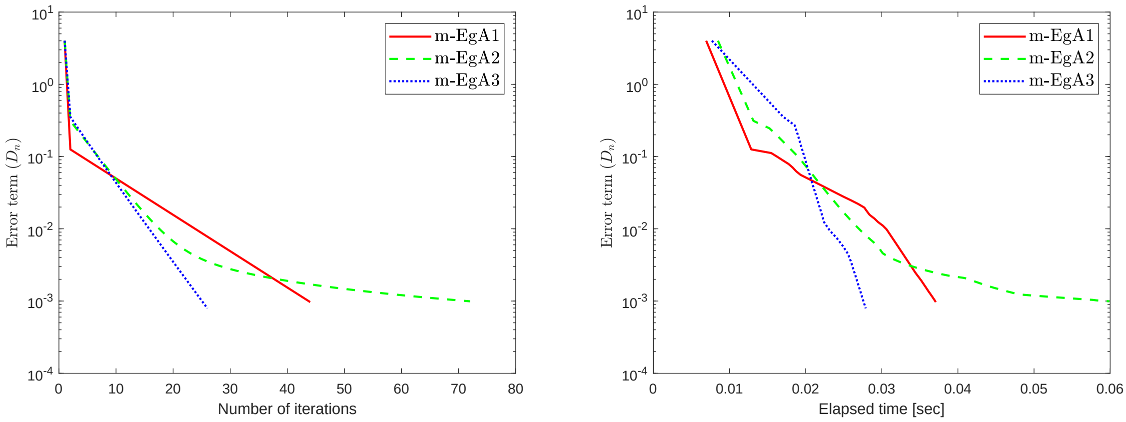

Figure 10. Numerical behaviour of Algorithm 1 compared to Algorithm 1 in [30] and Algorithm 2 in [30] for Example 1, when $u_0 = \sin(t)$.

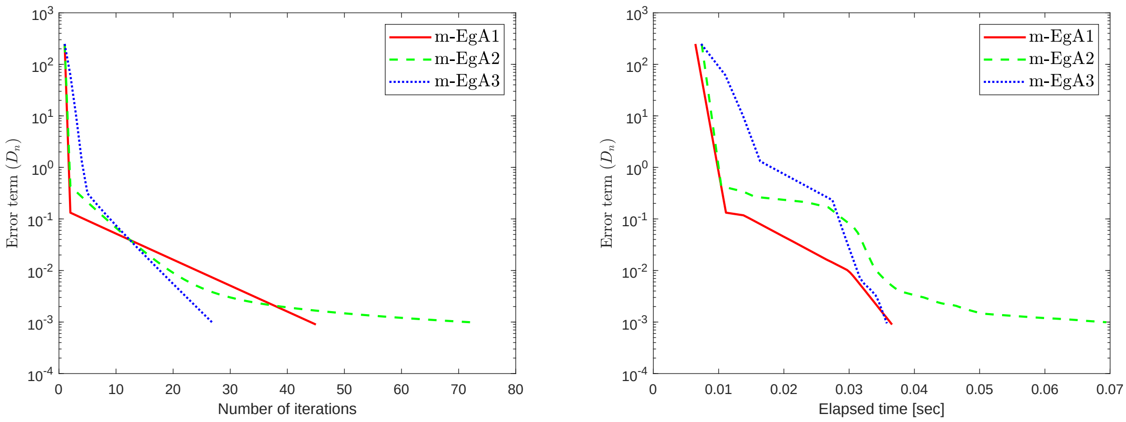

Figure 11. Numerical behaviour of Algorithm 1 compared to Algorithm 1 in [30] and Algorithm 2 in [30] for Example 1, when $u_0 = \cos(t)$.

Table 2. Numerical comparison values for Figures 1–8.

	m-EgA1 [30]		**m-EgA2 [30]**		**m-EgA3**	
u_0	**Iter.**	**Time**	**Iter.**	**Time**	**Iter.**	**Time**
t	44	0.0342	72	0.0609	27	0.0390
$\sin(t)$	44	0.0876	72	0.0569	40	0.0569
$\cos(t)$	45	0.0366	72	0.0358	27	0.0358

Example 3. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by*

$$F\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 + u_2 + \sin(u_1) \\ -u_1 + u_2 + \sin(u_2) \end{pmatrix}, \quad \forall \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2$$

and $\mathcal{C}$ is taken as

$$\mathcal{C} = \{u = (u_1, u_1)^T \in \mathbb{R}^2 : 0 \leq u_i \leq 10,\ i = 1, 2\}.$$

This problem was proposed in [43], where F is L-Lipschitz continuous with Lipschitz constant $L = \sqrt{10}$ and monotone. In this experiment, we take the different initial points u_0 and $D_n = \|u_n - v_n\| \leq TOL$. Moreover, the control parameters $\zeta_0 = \frac{0.7}{L}$ and $\mu = 0.50$ for Algorithm 1 (m-EgA1) in [30]; $\zeta_0 = \frac{0.7}{L}$, $\mu = 0.50$ and $\beta_n = \frac{1}{100(n+2)}$ for Algorithm 2 (m-EgA2) in [30]; $\zeta_0 = \frac{0.7}{L}$, $\mu = 0.50$, $\beta_n = \frac{1}{100(n+2)}$ and $f(u) = \frac{u}{4}$ for Algorithm 1 (m-EgA3). Table 3 reports the numerical results by using different tolerance and initial points.

Table 3. Numerical behaviour of Algorithm 1 compared to Algorithm 1 in [30] and Algorithm 2 in [30] for Example 3 by using different initial points u_0.

	TOL u_0	0.01 Iter.	0.001 Iter.	0.0001 Iter.	0.00001 Iter.	0.01 Time	0.001 Time	0.0001 Time	0.00001 Time
Algorithm 1 in [30]									
	$[10,20]^T$	29	41	83	277	0.4668	0.6234	1.5395	3.0415
	$[-10,-10]^T$	45	57	117	345	0.9234	1.1440	1.7387	3.4382
	$[10,20]^T$	59	71	143	389	1.0806	1.4264	1.8271	3.9269
Algorithm 2 in [30]									
	$[10,20]^T$	31	42	87	290	0.4743	0.5981	1.4921	3.2051
	$[-10,-10]^T$	45	61	115	360	0.8976	1.2081	1.5891	3.7891
	$[10,20]^T$	69	73	151	407	1.2711	1.3910	2.0810	4.1981
Algorithm 1									
	$[10,20]^T$	19	26	49	119	0.2391	0.3871	0.7716	1.6781
	$[-10,-10]^T$	25	39	64	123	0.2991	0.5192	0.9981	1.7021
	$[10,20]^T$	31	45	73	189	0.3018	0.7610	1.1012	2.4071

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