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Inverse Problem for a Mixed Type Integro-Differential Equation with Fractional Order Caputo Operators and Spectral Parameters

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Abstract: The questions of the one-value solvability of an inverse boundary value problem for a mixed type integro-differential equation with Caputo operators of different fractional orders and spectral parameters are considered. The mixed type integro-differential equation with respect to the main unknown function is an inhomogeneous partial integro-differential equation of fractional order in both positive and negative parts of the multidimensional rectangular domain under consideration. This mixed type of equation, with respect to redefinition functions, is a nonlinear Fredholm type integral equation. The fractional Caputo operators' orders are smaller in the positive part of the domain than the orders of Caputo operators in the negative part of the domain under consideration. Using the method of Fourier series, two systems of countable systems of ordinary fractional integro-differential equations with degenerate kernels and different orders of integro-differentation are obtained. Furthermore, a method of degenerate kernels is used. In order to determine arbitrary integration constants, a linear system of functional algebraic equations is obtained. From the solvability condition of this system are calculated the regular and irregular values of the spectral parameters. The solution of the inverse problem under consideration is obtained in the form of Fourier series. The unique solvability of the problem for regular values of spectral parameters is proved. During the proof of the convergence of the Fourier series, certain properties of the Mittag–Leffler function of two variables, the Cauchy–Schwarz inequality and Bessel inequality, are used. We also studied the continuous dependence of the solution of the problem on small parameters for regular values of spectral parameters. The existence and uniqueness of redefined functions have been justified by solving the systems of two countable systems of nonlinear integral equations. The results are formulated as a theorem.

Keywords: integro-differential equation; mixed type equation; small parameter; spectral parameters; Caputo operators of different fractional orders; inverse problem; one value solvability

1. Introduction

Fractional calculus plays an important role in the mathematical modeling of many natural and engineering processes (see [1]). We can gladly refer to many examples of applied research works, where fractional integro-differential operators are successfully and widely used. For example, in [2] some applications of basic problems in continuum and statistical mechanics are considered. In [3], the mathematical problems of an Ebola epidemic model by fractional order equations are studied. In [4,5], fractional models of the dynamics of tuberculosis infection and novel coronavirus (nCoV-2019) are studied, respectively. The construction of various models for studying problems of theoretical physics by the aid of fractional calculus is described in [6] (vol. 4, 5), [7,8]. A specific physical interpretation of the fractional derivatives, describing the random motion of a particle moving on the real line at Poisson-paced times with finite velocity, is given in [9]. A detailed review of the applications of fractional calculus in solving practical problems is given in [6] (vol. 6–8), [10]. More detailed information, as well as a bibliography related to the theory of fractional integro-differentiation and fractional derivatives, can also be found in [11–18].

We also note the special role of generalized special functions, such as polynomials, in solving fractional differential equations. In [19], using Hermite polynomials of higher and fractional order, some operational techniques to find general solutions of extended forms to d'Alembert and Fourier equations. In [20], the solutions of various generalized forms of the Heat Equation, by means of different tools ranging from the use of Hermite–Kampé de Fériet polynomials of higher and fractional order to operational techniques, are discussed. In [21], the combined use of integral transforms and special polynomials provides a powerful tool to deal with fractional derivatives and integrals. The real need to know the properties of such special functions in solving direct and inverse problems for fractional partial differential equations has been shown in [22].

Applications for equations of mixed type are studied in the works of many researchers. For example, in [23], an example of gas motion in a channel surrounded by a porous medium was studied, with the gas motion in a channel being described by a wave equation, while—outside the channel—a diffusion equation was posed. In [24], a problem related to the propagation of electric oscillations in compound lines, when the losses on a semi-infinite line were neglected and the rest of the line was treated as a cable with no leaks, was investigated. This reduced the problem under consideration to a mixed parabolic–hyperbolic type equation. In [25], a hyperbolic–parabolic system, in relation to pulse combustion, is investigated. Mixed type fractional differential equations are studied in many works by scientists—particularly in [26–35].

The theories of integral and integro-differential equations are important in studying the large directions of the general theory of equations of mathematical physics. The presence of an integral term in differential equations of the first and second order has an important role in the theory of dynamical systems of automatic control [36,37]. Boundary value problems for integro-differential equations with spectral parameters have singularities in studying the questions of one-value solvability [38,39]. Mixed type integer order integro-differential equations with degenerate kernels and spectral parameters are studied in [40,41].

To find the solutions of direct mixed and boundary value problems of mathematical physics, it is required to set the coefficients of the equation, the boundary of the domain under consideration, and the initial and boundary data. It usually happens that, in solving the applied problems experimentally, the quantitative characteristics of the object under study are not available for direct observation, or it is impossible to carry out the experiment itself for one reason or another. Then, in practice, the researchers can obtain some indirect information and draw a conclusion about the properties of the studied object. This information is determined by the nature of the object under study and here requires mathematical processing and the interpretation of research results. Nonlocal integral conditions often arise when the experiment gives averaged information about this object. When the structure of the mathematical model of the studying process is known, the problem of redefining the mathematical model is posed. Such problems belong to the class of inverse problems. By inverse problems we mean problems whose solution consists of determining the parameters of a model based on the available observation results and other experimental information. Inverse problems for equations of mixed type are studied relatively rarely due to the complexity of the studying process.

In the present paper, we study the questions of the one-value solvability of an inverse boundary value problem for a mixed type integro-differential equation with Caputo operators of different fractional orders and spectral parameters in a multidimensional rectangular domain.

The rest of this paper is organized as follows. In Section 2, we state the problem, which we will investigate in this work. Section 3 is devoted to formally expanding the solution of the direct problem into Fourier series. In Section 4, we formally determine the redefinition functions. Section 5 contains the proof of existence and uniqueness of Fourier coefficients of redefinition functions from a countable system of nonlinear integral equations. Section 6 is devoted to the justification of convergence and the possibility of the term by term differentiation of the obtained Fourier series. Section 7 contains the proof of the continuous dependence of the solution on the small parameter. In the last Section 8, as a conclusion, we formulate the theorem, which we have proved in this paper.

2. Statement of the Problem

We recall that the Caputo differential operator of fractional order $m-1<\alpha<m$ has the form

$$_{C}D_{a\,t}^{\alpha}f\,(t)=\frac{1}{\Gamma\,(m-\alpha)}\int\limits_{a}^{t}(t-s)^{m-\alpha-1}f^{\,(m)}\,(s)\,d\,s,$$

where $\Gamma\,(z)$ is Euler gamma function.

In the multidimensional domain $\Omega=\{-T<t<T,\ 0<x_1,\ldots,x_m<l\}$, a mixed type integro-differential equation of the following form is considered:

$$A_{\varepsilon}(U)-B_{\omega}(U)=\begin{cases}\nu\int\limits_{0}^{T}K_1(t,\,s)\,U(s,\,x)\,d\,s+F_1\,(t,\,x), & t>0,\\ \nu\int\limits_{-T}^{0}K_2(t,\,s)\,U(s,\,x)\,d\,s+F_2\,(t,\,x), & t<0,\end{cases}\tag{1}$$

where

$$F_i\,(t,\,x)=k_i\,(t)\left[g_i\,(x)+f_i\left(x,\,\int\limits_{\Omega_l^m}\Theta_i(y)\,g_i\,(y)\,d\,y\right)\right],\ i=1,\,2,$$

$$A_{\varepsilon}(U)=\frac{1+\operatorname{sgn}\,(t)}{2}\left[{}_{C}D_{0\,t}^{\alpha_1}-\varepsilon\sum_{i=1}^{m}\frac{\partial^2}{\partial\,x_i\,\partial\,x_i}{}_{C}D_{0\,t}^{\beta_1}\right]U\,(t,\,x)$$

$$+\frac{1-\operatorname{sgn}\,(t)}{2}\left[{}_{C}D_{0\,t}^{\alpha_2}-\varepsilon\sum_{i=1}^{m}\frac{\partial^2}{\partial\,x_i\,\partial\,x_i}{}_{C}D_{0\,t}^{\beta_2}\right]U\,(t,\,x),$$

$$B_{\omega}(U)=\begin{cases}\sum\limits_{i=1}^{m}U_{x_i\,x_i}, & t>0,\\ \omega^2\sum\limits_{i=1}^{m}U_{x_i\,x_i}, & t<0,\end{cases}$$

T and l are given as positive real numbers, ω is a positive spectral parameter, ε is a positive small parameter, ν is a real non-zero spectral parameter, $0\neq K_j(t,\,s)=a_j(t)\,b_j(s)$, $a_j(t)\in C^2[-T;\,T]$, $b_j(s)\in C\,[-T;\,T]$, $f_i\in C_x^2\left(\Omega_l^m\times\mathbb{R}\right)$, $\int\limits_{\Omega_l^m}|\Theta_i(y)|\,d\,y<\infty$, $\int\limits_{\Omega_l^m}|\Theta_i(y)|\,d\,y=\int\limits_{0}^{l}\ldots\int\limits_{0}^{l}|\Theta_i(y)|\,d\,y_1\cdot\ldots\cdot d\,y_m$, $i,j=1,\,2$, $k_1(t)\in C^2[0;\,T]$, $k_2(t)\in C^2[-T;\,0]$, while $g_1(x)$ and $g_2(x)$ are redefinition functions, $\mathbb{R}\equiv(-\infty;\,\infty)$, $x\in\Omega_l^m\equiv[0;\,l]^m$, $0<\beta_1<\alpha_1\leq 1$, $1<\beta_2<\alpha_2\leq 2$.

Problem. Find in the domain Ω a triple of unknown function

$$\begin{aligned}U\,(t,\,x)\in C\,(\overline{\Omega})\cap C^{0,1}\,(\Omega')\cap C^{\alpha_1,2}(\Omega_+)\cap C^{\alpha_2,2}(\Omega_-)\cap C_{t,x}^{\alpha_1+2}(\Omega_+)\cap C_{t,x}^{\alpha_2+2}(\Omega_-)\\ \cap C_{t,x_1,x_2,\ldots,x_m}^{\alpha_1+2+0+\ldots+0}(\Omega_+)\cap C_{t,x_1,x_2,\ldots,x_m}^{\alpha_2+2+0+\ldots+0}(\Omega_-)\cap C_{t,x_1,x_2,x_3,\ldots,x_m}^{\alpha_1+0+2+0+\ldots+0}(\Omega_+)\\ \cap C_{t,x_1,x_2,x_3,\ldots,x_m}^{\alpha_2+0+2+0+\ldots+0}(\Omega_-)\cap\ldots\cap C_{t,x_1,\ldots,x_{m-1},x_m}^{\alpha_1+0+\ldots+0+2}(\Omega_+)\cap C_{t,x_1,\ldots,x_{m-1},x_m}^{\alpha_2+0+\ldots+0+2}(\Omega_-)\end{aligned}\tag{2}$$

and redefinition functions $g_i(x) \in C\left(\Omega_l^m\right)$, $i = 1, 2$, satisfying the mixed integro-differential Equation (1) and the following boundary conditions

$$U(-T, x) = \varphi_1(x), \ \ _C D_{0t}^{\theta} U(-T, x) = \varphi_2(x), \ \ x \in \Omega_l^m, \tag{3}$$

$$U(t, 0) = U(t, l) = 0, \quad -T < t < T \tag{4}$$

and additional conditions

$$\int_0^T \Phi_1(t)\, U(t, x) = \psi_1(x), \quad x \in \Omega_l^m, \tag{5}$$

$$\int_{-T}^0 \Phi_2(t)\, U(t, x) = \psi_2(x), \quad x \in \Omega_l^m, \tag{6}$$

where $0 < \theta < 1$, $\varphi_i(x)$, $\psi_i(x)$ are given smooth functions, $\varphi_i(0) = \varphi_i(l) = 0$, $\psi_i(0) = \psi_i(l) = 0$, $i = 1, 2$, $C^r(\Omega)$ is a class of functions $U(t, x_1, \ldots, x_m)$ with continuous derivatives $\frac{\partial^r U}{\partial t^r}, \frac{\partial^r U}{\partial x_1^r}, \ldots, \frac{\partial^r U}{\partial x_m^r}$ in Ω, $C_{t,x}^{r,s}(\Omega)$ is a class of functions $U(t, x_1, \ldots, x_m)$ with continuous derivatives $\frac{\partial^r U}{\partial t^r}, \frac{\partial^s U}{\partial x_1^s}, \ldots, \frac{\partial^s U}{\partial x_m^s}$ in Ω, $C_{t,x_1,x_2,\ldots,x_m}^{r+r+0+\ldots+0}(\Omega)$ is a class of functions $U(t, x_1, \ldots, x_m)$ with continuous derivative $\frac{\partial^{2r} U}{\partial t^r \partial x_1^r}$ in $\Omega, \ldots,$ $C_{t,x_1,\ldots,x_{m-1},x_m}^{r+0+\ldots+0+r}(\Omega)$ is a class of functions $U(t, x_1, \ldots, x_m)$ with continuous derivative $\frac{\partial^{2r} U}{\partial t^r \partial x_m^r}$ in Ω, r, s are positive real numbers, $\overline{\Omega} = \left\{-T \le t \le T, \ x \in \Omega_l^m\right\}$, $\Omega' = \Omega \cup \{x_1, \ldots, x_m = 0\} \cup \{x_1, \ldots, x_m = l\}$, $\Omega_- = \{-T < t < 0, \ 0 < x_1, \ldots, x_m < l\}$, $\Omega_+ = \{0 < t < T, \ 0 < x_1, \ldots, x_m < l\}$.

3. Expansion of the Solution of the Direct Problem (1)–(4) into Fourier Series

Our investigation is based on the application of sine Fourier series to the mixed type integro-differential Equation (1) of the complicated form. Hence, the solution of the mixed integro-differential Equation (1) in domain Ω is sought in the form of the following Fourier series

$$U(t, x) = \sum_{n_1, \ldots, n_m = 1}^{\infty} u_{n_1, \ldots, n_m}^{\pm}(t)\, \vartheta_{n_1, \ldots, n_m}(x), \tag{7}$$

where

$$u_{n_1, \ldots, n_m}^{\pm}(t) = \begin{cases} u_{n_1, \ldots, n_m}^{+}(t) = \int\limits_{\Omega_l^m} U(t, x)\, \vartheta_{n_1, \ldots, n_m}(x)\, dx, & t > 0, \\ u_{n_1, \ldots, n_m}^{-}(t) = \int\limits_{\Omega_l^m} U(t, x)\, \vartheta_{n_1, \ldots, n_m}(x)\, dx, & t < 0, \end{cases} \tag{8}$$

$$\int_{\Omega_l^m} U(t, x)\, \vartheta_{n_1, \ldots, n_m}(x)\, dx = \int_0^l \ldots \int_0^l U(t, x)\, \vartheta_{n_1, \ldots, n_m}(x)\, dx_1 \cdot \ldots \cdot dx_m,$$

$$\vartheta_{n_1, \ldots, n_m}(x) = \left(\sqrt{\frac{2}{l}}\right)^m \sin \frac{\pi n_1}{l} x_1 \cdot \ldots \cdot \sin \frac{\pi n_m}{l} x_m, \ \ n_1, \ldots, n_m = 1, 2, \ldots$$

In this order, we also suppose that the redefinition functions and nonlinear functions on the right-hand side of the integro-differential Equation (1) are representable as the following Fourier series

$$g_i(x) = \sum_{n_1, \ldots, n_m = 1}^{\infty} g_{i\, n_1, \ldots, n_m} \vartheta_{n_1, \ldots, n_m}(x), \ \ f_i(x, V_i) = \sum_{n_1, \ldots, n_m = 1}^{\infty} f_{i\, n_1, \ldots, n_m}(V_i)\, \vartheta_{n_1, \ldots, n_m}(x), \tag{9}$$

where

$$g_{i\,n_1,\ldots,n_m} = \int\limits_{\Omega_l^m} g_i(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx, \quad f_{i\,n_1,\ldots,n_m}(V_i) = \int\limits_{\Omega_l^m} f_i(y,V_i)\,\vartheta_{n_1,\ldots,n_m}(y)\,dy,$$

$$f_i(y,V_i) = f_i\left(y, \int\limits_{\Omega_l^m} \Theta_i(z)\,g_i(z)\,dz\right), \quad i=1,2.$$

Substituting series (7) and (9) into mixed Equation (1), we obtain two fractional countable systems of ordinary integro-differential equations

$$\begin{gathered} {}_C D_{0t}^{\alpha_1} u^+_{n_1,\ldots,n_m}(t) + \varepsilon\,\mu^2_{n_1,\ldots,n_m}\,{}_C D_{0t}^{\beta_1} u^+_{n_1,\ldots,n_m}(t) + \mu^2_{n_1,\ldots,n_m}\,u^+_{n_1,\ldots,n_m}(t) \\ = \nu \int\limits_0^T a_1(t)\,b_1(s)\,u^+_{n_1,\ldots,n_m}(s)\,ds + F_{1\,n_1,\ldots,n_m}(t), \quad t>0, \end{gathered} \tag{10}$$

$$\begin{gathered} {}_C D_{0t}^{\alpha_2} u^-_{n_1,\ldots,n_m}(t) + \varepsilon\,\mu^2_{n_1,\ldots,n_m}\,{}_C D_{0t}^{\beta_2} u^-_{n_1,\ldots,n_m}(t) + \mu^2_{n_1,\ldots,n_m}\omega^2 u^-_{n_1,\ldots,n_m}(t) \\ = \nu \int\limits_{-T}^0 a_2(t)\,b_2(s)\,u^-_{n_1,\ldots,n_m}(s)\,ds + F_{2\,n_1,\ldots,n_m}(t), \quad t<0, \end{gathered} \tag{11}$$

where $\mu_{n_1,\ldots,n_m} = \frac{\pi}{l}\sqrt{n_1^2 + \ldots + n_m^2}$,

$$F_{i\,n_1,\ldots,n_m}(t) = k_i(t)\left[g_{i\,n_1,\ldots,n_m} + f_{i\,n_1,\ldots,n_m}(V_i)\right], \quad i=1,2. \tag{12}$$

We use the method of degenerate kernels. In this order, by the aid of designations

$$\tau^+_{n_1,\ldots,n_m} = \int\limits_0^T b_1(s)\,u^+_{n_1,\ldots,n_m}(s)\,ds, \tag{13}$$

$$\tau^-_{n_1,\ldots,n_m} = \int\limits_{-T}^0 b_2(s)\,u^-_{n_1,\ldots,n_m}(s)\,ds \tag{14}$$

we present the countable systems of ordinary integro-differential Equations (10) and (11) as follows

$$\begin{gathered} {}_C D_{0t}^{\alpha_1} u^+_{n_1,\ldots,n_m}(t) + \varepsilon\,\mu^2_{n_1,\ldots,n_m}\,{}_C D_{0t}^{\beta_1} u^+_{n_1,\ldots,n_m}(t) + \mu^2_{n_1,\ldots,n_m}\,u^+_{n_1,\ldots,n_m}(t) \\ = \nu\,a_1(t)\,\tau^+_{n_1,\ldots,n_m} + F_{1\,n_1,\ldots,n_m}(t), \quad t>0, \end{gathered} \tag{15}$$

$$\begin{gathered} {}_C D_{0t}^{\alpha_2} u^-_{n_1,\ldots,n_m}(t) + \varepsilon\,\mu^2_{n_1,\ldots,n_m}\,{}_C D_{0t}^{\beta_2} u^-_{n_1,\ldots,n_m}(t) + \mu^2_{n_1,\ldots,n_m}\omega^2 u^-_{n_1,\ldots,n_m}(t) \\ = \nu\,a_2(t)\,\tau^-_{n_1,\ldots,n_m} + F_{2\,n_1,\ldots,n_m}(t), \quad t<0. \end{gathered} \tag{16}$$

The solutions of the countable systems of differential Equations (15) and (16), satisfying conditions

$$u^+_{n_1,\ldots,n_m}(0) = C^+_{1\,n_1,\ldots,n_m}, \quad u^-_{n_1,\ldots,n_m}(0) = C^-_{1\,n_1,\ldots,n_m}, \quad \frac{d}{dt}u^-_{n_1,\ldots,n_m}(0) = C^-_{2\,n_1,\ldots,n_m}$$

have the following form:

$$u^+_{n_1,\ldots,n_m}(t) = \nu\,\tau^+_{n_1,\ldots,n_m}\Psi_{11n_1,\ldots,n_m}(t,\varepsilon) + \Psi_{12n_1,\ldots,n_m}(t,\varepsilon) + C^+_{1n_1,\ldots,n_m}\Psi_{13n_1,\ldots,n_m}(t,\varepsilon), \; t>0, \tag{17}$$

$$\begin{aligned} u^{-}_{n_1,\ldots,n_m}(t) = \;& \nu\, \tau^{-}_{n_1,\ldots,n_m} \Psi_{21\, n_1,\ldots,n_m}(t,\varepsilon,\omega) + \Psi_{22\, n_1,\ldots,n_m}(t,\varepsilon,\omega) \\ & + C^{-}_{1\, n_1,\ldots,n_m} \Psi_{23\, n_1,\ldots,n_m}(t,\varepsilon,\omega) - C^{-}_{2\, n_1,\ldots,n_m} \Psi_{24\, n_1,\ldots,n_m}(t,\varepsilon,\omega), \;\; t<0, \end{aligned} \tag{18}$$

where $C^{+}_{1\, n_1,\ldots,n_m}$, $C^{-}_{i\, n_1,\ldots,n_m}$, $(i=1,2)$ are for unknown constants to be uniquely determined,

$$\Psi_{11\, n_1,\ldots,n_m}(t,\varepsilon) = \int_0^t a_1(t-s)\, s^{\alpha_1-1} E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1}\left(-\varepsilon\, \mu^2_{n_1,\ldots,n_m} s^{\alpha_1-\beta_1},\; -\mu^2_{n_1,\ldots,n_m} s^{\alpha_1}\right) d\,s,$$

$$\Psi_{12\, n_1,\ldots,n_m}(t,\varepsilon) = \int_0^t F_{1\, n_1,\ldots,n_m}(t-s)\, s^{\alpha_1-1} E_{(\alpha_1-\beta_1,\,\alpha_1),\,\alpha_1}\left(-\varepsilon\, \mu^2_{n_1,\ldots,n_m} s^{\alpha_1-\beta_1},\; -\mu^2_{n_1,\ldots,n_m} s^{\alpha_1}\right) d\,s,$$

$$\Psi_{13\, n_1,\ldots,n_m}(t,\varepsilon) = E_{(\alpha_1-\beta_1,\,\alpha_1),1}\left(-\varepsilon\, \mu^2_{n_1,\ldots,n_m} t^{\alpha_1-\beta_1},\; -\mu^2_{n_1,\ldots,n_m} t^{\alpha_1}\right),$$

$$\Psi_{21\, n_1,\ldots,n_m}(t,\varepsilon,\omega) = \int_t^0 a_2(s-t)\,(-s)^{\alpha_2-1} \Psi_{25\, n_1,\ldots,n_m}(t,\varepsilon,\omega) d\,s,$$

$$\Psi_{22\, n_1,\ldots,n_m}(t,\varepsilon,\omega) = \int_t^0 F_{2\, n_1,\ldots,n_m}(s-t)\,(-s)^{\alpha_2-1} \Psi_{25\, n_1,\ldots,n_m}(t,\varepsilon,\omega) d\,s,$$

$$\Psi_{23\, n_1,\ldots,n_m}(t,\varepsilon,\omega) = E_{(\alpha_2-\beta_2,\,\alpha_2),1}\left(-\varepsilon\, \mu^2_{n_1,\ldots,n_m} (-t)^{\alpha_2-\beta_2},\; -\mu^2_{n_1,\ldots,n_m} \omega^2 (-t)^{\alpha_2}\right),$$

$$\Psi_{24\, n_1,\ldots,n_m}(t,\varepsilon,\omega) = t\, E_{(\alpha_2-\beta_2,\,\alpha_2),2}\left(-\varepsilon\, \mu^2_{n_1,\ldots,n_m} (-t)^{\alpha_2-\beta_2},\; -\mu^2_{n_1,\ldots,n_m}\, \omega^2 (-t)^{\alpha_2}\right),$$

$$\Psi_{25\, n_1,\ldots,n_m}(t,\varepsilon,\omega) = E_{(\alpha_2-\beta_2,\alpha_2),\,\alpha_2}\left(-\varepsilon\, \mu^2_{n_1,\ldots,n_m} (-s)^{\alpha_2-\beta_2},\; -\mu^2_{n_1,\ldots,n_m}\, \omega^2 (-s)^{\alpha_2}\right),$$

The function $E_{(\alpha,\beta),\gamma}(z_1, z_2)$ is a Mittag–Leffler function of two variables:

$$E_{(\alpha,\beta),\gamma}(z_1,z_2) = \sum_{m_1,m_2=0}^{\infty} \frac{z_1^{m_1} z_2^{m_2}}{\Gamma\left(\gamma + \alpha\, m_1 + \beta\, m_2\right)},$$

where $z_i,\ \alpha,\ \beta,\ \gamma \in \mathbb{C}$, $\mathrm{Re}\,(\alpha) > 0$, $\mathrm{Re}\,(\beta) > 0$.

From the statement of the problem (properties in (2)), it follows that the continuous conjugation condition is fulfilled for the main unknown function: $U\,(0+0,\,x) = U\,(0-0,\,x)$. Therefore, by taking Formula (6) into account, we have the conditions for Fourier coefficients of the main unknown function

$$\begin{aligned} u^{+}_{n_1,\ldots,n_m}(0+0) &= \int_{\Omega^m_l} U\,(0+0,\,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,d\,x \\ &= \int_{\Omega^m_l} U\,(0-0,\,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,d\,x = u^{-}_{n_1,\ldots,n_m}(0-0). \end{aligned} \tag{19}$$

We put

$$\varphi_{i\, n_1,\ldots,n_m} = \int_{\Omega^m_l} \varphi_i(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,d\,x,\;\; i=1,2.$$

Then, taking (8) into account, from the conditions in (3), we obtain

$$u^{-}_{n_1,\ldots,n_m}(-T) = \int_{\Omega^m_l} U\,(-T,\,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,d\,x = \int_{\Omega^m_l} \varphi_1(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,d\,x = \varphi_{1\, n_1,\ldots,n_m}, \tag{20}$$

$$
\begin{aligned}
{}_C D^{\theta}_{0\,t} u^{-}_{n_1,\ldots,n_m}(-T) &= \int\limits_{\Omega^m_l} {}_C D^{\theta}_{0\,t} U(-T,x)\, \vartheta_{n_1,\ldots,n_m}(x)\, dx \\
&= \int\limits_{\Omega^m_l} \varphi_2(x)\, \vartheta_{n_1,\ldots,n_m}(x)\, dx = \varphi_{2\,n_1,\ldots,n_m}.
\end{aligned} \tag{21}
$$

By the aid of the continuous conjugation condition (19) from (17) and (18), we have the relation that $C^{+}_{1\,n_1,\ldots,n_m} = C^{-}_{1\,n_1,\ldots,n_m}$. To find the unknown coefficients of the integration $C^{-}_{1\,n_1,\ldots,n_m}$ and $C^{-}_{2\,n_1,\ldots,n_m}$ in (18), we use the conditions (20) and (21) and deduce the following system of linear algebraic equations:

$$
\begin{cases}
\nu\,\tau^{-}_{n_1,\ldots,n_m}\Psi_{21\,n_1,\ldots,n_m}(-T,\varepsilon,\omega) + \Psi_{22\,n_1,\ldots,n_m}(-T,\varepsilon,\omega) + \\
+C^{-}_{1\,n_1,\ldots,n_m}\Psi_{23\,n_1,\ldots,n_m}(-T,\varepsilon,\omega) - C^{-}_{2\,n_1,\ldots,n_m}\Psi_{24\,n_1,\ldots,n_m}(-T,\varepsilon,\omega) = \varphi_{1\,n_1,\ldots,n_m}, \\
\nu\,\tau^{-}_{n_1,\ldots,n_m} D^{\theta}_{0\,t}\Psi_{21\,n_1,\ldots,n_m}(-T,\varepsilon,\omega) + D^{\theta}_{0\,t}\Psi_{22\,n_1,\ldots,n_m}(-T,\varepsilon,\omega) + \\
+C^{-}_{1n_1,\ldots,n_m} D^{\theta}_{0t}\Psi_{23n_1,\ldots,n_m}(-T,\varepsilon,\omega) - C^{-}_{2n_1,\ldots,n_m} D^{\theta}_{0t}\Psi_{24n_1,\ldots,n_m}(-T,\varepsilon,\omega) = \varphi_{2n_1,\ldots,n_m},
\end{cases} \tag{22}
$$

where by $D^{\theta}_{0\,t}\Psi(-T)$ is denoted $D^{\theta}_{0\,t}\Psi(t)_{|\,t=-T}$. We assume that

$$
\begin{aligned}
\sigma_{n_1,\ldots,n_m}(\omega) = \;& \Psi_{24\,n_1,\ldots,n_m}(-T,\varepsilon,\omega)\cdot D^{\theta}_{0\,t}\Psi_{23\,n_1,\ldots,n_m}(-T,\varepsilon,\omega) \\
& -\Psi_{23\,n_1,\ldots,n_m}(-T,\varepsilon,\omega)\cdot D^{\theta}_{0\,t}\Psi_{24\,n_1,\ldots,n_m}(-T,\varepsilon,\omega) \neq 0.
\end{aligned} \tag{23}
$$

If the condition (23) is fulfilled, then the system (22) with respect to $C^{-}_{1\,n_1,\ldots,n_m}$ and $C^{-}_{2\,n_1,\ldots,n_m}$ is uniquely solvable. By solving this system (22), we arrive at the following presentations for these unknown coefficients

$$
C^{-}_{1\,n_1,\ldots,n_m} = \frac{1}{\sigma_{n_1,\ldots,n_m}(\omega)}
$$

$$
\times\Big[\varphi_{1\,n_1,\ldots,n_m} D^{\theta}_{0\,t}\Psi_{24\,n_1,\ldots,n_m}(-T,\varepsilon,\omega) + \varphi_{2\,n_1,\ldots,n_m}\Psi_{24\,n_1,\ldots,n_m}(-T,\varepsilon,\omega) - \nu\,\tau^{-}_{n_1,\ldots,n_m}
$$

$$
\times\Big(\Psi_{24n_1,\ldots,n_m}(-T,\varepsilon,\omega)\, D^{\theta}_{0t}\Psi_{21n_1,\ldots,n_m}(-T,\varepsilon,\omega) - \Psi_{21n_1,\ldots,n_m}(-T,\varepsilon,\omega)\, D^{\theta}_{0t}\Psi_{24n_1,\ldots,n_m}(-T,\varepsilon,\omega)\Big)
$$

$$
+\,\Psi_{22n_1,\ldots,n_m}(-T,\varepsilon,\omega)\, D^{\theta}_{0t}\Psi_{24n_1,\ldots,n_m}(-T,\varepsilon,\omega) - \Psi_{24n_1,\ldots,n_m}(-T,\varepsilon,\omega)\, D^{\theta}_{0t}\Psi_{22n_1,\ldots,n_m}(-T,\varepsilon,\omega)\Big],
$$

$$
C^{-}_{2\,n_1,\ldots,n_m} = \frac{1}{\sigma_{n_1,\ldots,n_m}(\omega)}
$$

$$
\times\Big[\varphi_{1n_1,\ldots,n_m} D^{\theta}_{0t}\Psi_{23n_1,\ldots,n_m}(-T,\varepsilon,\omega) + \varphi_{2n_1,\ldots,n_m}\Psi_{23n_1,\ldots,n_m}(-T,\varepsilon,\omega) - \nu\,\tau^{-}_{n_1,\ldots,n_m}
$$

$$
\times\Big(\Psi_{23n_1,\ldots,n_m}(-T,\varepsilon,\omega)\, D^{\theta}_{0t}\Psi_{21n_1,\ldots,n_m}(-T,\varepsilon,\omega) - \Psi_{21n_1,\ldots,n_m}(-T,\varepsilon,\omega)\, D^{\theta}_{0t}\Psi_{23n_1,\ldots,n_m}(-T,\varepsilon,\omega)\Big)
$$

$$
+\,\Psi_{22n_1,\ldots,n_m}(-T,\varepsilon,\omega)\, D^{\theta}_{0t}\Psi_{23n_1,\ldots,n_m}(-T,\varepsilon,\omega) - \Psi_{23n_1,\ldots,n_m}(-T,\varepsilon,\omega)\, D^{\theta}_{0t}\Psi_{22n_1,\ldots,n_m}(-T,\varepsilon,\omega)\Big].
$$

By substituting these results into (18) and taking into account $C^{+}_{1\,n_1,\ldots,n_m} = C^{-}_{1\,n_1,\ldots,n_m}$ in (17) and designation (12), we obtain the following representations for the Fourier coefficients of the main unknown functions in the positive and negative parts of the domain:

$$
\begin{aligned}
& u^{+}_{n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) = \left[\varphi_{1\,n_1,\ldots,n_m} + \varphi_{2\,n_1,\ldots,n_m}\right] N_{11\,n_1,\ldots,n_m}(t,\varepsilon,\omega) \\
& +\nu\,\tau^{+}_{n_1,\ldots,n_m} N_{12\,n_1,\ldots,n_m}(t,\varepsilon) - \nu\,\tau^{-}_{n_1,\ldots,n_m} N_{13\,n_1,\ldots,n_m}(t,\varepsilon,\omega) + \left[g_{1\,n_1,\ldots,n_m} + f_{1\,n_1,\ldots,n_m}(V_1)\right] \\
& \times N_{14\,n_1,\ldots,n_m}(t,\varepsilon) + \left[g_{2\,n_1,\ldots,n_m} + f_{2\,n_1,\ldots,n_m}(V_2)\right] N_{15\,n_1,\ldots,n_m}(t,\varepsilon,\omega), \;\; t > 0,
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
& u^{-}_{n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) = \varphi_{1\,n_1,\ldots,n_m} N_{21\,n_1,\ldots,n_m}(t,\varepsilon,\omega) + \varphi_{2\,n_1,\ldots,n_m} N_{22\,n_1,\ldots,n_m}(t,\varepsilon,\omega) \\
& +\nu\,\tau^{-}_{n_1,\ldots,n_m} N_{23\,n_1,\ldots,n_m}(t,\varepsilon,\omega) + \left[g_{2\,n_1,\ldots,n_m} + f_{2\,n_1,\ldots,n_m}(V_2)\right] N_{24\,n_1,\ldots,n_m}(t,\varepsilon,\omega), \;\; t < 0,
\end{aligned} \tag{25}
$$

where

$$N_{11\,n_1,\dots,n_m}(t,\varepsilon,\omega)=\frac{1}{\sigma_{n_1,\dots,n_m}(\omega)}\,\Psi_{13\,n_1,\dots,n_m}(t,\varepsilon)\,\Psi_{24\,n_1,\dots,n_m}(-T,\varepsilon,\omega),$$

$$N_{12\,n_1,\dots,n_m}(t,\varepsilon)=\Psi_{11\,n_1,\dots,n_m}(t,\varepsilon),$$

$$N_{13\,n_1,\dots,n_m}(t,\varepsilon,\omega)=\frac{1}{\sigma_{n_1,\dots,n_m}(\omega)}\Big[\Psi_{24\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\Psi_{21\,n_1,\dots,n_m}(-T,\varepsilon,\omega)$$
$$-\Psi_{21\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\Psi_{24\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\Big]\,\Psi_{13\,n_1,\dots,n_m}(t,\varepsilon),$$

$$N_{14\,n_1,\dots,n_m}(t,\varepsilon)=\overline{\Psi}_{12\,n_1,\dots,n_m}(t,\varepsilon),$$

$$N_{15\,n_1,\dots,n_m}(t,\varepsilon,\omega)=\frac{1}{\sigma_{n_1,\dots,n_m}(\omega)}\Big[\overline{\Psi}_{22\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\Psi_{24\,n_1,\dots,n_m}(-T,\varepsilon,\omega)$$
$$-\Psi_{24\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\overline{\Psi}_{22\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\Big]\,\Psi_{13\,n_1,\dots,n_m}(t,\varepsilon),$$

$$N_{21\,n_1,\dots,n_m}(t,\varepsilon,\omega)=\frac{1}{\sigma_{n_1,\dots,n_m}(\omega)}\Big[\Psi_{23\,n_1,\dots,n_m}(t,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\Psi_{24\,n_1,\dots,n_m}(-T,\varepsilon,\omega)$$
$$-\Psi_{24\,n_1,\dots,n_m}(t,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\Psi_{23\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\Big],$$

$$N_{22\,n_1,\dots,n_m}(t,\varepsilon,\omega)=\frac{1}{\sigma_{n_1,\dots,n_m}(\omega)}\big[\Psi_{23\,n_1,\dots,n_m}(t,\varepsilon,\omega)\,\Psi_{24\,n_1,\dots,n_m}(-T,\varepsilon,\omega)$$
$$+\Psi_{24\,n_1,\dots,n_m}(t,\varepsilon,\omega)\,\Psi_{23\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\big],\ N_{23\,n_1,\dots,n_m}(t,\varepsilon,\omega)=\Psi_{21\,n_1,\dots,n_m}(t,\varepsilon,\omega)$$
$$-\frac{1}{\sigma_{n_1,\dots,n_m}(\omega)}\Big[\Psi_{24\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\Psi_{21\,n_1,\dots,n_m}(-T,\varepsilon,\omega)$$
$$-\Psi_{21\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\Psi_{24\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\Big]\,\Psi_{23\,n_1,\dots,n_m}(t,\varepsilon,\omega)$$
$$+\frac{1}{\sigma_{n_1,\dots,n_m}(\omega)}\Big[\Psi_{23\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\Psi_{21\,n_1,\dots,n_m}(-T,\varepsilon,\omega)$$
$$-\Psi_{21\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\Psi_{23\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\Big]\,\Psi_{24\,n_1,\dots,n_m}(t,\varepsilon,\omega),$$

$$N_{24\,n_1,\dots,n_m}(t,\varepsilon,\omega)=\overline{\Psi}_{22\,n_1,\dots,n_m}(t,\varepsilon,\omega)$$
$$+\frac{1}{\sigma_{n_1,\dots,n_m}(\omega)}\Big[\overline{\Psi}_{22\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\Psi_{24\,n_1,\dots,n_m}(-T,\varepsilon,\omega)$$
$$-\Psi_{24\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\overline{\Psi}_{22\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\Big]\,\Psi_{23\,n_1,\dots,n_m}(t,\varepsilon,\omega)$$
$$+\frac{1}{\sigma_{n_1,\dots,n_m}(\omega)}\Big[\overline{\Psi}_{22\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\Psi_{23\,n_1,\dots,n_m}(-T,\varepsilon,\omega)$$
$$-\Psi_{23\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\,D^{\theta}_{0\,t}\,\overline{\Psi}_{22\,n_1,\dots,n_m}(-T,\varepsilon,\omega)\Big]\,\Psi_{24\,n_1,\dots,n_m}(t,\varepsilon,\omega),$$

$$\overline{\Psi}_{12\,n_1,\dots,n_m}(t,\varepsilon)=\int\limits_0^t k_1(t-s)\,s^{\alpha_1-1}\,E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1}\left(-\varepsilon\,\mu^2_{n_1,\dots,n_m}s^{\alpha_1-\beta_1},\ -\mu^2_{n_1,\dots,n_m}s^{\alpha_1}\right)ds,$$

$$\overline{\Psi}_{22\,n_1,\dots,n_m}(t,\varepsilon,\omega)=\int\limits_t^0 k_2(s-t)\,(-s)^{\alpha_2-1}\,\Psi_{25\,n_1,\dots,n_m}(t,\varepsilon,\omega)\,ds.$$

According to the degenerate kernels method, we substitute these presentations, (24) and (25), into designations (13) and (14):

$$
\begin{aligned}
&\tau^{+}_{n_1,\ldots,n_m}\left[1-\nu\,\chi_{12\,n_1,\ldots,n_m}(\varepsilon,\,\omega)\right]+\nu\,\tau^{-}_{n_1,\ldots,n_m}\,\chi_{13\,n_1,\ldots,n_m}(\varepsilon,\,\omega)\\
&=\left[\varphi_{1\,n_1,\ldots,n_m}+\varphi_{2\,n_1,\ldots,n_m}\right]\chi_{11\,n_1,\ldots,n_m}(\varepsilon,\,\omega)+\left[g_{1\,n_1,\ldots,n_m}+f_{1\,n_1,\ldots,n_m}(V_1)\right]\chi_{14\,n_1,\ldots,n_m}(\varepsilon,\,\omega)\\
&+\left[g_{2\,n_1,\ldots,n_m}+f_{2\,n_1,\ldots,n_m}(V_2)\right]\chi_{15\,n_1,\ldots,n_m}(\varepsilon,\,\omega),
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&\tau^{-}_{n_1,\ldots,n_m}\left[1-\nu\,\chi_{23\,n_1,\ldots,n_m}(\varepsilon,\,\omega)\right]=\varphi_{1\,n_1,\ldots,n_m}\chi_{21\,n_1,\ldots,n_m}(\varepsilon,\,\omega)+\varphi_{2\,n_1,\ldots,n_m}\chi_{22\,n_1,\ldots,n_m}(\varepsilon,\,\omega)\\
&+\left[g_{2\,n_1,\ldots,n_m}+f_{2\,n_1,\ldots,n_m}(V_2)\right]\chi_{24\,n_1,\ldots,n_m}(\varepsilon,\,\omega),
\end{aligned}
\tag{27}
$$

where

$$\chi_{1i\,n_1,\ldots,n_m}(\varepsilon,\,\omega)=\int\limits_0^T b_1(s)\,N_{1i\,n_1,\ldots,n_m}(s,\,\varepsilon,\,\omega)\,ds,\;\; i=\overline{1,\,5},$$

$$\chi_{2i\,n_1,\ldots,n_m}(\varepsilon,\,\omega)=\int\limits_{-T}^0 b_2(s)\,N_{2i\,n_1,\ldots,n_m}(s,\,\varepsilon,\,\omega)\,ds,\;\; i=\overline{1,\,4}.$$

We solve the linear algebraic Equations (26) and (27) as a system of algebraic equations with respect to quantities $\tau^{+}_{n_1,\ldots,n_m}$ and $\tau^{-}_{n_1,\ldots,n_m}$. If the following conditions are fulfilled

$$\nu\,\chi_{12\,n_1,\ldots,n_m}(\varepsilon,\,\omega)\neq 1,\;\;\nu\,\chi_{23\,n_1,\ldots,n_m}(\varepsilon,\,\omega)\neq 1, \tag{28}$$

then, from (26) and (27), we derive

$$
\begin{aligned}
\tau^{+}_{n_1,\ldots,n_m}&=\varphi_{1\,n_1,\ldots,n_m}M_{11\,n_1,\ldots,n_m}(\varepsilon,\,\omega)+\varphi_{2\,n_1,\ldots,n_m}M_{12\,n_1,\ldots,n_m}(\varepsilon,\,\omega)\\
&+[g_{1n_1,\ldots,n_m}+f_{1n_1,\ldots,n_m}(V_1)]\,M_{13n_1,\ldots,n_m}(\varepsilon,\omega)+[g_{2n_1,\ldots,n_m}+f_{2n_1,\ldots,n_m}(V_2)]\,M_{14n_1,\ldots,n_m}(\varepsilon,\omega),
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
\tau^{-}_{n_1,\ldots,n_m}&=\varphi_{1\,n_1,\ldots,n_m}M_{21\,n_1,\ldots,n_m}(\varepsilon,\,\omega)+\varphi_{2\,n_1,\ldots,n_m}M_{22\,n_1,\ldots,n_m}(\varepsilon,\,\omega)\\
&+\left[g_{2\,n_1,\ldots,n_m}+f_{2\,n_1,\ldots,n_m}(V_2)\right]M_{23\,n_1,\ldots,n_m}(\varepsilon,\,\omega),
\end{aligned}
\tag{30}
$$

where

$$M_{1in_1,\ldots,n_m}(\varepsilon,\omega)=\frac{1}{1-\nu\,\chi_{12n_1,\ldots,n_m}(\varepsilon,\omega)}\left[\chi_{11n_1,\ldots,n_m}(\varepsilon,\omega)-\nu\,\frac{\chi_{13n_1,\ldots,n_m}(\varepsilon,\omega)\,\chi_{2in_1,\ldots,n_m}(\varepsilon,\omega)}{1-\nu\,\chi_{23n_1,\ldots,n_m}(\varepsilon,\omega)}\right],$$

$$i=1,\,2,\quad M_{13\,n_1,\ldots,n_m}(\varepsilon,\,\omega)=\frac{\chi_{14\,n_1,\ldots,n_m}(\varepsilon,\,\omega)}{1-\nu\,\chi_{12\,n_1,\ldots,n_m}(\varepsilon,\,\omega)},$$

$$M_{14n_1,\ldots,n_m}(\varepsilon,\omega)=\frac{1}{1-\nu\,\chi_{12n_1,\ldots,n_m}(\varepsilon,\omega)}\left[\chi_{15n_1,\ldots,n_m}(\varepsilon,\omega)-\nu\,\frac{\chi_{13n_1,\ldots,n_m}(\varepsilon,\omega)\,\chi_{24n_1,\ldots,n_m}(\varepsilon,\omega)}{1-\nu\,\chi_{23n_1,\ldots,n_m}(\varepsilon,\omega)}\right],$$

$$M_{2i\,n_1,\ldots,n_m}(\varepsilon,\,\omega)=\frac{\chi_{2i\,n_1,\ldots,n_m}(\varepsilon,\,\omega)}{1-\nu\,\chi_{23\,n_1,\ldots,n_m}(\varepsilon,\,\omega)},\;\; i=1,\,2,\;\; M_{23\,n_1,\ldots,n_m}(\varepsilon,\,\omega)=\frac{\chi_{24\,n_1,\ldots,n_m}(\varepsilon,\,\omega)}{1-\nu\,\chi_{23\,n_1,\ldots,n_m}(\varepsilon,\,\omega)}.$$

Substituting presentations (29) and (30) of $\tau^{\pm}_{n_1,\ldots,n_m}$ into (24) and (25), we derive

$$
\begin{aligned}
u^{+}_{n_1,\ldots,n_m}(t,\,\varepsilon,\,\omega,\,\nu)&=\varphi_{1\,n_1,\ldots,n_m}Q_{11\,n_1,\ldots,n_m}(t,\,\varepsilon,\,\omega,\,\nu)+\varphi_{2\,n_1,\ldots,n_m}Q_{12\,n_1,\ldots,n_m}(t,\,\varepsilon,\,\omega,\,\nu)\\
&+\left[g_{1\,n_1,\ldots,n_m}+f_{1\,n_1,\ldots,n_m}(V_1)\right]Q_{13\,n_1,\ldots,n_m}(t,\,\varepsilon,\,\omega,\,\nu)\\
&+\left[g_{2\,n_1,\ldots,n_m}+f_{2\,n_1,\ldots,n_m}(V_2)\right]Q_{14\,n_1,\ldots,n_m}(t,\,\varepsilon,\,\omega,\,\nu),\;\; t>0,
\end{aligned}
\tag{31}
$$

$$\begin{aligned} u^{-}_{n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) &= \varphi_{1\,n_1,\ldots,n_m} Q_{21\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) + \varphi_{2\,n_1,\ldots,n_m} Q_{22\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) \\ &+ \left[g_{2\,n_1,\ldots,n_m} + f_{2\,n_1,\ldots,n_m}(V_2)\right] Q_{23\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu), \;\; t<0, \end{aligned} \tag{32}$$

where

$$Q_{1i\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) = N_{11\,n_1,\ldots,n_m}(t,\varepsilon,\omega) + \nu\, N_{12\,n_1,\ldots,n_m}(t,\varepsilon,\omega)\, M_{1i\,n_1,\ldots,n_m}(\varepsilon,\omega)$$

$$-\nu\, N_{13\,n_1,\ldots,n_m}(t,\varepsilon,\omega)\, M_{2i\,n_1,\ldots,n_m}(\varepsilon,\omega), \;\; i=1,2,$$

$$Q_{13\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) = N_{14\,n_1,\ldots,n_m}(t,\varepsilon,\omega) + \nu\, N_{12\,n_1,\ldots,n_m}(t,\varepsilon,\omega)\, M_{13\,n_1,\ldots,n_m}(\varepsilon,\omega),$$

$$Q_{14\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) = N_{15\,n_1,\ldots,n_m}(t,\varepsilon,\omega) + \nu\, N_{12\,n_1,\ldots,n_m}(t,\varepsilon,\omega)\, M_{14\,n_1,\ldots,n_m}(\varepsilon,\omega)$$

$$-\nu\, N_{13\,n_1,\ldots,n_m}(t,\varepsilon,\omega)\, M_{23\,n_1,\ldots,n_m}(\varepsilon,\omega),$$

$$Q_{2i\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) = N_{2i\,n_1,\ldots,n_m}(t,\varepsilon,\omega) + \nu\, N_{23\,n_1,\ldots,n_m}(t,\varepsilon,\omega)\, M_{2i\,n_1,\ldots,n_m}(\varepsilon,\omega), \;\; i=1,2,$$

$$Q_{23\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) = N_{24\,n_1,\ldots,n_m}(t,\varepsilon,\omega) + \nu\, N_{23\,n_1,\ldots,n_m}(t,\varepsilon,\omega)\, M_{23\,n_1,\ldots,n_m}(\varepsilon,\omega).$$

Now, we substitute presentations (31) and (32) into the Fourier series (7) and obtain the following formal solution of the direct problem (1)–(4)

$$\begin{aligned} U(t,x,\varepsilon,\omega,\nu) &= \sum_{n_1,\ldots,n_m=1}^{\infty} \vartheta_{n_1,\ldots,n_m}(x) \left[\varphi_{1\,n_1,\ldots,n_m} Q_{11\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\right. \\ &+\varphi_{2\,n_1,\ldots,n_m} Q_{12\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) + \left(g_{1\,n_1,\ldots,n_m} + f_{1\,n_1,\ldots,n_m}(V_1)\right) Q_{13\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) \\ &\left.+ \left(g_{2\,n_1,\ldots,n_m} + f_{2\,n_1,\ldots,n_m}(V_2)\right) Q_{14\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\right], \;\; t>0, \end{aligned} \tag{33}$$

$$\begin{aligned} U(t,x,\varepsilon,\omega,\nu) &= \sum_{n_1,\ldots,n_m=1}^{\infty} \vartheta_{n_1,\ldots,n_m}(x) \left[\varphi_{1\,n_1,\ldots,n_m} Q_{21\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) + \varphi_{2\,n_1,\ldots,n_m}\right. \\ &\left.\times Q_{22\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) + \left(g_{2\,n_1,\ldots,n_m} + f_{2\,n_1,\ldots,n_m}(V_2)\right) Q_{23\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\right], \;\; t<0. \end{aligned} \tag{34}$$

We suppose that the conditions of (23) were violated for some values of spectral parameter ω. So, we have to consider the algebraic equation with respect to spectral parameter ω

$$\begin{aligned} \sigma_{n_1,\ldots,n_m}(\omega) &= \Psi_{24n_1,\ldots,n_m}(-T,\varepsilon,\omega)\cdot D^{\theta}_{0\,t}\Psi_{23n_1,\ldots,n_m}(-T,\varepsilon,\omega) \\ &-\Psi_{23n_1,\ldots,n_m}(-T,\varepsilon,\omega)\cdot D^{\theta}_{0\,t}\Psi_{24n_1,\ldots,n_m}(-T,\varepsilon,\omega) = 0. \end{aligned} \tag{35}$$

The set of positive solutions of this algebraic Equation (35) with respect to the spectral parameter ω, we denote by $\Im_1$. We call these values $\omega \in \Im_1$ as irregular values and, for these values, the condition (23) is violated. Another set $\Lambda_1 = (0;\infty) \setminus \Im_1$ is called the set of regular values of the spectral parameter ω and, for these regular values, the condition (23) is fulfilled.

Now, we assume that the conditions in (28) are violated $\nu\,\chi_{12\,n_1,\ldots,n_m}(\varepsilon,\omega) = 1$, $\;\nu\,\chi_{23\,n_1,\ldots,n_m}(\varepsilon,\omega) = 1$. Hence, we have

$$\nu_1 = \frac{1}{\chi_{12\,n_1,\ldots,n_m}(\varepsilon,\omega)}, \;\; \nu_2 = \frac{1}{\chi_{23\,n_1,\ldots,n_m}(\varepsilon,\omega)}.$$

For regular values $\omega \in \Lambda_1$ there hold $\chi_{12\,n_1,\ldots,n_m}(\varepsilon,\omega) \neq 0$, $\;\chi_{23\,n_1,\ldots,n_m}(\varepsilon,\omega) \neq 0$. So, we denote the set $\{\nu_1, \nu_2\}$ by $\Im_2$. Then a set $\Lambda_2 = (-\infty;0) \cup (0;\infty) \setminus \Im_2$ is called the set of regular values of the spectral parameter ν. Therefore, for all values of $\nu \in \Lambda_2$, condition (28) is satisfied. We use the following notation $\aleph = \{n_1, \ldots, n_m \in \mathbb{N};\, \omega \in \Lambda_1;\, \nu \in \Lambda_2\}$, where $\mathbb{N}$ is the set of natural numbers. This is the set on which all values of the spectral parameters ω and ν are regular. Therefore, in this case, we study the solution of the direct problem (1)–(4) in the domain Ω as Fourier series (33) and (34).

4. Redefinition Functions

Suppose that the functions $\psi_i(x)$ expand in the Fourier series

$$\psi_i(x) = \sum_{n_1,\ldots,n_m=1}^{\infty} \psi_{i\,n_1,\ldots,n_m} \vartheta_{n_1,\ldots,n_m}(x), \tag{36}$$

where

$$\psi_{i\,n_1,\ldots,n_m} = \int_{\Omega_l^m} \psi_i(x)\, \vartheta_{n_1,\ldots,n_m}(x)\, d\,x, \quad i = 1, 2, \quad n_1, \ldots, n_m = 1, 2, \ldots$$

By virtue of series (33), (34) and (36), we apply conditions (5) and (6):

$$\sum_{n_1,\ldots,n_m=1}^{\infty} \psi_{1\,n_1,\ldots,n_m} \vartheta_{n_1,\ldots,n_m}(x) = \sum_{n_1,\ldots,n_m=1}^{\infty} \vartheta_{n_1,\ldots,n_m}(x)$$

$$\times \int_0^T \Phi_1(t)\, \left[\varphi_{1\,n_1,\ldots,n_m} Q_{11\,n_1,\ldots,n_m}(t, \varepsilon, \omega, \nu) + \varphi_{2\,n_1,\ldots,n_m} Q_{12\,n_1,\ldots,n_m}(t, \varepsilon, \omega, \nu)\right.$$

$$\left. + (g_{1n_1,\ldots,n_m} + f_{1n_1,\ldots,n_m}(V_1))\, Q_{13n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) + (g_{2n_1,\ldots,n_m} + f_{2n_1,\ldots,n_m}(V_2))\, Q_{14n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\right] dt,$$

$$\sum_{n_1,\ldots,n_m=1}^{\infty} \psi_{2\,n_1,\ldots,n_m} \vartheta_{n_1,\ldots,n_m}(x) = \sum_{n_1,\ldots,n_m=1}^{\infty} \vartheta_{n_1,\ldots,n_m}(x)$$

$$\times \int_{-T}^0 \Phi_2(t)\, \left[\varphi_{1\,n_1,\ldots,n_m} Q_{21\,n_1,\ldots,n_m}(t, \varepsilon, \omega, \nu) + \varphi_{2\,n_1,\ldots,n_m}\, Q_{22\,n_1,\ldots,n_m}(t, \varepsilon, \omega, \nu)\right.$$

$$\left. + \left(g_{2\,n_1,\ldots,n_m} + f_{2\,n_1,\ldots,n_m}(V_2)\right)\; Q_{23\,n_1,\ldots,n_m}(t, \varepsilon, \omega, \nu)\right] d\,t.$$

Hence, we obtain

$$\begin{aligned} \psi_{1n_1,\ldots,n_m} &= \varphi_{1n_1,\ldots,n_m} \Upsilon_{11n_1,\ldots,n_m}(\varepsilon,\omega,\nu) + \varphi_{2n_1,\ldots,n_m} \Upsilon_{12n_1,\ldots,n_m}(\varepsilon,\omega,\nu) \\ &+ \left(g_{1\,n_1,\ldots,n_m} + f_{1\,n_1,\ldots,n_m}(V_1)\right)\; \Upsilon_{13\,n_1,\ldots,n_m}(\varepsilon, \omega, \nu) \\ &+ \left(g_{2\,n_1,\ldots,n_m} + f_{2\,n_1,\ldots,n_m}(V_2)\right)\; \Upsilon_{14\,n_1,\ldots,n_m}(\varepsilon, \omega, \nu), \end{aligned} \tag{37}$$

$$\begin{aligned} \psi_{2\,n_1,\ldots,n_m} &= \varphi_{1\,n_1,\ldots,n_m} \Upsilon_{21\,n_1,\ldots,n_m}(\varepsilon, \omega, \nu) + \varphi_{2\,n_1,\ldots,n_m} \Upsilon_{22\,n_1,\ldots,n_m}(\varepsilon, \omega, \nu) \\ &+ \left(g_{2\,n_1,\ldots,n_m} + f_{2\,n_1,\ldots,n_m}(V_2)\right)\; \Upsilon_{23\,n_1,\ldots,n_m}(\varepsilon, \omega, \nu), \end{aligned} \tag{38}$$

where

$$\Upsilon_{1\,i\,n_1,\ldots,n_m}(\varepsilon, \omega, \nu) = \int_0^T \Phi_1(t)\, Q_{1\,i\,n_1,\ldots,n_m}(t, \varepsilon, \omega, \nu)\, d\,t, \quad i = \overline{1, 4},$$

$$\Upsilon_{2\,i\,n_1,\ldots,n_m}(\varepsilon, \omega, \nu) = \int_{-T}^0 \Phi_2(t)\, Q_{2\,i\,n_1,\ldots,n_m}(t, \varepsilon, \omega, \nu)\, d\,t, \quad i = \overline{1, 3}.$$

The relations of (37) and (38) we consider as a system of functional algebraic equations with respect to coefficients of redefinition functions. By solving this system, we obtain the following representations

$$\begin{aligned} g_{1n_1,\ldots,n_m}(\varepsilon,\omega,\nu) + f_{1n_1,\ldots,n_m}(V_1) &= \psi_{1n_1,\ldots,n_m} \Delta_{11n_1,\ldots,n_m}(\varepsilon,\omega,\nu) + \psi_{2n_1,\ldots,n_m} \Delta_{12n_1,\ldots,n_m}(\varepsilon,\omega,\nu) \\ &+ \varphi_{1\,n_1,\ldots,n_m} \Delta_{13\,n_1,\ldots,n_m}(\varepsilon, \omega, \nu) + \varphi_{2\,n_1,\ldots,n_m} \Delta_{14\,n_1,\ldots,n_m}(\varepsilon, \omega, \nu), \end{aligned} \tag{39}$$

$$g_{2\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) + f_{2\,n_1,\,\dots,\,n_m}(V_2) = \psi_{2\,n_1,\,\dots,\,n_m}\Delta_{21\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)$$
$$+\varphi_{1\,n_1,\,\dots,\,n_m}\Delta_{22\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) + \varphi_{2\,n_1,\,\dots,\,n_m}\Delta_{23\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu), \tag{40}$$

where

$$\Delta_{11\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) = \left(\Upsilon_{13\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)\right)^{-1},$$
$$\Delta_{12\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) = -\Upsilon_{23\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)\left(\Upsilon_{13\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)\right)^{-1},$$
$$\Delta_{13n_1,\dots,n_m}(\varepsilon,\omega,\nu) = [-\Upsilon_{11n_1,\dots,n_m}(\varepsilon,\omega,\nu) + \Upsilon_{21n_1,\dots,n_m}(\varepsilon,\omega,\nu)\,\Upsilon_{23n_1,\dots,n_m}(\varepsilon,\omega,\nu)]\,(\Upsilon_{13n_1,\dots,n_m}(\varepsilon,\omega,\nu))^{-1},$$
$$\Delta_{14n_1,\dots,n_m}(\varepsilon,\omega,\nu) = [-\Upsilon_{12n_1,\dots,n_m}(\varepsilon,\omega,\nu) + \Upsilon_{22n_1,\dots,n_m}(\varepsilon,\omega,\nu)\,\Upsilon_{23n_1,\dots,n_m}(\varepsilon,\omega,\nu)]\,(\Upsilon_{13n_1,\dots,n_m}(\varepsilon,\omega,\nu))^{-1},$$
$$\Delta_{21\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) = \left(\Upsilon_{23\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)\right)^{-1},$$
$$\Delta_{22\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) = -\Upsilon_{21\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)\left(\Upsilon_{23\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)\right)^{-1},$$
$$\Delta_{23\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) = -\Upsilon_{22\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)\left(\Upsilon_{23\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)\right)^{-1}.$$

We rewrite Formulas (39) and (40) in the form of countable systems of nonlinear integral equations (CSNIE)

$$g_{i\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) = I_i\left(g_{i\,n_1,\,\dots,\,n_m}\right) \equiv c_{i\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)$$
$$-\int\limits_{\Omega_l^m} f_i\left(y,\,\int\limits_{\Omega_l^m}\Theta_i(z)\sum_{n_1,\,\dots,\,n_m=1}^{\infty} g_{i\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)\,\vartheta_{n_1,\,\dots,\,n_m}(z)\,d\,z\right)\vartheta_{n_1,\,\dots,\,n_m}(y)\,d\,y,\ \ i = 1,\,2, \tag{41}$$

where

$$c_{1\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) = \psi_{1\,n_1,\,\dots,\,n_m}\Delta_{11\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) + \psi_{2\,n_1,\,\dots,\,n_m}\Delta_{12\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)$$
$$+\varphi_{1\,n_1,\,\dots,\,n_m}\Delta_{13\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) + \varphi_{2\,n_1,\,\dots,\,n_m}\Delta_{14\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu),$$
$$c_{2\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) = \psi_{2\,n_1,\,\dots,\,n_m}\Delta_{21\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu)$$
$$+\varphi_{1\,n_1,\,\dots,\,n_m}\Delta_{22\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu) + \varphi_{2\,n_1,\,\dots,\,n_m}\Delta_{23\,n_1,\,\dots,\,n_m}(\varepsilon,\,\omega,\,\nu).$$

5. Unique Solvability of CSNIE (41)

We use the concepts of the following well-known Banach spaces, including a Hilbert coordinate space ℓ_2 of number sequences $\{b_{n_1,\,\dots,\,n_m}\}_{n_1,\,\dots,\,n_m=1}^{\infty}$ with the norm

$$\|\,b\,\|_{\ell_2} = \sqrt{\sum_{n_1,\,\dots,\,n_m=1}^{\infty}|\,b_{\,n_1,\,\dots,\,n_m}|^2} < \infty.$$

We also use the space $L_2(\Omega_l^m)$ of square-summable functions on the domain Ω_l^m with the norm

$$\|\,\vartheta\,(x)\,\|_{L_2(\Omega_l^m)} = \sqrt{\int\limits_{\Omega_l^m}|\,\vartheta\,(x)\,|^2\,d\,x} < \infty.$$

In the process of proofing the unique solvability of CSNIE (41), we need the following conditions.

Smoothness conditions. Let functions

$$\varphi_i(x),\ \psi_i(x) \in C^2(\Omega_l^m),\ f_i\left(x, \int\limits_{\Omega_l^m} \Theta_i(y)\, g_i(y)\, dy\right) \in C_x^2(\Omega_l^m \times \mathbb{R}),\ i = 1, 2$$

in the domain Ω_l^m have piecewise continuous third order derivatives.

Then, by integrating them in parts three times over all variables $x_1, x_2, \ldots, x_m$, we obtain the following formulas [41]

$$\sum_{n_1,\ldots,n_m=1}^{\infty} \left[\varphi_{i\,n_1,\ldots,n_m}^{(3m)}\right]^2 \leq \left(\frac{2}{l}\right)^m \int\limits_{\Omega_l^m} \left[\frac{\partial^{3m}\varphi_i(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3}\right]^2 dx, \tag{42}$$

$$\sum_{n_1,\ldots,n_m=1}^{\infty} \left[\psi_{i\,n_1,\ldots,n_m}^{(3m)}\right]^2 \leq \left(\frac{2}{l}\right)^m \int\limits_{\Omega_l^m} \left[\frac{\partial^{3m}\psi_i(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3}\right]^2 dx, \tag{43}$$

$$\left|\varphi_{i\,n_1,\ldots,n_m}\right| = \left(\frac{l}{\pi}\right)^{3m} \frac{\left|\varphi_{i\,n_1,\ldots,n_m}^{(3m)}\right|}{n_1^3 \ldots n_m^3},\ \left|\psi_{i\,n_1,\ldots,n_m}\right| = \left(\frac{l}{\pi}\right)^{3m} \frac{\left|\psi_{i\,n_1,\ldots,n_m}^{(3m)}\right|}{n_1^3 \ldots n_m^3}, \tag{44}$$

where

$$\varphi_{i\,n_1,\ldots,n_m}^{(3m)} = \int\limits_{\Omega_l^m} \frac{\partial^{3m}\varphi_i(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \vartheta_{n_1,\ldots,n_m}(x)\, dx,$$

$$\psi_{i\,n_1,\ldots,n_m}^{(3m)} = \int\limits_{\Omega_l^m} \frac{\partial^{5m}\psi_i(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \vartheta_{n_1,\ldots,n_m}(x)\, dx, \quad i = 1, 2.$$

We obtain also that

$$\left| f_{i\,n_1,\ldots,n_m}(V_i) \right| = \left(\frac{l}{\pi}\right)^{3m} \frac{\left| f_{i\,n_1,\ldots,n_m}^{(3m)}(x, V_i) \right|}{n_1^3 \ldots n_m^3}, \tag{45}$$

$$\sum_{n_1,\ldots,n_m=1}^{\infty} \left[f_{i\,n_1,\ldots,n_m}^{(3m)}(V_i)\right]^2 \leq \left(\frac{2}{l}\right)^m \int\limits_{\Omega_l^m} \left[\frac{\partial^{3m} f_i(x, V_i)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3}\right]^2 dx, \tag{46}$$

where

$$f_{i\,n_1,\ldots,n_m}^{(3m)}(V_i) = \int\limits_{\Omega_l^m} \frac{\partial^{3m} f_i(x, V_i)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3}\, \vartheta_{n_1,\ldots,n_m}(x)\, dx, \quad i = 1, 2.$$

We use also the following well known properties of the Mittag–Leffler function:

(1) For all $k > 0$, $\alpha_0, \beta_0, \gamma_0 \in (0; 2]$, $\alpha_0 \leq \beta_0 \leq \gamma_0$, $t \geq 0$ the function $t^{\beta_0 - 1} E_{\alpha_0, \beta_0, \gamma_0}\left(-k\,t^{\alpha}, -k\,t^{\beta}\right)$ is complete and monotonous and there holds

$$(-1)^s \left[t^{\beta_0 - 1} E_{(\alpha_0, \beta_0), \gamma_0}\left(-k\,t^{\alpha_0}, -k\,t^{\beta_0}\right)\right]^{(s)} \geq 0, \quad s = 0, 1, 2, \ldots \tag{47}$$

(2) For all $\alpha_0, \beta_0 \in (0, 2)$, $\gamma \in \mathbb{R}$ and $\arg z_1 = \pi$, there hold the following estimates

$$\left| E_{(\alpha_0, \beta_0), \gamma_0}(z_1, z_2) \right| \leq \frac{C_1}{1 + |z_1|}, \tag{48}$$

$$\left| E_{(\alpha_0, \beta_0), \gamma_0}(\varepsilon_1 z_1, z_2) - E_{(\alpha_0, \beta_0), \gamma_0}(\varepsilon_2 z_1, z_2) \right| \leq |\varepsilon_1 - \varepsilon_2| \frac{C_2}{1 + |z_1|}, \tag{49}$$

where $0 < C_i = \text{const}$ does not depend on z, $\varepsilon_i \in (0;\,\varepsilon_0)$, $0 < \varepsilon_0 = \text{const}$, $i = 1, 2$.

According to the properties of the Mittag–Leffler function (Formulas (47) and (48)) the quantities $\Delta_{1,j\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu)$ $(j = \overline{1,4})$ and $\Delta_{2,j\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu)$ $(j = \overline{1,3})$ are uniformly bounded. So, for any positive integers $n_1, \dots, n_m$, there exist finite constant numbers C_{0i} $(i = 1, 2)$, by which the following estimates take place

$$\max_{n_1,\dots,n_m\in\mathbb{N}} \max_j \left|\,\Delta_{1,j\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu)\,\right| \le C_{01},\;\; j = \overline{1,4}, \tag{50}$$

$$\max_{n_1,\dots,n_m\in/mathbbN} \max_j \left|\,\Delta_{2,j\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu)\,\right| \le C_{02},\;\; j = \overline{1,3}, \tag{51}$$

where $0 < C_{0i} = \text{const}$, $i = 1, 2$.

Lemma 1. *Suppose that the smoothness conditions are fulfilled and*

$$|\,f_i(x,\,V_{1i}) - f_i(x,\,V_{2i})\,| \le K_{1i}(x)\,|\,V_{1i} - V_{2i}\,|,\;\; \rho < 1,$$

where $\rho = C_{04}\gamma_3\,\|\,\Theta_i(x)\,\|_{L_2(\Omega_l^m)}$, $\gamma_3 = C_{03}\left(\frac{l}{\pi}\right)^{3m}\left(\frac{2}{l}\right)^{m}$,

$$C_{03} = \sqrt{\sum_{n_1,\dots,n_m=1}^{\infty} \frac{1}{n_1^6\,\dots n_m^6}} < \infty;\;\; \max_i \left\|\,\frac{\partial^{3m}K_{1i}(x)}{\partial\,x_1^3\partial\,x_2^3\,\dots\partial\,x_m^3}\,\right\|_{L_2(\Omega_l^m)} \le C_{04} < \infty,\;\; i = 1, 2.$$

Then, for regular values of spectral parameters ω and ν, CSNIE (41) is uniquely solvable in the space ℓ_2. In this case, successive approximations are defined as follows:

$$g^0_{i\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu) = c_{i\,n_1,\dots,n_m},\;\; g^{k+1}_{i\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu) = I_{i\,n_1,\dots,n_m}(g_i^k),\;\; i = 1, 2. \tag{52}$$

Proof. We apply the method of successive approximations and the method of compressive mappings. We use Formulas (42)–(44) and estimates (50) and (51). By the aid of the Cauchy–Schwartz inequality and the Bessel inequality for the zeroth approximation of the coefficients of the redefinition functions from successive approximations (52), we obtain

$$\begin{aligned}
&\|\,g_1^0(\varepsilon,\,\omega,\,\nu)\,\|_{\ell_2} \le \sum_{n_1,\dots,n_m=1}^{\infty} |\,c_{1\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu)\,| \\
&\le \sum_{n_1,\dots,n_m=1}^{\infty} |\,\psi_{1\,n_1,\dots,n_m}\Delta_{11\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu)\,| + \sum_{n_1,\dots,n_m=1}^{\infty} |\,\psi_{2\,n_1,\dots,n_m}\Delta_{12\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu)\,| \\
&+ \sum_{n_1,\dots,n_m=1}^{\infty} |\,\varphi_{1\,n_1,\dots,n_m}\Delta_{13\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu)\,| + \sum_{n_1,\dots,n_m=1}^{\infty} |\,\varphi_{2\,n_1,\dots,n_m}\Delta_{14\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu)\,| \\
&\le C_{01}\left[\sum_{n_1,\dots,n_m=1}^{\infty} |\psi_{1n_1,\dots,n_m}| + \sum_{n_1,\dots,n_m=1}^{\infty} |\psi_{2n_1,\dots,n_m}| + \sum_{n_1,\dots,n_m=1}^{\infty} |\varphi_{1n_1,\dots,n_m}| + \sum_{n_1,\dots,n_m=1}^{\infty} |\varphi_{2n_1,\dots,n_m}|\right] \\
&\le C_{01}\left(\frac{l}{\pi}\right)^{3m}\left[\sum_{n_1,\dots,n_m=1}^{\infty} \frac{\left|\psi^{(3m)}_{1\,n_1,\dots,n_m}\right|}{n_1^3\dots n_m^3} + \sum_{n_1,\dots,n_m=1}^{\infty} \frac{\left|\psi^{(3m)}_{2\,n_1,\dots,n_m}\right|}{n_1^3\dots n_m^3}\right. \\
&\left. + \sum_{n_1,\dots,n_m=1}^{\infty} \frac{\left|\varphi^{(3m)}_{1n_1,\dots,n_m}\right|}{n_1^3\dots n_m^3} + \sum_{n_1,\dots,n_m=1}^{\infty} \frac{\left|\varphi^{(3m)}_{2n_1,\dots,n_m}\right|}{n_1^3\dots n_m^3}\right] \\
&\le C_{01}\left(\frac{l}{\pi}\right)^{3m}\sqrt{\sum_{n_1,\dots,n_m=1}^{\infty} \frac{1}{n_1^6\dots n_m^6}}\left[\left\|\,\psi_1^{(3m)}\right\|_{\ell_2} + \left\|\,\psi_2^{(3m)}\right\|_{\ell_2} + \left\|\,\varphi_1^{(3m)}\right\|_{\ell_2} + \left\|\,\varphi_2^{(3m)}\right\|_{\ell_2}\right] \\
&\le \gamma_1\left[\left\|\,\frac{\partial^{3m}\psi_1(x)}{\partial\,x_1^3\partial\,x_2^3\dots\partial\,x_m^3}\,\right\|_{L_2\left(\Omega_l^m\right)} + \left\|\,\frac{\partial^{3m}\psi_2(x)}{\partial\,x_1^3\partial\,x_2^3\dots\partial\,x_m^3}\,\right\|_{L_2\left(\Omega_l^m\right)}\right. \\
&\left. + \left\|\,\frac{\partial^{3m}\varphi_1(x)}{\partial\,x_1^3\partial\,x_2^3\dots\partial\,x_m^3}\,\right\|_{L_2\left(\Omega_l^m\right)} + \left\|\,\frac{\partial^{3m}\varphi_2(x)}{\partial\,x_1^3\partial\,x_2^3\dots\partial\,x_m^3}\,\right\|_{L_2\left(\Omega_l^m\right)}\right] < \infty,
\end{aligned} \tag{53}$$

where $\gamma_1 = C_{01}C_{03}\left(\frac{2}{l}\right)^m\left(\frac{l}{\pi}\right)^{3m}$, $C_{03} = \sqrt{\sum\limits_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^6\ldots n_m^6}} < \infty$;

$$\begin{aligned}
&\| g_2^0(\varepsilon,\omega,\nu)\|_{\ell_2} \le \sum_{n_1,\ldots,n_m=1}^{\infty} |c_{2\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu)| \le \sum_{n_1,\ldots,n_m=1}^{\infty} |\psi_{2\,n_1,\ldots,n_m}\Delta_{21\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu)| \\
&+ \sum_{n_1,\ldots,n_m=1}^{\infty} |\varphi_{1\,n_1,\ldots,n_m}\Delta_{22\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu)| + \sum_{n_1,\ldots,n_m=1}^{\infty} |\varphi_{2\,n_1,\ldots,n_m}\Delta_{23\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu)| \\
&\le C_{02}\left(\frac{l}{\pi}\right)^{3m}\left[\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{\left|\psi^{(3m)}_{2n_1,\ldots,n_m}\right|}{n_1^3\ldots n_m^3} + \sum_{n_1,\ldots,n_m=1}^{\infty}\frac{\left|\varphi^{(3m)}_{1n_1,\ldots,n_m}\right|}{n_1^3\ldots n_m^3} + \sum_{n_1,\ldots,n_m=1}^{\infty}\frac{\left|\varphi^{(3m)}_{2n_1,\ldots,n_m}\right|}{n_1^3\ldots n_m^3}\right] \\
&\le \gamma_2\left[\left\|\frac{\partial^{3m}\psi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)} + \left\|\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)} + \left\|\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)}\right] < \infty,
\end{aligned} \tag{54}$$

where $\gamma_2 = C_{02}C_{03}\left(\frac{2}{l}\right)^m\left(\frac{l}{\pi}\right)^{3m}$.

By Formulas (45) and (46), using the Cauchy–Schwartz inequality and Bessel inequality for the first difference of approximation (52), we obtain

$$\begin{aligned}
&\|g_i^1(\varepsilon,\omega,\nu) - g_i^0(\varepsilon,\omega,\nu)\|_{\ell_2} \le \sum_{n_1,\ldots,n_m=1}^{\infty}\left|\int_{\Omega_l^m} f_i\left(x, \int_{\Omega_l^m}\Theta_i(y)\,g_i^0(y,\varepsilon,\omega,\nu)\,dy\right)\vartheta_{n_1,\ldots,n_m}(x)\,dx\right| \\
&\le \left(\frac{l}{\pi}\right)^{3m}\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{\left|f^{(3m)}_{i\,n_1,\ldots,n_m}\left(x,V_i^0\right)\right|}{n_1^3\ldots n_m^3} \le \gamma_3\left\|\frac{\partial^{3m}f_i\left(x,V_i^0\right)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)} < \infty,
\end{aligned} \tag{55}$$

where $\gamma_3 = C_{03}\left(\frac{l}{\pi}\right)^{3m}\left(\frac{2}{l}\right)^m$, $V_i^0 = \int\limits_{\Omega_l^m}\Theta_i(x)\,g_i^0(x,\varepsilon,\omega,\nu)\,dx$, $i = 1, 2$.

Analogously, by the condition of the lemma and expansion (9), using the Cauchy–Schwartz inequality and Bessel inequality for an arbitrary difference of approximation (52), we obtain

$$\begin{aligned}
&\left\|g_i^{k+1}(\varepsilon,\omega,\nu) - g_i^k(\varepsilon,\omega,\nu)\right\|_{\ell_2} \\
&\le \gamma_3\left\|\frac{\partial^{3m}}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\left|f_i(x,V_i^k) - f_i(x,V_i^{k-1})\right|\right\|_{L_2(\Omega_l^m)} \\
&\le \gamma_3\int_{\Omega_l^m}|\Theta_i(y)|\cdot\left|g_i^k(y,\varepsilon,\omega,\nu) - g_i^{k-1}(y,\varepsilon,\omega,\nu)\right|dy\left\|\frac{\partial^{3m}K_{1i}(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)} \\
&\le C_{04}\gamma_3\int_{\Omega_l^m}|\Theta_i(y)|\sum_{n_1,\ldots,n_m=1}^{\infty}\left|g^k_{i\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu) - g^{k-1}_{i\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu)\right|\cdot|\vartheta_{n_1,\ldots,n_m}(y)|\,dy \\
&\le \rho\left\|g_i^k(\varepsilon,\omega,\nu) - g_i^{k-1}(\varepsilon,\omega,\nu)\right\|_{\ell_2}, \quad i = 1, 2,
\end{aligned} \tag{56}$$

where

$$\rho = C_{04}\gamma_3\|\Theta_i(x)\|_{L_2(\Omega_l^m)}, \ \max_i\left\|\frac{\partial^{3m}K_{1i}(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)} \le C_{04} < \infty, \ \ i = 1, 2.$$

By the condition of the lemma, $\rho < 1$. Therefore, it follows from estimate (56) that the operators on the right-hand side of (41) are contracting. From the estimates (53)–(56), it is implied that there exists a unique pair of fixed points $\{g_{1n_1,\ldots,n_m}(\varepsilon,\omega,\nu);\ g_{2n_1,\ldots,n_m}(\varepsilon,\omega,\nu)\}$, which is a solution of CSNIE (41) in the space ℓ_2. The Lemma 1 is proved. □

6. Convergence of Fourier Series (57)

Now, we determine the redefinition functions. In this order, we substitute representations (41) into the Fourier series (9) and obtain

$$
\begin{gathered}
g_i(x,\varepsilon,\omega,\nu)=\sum_{n_1,\ldots,n_m=1}^{\infty}\vartheta_{n_1,\ldots,n_m}(x)\,\Big[c_{i\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu)\\
-\int\limits_{\Omega_l^m} f_i\left(y,\int\limits_{\Omega_l^m}\Theta_i(z)\sum_{n_1,\ldots,n_m=1}^{\infty} g_{i\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu)\,\vartheta_{n_1,\ldots,n_m}(z)\,dz\right)\vartheta_{n_1,\ldots,n_m}(y)\,dy\Big],\ i=1,2.
\end{gathered}
\tag{57}
$$

We prove that the following lemma holds.

Lemma 2. *Assume that the conditions of Lemma 1 are satisfied. Then for regular values of spectral parameters ω and ν, the series (57) converge absolutely.*

Proof. We use estimates (53)–(55). Using the Cauchy–Schwartz inequality and Bessel inequality for series (57), we obtain the following estimates

$$
\begin{gathered}
|g_1(x,\varepsilon,\omega,\nu)|\le\sum_{n_1,\ldots,n_m=1}^{\infty}|\vartheta_{n_1,\ldots,n_m}(x)|\,\Big[|c_{1\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu)|\\
+\left|\int\limits_{\Omega_l^m} f_1\left(y,\int\limits_{\Omega_l^m}\Theta_1(z)\sum_{n_1,\ldots,n_m=1}^{\infty} g_{1\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu)\,\vartheta_{n_1,\ldots,n_m}(z)\,dz\right)\vartheta_{n_1,\ldots,n_m}(y)\,dy\right|\Big]\\
\le C_{03}\left(\tfrac{2}{l}\right)^{\frac{3m}{2}}\left(\tfrac{l}{\pi}\right)^{3m}\Big[C_{01}\left\|\psi_1^{(3m)}\right\|_{\ell_2}+C_{01}\left\|\psi_2^{(3m)}\right\|_{\ell_2}\\
+C_{01}\left\|\varphi_1^{(3m)}\right\|_{\ell_2}+C_{01}\left\|\varphi_2^{(3m)}\right\|_{\ell_2}+\left\|f_1^{(3m)}(V_1)\right\|_{\ell_2}\Big]\\
\le\gamma_4\Big[\left\|\frac{\partial^{3m}\psi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)}+\left\|\frac{\partial^{3m}\psi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)}+\left\|\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)}\\
+\left\|\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)}+\left\|\frac{\partial^{3m}f_1(x,V_1)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)}\Big]<\infty,
\end{gathered}
\tag{58}
$$

$$
\begin{gathered}
|g_2(x,\varepsilon,\omega,\nu)|\le\sum_{n_1,\ldots,n_m=1}^{\infty}|\vartheta_{n_1,\ldots,n_m}(x)|\,\Big[|c_{2\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu)|\\
+\left|\int\limits_{\Omega_l^m} f_2\left(y,\int\limits_{\Omega_l^m}\Theta_2(z)\sum_{n_1,\ldots,n_m=1}^{\infty} g_{2\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu)\,\vartheta_{n_1,\ldots,n_m}(z)\,dz\right)\vartheta_{n_1,\ldots,n_m}(y)\,dy\right|\Big]\\
\le\gamma_4\Big[\left\|\frac{\partial^{3m}\psi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)}+\left\|\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)}\\
+\left\|\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)}+\left\|\frac{\partial^{3m}f_2(x,V_2)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)}\Big]<\infty,
\end{gathered}
\tag{59}
$$

where

$$
\left\|\frac{\partial^{3m}f_i(x,V_2)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right\|_{L_2(\Omega_l^m)}=\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}f_i(x,V_i)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx},
$$

$$
V_i=\int\limits_{\Omega_l^m}\Theta_i(y)\sum_{n_1,\ldots,n_m=1}^{\infty}g_{i\,n_1,\ldots,n_m}(\varepsilon,\omega,\nu)\,\vartheta_{n_1,\ldots,n_m}(z)\,dz,\ \ i=1,2,
$$

$$
\gamma_4=C_{03}C_{05}\left(\frac{2}{l}\right)^{\frac{3m}{2}}\left(\frac{l}{\pi}\right)^{3m},\ \ C_{05}=\max\left\{C_{01};\ C_{02};\ 1\right\}.
$$

From (58) and (59) the convergence of series (57) is implied. Lemma 2 is proved. □

So, we determined the redefinition functions as a Fourier series (57). Using representations (39) and (40),we can present Fourier series (33) and (34) of the main unknown functions as

$$\begin{aligned} U(t,x,\varepsilon,\omega,\nu) &= \sum_{n_1,\dots,n_m=1}^{\infty} \vartheta_{n_1,\dots,n_m}(x)\ [\psi_{1n_1,\dots,n_m} W_{11n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) + \psi_{2n_1,\dots,n_m} W_{12n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) \\ &\quad + \varphi_{1\,n_1,\dots,n_m} W_{13\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu) + \varphi_{2\,n_1,\dots,n_m} W_{14\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu)]\,,\ \ t>0, \end{aligned} \tag{60}$$

$$\begin{aligned} U\,(t,\,x,\,\varepsilon,\,\omega,\,\nu) &= \sum_{n_1,\dots,n_m=1}^{\infty} \vartheta_{\,n_1,\dots,n_m}(x)\,[\psi_{2\,n_1,\dots,n_m} W_{21\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu) \\ &\quad + \varphi_{1\,n_1,\dots,n_m} W_{22\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu) + \varphi_{2\,n_1,\dots,n_m} W_{23\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu)]\,,\ \ t<0, \end{aligned} \tag{61}$$

where

$$W_{i1\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu) = \Delta_{\,i1\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu)\,Q_{\,i3\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu),\ \ i=1,\,2,$$

$$W_{12n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) = \Delta_{12n_1,\dots,n_m}(\varepsilon,\omega,\nu)\,Q_{13n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) + \Delta_{21n_1,\dots,n_m}(\varepsilon,\omega,\nu)\,Q_{14n_1,\dots,n_m}(t,\varepsilon,\omega,\nu),$$

$$W_{1j\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu) = Q_{\,1j-2\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu) + \Delta_{\,1j\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu)\,Q_{\,13\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu)$$

$$+\Delta_{\,2j-1\,n_1,\dots,n_m}(\varepsilon,\,\omega,\,\nu)\,Q_{\,14\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu),\ \ j=3,\,4,$$

$$W_{2kn_1,\dots,n_m}(t,\varepsilon,\omega,\nu) = Q_{2k-1n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) + \Delta_{2kn_1,\dots,n_m}(\varepsilon,\omega,\nu)\,Q_{23n_1,\dots,n_m}(t,\varepsilon,\omega,\nu),\ k=2,3.$$

To establish the uniqueness of the function $U\,(t,\,x,\,\varepsilon,\,\omega,\,\nu)$, we suppose that there are two solutions U_1 and U_2 to this problem. Then, their difference $U = U_1 - U_2$ is a solution of Equation (1), satisfying conditions (2)–(6) with functions $\varphi_i(x) \equiv 0,\ \psi_i(x) \equiv 0\ (i=1,\,2)$. Then, for $\varphi_{i\,n_1,\dots,n_m} = \psi_{i\,n_1,\dots,n_m} = 0\ \ (i=1,\,2)$, it follows from Formulas (60) and (61) in the domain Ω that

$$\int\limits_{\Omega_l^m} U\,(t,\,x,\,\varepsilon,\,\omega,\,\nu)\,\vartheta_{n_1,\dots,n_m}(x)\,d\,x = 0.$$

Hence, by virtue of the completeness of the systems of eigenfunctions $\left\{\sqrt{\frac{2}{l}}\sin\,\frac{\pi\,n_1}{l}x_1\right\}$, $\left\{\sqrt{\frac{2}{l}}\sin\,\frac{\pi\,n_2}{l}x_2\right\},\,\dots,\ \left\{\sqrt{\frac{2}{l}}\sin\,\frac{\pi\,n_m}{l}x_m\right\}$ in $L_2\left(\Omega_l^m\right)$, we deduce that $U\,(t,\,x,\,\varepsilon,\,\omega,\,\nu) \equiv 0$ for all $x \in \Omega_l^m \equiv [0;\,l]^{\,m}$ and $t \in [-T;\,T]$.

Therefore, for regular values of spectral parameters ω and ν, the function $U\,(t,\,x,\,\varepsilon,\,\omega,\,\nu)$ is a unique solution tp the mixed type integro-differential Equation (1) with conditions (2)–(6), if this function exists in the domain Ω.

Lemma 3. *Let smoothness conditions hold. Then, for regular values of spectral parameters ω and ν, series (60) and (61) converge. At the same time, their term by term differentiation is possible.*

Proof. According to the properties of the Mittag–Leffler function (Formulas (47) and (48)), the functions $W_{1i\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu)\ (i=\overline{1,\,4})$ and $W_{2j\,n_1,\dots,n_m}(t,\,\varepsilon,\,\omega,\,\nu)\ (j=\overline{1,\,3})$ are uniformly bounded on the segment $[-T;\,T]$. So, for any positive integers $n_1,\,\dots,\,n_m$, there exist finite constant numbers $C_{\,1k}\,(k=1,\,2)$; then, the following estimates take place

$$\max_{n_1,\dots,n_m\in\mathbb{N}}\ \max_{i=\overline{1,4}} \left|\,W_{1in_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\,\right| \le C_{11},\quad \max_{n_1,\dots,n_m\in\mathbb{N}}\ \max_{j=\overline{1,3}} \left|\,W_{1jn_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\,\right| \le C_{12}, \tag{62}$$

where $C_{\,1k} = \text{const},\ \ k=1,\,2.$

Analogously to the estimates (58) and (59), by applying estimates (62), the Cauchy–Schwartz inequality and Bessel inequality for series (60) and (61), we

$$\begin{aligned} |U(t,x,\varepsilon,\omega,\nu)| &\le \sum_{n_1,\dots,n_m=1}^{\infty} |u^{+}_{n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)|\cdot|\vartheta_{n_1,\dots,n_m}(x)| \\ &\le \left(\sqrt{\tfrac{2}{l}}\right)^m C_{11} \sum_{n_1,\dots,n_m=1}^{\infty} \left[|\psi_{1\,n_1,\dots,n_m}|+|\psi_{2\,n_1,\dots,n_m}|+|\varphi_{1\,n_1,\dots,n_m}|+|\varphi_{2\,n_1,\dots,n_m}|\right] \\ &\le \gamma_5 \left[\sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\psi_1(x)}{\partial x_1^3 \partial x_2^3 \dots \partial x_m^3}\right]^2 dx} + \sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\psi_2(x)}{\partial x_1^3 \partial x_2^3 \dots \partial x_m^3}\right]^2 dx} \right. \\ &\left. + \sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3 \partial x_2^3 \dots \partial x_m^3}\right]^2 dx} + \sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3 \partial x_2^3 \dots \partial x_m^3}\right]^2 dx} \right] < \infty, \end{aligned} \tag{63}$$

where $\gamma_5 = \left(\sqrt{\frac{2}{l}}\right)^{\frac{3m}{2}} C_{11} C_{03} \left(\frac{l}{\pi}\right)^{3m}$,

$$\begin{aligned} |U(t,x,\varepsilon,\omega,\nu)| &\le \sum_{n_1,\dots,n_m=1}^{\infty} |u^{-}_{n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)|\cdot|\vartheta_{n_1,\dots,n_m}(x)| \\ &\le \gamma_6 \left[\sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\psi_2(x)}{\partial x_1^3 \partial x_2^3 \dots \partial x_m^3}\right]^2 dx} \right. \\ &\left. + \sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3 \partial x_2^3 \dots \partial x_m^3}\right]^2 dx} + \sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3 \partial x_2^3 \dots \partial x_m^3}\right]^2 dx} \right] < \infty, \end{aligned} \tag{64}$$

where $\gamma_6 = \left(\sqrt{\frac{2}{l}}\right)^{\frac{3m}{2}} C_{12} C_{03} \left(\frac{l}{\pi}\right)^{3m}$.

It follows from estimates (63) and (64) that the series (60) and (61) are convergent absolutely and uniformly in the domain $\overline{\Omega}$ for the

$$(n_1, \dots, n_m, \omega, \nu) \in \aleph = \{n_1, \dots, n_m \in \mathbb{N};\ \omega \in \Lambda_1;\ \nu \in \Lambda_2\}.$$

Therefore, for the $(n_1, \dots, n_m, \omega, \nu) \in \aleph$ functions, (63) and (64) formally differentiate in $\overline{\Omega}$ the required number of times

$$\begin{aligned} &{}_C D_{0t}^{\alpha_1} U(t,x,\varepsilon,\omega,\nu) = \sum_{n_1,\dots,n_m=1}^{\infty} \vartheta_{n_1,\dots,n_m}(x) \\ &\times \left[\psi_{1\,n_1,\dots,n_m}\, {}_C D_{0t}^{\alpha_1} W_{11\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) + \psi_{2\,n_1,\dots,n_m}\, {}_C D_{0t}^{\alpha_1} W_{12\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) \right. \\ &\left. + \varphi_{1\,n_1,\dots,n_m}\, {}_C D_{0t}^{\alpha_1} W_{13\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) + \varphi_{2\,n_1,\dots,n_m}\, {}_C D_{0t}^{\alpha_1} W_{14\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\right], \quad t > 0, \end{aligned} \tag{65}$$

$$\begin{aligned} &{}_C D_{0t}^{\alpha_2} U(t,x,\varepsilon,\omega,\nu) = \sum_{n_1,\dots,n_m=1}^{\infty} \vartheta_{n_1,\dots,n_m}(x) \left[\psi_{2\,n_1,\dots,n_m}\, {}_C D_{0t}^{\alpha_2} W_{21\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) \right. \\ &\left. + \varphi_{1\,n_1,\dots,n_m}\, {}_C D_{0t}^{\alpha_2} W_{22\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) + \varphi_{2\,n_1,\dots,n_m}\, {}_C D_{0t}^{\alpha_2} W_{23\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\right], \quad t < 0, \end{aligned} \tag{66}$$

$$\begin{aligned} &U_{x_1 x_1}(t,x,\varepsilon,\omega,\nu) = -\sum_{n_1,\dots,n_m=1}^{\infty} \left(\frac{\pi n_1}{l}\right)^2 \vartheta_{n_1,\dots,n_m}(x) \left[\psi_{1\,n_1,\dots,n_m} W_{11\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) \right. \\ &+ \psi_{2\,n_1,\dots,n_m} W_{12\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) + \varphi_{1\,n_1,\dots,n_m} W_{13\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) \\ &\left. + \varphi_{2\,n_1,\dots,n_m} W_{14\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\right], \quad t > 0, \end{aligned} \tag{67}$$

$$\begin{aligned} &U_{x_1 x_1}(t,x,\varepsilon,\omega,\nu) = -\sum_{n_1,\dots,n_m=1}^{\infty} \left(\frac{\pi n_1}{l}\right)^2 \vartheta_{n_1,\dots,n_m}(x) \left[\psi_{2\,n_1,\dots,n_m} W_{21\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) \right. \\ &\left. + \varphi_{1\,n_1,\dots,n_m} W_{22\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) + \varphi_{2\,n_1,\dots,n_m} W_{23\,n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\right], \quad t < 0, \end{aligned} \tag{68}$$

$$U_{x_2x_2}(t,x,\varepsilon,\omega,\nu)=-\sum_{n_1,\ldots,n_m=1}^{\infty}\left(\frac{\pi n_2}{l}\right)^2\vartheta_{n_1,\ldots,n_m}(x)\,\left[\psi_{1\,n_1,\ldots,n_m}W_{11\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\right.$$
$$+\psi_{2\,n_1,\ldots,n_m}W_{12\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)+\varphi_{1n_1,\ldots,n_m}W_{13\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) \tag{69}$$
$$\left.+\varphi_{2\,n_1,\ldots,n_m}W_{14\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\right],\quad t>0,$$

$$U_{x_2x_2}(t,x,\varepsilon,\omega,\nu)=-\sum_{n_1,\ldots,n_m=1}^{\infty}\left(\frac{\pi n_2}{l}\right)^2\vartheta_{n_1,\ldots,n_m}(x)\left[\psi_{2\,n_1,\ldots,n_m}W_{21\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\right. \tag{70}$$
$$\left.+\varphi_{1\,n_1,\ldots,n_m}W_{22\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)+\varphi_{2\,n_1,\ldots,n_m}W_{23\,n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\right],\quad t<0.$$

The expansions of the following functions into Fourier series are defined in the domain Ω in a similar way

$$U_{x_3x_3}(t,x,\varepsilon,\omega,\nu),\ \ldots,\ U_{x_mx_m}(t,x,\varepsilon,\omega,\nu),\ {}_CD_{0t}^{\alpha_1}U_{x_1x_1}(t,x,\varepsilon,\omega,\nu),$$
$${}_CD_{0t}^{\alpha_2}U_{x_1x_1}(t,x,\varepsilon,\omega,\nu),\ {}_CD_{0t}^{\alpha_1}U_{x_2x_2}(t,x,\varepsilon,\omega,\nu),\ \ldots,\ {}_CD_{0t}^{\alpha_2}U_{x_2x_2}(t,x,\varepsilon,\omega,\nu),\ \ldots,$$
$${}_CD_{0t}^{\alpha_1}U_{x_mx_m}(t,x,\varepsilon,\omega,\nu),\ {}_CD_{0t}^{\alpha_2}U_{x_mx_m}(t,x,\varepsilon,\omega,\nu).$$

The convergence of series (65) and (66) is proved similarly to the proof of the convergence of series (60) and (61). So, it is enough to show the convergence of series (67) and (70). Taking into account Formulas (42)–(44) and estimates (62) and applying the Cauchy–Schwartz inequality and Bessel inequality, we obtain

$$|\,U_{x_1x_1}(t,x,\varepsilon,\omega,\nu)\,|\le\sum_{n_1,\ldots,n_m=1}^{\infty}\left(\frac{\pi n_1}{l}\right)^2|\,u^{+}_{n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\,|\cdot|\,\vartheta_{n_1,\ldots,n_m}(x)\,|$$
$$\le\left(\sqrt{\frac{2}{l}}\right)^m\left(\frac{\pi}{l}\right)^2C_{11}\sum_{n_1,\ldots,n_m=1}^{\infty}n_1^2\left[|\,\psi_{1\,n_1,\ldots,n_m}\,|+|\,\psi_{2\,n_1,\ldots,n_m}\,|+|\,\varphi_{1\,n_1,\ldots,n_m}\,|+|\varphi_{2\,n_1,\ldots,n_m}|\right]$$
$$\le\left(\sqrt{\frac{2}{l}}\right)^m C_{11}\left(\frac{l}{\pi}\right)^{3m-2}\left[\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1n_2^3\ldots n_m^3}\left|\psi^{(3m)}_{1n_1,\ldots,n_m}\right|+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1n_2^3\ldots n_m^3}\left|\psi^{(3m)}_{2\,n_1,\ldots,n_m}\right|\right.$$
$$\left.+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1n_2^3\ldots n_m^3}\left|\varphi^{(3m)}_{1n_1,\ldots,n_m}\right|+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1n_2^3\ldots n_m^3}\left|\varphi^{(3m)}_{2\,n_1,\ldots,n_m}\right|\right]$$
$$\le\gamma_7\left[\sqrt{\int_{\Omega_l^m}\left[\frac{\partial^{3m}\psi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}+\sqrt{\int_{\Omega_l^m}\left[\frac{\partial^{3m}\psi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}\right.$$
$$\left.+\sqrt{\int_{\Omega_l^m}\left[\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}+\sqrt{\int_{\Omega_l^m}\left[\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}\right]<\infty,$$

where $\gamma_7=\left(\sqrt{\frac{2}{l}}\right)^{\frac{3m}{2}}C_{11}C_{06}\left(\frac{l}{\pi}\right)^{3m-2}$, $C_{06}=\sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^2n_2^6\ldots n_m^6}}$;

$$|\,U_{x_2x_2}(t,x,\varepsilon,\omega,\nu)\,|\le\sum_{n_1,\ldots,n_m=1}^{\infty}\left(\frac{\pi n_2}{l}\right)^2|\,u^{-}_{n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\,|\cdot|\,\vartheta_{n_1,\ldots,n_m}(x)\,|$$
$$\le\left(\sqrt{\frac{2}{l}}\right)^m\left(\frac{\pi}{l}\right)^2C_{12}\sum_{n_1,\ldots,n_m=1}^{\infty}n_2^2\left[|\,\psi_{2\,n_1,\ldots,n_m}\,|+|\,\varphi_{1\,n_1,\ldots,n_m}\,|+|\,\varphi_{2\,n_1,\ldots,n_m}\,|\right]$$

$$\leq \left(\sqrt{\frac{2}{l}}\right)^{m} C_{12} \left(\frac{l}{\pi}\right)^{3m-2} \left[\sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^3 n_2 n_3^3 \ldots n_m^3} \left| \psi_{2n_1,\ldots,n_m}^{(3m)} \right| \right.$$

$$\left. + \sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^3 n_2 n_3^3 \ldots n_m^3} \left| \varphi_{1n_1,\ldots,n_m}^{(3m)} \right| + \sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^3 n_2 n_3^3 \ldots n_m^3} \left| \varphi_{2n_1,\ldots,n_m}^{(3m)} \right| \right]$$

$$\leq \gamma_8 \left[\sqrt{\int\limits_{\Omega_l^m} \left[\frac{\partial^{3m} \psi_2(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \right]^2 dx} \right.$$

$$\left. + \sqrt{\int\limits_{\Omega_l^m} \left[\frac{\partial^{3m} \varphi_1(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \right]^2 dx} + \sqrt{\int\limits_{\Omega_l^m} \left[\frac{\partial^{3m} \varphi_2(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \right]^2 dx} \right] < \infty,$$

where $\gamma_8 = \left(\sqrt{\frac{2}{l}}\right)^{\frac{3m}{2}} C_{12} C_{07} \left(\frac{l}{\pi}\right)^{3m-2}$, $C_{07} = \sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^6 n_2^2 \ldots n_m^6}}$.

The convergence of series (68) and (69) is similar to the convergence of series (67) and (70). The convergence of Fourier series for functions

$$U_{x_3x_3}(t, x, \varepsilon, \omega, \nu), \ldots, U_{x_mx_m}(t, x, \varepsilon, \omega, \nu), \; {}_C D_{0t}^{\alpha_1} U_{x_1x_1}(t, x, \varepsilon, \omega, \nu),$$

$${}_C D_{0t}^{\alpha_2} U_{x_1x_1}(t, x, \varepsilon, \omega, \nu), \; {}_C D_{0t}^{\alpha_1} U_{x_2x_2}(t, x, \varepsilon, \omega, \nu), \ldots, \; {}_C D_{0t}^{\alpha_2} U_{x_2x_2}(t, x, \varepsilon, \omega, \nu), \ldots,$$

$${}_C D_{0t}^{\alpha_1} U_{x_mx_m}(t, x, \varepsilon, \omega, \nu), \; {}_C D_{0t}^{\alpha_2} U_{x_mx_m}(t, x, \varepsilon, \omega, \nu)$$

is proved in a similar way in the domain Ω. It follows from these last estimates that functions (60) and (61) possess the properties of (2) for the regular values of spectral parameters ω and ν. □

7. Continuous Dependence of Solution on the Small Parameter

We consider the continuous dependence of the solution to the problem (1)–(4) on small-parameter $\varepsilon > 0$ for regular values of spectral parameters ω and ν. Let ε_1 and ε_2 be two different values of small positive parameter ε. It is easy to check from (47)–(49) that the following estimates hold

$$\max_{n_1,\ldots,n_m\in\mathbb{N}} \max_{t\in[0;T]} \left| W_{1in_1,\ldots,n_m}(t,\varepsilon_1,\omega,\nu) - W_{1in_1,\ldots,n_m}(t,\varepsilon_2,\omega,\nu) \right| \leq C_{21} \left| \varepsilon_1 - \varepsilon_2 \right|, \; i = \overline{1,4}, \tag{71}$$

$$\max_{n_1,\ldots,n_m\in\mathbb{N}} \max_{t\in[-T;0]} \left| W_{2in_1,\ldots,n_m}(t,\varepsilon_1,\omega,\nu) - W_{2in_1,\ldots,n_m}(t,\varepsilon_2,\omega,\nu) \right| \leq C_{22} \left| \varepsilon_1 - \varepsilon_2 \right|, \; i = \overline{1,3}, \tag{72}$$

where $0 < C_{2i} = \text{const}$, $\varepsilon_i \in (0; \varepsilon_0)$, $0 < \varepsilon_0 = \text{const}$, $i = 1, 2$.

Then, taking estimates (63), (64), (71) and (72) into account and applying the Cauchy–Schwartz inequality and Bessel inequality, from series (60) and (61), we obtain

$$\begin{aligned}
&| U(t, x, \varepsilon_1, \omega, \nu) - U(t, x, \varepsilon_2, \omega, \nu | \\
&\leq \sum_{n_1,\ldots,n_m=1}^{\infty} \left| u_{n_1,\ldots,n_m}^{+}(t, \varepsilon_1, \omega, \nu) - u_{n_1,\ldots,n_m}^{+}(t, \varepsilon_2, \omega, \nu) \right| \cdot \left| \vartheta_{n_1,\ldots,n_m}(x) \right| \\
&\leq \left(\sqrt{\tfrac{2}{l}}\right)^{m} C_{21} \left| \varepsilon_1 - \varepsilon_2 \right| \sum_{n_1,\ldots,n_m=1}^{\infty} \left[\left| \psi_{1n_1,\ldots,n_m} \right| + \left| \psi_{2n_1,\ldots,n_m} \right| + \left| \varphi_{1n_1,\ldots,n_m} \right| + \left| \varphi_{2n_1,\ldots,n_m} \right| \right] \\
&\leq \gamma_9 \left| \varepsilon_1 - \varepsilon_2 \right| \left[\sqrt{\int\limits_{\Omega_l^m} \left[\tfrac{\partial^{3m} \psi_1(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \right]^2 dx} + \sqrt{\int\limits_{\Omega_l^m} \left[\tfrac{\partial^{3m} \psi_2(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \right]^2 dx} \right. \\
&\left. + \sqrt{\int\limits_{\Omega_l^m} \left[\tfrac{\partial^{3m} \varphi_1(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \right]^2 dx} + \sqrt{\int\limits_{\Omega_l^m} \left[\tfrac{\partial^{3m} \varphi_2(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \right]^2 dx} \right] = \left| \varepsilon_1 - \varepsilon_2 \right| \cdot C_{31},
\end{aligned} \tag{73}$$

where $\gamma_9 = \left(\sqrt{\frac{2}{l}}\right)^{\frac{3m}{2}} C_{21} C_{03} \left(\frac{l}{\pi}\right)^{3m}$, $0 < C_{31} = \text{const} < \infty$;

$$
\begin{aligned}
& |\,U(t, x, \varepsilon_1, \omega, \nu) - U(t, x, \varepsilon_2, \omega, \nu)\,| \\
& \le \sum_{n_1, \ldots, n_m = 1}^{\infty} |\,u^{-}_{n_1, \ldots, n_m}(t, \varepsilon_1, \omega, \nu) - u^{-}_{n_1, \ldots, n_m}(t, \varepsilon_2, \omega, \nu)\,| \cdot |\,\vartheta_{n_1, \ldots, n_m}(x)\,| \\
& \le \gamma_{10}\,|\,\varepsilon_1 - \varepsilon_2\,| \left[\sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\psi_2(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3}\right]^2 dx} + \right. \\
& \left. + \sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3}\right]^2 dx} + \sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3}\right]^2 dx} \right] = |\,\varepsilon_1 - \varepsilon_2\,| \cdot C_{32},
\end{aligned}
\tag{74}
$$

where $\gamma_{10} = \left(\sqrt{\frac{2}{l}}\right)^{\frac{3m}{2}} C_{22} C_{03} \left(\frac{l}{\pi}\right)^{3m}$, $0 < C_{32} = \text{const} < \infty$.

It follows from estimates (73) and (74) that $|\,U(t, x, \varepsilon_1, \omega, \nu) - U(t, x, \varepsilon_2, \omega, \nu)\,|$ is small if $|\,\varepsilon_1 - \varepsilon_2\,|$ is small in the domain $\overline{\Omega}$ for the $(n_1, \ldots, n_m, \omega, \nu) \in \aleph$.

8. Conclusions and Statement of the Theorem

In the present paper, we study the questions of the one-value solvability of an inverse boundary value problem (1)–(6) for a mixed type integro-differential equation with Caputo operators of different fractional orders and spectral parameters in a multidimensional rectangular domain. For $(n_1, \ldots, n_m, \omega, \nu) \in \aleph$, we proved four lemmas under the following conditions **A**: Let functions

$$
\varphi_i(x),\ \psi_i(x) \in C^2(\Omega_l^m),\ f_i\left(x, \int_{\Omega_l^m} \Theta_i(y)\, g_i(y)\, dy\right) \in C_x^2(\Omega_l^m \times \mathbb{R}),\ \ i = 1, 2
$$

in the domain Ω_l^m have piecewise continuous third order derivatives.

We will formulate a theorem as generalizing the above four proved lemmas. Thus, the following theorem is true.

Theorem 1. *Let the conditions of **A** be fulfilled. Then, for the possible numbers $n_1, \ldots, n_m$ and regular values of spectral parameters ω and ν from the set $\aleph$, the inverse boundary value problem (1)–(6) is uniquely solvable in the domain Ω and the triple of solutions is represented in the form of series (57), (60) and (61). Moreover, it is true that*

$$
\lim_{\varepsilon \to 0} U(t, x, \varepsilon, \omega, \nu) = U(t, x, 0, \omega, \nu),
$$

where $U(t, x, 0, \omega, \nu)$ is the solution of the mixed type fractional integro-differential equation of the form

$$
A_0(U) - B_\omega(U) = \begin{cases} \nu \int_0^T K_1(t, s)\, U(s, x)\, ds + F_1(t, x), & t > 0, \\ \nu \int_{-T}^0 K_2(t, s)\, U(s, x)\, ds + F_2(t, x), & t < 0, \end{cases}
$$

$$
A_0(U) = \left[\frac{1 + \operatorname{sgn}(t)}{2}\, {}_C D_{0t}^{\alpha_1} + \frac{1 - \operatorname{sgn}(t)}{2}\, {}_C D_{0t}^{\alpha_2}\right] U(t, x),\ \ B_\omega(U) = \begin{cases} \sum_{i=1}^m U_{x_i x_i}, & t > 0, \\ \omega^2 \sum_{i=1}^m U_{x_i x_i}, & t < 0 \end{cases}
$$

with boundary value conditions (3)–(6) under consideration,

$$F_i(t,\,x) = k_i(t)\left[g_i(x) + f_i\left(x,\,\int\limits_{\Omega_l^m}\Theta_i(y)\,g_i(y)\,dy\right)\right],\ \ i = 1,\,2.$$

As a conclusion, we say that the numerical methods for solving fractional differential equations are important in the implementation of applied problems. In the future, we will also try to consider the applications of the numerical solution to the problems that we are solving. There are many methods for the numerical implementation of fractional differential equations. In this regard, we note the work done in [42]. In this paper, a new class of (C, G_f)-invex functions is introduced and given nontrivial numerical examples, which justify the existence of such functions. Moreover, we construct generalized convexity definitions (such as, (F, G_f)-invexity, C-convex etc.).

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