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Shifting Operators in Geometric Quantization

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Abstract: The original Bohr-Sommerfeld theory of quantization did not give operators of transitions between quantum quantum states. This paper derives these operators, using the first principles of geometric quantization.

Keywords: Bohr-Sommerfeld; geometric quantization;shifting operator

1. Introduction

Even though the Bohr–Sommerfeld theory was very successful in predicting some physical results, it was never accepted by physicists as a valid quantum theory in the same class as the Schrödinger theory or the Bargmann–Fock theory. The reason for this was that the original Bohr–Sommerfeld theory did not provide operators of transition between quantum states. The need for such operators in the Bohr–Sommerfeld quantization was already pointed out by Heisenberg [1]. The aim of this paper is to derive operators of transition between quantum states in the Bohr–Sommerfeld theory, which we call shifting operators, from the first principles of geometric quantization.

The first step of geometric quantization of a symplectic manifold (P, ω) is called prequantization. It consists of the construction of a complex line bundle $\pi : L \to P$ with connection whose curvature form satisfies a prequantization condition relating it to the symplectic form ω. A comprehensive study of prequantization, from the point of view of representation theory, was given by Kostant in [2]. The work of Souriau [3] was aimed at quantization of physical systems, and studied a circle bundle over phase space. In Souriau's work, the prequantization condition explicitly involved Planck's constant h. In [4], Blattner combined the approaches of Kostant and Souriau by using the complex line bundle with the prequantization condition involving Planck's constant. Since then, geometric quantization has been an effective tool in quantum theory.

We find it convenient to deal with connection and curvature of complex line bundles using the theory of principal and associated bundles [5]. In this framework, the prequantization condition reads

$$\mathrm{d}\beta = (\pi^{\times})^*(-\tfrac{1}{\mathrm{h}}\,\omega),$$

where β is the connection 1-form on the principal $\mathbb{C}^{\times}$-bundle $\pi^{\times} : L^{\times} \to P$ associated to the complex line bundle $\pi : L \to P$, and $\mathbb{C}^{\times}$ is the multiplicative group of nonzero complex numbers.

The aim of prequantization is to construct a representation of the Poisson algebra $(C^{\infty}(P), \{\ ,\ \}, \cdot)$ of (P, ω) on the space of sections of the line bundle L. Each Hamiltonian vector field X_f on P lifts to a unique $\mathbb{C}^{\times}$-invariant vector field Z_f on $L^{\times}$ that preserves the principal connection β on $L^{\times}$. If the vector field X_f is complete, then it generates a 1-parameter group e^{tX_f} of symplectomorphisms of (P, ω). Then, the vector field Z_f is complete and it generates a 1-parameter group e^{tZ_f} of connection preserving diffeomorphisms of the bundle $(L^{\times}, \beta)$, called quantomorphisms, which cover the 1-parameter group e^{tX_f}. The term quantomorphism was introduced by Souriau [3] in the context of SU(n)-principal bundles and discussed in detail in his book [6]. The construction discussed here follows [7], where the term quantomorphism was not used. In this case, e^{tX_f} and e^{tZ_f} are 1-parameter groups of diffeomorphisms of P and $L^\times$, respectively. We refer to e^{tX_f} and e^{tZ_f} as flows of X_f and Z_f. Since L is an associated bundle of $L^\times$, the action $\mathrm{e}^{tZ_f} : L^\times \to L^\times$, induces an action $\widehat{\mathrm{e}}^{tZ_f} : L \to L$, which gives rise to an action on smooth sections σ of L by push forwards, $\sigma \mapsto \widehat{\mathrm{e}}_*^{tZ_f}\sigma = \widehat{\mathrm{e}}^{tZ_f} \circ \sigma \circ \mathrm{e}^{-tX_f}$. Although $\widehat{\mathrm{e}}_*^{tZ_f}\sigma$ may not be defined for all sections σ and all t, its derivative at $t = 0$ is defined for all smooth sections. The prequantization operator

$$\mathcal{P}_f\sigma = i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \widehat{\mathrm{e}}_*^{tZ_f}\sigma, \tag{1}$$

where $\hbar$ is Planck's constant h divided by 2π, is a symmetric operator on the Hilbert space $\mathfrak{H}$ of square integrable sections of L. The operator $\mathcal{P}_f$ is self adjoint if X_f is complete.

The whole analysis of prequantization is concerned with *global* Hamiltonian vector fields. Since every vector field on (P, ω) that preserves the symplectic form is locally Hamiltonian, it is of interest to understand how much of prequantization can be extended to this case. In particular, we are interested in the case where the locally Hamiltonian vector field is the vector field X_ϑ of the integer angle variable ϑ that is defined up to an additive term n, where $n \in \mathbb{Z}$. For a globally Hamiltonian vector field X_f,

$$\widehat{\mathrm{e}}_*^{tZ_f}\sigma = \mathrm{e}^{-2\pi i\, t f/\mathrm{h}}\, \widehat{\mathrm{e}}_*^{t\,\mathrm{lift}X_f}\sigma, \tag{2}$$

where $\widehat{\mathrm{e}}_*^{t\,\mathrm{lift}X_f}\sigma$ is the horizontal transport of section σ by parameter t along integral curves of X_f. Replacing f by a multivalued function ϑ, defined up to an additive n, yields the multivalued expression

$$\widehat{\mathrm{e}}_*^{tZ_\vartheta}\sigma = \mathrm{e}^{-it\vartheta/\mathrm{h}}\, \widehat{\mathrm{e}}_*^{t\,\mathrm{lift}X_\vartheta}\sigma. \tag{3}$$

We observe that, for $t = \mathrm{h}$, Equation (3) gives a single valued expression

$$\widehat{\mathrm{e}}_*^{\mathrm{h}\,Z_\vartheta}\sigma = \mathrm{e}^{-i\,\vartheta}\, \widehat{\mathrm{e}}_*^{\mathrm{h}\,\mathrm{lift}X_\vartheta}\sigma. \tag{4}$$

The shifting operator

$$\boldsymbol{a}_{X_\vartheta} = \widehat{\mathrm{e}}_*^{\mathrm{h}\,Z_\vartheta} = \mathrm{e}^{-i\,\vartheta}\, \widehat{\mathrm{e}}_*^{\mathrm{h}\,\mathrm{lift}X_\vartheta} \tag{5}$$

is an operator on $\mathfrak{H}$, which shifts the support of $\sigma \in \mathfrak{H}$ by h along the integral curve of X_ϑ. If the vector field X_ϑ is complete, then $\boldsymbol{a}_{X_\vartheta}^n = \widehat{\mathrm{e}}_*^{n\mathrm{h}Z_\vartheta}$ for every $n \in \mathbb{Z}$.

Our results provide an answer to Heisenberg's criticism that in Bohr–Sommerfeld theory there are not enough operators to describe transitions between quantum states [1].

Superficially, the shifting operator $\boldsymbol{a}_{X_\theta} = \mathrm{e}^{i\,\theta}\widehat{\mathrm{e}}^{\hbar\,\mathrm{lift}\,X_\theta}{}_*$, see Equation (5), appears to be a quantization of an angle $\theta = 2\pi\vartheta$. It depends on θ and has the factor $\mathrm{e}^{-i\theta}$ considered by Dirac [8]. However, the factor $\widehat{\mathrm{e}}_*^{\hbar\,\mathrm{lift}X_\theta}$, describing the parallel translation by $\hbar$ along integral curves of X_θ, makes it nonlocal in the phase space. Therefore, $\boldsymbol{a}_{X_\theta}$ cannot satisfy local commutation relations with any local quantum variable that is described by a differential operator. Hence, it cannot be the canonical conjugate of the corresponding action operator, or any other operator, which is local in the phase space.

In our earlier papers [9–12], we followed an algebraic analysis, similar to that used by Dirac [8], supplemented by heuristic guesses about the behaviour of the shifting operators at the points of singularity of the polarization. In particular, we assumed that $\boldsymbol{a}_{X_\vartheta}$ vanishes on the states concentrated on a set of limit points of $\mathrm{e}^{t\,X_\vartheta}(p)$ as $t \to \infty$. In the present paper, we derive shifting operators in the framework of geometric quantization, and extend our result to cases with a variable rank polarization.

The second stage in geometric quantization consists of the choice of a polarization, which is an involutive complex Lagrangian distribution F on the phase space. Suppose that P is the cotangent bundle space of the configuration space. In this case, the choice of F containing the vertical directions, leads the quantum mechanics of Schrödinger. If F leads to complex analytic structure on P, we have the Bargmann–Fock theory. If F is spanned by the Hamiltonian vector fields of a completely integrable system, we have Bohr–Sommerfeld theory. Each of these theories have specific structure, which is helpful in formulation and solving problems. In the following, we restrict our investigation to the Bohr–Sommerfeld theory in order to emphasize its membership in the class of quantum theories corresponding to different polarization.

A common problem in arising in quantum theories is occurrence of singularities. Usually, one studies the geometric structure of the theory in the language of differential geometry of smooth manifolds, and then investigates the structure of singularities separately. The theory of differential spaces, introduced by Sikorski [13,14], is a powerful tool in the study of the geometry of spaces with singularities [15]. The main singularity encountered here corresponds to the fact that the polarization F spanned by the Hamiltonian vector fields of a completely integrable system does not have constant rank. This singularity is so well known that we do not have to use the language of differential spaces to get results. It should be noted that the results in [9,11] rely on the theory of differential spaces.

In conclusion, it should be mentioned that the scientists, who used visual presentation of the Bohr–Sommerfeld spectra in terms of dots on the space of the action variables, are familiar with handling shifting operators. The line segments joining two dots corresponding to quantum states represent the shifting operators between these states.

To make the paper more accessible to the reader, we have provided an introductory section with a comprehensive review of geometric quantization. Experts may omit this section and proceed directly to the next section on Bohr–Sommerfeld theory.

2. Elements of Geometric Quantization

Let (P, ω) be a symplectic manifold. Geometric quantization can be divided into three steps: prequantization, polarization, and unitarization.

2.1. Principal Line Bundles with a Connection

We begin with a brief review of connections on complex line bundles.

Let $\mathbb{C}^{\times}$ denote the multiplicative group of nonzero complex numbers. Its Lie algebra $\mathfrak{c}^{\times}$ is isomorphic to the abelian Lie algebra $\mathbb{C}$ of complex numbers. Different choices of the isomorphism $\iota : \mathbb{C} \to \mathfrak{c}^{\times}$ lead to different factors in various expressions. Here, to each $c \in \mathbb{C}$ we associate the 1-parameter subgroup $t \mapsto \mathrm{e}^{2\pi i\, tc}$ of $\mathbb{C}^{\times}$. In other words, we take

$$\iota : \mathbb{C} \to \mathfrak{c}^{\times} : c \mapsto \iota(c) = 2\pi i\, c. \tag{6}$$

The prequantization structure for (P, ω) consists of a principal $\mathbb{C}^{\times}$ bundle $\pi^{\times} : L^{\times} \to P$ and a $\mathfrak{c}^{\times}$-valued $\mathbb{C}^{\times}$-invariant connection 1-form β satisfying

$$\mathrm{d}\beta = (\pi^{\times})^{*}(-\tfrac{1}{h}\,\omega), \tag{7}$$

where h is Planck's constant. The *prequantization condition* requires that the cohomology class $[-\frac{1}{h}\,\omega]$ is integral, that is, it lies in $\mathrm{H}^2(P, \mathbb{Z})$, otherwise the $\mathbb{C}^{\times}$ principal bundle $\pi^{\times} : L^{\times} \to P$ would not exist.

Let Y_c be the vector field on $L^{\times}$ generating the action of $\mathrm{e}^{2\pi i\, tc}$ on $L^{\times}$. In other words, the 1-parameter group e^{tY_c} of diffeomorphisms of $L^{\times}$ generated by Y_c is

$$\mathrm{e}^{tY_c} : L^{\times} \to L^{\times} : \ell^{\times} \to \ell^{\times}\mathrm{e}^{2\pi i\, tc}. \tag{8}$$

The connection 1-form β is normalized by the requirement

$$\langle \beta | Y_c \rangle = c. \tag{9}$$

For each $c \neq 0$, the vector field, Y_c spans the vertical distribution $\mathrm{ver}\, TL^{\times}$ tangent to the fibers of $\pi^{\times} : L^{\times} \to P$. The horizontal distribution $\mathrm{hor}\, TL^{\times}$ on $L^{\times}$ is the kernel of the connection 1-form β, that is,

$$\mathrm{hor}\, TL^{\times} = \ker \beta. \tag{10}$$

The vertical and horizontal distributions of $L^{\times}$ give rise to the direct sum $TL^{\times} = \mathrm{ver}\, TL^{\times} \oplus \mathrm{hor}\, TL^{\times}$, which is used to decompose any vector field Z on $L^{\times}$ into its vertical and horizontal components, $Z = \mathrm{ver}\, Z + \mathrm{hor}\, Z$. Here, the vertical component $\mathrm{ver}\, Z$ has range in $\mathrm{ver}\, TL^{\times}$ and the horizontal component has range in $\mathrm{hor}\, TL^{\times}$.

If X is a vector field on P, the unique horizontal vector field on $L^{\times}$, which is $\pi^{\times}$-related to X, is called the *horizontal lift* of X and is denoted by $\mathrm{lift}\, X$. In other words, $\mathrm{lift}\, X$ has range in the horizontal distribution $\mathrm{hor}\, TL^{\times}$ and satisfies

$$T\pi^{\times} \circ \mathrm{lift}\, X = X \circ \pi^{\times}. \tag{11}$$

Claim 1. *A vector field Z on $L^{\times}$ is invariant under the action of $\mathbb{C}^{\times}$ on $L^{\times}$ if and only if the horizontal component of Z is the horizontal lift of its projection X to P, that is,* $\mathrm{hor}\, Z = \mathrm{lift}\, X$ *and there is a smooth function $\kappa : P \to \mathbb{C}$ such that* $\mathrm{ver}\, Z = Y_{\kappa(p)}$ *on $L^{\times}_p = (\pi^{\times})^{-1}(p)$.*

Proof. Since the direct sum $TL^{\times} = \mathrm{ver}\, TL^{\times} \oplus \mathrm{hor}\, TL^{\times}$ is invariant under the $\mathbb{C}^{\times}$ action on $L^{\times}$, it follows that the vector field Z is invariant under the action of $\mathbb{C}^{\times}$ if and only if $\mathrm{hor}\, Z$ and $\mathrm{ver}\, Z$ are $\mathbb{C}^{\times}$-invariant. However, $\mathrm{hor}\, Z$ is $\mathbb{C}^{\times}$ invariant if $T\pi^{\times} \circ \mathrm{hor}\, Z = X \circ \pi^{\times}$ for some vector field X on P, that is, $\mathrm{hor}\, Z = \mathrm{lift}\, X$. However, this holds by definition. On the other hand, the vertical distribution $\mathrm{ver}\, TL^{\times}$ is spanned by the vector fields Y_c for $c \in \mathbb{C}$. Hence, $\mathrm{ver}\, Z$ is $\mathbb{C}^{\times}$-invariant if and only if for every fiber $L^{\times}_p$ the restriction of $\mathrm{ver}\, Z$ to $L^{\times}_p$ coincides with the restriction of Y_c to $L^{\times}_p$ for some $c \in \mathbb{C}$, that is, there is a smooth complex valued function κ on P such that $c = \kappa(p)$. □

Let U be an open subset of P. A local smooth section $\tau : U \subseteq P \to L^{\times}$ of the bundle $\pi^{\times} : L^{\times} \to P$ gives rise to a diffeomorphism

$$\eta_{\tau} : L^{\times}_{|U} = \bigcup_{p \in U} \big((\pi^{\times})^{-1}(p) \big) \to U \times \mathbb{C}^{\times} : \ell^{\times} \mapsto (\pi^{\times}(\ell^{\times}), b) = (p, b),$$

where $b \in \mathbb{C}^{\times}$ is the unique complex number such that $\ell^{\times} = \tau(p) b$. In the general theory of principal bundles the structure group of the principal bundle acts on the right. In the theory of $\mathbb{C}^{\times}$ principal bundles, elements of $L^{\times}$ are considered to be one-dimensional frames, which are usually written on the right, see [2]. The diffeomorphism η_{τ} is called a *trivialization* of $L^{\times}_{|U}$. It intertwines the action of $\mathbb{C}^{\times}$ on the principal bundle $L^{\times}$ with the right action of $\mathbb{C}^{\times}$ on $U \times \mathbb{C}^{\times}$, given by multiplication in $\mathbb{C}^{\times}$. If a local section $\sigma : U \to L$ of $\pi : L \to P$ is nowhere zero, then it determines a trivialization $\eta_{\tau} : L^{\times}_{|U} \to U \times \mathbb{C}^{\times}$. Conversely, a local smooth section τ such that η_{τ} is a trivialization of $L^{\times}$ may be considered as a local nowhere zero section of L.

In particular, for every $c \in \mathbb{C}$, which is identified with the Lie algebra $\mathfrak{c}^{\times}$ of $\mathbb{C}^{\times}$, Equation (7) gives $\mathrm{e}^{t\, Y_c} \circ \tau = \mathrm{e}^{2\pi i\, tc}\, \tau$. Differentiating with respect to t and then setting $t = 0$ gives

$$Y_c \circ \tau = 2\pi i\, c\, \tau. \tag{12}$$

For every smooth complex valued function $\kappa : P \to \mathbb{C}$, consider the vertical vector field Y_{κ} such that $Y_{\kappa}(\ell^{\times}) = Y_{\kappa(\pi^{\times}(\ell^{\times}))}$ for every $\ell^{\times} \in L^{\times}$. The vector field Y_{κ} is complete and the 1-parameter group of diffeomorphisms it generates is

$$\mathrm{e}^{t\, Y_{\kappa}} : L^{\times} \to L^{\times} : \ell^{\times} \mapsto \ell^{\times} \mathrm{e}^{2\pi i\, t \kappa(\pi^{\times}(\ell^{\times}))}.$$

For every smooth section τ of the bundle $\pi^\times$, we have $\mathrm{e}^{t\,Y_\kappa} \circ \tau = \mathrm{e}^{2\pi i\,t\kappa}\,\tau$ so that

$$Y_\kappa \circ \tau = 2\pi i\,\kappa\,\tau. \tag{13}$$

Let X be a vector field on P and let $\mathrm{lift}\,X$ be its horizontal lift to $L^\times$. The local 1-parameter group $\mathrm{e}^{t\,\mathrm{lift}\,X}$ of local diffeomorphisms of $L^\times$ generated by $\mathrm{lift}\,X$ commutes with the action of $\mathbb{C}^\times$ on $L^\times$. For every $\ell^\times$, $\mathrm{e}^{t\,\mathrm{lift}\,X}(\ell^\times)$ is called *parallel transport* of $\ell^\times$ along the integral curve of X starting at $p = \pi^\times(\ell^\times)$. For every $p \in P$ the map $\mathrm{e}^{t\,\mathrm{lift}\,X}$ sends the fiber $L^\times_p$ to the fiber $L_{\mathrm{e}^{tX}(p)}$.

There are several equivalent definitions of covariant derivative of a smooth section of the bundle $\pi^\times$ in the direction of a vector field X on P. We use the following one. The *covariant derivative* of the smooth section τ of the bundle $\pi^\times : L^\times \to P$ in the direction X is

$$\nabla_X \tau = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\mathrm{e}^{t\,\mathrm{lift}\,X})^*\tau. \tag{14}$$

Claim 2. *The covariant derivative of a smooth local section of the bundle $\pi^\times : L^\times \to P$ in the direction X is given by*

$$\nabla_X \tau = 2\pi i \langle \tau^*\beta | X\rangle\,\tau. \tag{15}$$

Proof. For every $p \in P$, we have

$$\begin{aligned}
\nabla_X \tau(p) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\mathrm{e}^{t\,\mathrm{lift}\,X})^*\tau(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\mathrm{e}^{-t\,\mathrm{lift}\,X} \circ \tau \circ \mathrm{e}^{t\,X})(p) \\
&= -\mathrm{lift}\,X(\tau(p)) + T\tau(X(p)) \\
&= -\mathrm{lift}\,X(\tau(p)) + \mathrm{hor}\,(T\tau)X(p) + \mathrm{ver}\,(T\tau)X(p) \\
&= \mathrm{ver}\,(T\tau)X(p).
\end{aligned}$$

The definition of the connection 1-form β and Equation (13) yield

$$\mathrm{ver}\,(T\tau(X(p)) = Y_{\langle \beta | T\tau \circ X\rangle}(\tau(p)) = 2\pi i\,\langle \beta | T\tau \circ X\rangle \tau(p).$$

Hence,

$$\nabla_X \tau = 2\pi i\,\langle \beta | T\tau \circ X\rangle\,\tau, \tag{16}$$

which is equivalent to Equation (15). □

2.2. Associated Line Bundles

The complex line bundle $\pi : L \to P$ associated to the $\mathbb{C}^\times$ principal bundle $\pi^\times : L^\times \to P$ is defined in terms of the action of $\mathbb{C}^\times$ on $(L^\times \times \mathbb{C})$ given by

$$\Phi : \mathbb{C}^\times \times (L^\times \times \mathbb{C}) \to L^\times \times \mathbb{C} : \big(b, (\ell^\times, c)\big) \mapsto (\ell^\times b, b^{-1}c). \tag{17}$$

Since the action Φ is free and proper, its orbit space $L = (L^\times \times \mathbb{C})/\mathbb{C}^\times$ is a smooth manifold. A point $\ell \in L$ is the $\mathbb{C}^\times$ orbit $[(\ell^\times, c)]$ through $(\ell^\times, c) \in (L^\times \times \mathbb{C})$, namely,

$$\ell = [(\ell^\times, c)] = \{(\ell^\times b, b^{-1}c) \in L^\times \times \mathbb{C} \,\big|\, b \in \mathbb{C}^\times\}. \tag{18}$$

The left action of $\mathbb{C}^\times$ on $\mathbb{C}$ gives rise to the left action

$$\widehat{\Phi} : \mathbb{C}^\times \times L \to L : \big(a, [(\ell^\times, c)]\big) \mapsto [(\ell^\times, ac)], \tag{19}$$

which is well defined because $[(\ell^\times, ac)] = [(\ell^\times b, b^{-1}(ac))] = [(\ell^\times b, a(b^{-1}c))]$ for every $\ell^\times \in L^\times$, every $a, b \in \mathbb{C}^\times$ and every $c \in \mathbb{C}$. The projection map $\pi^\times : L^\times \to P$ induces the projection map

$$\pi : L \to L/\mathbb{C}^\times = P : \ell = [(\ell^\times, c)] \mapsto \pi(\ell) = \pi([(\ell^\times, c)]) = \pi^\times(\ell^\times).$$

Claim 3. *A local smooth section $\sigma : U \to L$ of the complex line bundle $\pi : L \to P$ corresponds to a unique mapping $\sigma^\sharp : L^\times_{|U} \to \mathbb{C}$ such that for every $p \in U$ and every $\ell^\times \in L^\times_p$*

$$\sigma(p) = [(\ell^\times, \sigma^\sharp(\ell^\times))], \tag{20}$$

which is $\mathbb{C}^\times$-equivariant, that is, $\sigma^\sharp(\ell^\times b) = b^{-1}\sigma^\sharp(\ell^\times)$.

Proof. Given $p \in U$ there exists $(\ell^\times, c) \in L^\times \times \mathbb{C}$ such that $\sigma(p) = [(\ell^\times, c)]$. Since the action of $\mathbb{C}^\times$ on $L^\times_p$ is free and transitive, it follows that the $\mathbb{C}^\times$ orbit $\{(\ell^\times b, b^{-1}c) \in L^\times_p \times \mathbb{C} \big| \, b \in \mathbb{C}^\times\}$ is the graph of a smooth function from $L^\times_p$ to $\mathbb{C}$, which we denote by $\sigma^\sharp_p$. In particular, $c = \sigma^\sharp_p(\ell^\times)$ so that $\sigma(p) = [(\ell^\times, c)] = [(\ell^\times, \sigma^\sharp_p(\ell^\times))]$. As p varies over U we get a map

$$\sigma^\sharp : L^\times_{|U} \to \mathbb{C} : \ell^\times \mapsto \sigma^\sharp(\ell^\times) = \sigma^\sharp_{\pi^\times(\ell^\times)}(\ell^\times),$$

which satisfies Equation (20). For every $b \in \mathbb{C}^\times$, Equations (18) and (20) imply that

$$\sigma(p) = [(\ell^\times, \sigma^\sharp(\ell^\times))] = [(\ell^\times b, b^{-1}\sigma^\sharp(\ell^\times))] = [(\ell^\times b, \sigma^\sharp(\ell^\times b))].$$

Hence, $\sigma^\sharp(\ell^\times b) = b^{-1}\sigma^\sharp(\ell^\times)$. Thus, the function $\sigma^\sharp$ is $\mathbb{C}^\times$-equivariant. $\square$

If $\tau : U \to L^\times$ is a local smooth section of the bundle $\pi^\times : L^\times \to P$, then for every $p \in P$ we have $\sigma(p) = [(\tau(p), \sigma^\sharp(\tau(p)))]$ or $\sigma = [(\tau, \sigma^\sharp \circ \tau)]$ suppressing the argument p. The function $\psi = \sigma^\sharp \circ \tau : U \to \mathbb{C}$ is the coordinate representation of the section τ in terms of the trivialization $\eta_\tau : L^\times_{|U} \to U \times \mathbb{C}$.

Let Z be a $\mathbb{C}^\times$-invariant vector field on $L^\times$. Then, Z is $\pi^\times$-related to a vector field X on P, that is, $T\pi^\times \circ Z = X \circ \pi^\times$. We denote by e^{tX} and e^{tZ} the local 1-parameter groups of local diffeomorphisms of P and $L^\times$ generated by X and Z, respectively. Because the vector fields X and Z are $\pi^\times$-related, we obtain $\pi^\times \circ \mathrm{e}^{tZ} = \mathrm{e}^{tX} \circ \pi^\times$. In other words, the flow e^{tZ} of Z covers the flow e^{tX} of X. The local group e^{tZ} of automorphisms of the principal bundle $L^\times$ act on the associated line bundle L by

$$\widehat{\mathrm{e}}^{tZ} : L \to L : \ell = [(\ell^\times, c)] \mapsto [(\mathrm{e}^{tZ}(\ell^\times), c)], \tag{21}$$

which holds for all $\ell = [(\ell^\times, c)]$ for which $\mathrm{e}^{tZ}(\ell^\times)$ is defined.

Lemma 1. *The map $\widehat{\mathrm{e}}^{tZ}$ is a local 1-parameter group of local automorphisms of the line bundle L, which covers the local 1-parameter group e^{tX} of the vector field X on P.*

Proof. We compute. For $\ell = [(\ell^\times, c)] \in L$ we have

$$\begin{aligned}\widehat{\mathrm{e}}^{(t+s)Z}(\ell) &= \widehat{\mathrm{e}}^{(t+s)Z}([(\ell^\times, c)]) = [(\mathrm{e}^{(t+s)Z}(\ell^\times), c)] = [(\mathrm{e}^{tZ}(\mathrm{e}^{sZ}(\ell^\times)), c)] \\ &= \widehat{\mathrm{e}}^{tZ}([(\mathrm{e}^{sZ}(\ell^\times, c)] = \widehat{\mathrm{e}}^{tZ} \circ \widehat{\mathrm{e}}^{sZ}([(\ell^\times, c)]) = \widehat{\mathrm{e}}^{tZ} \circ \widehat{\mathrm{e}}^{sZ}(\ell).\end{aligned}$$

Hence, $\widehat{\mathrm{e}}^{tZ}$ is a local 1-parameter group of local diffeomorphisms. Moreover,

$$\pi \circ \widehat{\mathrm{e}}^{tZ}(\ell) = \pi([(\mathrm{e}^{tZ}(\ell^\times), c)]) = \pi^\times(\mathrm{e}^{tZ}(\ell^\times)) = \mathrm{e}^{tX}(\pi^\times(\ell^\times));$$

while

$$\mathrm{e}^{tX} \circ \pi(\ell) = \mathrm{e}^{tX}(\pi([(\ell^\times, c)])) = \mathrm{e}^{tX}(\pi^\times(\ell^\times)).$$

This shows that e^{tZ} covers e^{tX}. Finally, for every $\ell = [(\ell^{\times}, c)] \in L$ and every $b \in \mathbb{C}^{\times}$

$$\widehat{\Phi}_b(\widehat{\mathrm{e}}^{tZ}(\ell)) = \widehat{\Phi}_b([(\mathrm{e}^{tZ}(\ell^{\times}), c)]) = [(\Phi_b(\mathrm{e}^{tZ}(\ell^{\times})), c)] = [(\mathrm{e}^{tZ}(\Phi_b(\ell^{\times})), c)],$$

since Z is a $\mathbb{C}^{\times}$-invariant vector field on $L^{\times}$. Therefore,

$$\widehat{\Phi}_b(\widehat{\mathrm{e}}^{tZ}(\ell)) = \widehat{\mathrm{e}}^{tZ}([(\Phi_b(\ell^{\times}), c)]) = \widehat{\mathrm{e}}^{tZ} \circ \widehat{\Phi}_b([(\ell^{\times}, c)]) = \widehat{\mathrm{e}}^{tZ} \circ \widehat{\Phi}_b(\ell).$$

This shows that $\widehat{\mathrm{e}}^{tZ}$ is a local group of automorphisms of the line bundle $\pi : L \to P$. $\square$

If $Z = \mathrm{hor}\, X$, then $\mathrm{e}^{t\,\mathrm{lift}\,X}(\ell^{\times})$ is parallel transport of $\ell^{\times}$ along the integral curve $\mathrm{e}^{tX}(p)$ of X starting at $p = \pi^{\times}(\ell^{\times})$. Similarly, if $\ell = [(\ell^{\times}, c)] \in L$, then

$$\widehat{\mathrm{e}}^{t\,\mathrm{lift}\,X}(\ell) = [(\mathrm{e}^{t\,\mathrm{lift}\,X}(\ell^{\times}), c)] \tag{22}$$

is parallel transport of $\ell \in L$ along the integral curve $\mathrm{e}^{tX}(p)$ of X starting at p. The covariant derivative of a section σ of the bundle $\pi : L \to P$ in the direction of the vector field X on P is

$$\nabla_X \sigma = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\widehat{\mathrm{e}}^{t\,\mathrm{lift}\,X})^* \sigma = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\widehat{\mathrm{e}}^{-t\,\mathrm{lift}\,X} \circ \sigma \circ \mathrm{e}^{tX}). \tag{23}$$

Since $\widehat{\mathrm{e}}^{-t\,\mathrm{lift}\,X}$ maps $\pi^{-1}(\mathrm{e}^{tX})$ onto $\pi^{-1}(p)$, Equations (22) and (23) are consistent with the definitions in [5].

Theorem 1. *Let σ be a smooth section of the complex line bundle $\pi : L \to P$ and let X be a vector field on P. For every $\ell^{\times} \in L^{\times}$*

$$\nabla_X \sigma(\pi^{\times}(\ell^{\times})) = [(\ell^{\times}, L_{\mathrm{lift}\,X}(\sigma^{\sharp}(\ell^{\times})))]. \tag{24}$$

Here, L_X is the Lie derivative with respect to the vector field X.

Proof. Let $p = \pi^{\times}(\ell^{\times})$. Equation (23) yields

$$\nabla_X \sigma(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\widehat{\mathrm{e}}^{-t\,\mathrm{lift}\,X} \circ \sigma \circ \mathrm{e}^{tX}(\sigma(p))).$$

Recall that $\sigma(p) = [(\ell^{\times}, \sigma^{\sharp}(\ell^{\times}))]$. Hence,

$$\sigma(\mathrm{e}^{tX}(p)) = [(\mathrm{e}^{t\,\mathrm{lift}\,X}(\ell^{\times}), \sigma^{\sharp}(\mathrm{e}^{t\,\mathrm{lift}\,X}(\ell^{\times})))].$$

By Equation (22),

$$\begin{aligned}
\widehat{\mathrm{e}}^{-t\,\mathrm{lift}\,X}(\sigma(\mathrm{e}^{tX}(p))) &= \widehat{\mathrm{e}}^{-t\,\mathrm{lift}\,X}[(\mathrm{e}^{t\,\mathrm{lift}\,X}(\ell^{\times}), \sigma^{\sharp}(\mathrm{e}^{t\,\mathrm{lift}\,X}(\ell^{\times})))] \\
&= [(\mathrm{e}^{-t\,\mathrm{lift}\,X}(\mathrm{e}^{t\,\mathrm{lift}\,X})(\ell^{\times}), \sigma^{\sharp}(\mathrm{e}^{t\,\mathrm{lift}\,X}(\ell^{\times})))] = [(\ell^{\times}, \sigma^{\sharp}(\mathrm{e}^{t\,\mathrm{lift}\,X}(\ell^{\times})))]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla_X \sigma(p) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \widehat{\mathrm{e}}^{-t\,\mathrm{lift}\,X}(\sigma(\mathrm{e}^{tX}(p))) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} [(\ell^{\times}, \sigma^{\sharp}(\mathrm{e}^{t\,\mathrm{lift}\,X}(\ell^{\times})))] \\
&= [(\ell^{\times}, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma^{\sharp}(\mathrm{e}^{t\,\mathrm{lift}\,X}(\ell^{\times})))] = [(\ell^{\times}, L_{\mathrm{lift}\,X}(\sigma^{\sharp}(\ell^{\times})))].
\end{aligned} \tag{25}$$

$\square$

2.3. Prequantization

Let $\pi : L \to P$ be the complex line bundle associated to the $\mathbb{C}^\times$ principal bundle $\pi^\times : L^\times \to P$. The space $S^\infty(L)$ of smooth sections of $\pi : L \to P$ is the representation space of prequantization. Since $\mathbb{C}^\times \subseteq \mathbb{C}$, we may identify $L^\times$ with the complement of the zero section in L. With this identification, if $\sigma : U \to L$ is a local smooth section of $\pi : L \to P$, which is nowhere vanishing, then it is a section of the bundle $\pi^\times_{|L^\times_{|U}} : L^\times_{|U} \to U$.

Theorem 2. *A $\mathbb{C}^\times$-invariant vector field Z on $L^\times$ preserves the connection 1-form β, on $L^\times$ if and only if there is a function $f \in C^\infty(P)$ such that*

$$Z = \mathrm{lift}\, X_f - Y_{f/\mathrm{h}}, \tag{26}$$

where h *is Planck's constant.*

Proof. The vector field Z on $L^\times$ preserves the connection 1-form, that is, $L_Z\beta = 0$, which is equivalent to

$$Z \mathbin{\lrcorner} \mathrm{d}\beta = -\mathrm{d}(Z \mathbin{\lrcorner} \beta). \tag{27}$$

Since $\mathrm{hor}\, Z \mathbin{\lrcorner} \beta = 0$, it follows that $Z \mathbin{\lrcorner} \beta = \mathrm{ver}\, Z \mathbin{\lrcorner} \beta$. The $\mathbb{C}^\times$-invariance of Z and β imply the $\mathbb{C}^\times$-invariance of $\mathrm{ver}\, Z \mathbin{\lrcorner} \beta$. Hence, $\mathrm{ver}\, Z \mathbin{\lrcorner} \beta$ pushes forward to a function $\pi_*(\mathrm{ver}\, Z \mathbin{\lrcorner} \beta) \in C^\infty(P)$. Thus, the right hand side of Equation (27) reads

$$-\mathrm{d}(Z \mathbin{\lrcorner} \beta) = -(\pi^\times)^*\big(\mathrm{d}(\pi^\times_*(\mathrm{ver}\, Z \mathbin{\lrcorner} \beta))\big). \tag{28}$$

By definition $Y_c \mathbin{\lrcorner} \beta = c$, for every $c \in \mathfrak{c}$. This implies

$$Y_c \mathbin{\lrcorner} \mathrm{d}\beta = L_{Y_c}\beta - \mathrm{d}(Y_c \mathbin{\lrcorner} \beta) = 0.$$

Thus, the left hand side of Equation (27) reads

$$Z \mathbin{\lrcorner} \mathrm{d}\beta = \mathrm{hor}\, Z \mathbin{\lrcorner} \mathrm{d}\beta. \tag{29}$$

The quantization condition (7) together with (28) and (29) allow us to rewrite Equation (27) in the form

$$\mathrm{lift}\, X \mathbin{\lrcorner} \big((\pi^\times)^*(-\tfrac{1}{\mathrm{h}}\omega)\big) = (\pi^\times)^*\big(\mathrm{d}(\pi_*(\mathrm{ver}\, Z \mathbin{\lrcorner} \beta))\big). \tag{30}$$

Equation (30) shows that X is the Hamiltonian vector field of the smooth function

$$f = -\mathrm{h}\, \pi_*(\mathrm{ver}\, Z \mathbin{\lrcorner} \beta)) \tag{31}$$

on P. We write $X = X_f$. This implies that

$$\mathrm{hor}\, Z = \mathrm{lift}\, X_f. \tag{32}$$

We still have to determine the vertical component $\mathrm{ver}\, Z$ of the vector field Z. For each $\ell^\times \in L^\times$ there is a $c \in \mathfrak{c}$ such that $\mathrm{ver}\, Z = Y_c$. Since Y_c is tangent to the fibers of the $\mathbb{C}^\times$ principal bundle $\pi^\times : L^\times \to P$, the element c of $\mathfrak{c}$ depends only on $\pi^\times(\ell^\times) = p \in P$. Therefore,

$$-(\pi^\times_*(\mathrm{ver}\, Z \mathbin{\lrcorner} \beta))(\ell^\times) = -(\pi^\times_*(Y_{c(p)} \mathbin{\lrcorner} \beta))(\ell^\times) = -c(p) = f(p)/\mathrm{h}$$

by Equation (31). In other words, for every point $\ell^\times \in L^\times$ we have $\mathrm{ver}\, Z(\ell^\times) = -Y_{f(p)/\mathrm{h}}(\ell^\times)$, where $p = \pi^\times(\ell^\times)$. Thus, we have shown that

$$Z_f = Z = \mathrm{lift}\, X_f - Y_{f/\mathrm{h}}. \tag{33}$$

Reversing the steps in the above argument proves the converse. □

To each $f \in C^\infty(P)$, we associate a prequantization operator

$$\mathcal{P}_f : S^\infty(L) \to S^\infty(L) : \sigma \mapsto \mathcal{P}_f\sigma = i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\widehat{\mathrm{e}}^{t\,Z_f})_*\sigma, \tag{34}$$

where $\widehat{\mathrm{e}}^{t\,Z_f}$ is the action of $\mathrm{e}^{t\,Z_f} : L^\times \to L^\times$ on L, see (22). Note that the definition of covariant derivative in Equation (23) is defined in terms of the pull back $(\widehat{\mathrm{e}}^{t\,Z_f})^*\sigma$ of the section σ by $\widehat{\mathrm{e}}^{t\,Z_f}$, while the prequantization operator in (34) is defined using the push forward $(\widehat{\mathrm{e}}^{t\,Z_f})_*\sigma$ of σ by $\widehat{\mathrm{e}}^{t\,Z_f}$.

Theorem 3. *For every $f \in C^\infty(P)$ and each $\sigma \in S^\infty(L)$*

$$\mathcal{P}_f\sigma = (-i\hbar\nabla_{X_f} + f)\sigma. \tag{35}$$

Proof. Since the horizontal distribution on $L^\times$ is $\mathbb{C}^\times$-invariant and the vector field Y_c generates multiplication on each fiber of $\pi^\times$ by $\mathrm{e}^{2\pi i\,c}$, it follows that $\mathrm{e}^{t\,\mathrm{lift}\,X_f}\,\mathrm{e}^{t\,Y_{f/\mathrm{h}}} = \mathrm{e}^{t\,Y_{f/\mathrm{h}}}\,\mathrm{e}^{t\,\mathrm{lift}\,X_f}$. Since f is constant along integral curves of X_f,

$$\begin{aligned}\mathrm{e}^{t\,Z_f} &= \mathrm{e}^{t(\mathrm{lift}\,X_f - Y_{f/\mathrm{h}})} = \mathrm{e}^{t\,\mathrm{lift}\,X_f}\,\mathrm{e}^{-t\,Y_{f/\mathrm{h}}} \\ &= \mathrm{e}^{t\,\mathrm{lift}\,X_f}\mathrm{e}^{-2\pi i t\,f/\mathrm{h}} = \mathrm{e}^{-2\pi i\,tf/\mathrm{h}}\mathrm{e}^{t\,\mathrm{lift}\,X_f},\end{aligned} \tag{36}$$

and

$$\begin{aligned}\mathcal{P}_f\sigma &= i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\widehat{\mathrm{e}}^{t\,Z_f})_*\sigma = i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\mathrm{e}^{t\,\mathrm{lift}\,X_f}\,\mathrm{e}^{t\,Y_{f/\mathrm{h}}})_*\sigma \\ &= i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\widehat{\mathrm{e}}^{t\,\mathrm{lift}\,X_f})_*\sigma + i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\widehat{\mathrm{e}}^{t\,Y_{-f/\mathrm{h}}})_*\sigma.\end{aligned} \tag{37}$$

Since $(\widehat{\mathrm{e}}^{t\,\mathrm{lift}\,X_f})_*\sigma = (\widehat{\mathrm{e}}^{-t\,\mathrm{lift}\,X_f})^*\sigma$, Equation (23) gives

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\widehat{\mathrm{e}}^{t\,\mathrm{lift}\,X_f})_*\sigma = i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\widehat{\mathrm{e}}^{-t\,\mathrm{lift}\,X_f})^*\sigma = -i\hbar\nabla_{X_f}\sigma. \tag{38}$$

Since $\pi \circ \widehat{\mathrm{e}}^{t\,Y_{-f/\mathrm{h}}} = \pi \circ \mathrm{id}_P$, where id_P is the identity map on P, it follows that

$$(\widehat{\mathrm{e}}^{tY_{-f/\mathrm{h}}})_*\sigma = \widehat{\mathrm{e}}^{t\,Y_{-f/\mathrm{h}}} \circ \sigma \circ \mathrm{id}_P = \widehat{\mathrm{e}}^{t\,Y_{-f/\mathrm{h}}} \circ \sigma.$$

Let $\tau : U \subseteq P \to L^\times$ be a smooth local section of $\pi^\times : L^\times \to P$, then $\sigma = [(\tau, \sigma^\sharp \circ \tau)]$. Thus, for every $p \in P$

$$\begin{aligned}\widehat{\mathrm{e}}^{-t\,Y_{f/\mathrm{h}}} \circ \sigma(p) &= \widehat{\mathrm{e}}^{-t\,Y_{f/\mathrm{h}}}[(\tau(p), \sigma^\sharp(\tau(p)))] = [(\mathrm{e}^{-t\,Y_{f/\mathrm{h}}}(\tau(p)), \sigma^\sharp(\tau(p)))] \\ &= [(\tau(p)\mathrm{e}^{-2\pi i\,tf(p)/\mathrm{h}}, \sigma^\sharp(\tau(p)))] = [(\tau(p), \mathrm{e}^{-2\pi i\,tf(p)/\mathrm{h}}\sigma^\sharp(\tau(p)))],\end{aligned}$$

since $[(\ell^\times b, c)] = [(\ell^\times b, b^{-1}(bc))] = [(\ell^\times, bc)]$ for every $\ell^\times \in L^\times$, $b \in \mathbb{C}^\times$ and $c \in \mathbb{C}$. It follows that

$$\widehat{\mathrm{e}}^{-t\,Y_{f/\mathrm{h}}} \circ \sigma(p) = [(\tau(p), \mathrm{e}^{-2\pi i\,tf(p)/\mathrm{h}}\sigma^\sharp(\tau(p)))] = \mathrm{e}^{-2\pi i\,tf(p)/\mathrm{h}}\sigma(p). \tag{39}$$

Therefore,

$$\begin{aligned}(\widehat{\mathrm{e}}^{t\,Z_f})_*\sigma &= (\widehat{\mathrm{e}}^{t\,(\mathrm{lift}\,X_f - Y_{f/\mathrm{h}})})_*\sigma \\ &= (\widehat{\mathrm{e}}^{t\,\mathrm{lift}\,X_f}\widehat{\mathrm{e}}^{-t\,Y_{f/\mathrm{h}}})_*\sigma = \mathrm{e}^{-2\pi i\,tf(p)/\mathrm{h}}\big(\widehat{\mathrm{e}}^{t\,\mathrm{lift}\,X_f}\big)_*\sigma.\end{aligned} \tag{40}$$

Since
$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \widehat{\mathrm{e}}_*^{\,tY_{-f/\mathrm{h}}}\sigma = i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\mathrm{e}^{-2\pi i t f/\mathrm{h}}\sigma) = i\hbar(-2\pi i f/\mathrm{h})\sigma = f\sigma \tag{41}$$
Equations (37), (38) and (41) imply Equation (35). □

A Hermitian scalar product $\langle\ |\ \rangle$ on the fibers of L that is invariant under parallel transport gives rise to a Hermitian scalar product on the space $S^\infty(L)$ of smooth sections of L. Since the dimension of (P,ω) is $2k$, the scalar product of the smooth sections σ_1 and σ_2 of L is

$$(\sigma_1|\sigma_2) = \int_P \langle\sigma_1|\sigma_2\rangle\,\omega^k. \tag{42}$$

The completion of the space $S_c^\infty(L)$ of smooth sections of L with compact support with respect to the norm $\|\sigma\| = \sqrt{(\sigma|\sigma)}$ is the Hilbert space $\mathfrak{H}$ of the prequantization representation.

Claim 4. *The prequantization operator $\mathcal{P}_f$ is a symmetric operator on the Hilbert space $\mathfrak{H}$ of square integrable sections of the line bundle $\pi : L \to P$ and satisfies Dirac's quantization commutation relations*

$$[\mathcal{P}_f, \mathcal{P}_g] = i\hbar\mathcal{P}_{\{f,g\}}. \tag{43}$$

for every $f, g \in C^\infty(P)$. Moreover, the operator $\mathcal{P}_f$ is self adjoint if the vector field X_f on (P,ω) is complete.

Proof. We only verify that the commutation relations (43) hold. Let $f, g \in C^\infty(P)$ and let $\sigma \in S^\infty(L)$. We compute.

$$\begin{aligned}
[\nabla_{X_f} - \tfrac{i}{\hbar}f, \nabla_{X_g} - \tfrac{i}{\hbar}g]\sigma &= [\nabla_{X_f}, \nabla_{X_g}]\sigma + \tfrac{i}{\hbar}(\nabla_{X_f}(g\sigma) - g\nabla_{X_f}\sigma) \\
&\quad - \tfrac{i}{\hbar}(\nabla_{X_g}(f\sigma) - f\nabla_{X_g}\sigma) \\
&= ([\nabla_{X_f}, \nabla_{X_g}] + \tfrac{i}{\hbar}(L_{X_f}g - L_{X_g}f))\sigma
\end{aligned}$$

The quantization condition

$$[\nabla_{X_f}, \nabla_{X_g}] - \nabla_{[X_f,X_g]} = -\tfrac{i}{\hbar}\omega(X_f, X_g)$$

yields
$$[\nabla_{X_f} - \tfrac{i}{\hbar}f, \nabla_{X_g} - \tfrac{i}{\hbar}g] = \nabla_{[X_f,X_g]} - \tfrac{i}{\hbar}\omega(X_f, X_g) + \tfrac{i}{\hbar}(L_{X_f}g - L_{X_g}f)$$
However, $\{f,g\} = L_{X_g}f = -\omega(X_f, X_g)$. Thus, $L_{X_f}g - L_{X_g}f = \{g,f\} - \{f,g\} = -2\{f,g\}$. Since $X_g \lrcorner\ \omega = -\mathrm{d}g$, it follows that

$$[X_f, X_g] \lrcorner\ \omega = L_{X_f}X_g \lrcorner\ \omega = -L_{X_f}\mathrm{d}g = -\mathrm{d}L_{X_f}g = \mathrm{d}\{f,g\}.$$

Consequently, $[X_f, X_g] = -X_{\{f,g\}}$. Thus,

$$[\nabla_{X_f} - \tfrac{i}{\hbar}f, \nabla_{X_g} - \tfrac{i}{\hbar}g] = \nabla_{X_{\{f,g\}}} - \tfrac{i}{\hbar}\{f,g\}.$$

□

2.4. Polarization

Prequantization is only the first step of geometric quantization. The prequantization operators do not satisfy Heisenberg's uncertainty relations. In the case of Lie groups, the prequantization representation fails to be irreducible. These apparently unrelated shortcomings lead to the next step of geometric quantization: the introduction of a polarization.

A complex distribution $F \subseteq T^{\mathbb{C}}P = \mathbb{C} \otimes TP$ on a symplectic manifold (P, ω) is *Lagrangian* if for every $p \in P$, the restriction of the symplectic form ω_p to the subspace $F_p \subseteq T^{\mathbb{C}}_p P$ vanishes identically, and $\mathrm{rank}_{\mathbb{C}} F = \frac{1}{2} \dim P$. If F is a complex distribution on P, let $\overline{F}$ be its complex conjugate. Let

$$D = F \cap \overline{F} \cap T^{\mathbb{C}}P \ \text{ and } \ E = (F + \overline{F}) \cap T^{\mathbb{C}}P.$$

A *polarization* of (P, ω) is an involutive complex Lagrangian distribution F on P such that D and E are involutive distributions on P. Let $C^\infty(P)_F$ be the space of smooth complex valued functions of P that are constant along F, that is,

$$C^\infty(P)_F = \{ f \in C^\infty(P) \otimes P \,\big|\, \langle \mathrm{d}f | u \rangle = 0 \text{ for every } u \in F \}. \tag{44}$$

The polarization F is *strongly admissible* if the spaces P/D and P/E of integral manifolds of D and E, respectively, are smooth manifolds and the natural projection $P/D \to P/E$ is a submersion. A strongly admissible polarization F is locally spanned by Hamiltonian vector fields of functions in $C^\infty(P)_F$. A polarization F is *positive* if $i\,\omega(u, \overline{u}) \geq 0$ for every $u \in F$. A positive polarization F is *semi-definite* if $\omega(u, \overline{u}) = 0$ for $u \in F$ implies that $u \in D^{\mathbb{C}}$.

Let F be a strongly admissible polarization on (P, ω). The space $S^\infty_F(L)$ of smooth sections of L that are covariantly constant along F is the *quantum space of states* corresponding to the polarization F.

The space $C^\infty_F(P)$ of smooth functions on P, whose Hamiltonian vector field preserves the polarization F, is a Poisson subalgebra of $C^\infty(P)$. Quantization in terms of the polarization F leads to *quantization map* $\mathcal{Q}$, which is the restriction of the *prequantization map*

$$\mathcal{P} : C^\infty(P) \times S^\infty(L) \to S^\infty(L) : (f, \sigma) \mapsto \mathcal{P}_f \sigma = (-i\hbar \nabla_{X_f} + f)\sigma$$

to the domain $C^\infty_F(P) \times S^\infty_F(L) \subseteq C^\infty(P) \times S^\infty(L)$ and the codomain $S^\infty_F(L) \subseteq S^\infty(L)$. In other words,

$$\mathcal{Q} : C^\infty_F(P) \times S^\infty(L) \to S^\infty_F(L) : (f, \sigma) \mapsto \mathcal{Q}_f \sigma = (-i\hbar \nabla_{X_f} + f)\sigma. \tag{45}$$

Quantization in terms of positive strongly admissible polarizations such that $E \cap \overline{E} = \{0\}$ lead to unitary representations. For other types of polarizations, unitarity may require additional structure.

3. Bohr–Sommerfeld Theory

3.1. Historical Background

Consider the cotangent bundle T^*Q of a manifold Q. Let $\pi_Q : T^*Q \to Q$ be the cotangent bundle projection map. The Liouville 1-form α_Q on T^*Q is defined as follows. For each $q \in Q$, $p \in T^*_q Q$ and $u_p \in T_p(T^*Q)$,

$$\langle \alpha_Q | u_p \rangle = \langle p | T\pi_Q(u_p) \rangle. \tag{46}$$

The exterior derivative of α_Q is the canonical symplectic form $\mathrm{d}\alpha_Q$ on T^*Q.

Let $\dim Q = k$. A Hamiltonian system on $(T^*Q, \mathrm{d}\alpha_Q)$ with Hamiltonian H_0 is *completely integrable* if there exists a collection of $k-1$ functions $H_1, \ldots, H_{k-1} \in C^\infty(T^*Q)$, which are integrals of X_{H_0}, that is, $\{H_0, H_i\} = 0$ for $i = 1, \ldots, k-1$, such that $\{H_i, H_j\} = 0$ for $i, j = 1, \ldots, k-1$. Assume that the functions $H_0, \ldots, H_{k-1}$ are independent on a dense open subset of T^*Q. For each $p \in T^*Q$, let M_p be the orbit of the family of Hamiltonian vector fields $\{X_{H_0}, \ldots, X_{H_{k-1}}\}$ passing through p. This orbit is the largest connected immersed submanifold in T^*Q with tangent space $T_{p'}(M_p)$ equal to $\mathrm{span}_{\mathbb{R}}\{X_{H_0}(p'), \ldots, X_{H_{k-1}}(p')\}$. The integral curve $t \mapsto \mathrm{e}^{t\,X_{H_0}}(p)$ of X_{H_0} starting at p is contained in M_p. Hence, knowledge of the family $\{M_p \,\big|\, p \in T^*Q\}$ of orbits provides information on the evolution of the Hamiltonian system with Hamiltonian H_0.

Bohr–Sommerfeld theory, see [16,17], asserts that the quantum states of the completely integrable system $(H_0, \ldots, H_{k-1}, T^*Q, \mathrm{d}\alpha_Q)$ are concentrated on the orbits $M \in \{M_p \,\big|\, p \in T^*Q\}$, which satisfy the

Bohr–Sommerfeld Condition: *For every closed loop $\gamma : S^1 \to M \subseteq T^*Q$, there exists an integer n such that*

$$\oint \gamma^*(\alpha_Q) = n\, h, \tag{47}$$

where h is Planck's constant.

This theory applied to the bounded states of the relativistic hydrogen atom yields results that agree exactly with the experimental data [17]. Attempts to apply Bohr–Sommerfeld theory to the helium atom, which is not completely integrable, failed to provide useful results. In his 1925 paper [1], Heisenberg criticized Bohr–Sommerfeld theory for not providing transition operators between different states. At present, the Bohr–Sommerfeld theory is remembered by physicists only for its agreement with the quasi-classical limit of Schrödinger theory. Quantum chemists have never stopped using it to describe the spectra of molecules.

3.2. Geometric Quantization in a Toric Polarization

To interpret Bohr–Sommerfeld theory in terms of geometric quantization, we consider a set $P \subseteq T^*Q$ consisting of points $p \in T^*Q$ where $X_{H_0}(p), \ldots, X_{H_{k-1}}(p)$ are linearly independent and the orbit M_p of the family $\{X_{H_0}, \ldots, X_{H_{k-1}}\}$ of Hamiltonian vector fields on $(T^*Q, \mathrm{d}\alpha_{T^*Q})$ is diffeomorphic to the k torus $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$. We assume that P is a $2k$-dimensional smooth manifold and that the set $B = \{M_p \big| p \in P\}$ is a quotient manifold of P with smooth projection map $\rho : P \to B$. This implies that the symplectic form $\mathrm{d}\alpha_Q$ on T^*Q restricts to a symplectic form on P, which we denote by ω. Let D be the distribution on P spanned by the Hamiltonian vector fields $X_{H_0}, \ldots, X_{H_{k-1}}$. Since $\{H_i, H_j\} = 0$ for $i, j = 0, 1, \ldots, k-1$, it follows that D is an involutive Lagrangian distribution on (P, ω). Moreover, $F = D^{\mathbb{C}}$ is a strongly admissible *polarization* of (P, ω).

Since the symplectic form $\mathrm{d}\alpha_Q$ on T^*Q is exact, we may choose a trivial prequantization line bundle

$$\pi^\times : L^\times_{T^*Q} = \mathbb{C}^\times \times T^*Q \to T^*Q : \big(b, (q,p)\big) \mapsto (q,p)$$

with connection 1-form $\beta_Q = \frac{1}{2\pi i}\frac{\mathrm{d}b}{b} + \frac{1}{\mathrm{h}}\alpha_Q$. Let $L^\times$ be the restriction of $L^\times_{T^*Q}$ to P and let α be the 1-form on P, which is the restriction of α_Q to P, that is, $\alpha = \alpha_{Q|P}$. Then, $L^\times = \mathbb{C}^\times \times P$ is a principal $\mathbb{C}^\times$ bundle over P with projection map

$$\pi^\times : L^\times = \mathbb{C}^\times \times P \to P : (b,p) \mapsto p$$

and connection 1-form $\beta = \frac{1}{2\pi i}\frac{\mathrm{d}b}{b} + \frac{1}{\mathrm{h}}\,\alpha$. The complex line bundle

$$\pi : L = \mathbb{C} \times P \to P : (c,p) \mapsto p$$

associated to the principal bundle $\pi^\times$ is also trivial. Prequantization of this system is obtained by adapting the results of Section 2.

Since integral manifolds of the polarization D are k-tori, we have to determine which of them admit nonzero covariantly constant sections of L.

Theorem 4. *An integral manifold M of the distribution D admits a section σ of the complex line bundle L, which is nowhere zero when restricted to M, if and only if it satisfies the Bohr–Sommerfeld condition (47).*

Proof. Suppose that an integral manifold M of D admits a nowhere zero section of $L_{|M}$. Since σ is nowhere zero, it is a section of $L^\times_{|M}$. Let $\gamma : S^1 \to M$ be a loop in M. For each $t \in S^1$, let $\dot\gamma(t) \in T_{\gamma(t)}M$ be the tangent vector to γ at t. Since σ is covariantly constant along M, Claim 2 applied to the section

$$\sigma : M \to L^\times_{|M} = \mathbb{C} \times M : p \mapsto (b(p), p)$$

gives

$$\nabla_{X(p)}\sigma(p) = 2\pi i \,\langle \sigma^*(\beta)(p)|X(p)\rangle\,\sigma(p) = 0$$

for every $p \in P$ and every $X(p) \in T_pM$. Taking $p = \gamma(t)$ and $X(p) = \dot\gamma(t)$ gives

$$2\pi i\,\langle \sigma^*\beta(\gamma(t))|\dot\gamma(t)\rangle\,\sigma(\gamma(t)) = 0. \tag{48}$$

Since $\beta = \frac{1}{2\pi i}\frac{\mathrm{d}b}{b} + \frac{1}{\mathrm{h}}\alpha$ and $(\sigma\circ\gamma)(t) = (b(\gamma(t),\gamma(t)))$, we get

$$\begin{aligned}2\pi i\,\langle \sigma^*\beta(\gamma(t))|\dot\gamma(t)\rangle &= 2\pi i\,\langle \beta(\sigma(\gamma(t)))|\dot\gamma(t)\rangle\\ &= \frac{1}{b(\gamma(t))}\frac{\mathrm{d}b(\gamma(t))}{\mathrm{d}t} + \frac{2\pi i}{\mathrm{h}}\,\langle\alpha|\dot\gamma(t)\rangle\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\ln b(\gamma(t)) + \frac{2\pi i}{\mathrm{h}}\,\langle\alpha(\gamma(t))|\dot\gamma(t)\rangle.\end{aligned}$$

Hence, Equation (48) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln b(\gamma(t)) + \frac{2\pi i}{\mathrm{h}}\langle\alpha(\gamma(t))|\dot\gamma(t)\rangle = 0,$$

which integrated from 0 to 2π gives

$$\ln b(\gamma(2\pi)) - \ln b(\gamma(0)) = -\frac{2\pi i}{\mathrm{h}}\int_0^{2\pi}\langle\alpha(\gamma(t))|\dot\gamma(t)\rangle\,\mathrm{d}t = -\frac{2\pi i}{\mathrm{h}}\oint\gamma^*\alpha.$$

If γ bounds a surface $\Sigma \subseteq M$, then Stokes' theorem together with Equation (47) and the quantization condition (7) yield

$$-\frac{2\pi i}{h}\oint\gamma^*\alpha = -\frac{2\pi i}{\mathrm{h}}\int_\Sigma \mathrm{d}\alpha = -\frac{2\pi i}{\mathrm{h}}\int_\Sigma\omega = 0,$$

because M is a Lagrangian submanifold of (P,ω). Thus, $\ln b(\gamma(2\pi)) = \ln b(\gamma(0))$, which implies that the nowhere zero section σ is parallel along γ. If γ does not bound a surface in M, but does satisfy the Bohr–Sommerfeld condition $\oint\gamma^*\alpha_Q = n\mathrm{h}$ (47) with α_Q replaced by its pull back α to P, then

$$\ln\Big(\frac{b(\gamma(2\pi))}{b(\gamma(0))}\Big) = -\frac{2\pi i}{\mathrm{h}}\oint\gamma^*\alpha = -\frac{2\pi i}{\mathrm{h}}\,n\mathrm{h} = -2\pi i\,n,$$

so that

$$\frac{b(\gamma(2\pi))}{b(\gamma(0))} = \mathrm{e}^{-2\pi i\,n} = 1.$$

Hence, $b(\gamma(2\pi)) = b(\gamma(0))$ and the nowhere zero section σ is parallel along γ. □

Note that the manifolds M that satisfy Bohr–Sommerfeld conditions (47) are k-dimensional toric submanifolds of P. We call them *Bohr–Sommerfeld tori*. Since Bohr–Sommerfeld tori have dimension $k = \frac{1}{2}\dim P$, there is no non-zero smooth section $\sigma_0 : P \to L$ that is covariantly constant along D. However, for each Bohr–Sommerfeld torus M, Theorem 4 guarantees the existence of a non-zero, covariantly constant along $D_{|M}$, smooth section $\sigma_M : M \to L_{|M}$, where $L_{|M}$ denotes the restriction of L to M.

Let $\mathcal{S} = \{M\}$ be the set of Bohr–Sommerfeld tori in P. For each $M \in \mathcal{S}$, there exists a non-zero, covariantly constant along $D_{|M}$, smooth section σ_M of L restricted to M determined up to a factor in $\mathbb{C}^\times$. The direct sum

$$\mathfrak{S} = \bigoplus_{M\in\mathcal{S}}\{\mathbb{C}\,\sigma_M\} \tag{49}$$

is the the space of quantum states of the Bohr–Sommerfeld theory. Thus, each Bohr–Sommerfeld torus M represents a 1-dimensional subspace $\{\mathbb{C}\,\sigma_M\}$ of quantum states. Moreover, $\{\mathbb{C}\,\sigma_M\}\cap\{\mathbb{C}\,\sigma_{M'}\} = \{0\}$

if $M \neq M'$ because Bohr–Sommerfeld tori are mutually disjoint. Hence, the collection $\{\sigma_M\}$ is a basis of $\mathfrak{S}$.

For our toral polarization $F = D^{\mathbb{C}}$, the space of smooth functions on P that are constant along F, see Equation (44), is $C^\infty_F(P) = \rho^*(C^\infty(B))$, see Lemma A3. For each $f \in C^\infty_F(P)$, the Hamiltonian vector field X_f is in D, that is, $\nabla_{X_f}\sigma_M = 0$ for every basic state $\sigma_M \in \mathfrak{S}$. Hence, the prequantization and quantization operators act on the basic states $\sigma_M \in \mathfrak{S}$ by multiplication by f, that is,

$$\mathcal{Q}_f\sigma_M = \mathcal{P}_f\sigma_M = f\,\sigma_M = f_{|M}\,\sigma_M. \tag{50}$$

Note that $f_{|M}$ is a constant because $f \in C^\infty_F(P)$. For a general quantum state $\sigma = \sum_{M\in\mathcal{S}} c_M\sigma_M \in \mathfrak{S}$,

$$\mathcal{Q}_f\sigma = \mathcal{Q}_f \sum_{M\in\mathcal{S}} c_M\sigma_M = \sum_{M\in\mathcal{S}} c_M\mathcal{Q}_f\sigma_M = \sum_{M\in\mathcal{S}} c_M f_{|M}\,\sigma_M.$$

We see that, for every function $f \in C^\infty(P)$, each basic quantum state σ_M is an eigenstate of $\mathcal{Q}_f$ corresponding to the eigenvalue $f_{|M}$. Since eigenstates corresponding to different eigenvalues of the same symmetric operator are mutually orthogonal, it follows that the basis $\{\sigma_M\}$ of $\mathfrak{S}$ is orthogonal. This is the only information we have about scalar product in $\mathfrak{S}$. Our results do not depend on other details about the scalar product in $\mathfrak{S}$.

3.3. Shifting Operators

3.3.1. The Simplest Case $P = T^*\mathbb{T}^k$

We begin by assuming that $P = T^*\mathbb{T}^k$ with canonical coordinates $(\boldsymbol{p}, \boldsymbol{\theta}) = (p_1, ..., p_k, \theta_1, ..., \theta_k)$ where, for each $i = 1, ..., k$, θ_i is the canonical angular coordinate on the ith torus and p_i is the conjugate momentum. The symplectic form is

$$\omega = \mathrm{d}\big(\sum_{i=1}^{k} p_i \mathrm{d}\theta_i\big) = \sum_{i=1}^{k} \mathrm{d}p_i \wedge \mathrm{d}\theta_i.$$

In this case, action–angle coordinates $(\mathbf{j}, \boldsymbol{\vartheta}) = (j_1, \ldots, j_k, \vartheta_1, \ldots, \vartheta_k)$ are obtained by rescaling the canonical coordinates so that, for every $i = 1, ..., k$, we have $j_i = 2\pi p_i$ and $\vartheta_i = \theta_i/2\pi$. Moreover, the rescaled angle coordinate $\vartheta_i : T^*\mathbb{T}^k \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is interpreted as a multi-valued real function, the symplectic form

$$\omega = \sum_{i=1}^{k} \mathrm{d}j_i \wedge \mathrm{d}\vartheta_i, \tag{51}$$

and the toric polarization of (P, ω) is given by $D = \mathrm{span}\left\{\frac{\partial}{\partial\vartheta_1}, \ldots, \frac{\partial}{\partial\vartheta_1}\right\}$.

In terms of action–angle coordinates, the Bohr–Sommerfeld tori in $T^*\mathbb{T}^k$ are given by equation

$$\boldsymbol{j} = (j_1, ..., j_k) = (n_1\mathrm{h}, \ldots, n_k\mathrm{h}) = \boldsymbol{n}\mathrm{h}, \tag{52}$$

where $\boldsymbol{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$. For each $\boldsymbol{n} \in \mathbb{Z}^k$, we denote by $\mathbb{T}^k_{\boldsymbol{n}}$ the corresponding Bohr–Sommerfeld torus in $\mathfrak{B}$. If $\beta = \frac{1}{2\pi i}\frac{\mathrm{d}b}{b} + \frac{1}{\mathrm{h}}\sum_{i=1}^{k} j_i\,\mathrm{d}\vartheta_i$ is the connection form in the principal line bundle $L^\times = \mathbb{C}^\times \times \mathbb{T}^k_{\mathbf{n}} \to \mathbb{T}^k_{\mathbf{n}}$, then sections

$$\sigma_{\boldsymbol{n}} : \mathbb{T}^k_{\boldsymbol{n}} \to L^\times : (\vartheta_1, \ldots, \vartheta_k) \mapsto \mathrm{e}^{-2\pi i(n_1\vartheta_1 + \ldots + n_k\vartheta_k)}, \tag{53}$$

form a basis in the space $\mathfrak{S}$ of quantum states.

For each $i = 1, ..., k$, the vector field $\frac{\partial}{\partial j_i}$ is transverse to D and $-\frac{\partial}{\partial j_i} \lrcorner\, \omega = -\mathrm{d}\vartheta_i$, so that $-\frac{\partial}{\partial j_i}$ is the Hamiltonian vector field of ϑ_i. In the following, we write

$$X_i = -\frac{\partial}{\partial j_i} = X_{\vartheta_i} \tag{54}$$

to describe the actual vector field X_i without referring to its relation to the action angle coordinates $(\mathbf{j}, \boldsymbol{\vartheta})$. Equation (36) in Section 2.1, for $f = \vartheta_i$, is multi-valued because the phase factor is multi-valued, and

$$\mathrm{e}^{tZ_{\vartheta_i}} = \mathrm{e}^{-2\pi i t\vartheta_i/\mathrm{h}}\mathrm{e}^{t\,\mathrm{lift}\,X_i}. \tag{55}$$

Claim 5. *If $t = \mathrm{h}$, then*

$$\mathrm{e}^{\mathrm{h}\,Z_{X_i}} = \mathrm{e}^{-2\pi i\vartheta_i}\mathrm{e}^{\mathrm{h}\,\mathrm{lift}\,X_i}. \tag{56}$$

is well defined.

Proof. For every $i = 1, ..., k$, consider an open interval (a_i, b_i) in $\mathbb{R}$ such that $0 < b_i - a_i < 1$. Let

$$W = \vartheta_1^{-1}(a_1, b_1) \cap \vartheta_2^{-1}(a_2, b_2) \cap ... \cap \vartheta_k^{-1}(a_k, b_k). \tag{57}$$

Since the action–angle coordinates $(j_1, \ldots, j_k, \vartheta_1, \ldots, \vartheta_k)$ are continuous, W is an open subset of P. Let η_i be a unique representative of $\vartheta_{i|W}$ with values in (a_i, b_i). With this notation,

$$\omega_{|W} = \sum_{i=1}^{k} \mathrm{d}j_{i|W} \wedge \mathrm{d}\vartheta_i. \tag{58}$$

The restriction to W of the vector field X_{ϑ_i} is the genuinely Hamiltonian vector field of η_i, namely,

$$X_{\vartheta_{i|W}} = X_{\eta_i}. \tag{59}$$

The vector field

$$Z_{\eta_i} = \mathrm{lift}\,X_{\eta_i} - Y_{\eta_i/\mathrm{h}} \tag{60}$$

is well defined. Equation (36) yields $\mathrm{e}^{t\,Z_{\eta_i}} = \mathrm{e}^{-2\pi i\,\eta_i/\mathrm{h}}\mathrm{e}^{t\,\mathrm{lift}\,X_{\eta_i}}$. Hence,

$$\mathrm{e}^{\mathrm{h}\,Z_{\eta_i}} = \mathrm{e}^{-2\pi i\,\eta_i}\mathrm{e}^{\mathrm{h}\,\mathrm{lift}\,X_{\eta_i}}. \tag{61}$$

If we make another choice of intervals (a_i', b_i') in $\mathbb{R}$ such that $0 < b_i' - a_i' < 1$ and let $W' = \cap_{i=1}^{k}\vartheta_i^{-1}(a_i', b_i')$. Then, η_i' with values in (a_i', b_i') differs from η_i by an integer, so that $\eta_i' = \eta_i + n_i$, and, in $W \cap W'$, we have

$$\mathrm{e}^{-2\pi i\,\eta_i'} = \mathrm{e}^{-2\pi i\,(\eta_i + n_i)} = \mathrm{e}^{-2\pi i\,\eta_i}.$$

Moreover, $X_{\theta_i|W\cap W'} = X_{\eta_i'|W\cap W'} = X_{i|W\cap W'}$, so that

$$(\mathrm{e}^{\mathrm{h}\,Z_{X_i}})_{|L^{\times}_{|W\cap W'}} = (\mathrm{e}^{h\,Z_{\eta_i}})_{|L^{\times}_{|W\cap W'}} = (\mathrm{e}^{h\,Z_{\eta_i'}})_{|L^{\times}_{|W\cap W'}}.$$

Since we can cover P by open contractible sets defined in Equation (57), we conclude that $\mathrm{e}^{\mathrm{h}\,Z_{X_i}}$ is well defined by Equation (56) and depends only on the vector field X_i. □

Consequently, there exists a connection preserving automorphism $\mathbf{A}_{X_i} : L^{\times} \to L^{\times}$ such that, if $\ell^{\times} \in L_{|W^{\times}}$, where $W \subseteq P$ is given by Equation (57), then

$$\boldsymbol{A}_{X_i}(\ell^{\times}) = \mathrm{e}^{\mathrm{h}\,Z_{X_i}}(\ell^{\times}). \tag{62}$$

Claim 6. *The connection preserving automorphism $\boldsymbol{A}_{X_i} : L^\times \to L^\times$, defined by Equation (62) depends only on the vector field X_i and not the original choice of the action–angle coordinates.*

Proof. If $(j'_1, \dots, j'_k, \vartheta'_1, \dots, \vartheta'_k)$ is another set of action–angle coordinates then

$$j_i = \sum_{l=1}^{k} a_{il}\, j'_l \text{ and } \vartheta_i = \sum_{l=1}^{k} b_{il}\, \vartheta'_l, \tag{63}$$

where the matrices $A = (a_{il})$ and $B = (b_{il})$ lie in $\mathrm{Sl}(k, \mathbb{Z})$ and $B = (A^{-1})^T$. In the new coordinates,

$$X_{\vartheta_i} = -\frac{\partial}{\partial j_i} = -\sum_{l=1}^{k} a_{il} \frac{\partial}{\partial j'_l} = -\sum_{l=1}^{k} b_{il} X_{\vartheta'_l} = X_{(b_{i1}\vartheta'_1 + \dots + b_{ik}\vartheta'_k)}.$$

Clearly,

$$\mathrm{e}^{t\, \mathrm{lift}\, X_{\vartheta_i}} = \mathrm{e}^{t\, \mathrm{lift}\, X_{(b_{i1}\vartheta'_1 + \dots + b_{ik}\vartheta'_k)}}. \tag{64}$$

To compare the phase factor entering Equation (55), we consider an open contractible set $W \subseteq P$. As before, for each $i = 1, ..., k$, choose a single-valued representative η'_i of $(\vartheta'_i)_{|W}$. Then,

$$\eta_i = \sum_{j=1}^{k} b_{ij}(\eta'_j + l_j) = \sum_{j=1}^{k} b_{ij}\eta'_j + \sum_{j=1}^{k} b_{ij} l_j = \sum_{j=1}^{k} b_{ij}\eta'_j + l, \tag{65}$$

where each l_j is an integer and thus $l = \sum_{j=1}^{k} b_{ij} l_j$ is also an integer. Hence,

$$\mathrm{e}^{-2\pi i\, \eta_i} = \mathrm{e}^{-2\pi i\, (b_{i1}\eta'_1 + \dots + b_{ik}\eta'_k + l)} = \mathrm{e}^{-2\pi i\, (b_{i1}\eta'_1 + \dots + b_{ik}\eta'_k)}, \tag{66}$$

where $b_{i1}, \dots, b_{ik}$ are integers. Since l is constant,

$$\begin{aligned} X_{\vartheta_{i|W}} &= X_{\eta_i} = X_{(b_{i1}\eta'_1 + \dots + b_{ik}\eta'_k + l)} \\ &= X_{(b_{i1}\eta'_1 + \dots + b_{ik}\eta'_k)} = X_{(b_{i1}\vartheta'_1 + \dots + b_{ik}\vartheta'_k)_{|W}}. \end{aligned} \tag{67}$$

Therefore,

$$\mathrm{e}^{\mathrm{h}\, Z_{\eta_i}} = \mathrm{e}^{-2\pi i\, \eta_i} \mathrm{e}^{\mathrm{h}\, \mathrm{lift}\, X_{\eta_i}} = \mathrm{e}^{-2\pi i\, (b_{i1}\eta'_1 + \dots + b_{ik}\eta'_k)} \mathrm{e}^{\mathrm{h}\, \mathrm{lift}\, X_{(b_{i1}\vartheta'_1 + \dots + b_{ik}\vartheta'_k)}}, \tag{68}$$

which shows that the automorphism $\mathbf{A}_{X_{\vartheta_i}} : L^\times \to L^\times$ depends on the vector field X_i and *not* on the action angle coordinates in which it is computed. □

Claim 7. *For each $i = 1, ..., k$, the symplectomorphism $\mathrm{e}^{hX_i} : P \to P$, where h is Planck's constant, preserves the set $\mathcal{B}$ of Bohr–Sommerfeld tori in P.*

Proof. Since X_i is complete, $\mathrm{e}^{tX_i} : P \to P$ is a 1-parameter group of symplectomorphisms of (P, ω). Hence, $\mathrm{e}^{\mathrm{h}\, X_i} : P \to P$ is well defined. By Equation (52), $j_{i|\mathbb{T}^k_{\boldsymbol{n}}} = n_i \mathrm{h}$ for every Bohr–Sommerfeld torus $\mathbb{T}^k_{\boldsymbol{n}}$, where $\boldsymbol{n} = (n_1, \dots, n_k)$.

Since $X_i = -\frac{\partial}{\partial j_i}$,

$$\begin{aligned} L_{X_i}(j_i\, \mathrm{d}\vartheta_i) &= X_i \,\lrcorner\, \mathrm{d}j_i \wedge \mathrm{d}\vartheta_i + \mathrm{d}(X_i \,\lrcorner\, j_i\, \mathrm{d}\vartheta_i) = -\mathrm{d}\vartheta_i, \\ L_{X_i}(j_l\, \mathrm{d}\vartheta_l) &= X_i \,\lrcorner\, \mathrm{d}j_l \wedge \mathrm{d}\vartheta_l + \mathrm{d}(X_i \,\lrcorner\, j_l\, \mathrm{d}\vartheta_l) = 0 \text{ for } l \neq i. \end{aligned}$$

This implies that, for every $l \neq i$, $(\mathrm{e}^{tX_i})^*(j_l\, \mathrm{d}\vartheta_l) = j_l\, \mathrm{d}\vartheta_l$ and $(\mathrm{e}^{tX_i})^*(j_i\, \mathrm{d}\vartheta_i) = (j_i - t)\mathrm{d}\vartheta_i$. Therefore, if $\boldsymbol{j} = \boldsymbol{n}\mathrm{h}$, then $(\mathrm{e}^{\mathrm{h}\, X_i})^* j_l = j_l = n_l$, if $l \neq i$, and $(\mathrm{e}^{tX_i})^* j_i = (j_i - \mathrm{h}) = (n_i - 1)\mathrm{h}$ if $\ell = i$. This implies that $\mathrm{e}^{\mathrm{h}X_{\vartheta_i}}(\mathbb{T}^k_{\boldsymbol{n}})$ is a Bohr–Sommerfeld torus. □

We denote by $\widehat{\mathbf{A}}_{X_i} : L \to L$ the action of $\mathbf{A}_{X_i} : L^\times \to L^\times$ on L. The automorphism $\widehat{\mathbf{A}}_{X_i}$ acts on sections of L by pull back and push forward, namely,

$$\begin{aligned} (\widehat{\boldsymbol{A}}_{X_i})_* \sigma &= (\widehat{\mathrm{e}}^{\mathrm{h}\, Z_i})_* \sigma = \widehat{\mathrm{e}}^{-\mathrm{h}\, Z_i} \circ \sigma \circ \mathrm{e}^{\mathrm{h}\, X_i}, \\ (\widehat{\boldsymbol{A}}_{X_i})^* \sigma &= (\widehat{\mathrm{e}}^{\mathrm{h}\, Z_i})^* \sigma = \widehat{\mathrm{e}}^{\mathrm{h}\, Z_i} \circ \sigma \circ \mathrm{e}^{-\mathrm{h}\, X_i}. \end{aligned} \tag{69}$$

Since $\boldsymbol{A}_{X_i} : L^\times \to L^\times$ is a connection preserving automorphism, it follows that, if σ satisfies the Bohr–Sommerfeld conditions, then $(\widehat{\boldsymbol{A}}_{X_i})_*\sigma$ and $(\widehat{\boldsymbol{A}}_{X_i})^*\sigma$ also satisfy the Bohr–Sommerfeld conditions. In other words, $(\widehat{\boldsymbol{A}}_{X_i})_*$ and $(\widehat{\boldsymbol{A}}_{X_i})^*$ preserve the space $\mathfrak{S}$ of quantum states. The *shifting operators* $\boldsymbol{a}_{X_i}$ and $\boldsymbol{b}_{X_i}$, corresponding to X_i, are the restrictions to $\mathfrak{S}$ of $(\widehat{\boldsymbol{A}}_{X_i})_*$ and $(\widehat{\boldsymbol{A}}_{X_i})^*$, respectively. For every $\boldsymbol{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, Equations (53) and (56) yield

$$\begin{aligned} \boldsymbol{a}_{X_i} \sigma_{\boldsymbol{n}} &= \widehat{\mathrm{e}}^{-\mathrm{h}\, Z_i} \circ \sigma_{\boldsymbol{n}} \circ \widehat{\mathrm{e}}^{\mathrm{h}\, X_i} = \sigma_{\boldsymbol{n}_i^-} = \mathrm{e}^{-2\pi i (\sum_{j \neq i} n_j \vartheta_j + (n_i - 1)\vartheta_i)} \sigma_{\mathbf{n}} \\ \boldsymbol{b}_{X_i} \sigma_{\boldsymbol{n}} &= \widehat{\mathrm{e}}^{\mathrm{h}\, Z_i} \circ \sigma_{\boldsymbol{n}} \circ \widehat{\mathrm{e}}^{-\mathrm{h}\, X_i} = \sigma_{\boldsymbol{n}_i^+} = \mathrm{e}^{-2\pi i (\sum_{j \neq i} n_j \vartheta_j + (n_i + 1)\vartheta_i)} \sigma_{\mathbf{n}}. \end{aligned} \tag{70}$$

For each $i = 1, \ldots, k$, $\boldsymbol{a}_{X_i} \circ \boldsymbol{b}_{X_i} = \boldsymbol{b}_{X_i} \circ \boldsymbol{a}_{X_i} = \mathrm{id}_{\mathfrak{S}}$. In addition, the operators $\boldsymbol{a}_{X_i}$, $\boldsymbol{b}_{X_j}$, for $i, j = 1, \ldots, k$, generate an abelian group $\mathfrak{A}$ of linear transformations of $\mathfrak{S}$ into itself, which acts transitively on the space of one-dimensional subspaces of $\mathfrak{S}$.

Given a non-zero section $\sigma \in \mathfrak{S}$ supported on a Bohr–Sommerfeld torus, the family of sections

$$\{(\boldsymbol{a}_{X_k}^{n_k} \cdots \boldsymbol{a}_{X_1}^{n_1} \sigma) \in \mathfrak{S} \,\big|\, n_1, \ldots n_k \in \mathbb{Z}\} \tag{71}$$

is a linear basis of $\mathfrak{S}$, invariant under the action of $\mathfrak{A}$. Since $\mathfrak{A}$ is abelian, there exists a positive, definite Hermitian scalar product $\langle \cdot \mid \cdot \rangle$ on $\mathfrak{S}$, which is invariant under the action of $\mathfrak{A}$, and such that the basis in (71) is orthonormal. It is defined up to a constant positive factor. The completion of $\mathfrak{S}$ with respect to this scalar product yields a Hilbert space $\mathfrak{H}$ of quantum states in the Bohr–Sommerfeld quantization of $T^*\mathbb{T}^k$. Elements of $\mathfrak{A}$ extend to unitary operators on $\mathfrak{H}$.

3.3.2. General Case of Toral Polarization

Hilbert Space and Operators

Let (P, ω) be a symplectic manifold with toroidal polarization D and a covering by domains of action–angle coordinates. If U and U' are the domain of the angle-action coordinates $(\boldsymbol{j}, \boldsymbol{\vartheta}) = (j_1, \ldots, j_k, \vartheta_1, \ldots, \vartheta_k)$ and $(\boldsymbol{j}', \boldsymbol{\vartheta}') = (j_1', \ldots, j_k', \vartheta_1', \ldots, \vartheta_k')$, respectively, and $U \cap U' \neq \varnothing$, then in $U \cap U'$ we have

$$j_i = \sum_{l=1}^{k} a_{il}\, j_l' \;\text{ and }\; \vartheta_i = \sum_{l=1}^{k} b_{il}\, \vartheta_l', \tag{72}$$

where the matrices $A = (a_{il})$ and $B = (b_{il})$ lie in $\mathrm{Sl}(k, \mathbb{Z})$ and $B = (A^{-1})^T$.

Consider a complete locally Hamiltonian vector field X on (P, ω) such that, for each angle-action coordinates $(\boldsymbol{j}, \boldsymbol{\vartheta})$ with domain U,

$$(X \mathbin{\lrcorner} \omega)_{|U} = -\mathrm{d}(\boldsymbol{c} \cdot \boldsymbol{\vartheta}) = -\mathrm{d}(c_1\vartheta_1 + \ldots + c_k\vartheta_k), \tag{73}$$

for some $\boldsymbol{c} = (c_1, \ldots, c_k) \in \mathbb{Z}^k$. Equation (72) shows that in $U \cap U'$, we have

$$c_1\vartheta_1 + \ldots + c_k\vartheta_k = c_1'\vartheta_1' + \ldots + c_k'\vartheta_k',$$

where $c_i' = \sum_{j=1}^{k} c_j b_{ji} \in \mathbb{Z}$, for $i = 1, \ldots, k$. As in the preceding section, Equation (36) with $f = \boldsymbol{c} \cdot \boldsymbol{\vartheta} = c_1\vartheta_1 + \ldots + c_k\vartheta_k$, which is multi-valued, gives

$$\mathrm{e}^{tZ_{\boldsymbol{c}\cdot\boldsymbol{\vartheta}}} = \mathrm{e}^{-2\pi i\, t\, \boldsymbol{c}\cdot\boldsymbol{\vartheta}/\mathrm{h}} \mathrm{e}^{t\, \mathrm{lift} X}, \tag{74}$$

which is multivalued, because the phase factor is multi-valued. As before, if we set $t = \mathrm{h}$, we would get a single-valued expression $\mathrm{e}^{\mathrm{h}\, Z_{c\cdot\vartheta}} = \mathrm{e}^{-2\pi i c\cdot\vartheta}\mathrm{e}^{\mathrm{h}\,\mathrm{lift} X}$ because $c_1, \dots, c_k \in \mathbb{Z}$. This would work along all integral curves $t \mapsto \mathrm{e}^{t\,X}(x)$ for $t \in [0,1]$, which are contained in U.

Now, consider the case when, for $x_0 \in U$, $\mathrm{e}^{hX}(x) \in U'$ and there exists $t_1 \in (0,h)$ such that $x_1 = \mathrm{e}^{t_1 X}(x_0) \in U \cap U'$, where U and U' are domains of action–angle variables $(\boldsymbol{j}, \boldsymbol{\vartheta})$ and $(\boldsymbol{j}', \boldsymbol{\vartheta}')$, respectively. Moreover, assume that $\mathrm{e}^{tX}(x_0) \in U$ for $t \in [0, t_1]$ and $\mathrm{e}^{tX}(x_1) \in U'$ for $t \in [0, \mathrm{h} - t_1]$. Using the multi-index notation, for $l^{\times} \in L^{\times}_{x_0}$, we write

$$\begin{aligned} \boldsymbol{A}_X(\ell^{\times}) &= \mathrm{e}^{(\mathrm{h}-t_1)Z_{c'\cdot\vartheta'}}(\mathrm{e}^{t_1 Z_{c\cdot\vartheta}}(\ell^{\times})) \\ &= \mathrm{e}^{-2\pi i(\mathrm{h}-t_1)c'\cdot\vartheta'/\mathrm{h}}\mathrm{e}^{(\mathrm{h}-t_1)\mathrm{lift}X}(\mathrm{e}^{-2\pi i t_1 c\cdot\vartheta/\mathrm{h}}\mathrm{e}^{t_1\,\mathrm{lift}X}(\ell^{\times})) \\ &= (\mathrm{e}^{-2\pi i(\mathrm{h}-t_1)c'\cdot\vartheta'/\mathrm{h}}\mathrm{e}^{-2\pi i t_1 c\cdot\vartheta/\mathrm{h}})\mathrm{e}^{(\mathrm{h}-t_1)\mathrm{lift}X}(\mathrm{e}^{t_1\mathrm{lift}X}(\ell^{\times})) \\ &= \mathrm{e}^{-2\pi i t_1(c\cdot\vartheta - c'\cdot\vartheta')/\mathrm{h}}\mathrm{e}^{-2\pi i c'\cdot\vartheta'}\mathrm{e}^{\mathrm{h}\,\mathrm{lift}X}(\ell^{\times}). \end{aligned} \tag{75}$$

Let W be a neighborhood of x_1 in P such that $U \cap W$ and $U' \cap W'$ are contractible. For each $i = 1, ..., k$, let θ_i be a single-valued representative of ϑ_i as in the proof of Claim 5. Similarly, we denote by η_i' a single-valued representative of ϑ_i'. Equation (73) shows that in $U \cap U' \cap W$, the functions $c_1\eta_1 + \cdots + c_k\eta_k$ and $c_1'\eta_1' + \cdots + c_k'\eta_k'$ are local Hamiltonians of the vector field X and are constant along the integral curve of $X_{|W}$. Hence, we have to make the choice of representatives η_i and η_i' so that

$$c_1\eta_1(x_1) + \cdots + c_k\eta_k(x_1) = c_1\eta_1'(x_1) + \cdots + c_k\eta_k'(x_1). \tag{76}$$

With this choice, $\mathrm{e}^{-2\pi i t_1(c\cdot\vartheta - c'\cdot\vartheta')/\mathrm{h}} = 1$, and

$$\boldsymbol{A}_X(l^{\times}) = \mathrm{e}^{-2\pi i c'\cdot\vartheta'}\mathrm{e}^{\mathrm{h}\,\mathrm{lift}X}(l^{\times}) \tag{77}$$

is well defined. It does not depend on the choice of the intermediate point x_1 in $U \cap U'$.

In the case when $m+1$, action–angle coordinate charts with domains $U_0, U_{1,} ..., U_m$ are needed to reach $x_m = \mathrm{e}^{\mathrm{h}\,X}(x_0) \in U_m$ from $x_0 \in U_0$; we choose $x_1 = \mathrm{e}^{t_1 X}(x_0) \in U_0 \cap U_1$, $x_2 = \mathrm{e}^{t_2 X}(x_1) \in U_1 \cap U_2, \dots, x_{m-1} = \mathrm{e}^{t_{m-1}X}(x_{m-2}) \in U_{m-1}$ and end with $x_m = \mathrm{e}^{(\mathrm{h}-t_1-\dots-t_{m-1})X}(x_{m-1}) \in U_m$. At each intermediate point $x_1, \dots, x_{m-1}$, we repeat the the argument of the preceding paragraph. We conclude that there is a connection preserving automorphism $\boldsymbol{A}_X : L^{\times} \to L^{\times}$ well defined by the procedure given here, and it depends only on the complete locally Hamiltonian vector field X satisfying condition (73). The automorphism $\boldsymbol{A}_X : L^{\times} \to L^{\times}$ of the principal bundle $L^{\times}$ leads to an automorphism $\widehat{\boldsymbol{A}}_X$ of the associated line bundle L. As in Equation (69), the shifting operators corresponded to the complete locally Hamiltonian vector field X are

$$\begin{aligned} \boldsymbol{a}_X &: \mathfrak{S} \to \mathfrak{S} : \sigma \mapsto (\widehat{\boldsymbol{A}}_X)_*\sigma, \\ \boldsymbol{b}_X &: \mathfrak{S} \to \mathfrak{S} : \sigma \mapsto (\widehat{\boldsymbol{A}}_X)^*\sigma. \end{aligned} \tag{78}$$

In absence of monodromy, if we have k independent, complete, locally Hamiltonian vector fields X_i on (P,ω) that satisfy the conditions leading to Equation (73), then the operators $\boldsymbol{a}_{X_i}$, $\boldsymbol{b}_{X_j}$ for $i, j = 1, ..., k$ generate an abelian group $\mathfrak{A}$ of linear transformations of $\mathfrak{S}$. If the local lattice $\mathfrak{S}$ of Bohr–Sommerfeld tori is regular, then $\mathfrak{A}$ acts transitively on the space of one-dimensional subspaces of $\mathfrak{S}$. This enables us to construct an $\mathfrak{A}$-invariant Hermitian scalar product on $\mathfrak{S}$, which is unique up to an arbitrary positive constant. The completion of $\mathfrak{S}$ with respect to this scalar product yields a Hilbert space $\mathfrak{H}$ of quantum states in the Bohr–Sommerfeld quantization of (P,ω).

Local Lattice Structure

The above discussion does not address the question of labeling the basic sections σ_b in $\mathfrak{H}$ by the quantum numbers $\mathbf{n} = (n_1, \ldots, n_k)$ associated to the Bohr–Sommerfeld k-torus $T = M_b$, the support of σ_b.

These quantum numbers *do* depend on the choice of action angle coordinates. If $(j', \vartheta') \in V \times \mathbb{T}^k$ is another choice of action angle coordinates in the trivializing chart (U', ψ'), where $T \subseteq U'$, then the quantum numbers $\mathbf{n}'$ of T in (j', ϑ') coordinates are related to the quantum numbers $\mathbf{n}$ of T in (j, ϑ) coordinates by a matrix $A \in \mathrm{Gl}(k, \mathbb{Z})$ such that $\mathbf{n}' = A\,\mathbf{n}$, because by Claim A2 in Appendix A on $U \cap U'$ the action coordinates j' is related to the action coordinate j by a constant matrix $A \in \mathrm{Gl}(k, \mathbb{Z})$. Let $\mathbb{L}_{|U} = \{\mathbf{n} \in \mathbb{Z}^k \big| T_{\mathbf{n}} \subseteq U\}$. Then, $\mathbb{L}_{|U}$ is the *local lattice structure* of the Bohr–Sommerfeld tori $T_{\mathbf{n}}$, which lie in the action angle chart (U, ψ). If (U, ψ) and (U', ψ') are action angle charts, then the set of Bohr–Sommerfeld tori in $U \cap U'$ are *compatible*. More precisely, on $U \cap U'$ the local lattices $\mathbb{L}_{|U}$ and $\mathbb{L}_{|U'}$ are compatible if there is a matrix $A \in \mathrm{Gl}(k, \mathbb{Z})$ such that $\mathbb{L}_{|U'} = A\,\mathbb{L}_{|U}$. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a *good* covering of P, that is, every finite intersection of elements of $\mathcal{U}$ is either contractible or empty, such that for each $i \in I$ we have a trivializing chart (U_i, ψ_i) for action angle coordinates for the toral bundle $\rho : P \to B$. Then, $\{\mathbb{L}_{U_u}\}_{i \in I}$ is a collection of pairwise compatible local lattice structures for the collection $\mathcal{S}$ of Bohr–Sommerfeld tori on P. We say that $\mathcal{S}$ has a *local lattice structure*.

The next result shows how the operator $(\widehat{\mathrm{e}}^{h\,Z_{\vartheta_i}})_*$ of Section 3.3 affects the quantum numbers of the Bohr–Sommerfeld torus $T = T_{\mathbf{n}}$.

Claim 8. *Let (U, ψ) be a chart in (P, ω) for action angle coordinates (j, ϑ). For every Bohr–Sommerfeld torus $T = T_{\mathbf{n}}$ in U with quantum numbers $\mathbf{n} = (n_1, \ldots, n_k)$, the torus $\mathrm{e}^{h\,X_{\vartheta_\ell}}(T)$ is also a Bohr–Sommerfeld torus $T'_{\mathbf{n}'}$, where $\mathbf{n}' = (n_1, \ldots, n_{\ell-1}, n_\ell - 1, n_{\ell+1}, \ldots, n_k)$.*

Proof. For simplicity, we assume that $\ell = 1$. Suppose that the image of the curve $\gamma : [0, \mathrm{h}] \to B : t \mapsto \mathrm{e}^{X_{\vartheta_1}}(\rho(x_0))$ lies in $V = \psi(U)$, where $x_0 \in T = T_{\mathbf{n}}$. For $x \in T$ and $t \in [0, \mathrm{h}]$ we have

$$X_{\vartheta_1} j_\ell = \begin{cases} X_{\vartheta_1} j_1 &= -\frac{\partial}{\partial j_1} j_1 \;= -1, \quad \text{if } \ell = 1 \\ X_{\vartheta_1} j_\ell &= -\frac{\partial}{\partial j_1} j_\ell \;= \;\;\;0, \quad \text{if } \ell = 2, \ldots, k \end{cases}$$

and $X_{\vartheta_1}\vartheta_\ell = -\frac{\partial}{\partial p_1}\vartheta_\ell = 0$. Since $x \in T$ has action angle coordinates $(j_1(x), \ldots, j_k(x), \vartheta_1(x), \ldots, \vartheta_k(x))$ in U, the point $\mathrm{e}^{t\,\vartheta_1}(x)$ has action angle coordinates $(j_1(x) - t, \ldots, j_k(x), \vartheta_1(x), \ldots, \vartheta_k(x))$. In particular, the point $\mathrm{e}^{t\,X_{\vartheta_1}}(x_0)$ has action angle coordinates $(j_1(x_0) - t, \ldots, j_k(x_0), \vartheta_1(x_0), \ldots, \vartheta_k(x_0))$. Thus,

$$(\mathrm{e}^{\mathrm{h}\,X_{\vartheta_1}})_* j_\ell = \begin{cases} j_1 - \mathrm{h}, & \text{if } \ell = 1 \\ j_\ell, & \text{if } \ell = 2, \ldots, k \end{cases}$$

and $(\mathrm{e}^{\mathrm{h}\,X_{\vartheta_1}})_*\vartheta_\ell = \vartheta_\ell$ for $\ell = 1, 2, \ldots, k$. Since T is the Bohr–Sommerfeld torus $T_{\mathbf{n}}$, we have $j_\ell = \int_0^1 j_\ell\, \mathrm{d}\vartheta_\ell = n_\ell\,\mathrm{h}$. Then,

$$\begin{aligned} \int_0^1 (\mathrm{e}^{\mathrm{h}\,X_{\vartheta_1}})_* j_1\, \mathrm{d}\big((\mathrm{e}^{\mathrm{h}\,X_{\vartheta_1}})_*\vartheta_1\big) &= \int_0^1 (j_1 - \mathrm{h})\mathrm{d}\vartheta_1 \\ &= j_1 - \mathrm{h} = (n_1 - 1)\mathrm{h}. \end{aligned}$$

Thus, the torus $\mathrm{e}^{\mathrm{h}\,X_{\vartheta_1}}(T)$ is a Bohr–Sommerfeld torus $T_{\mathbf{n}'}$ with $\mathbf{n}' = (n_1, \ldots, n_{\ell-1}, n_\ell - 1, n_{\ell+1}, \ldots, n_k)$.

Now, consider the case when the image of the curve $\gamma : [0, \mathrm{h}] \to B : t \mapsto \mathrm{e}^{t\,X_{\vartheta_1}}(\rho(x_0))$ is not contained in V. This means that $\mathrm{e}^{t\,X_{\vartheta_1}}(U)$, where $U = \rho^{-1}(V)$, does not contain the torus T. Since $\mathrm{e}^{t\,X_{\vartheta_1}}$ is a 1-parameter group of symplectomorphisms of (P, ω), for every $t \in \mathbb{R}$, the functions $\big((\mathrm{e}^{t\,X_{\vartheta_1}})_* j_\ell$, with $\ell = 1, \ldots, k$ and $(\mathrm{e}^{t\,X_{\vartheta_1}})_*\vartheta_\ell$, $\ell = 1, \ldots, k$ are action angle coordinates on $(\mathrm{e}^{t\,X_{\vartheta_1}})_*(U)$. Choose

$\tau > 0$ so that $\mathrm{e}^{\tau X_{\vartheta_1}}(T) \subseteq U$. Suppose that $\mathrm{h} = \tau + \eta$, where $\eta \in [0, \tau)$. Observe that for $t \in [0, \tau)$ the action angle coordinates $(j_1, \ldots, p_k, \vartheta_1, \ldots, \vartheta_k)$ in U satisfy

$$(\mathrm{e}^{tX_{\vartheta_1}})_* j_\ell = \begin{cases} j_1 - t & \text{if } \ell = 1 \\ j_\ell & \text{if } \ell = 2, 3, \ldots, k \end{cases} \quad \text{and} \quad (\mathrm{e}^{t\,X_{\vartheta_1}})_* \vartheta_\ell = \theta_\ell.$$

Hence, $(\mathrm{e}^{\tau X_{\vartheta_1}})_* j_1 = j_1 - \tau$ and

$$\begin{aligned}(\mathrm{e}^{h\,X_{\vartheta_1}})_* j_1 &= (\mathrm{e}^{(\tau+\eta)X_{\vartheta_1}})_* j_1 = (\mathrm{e}^{\tau\,X_{\vartheta_1}})_* \, (\mathrm{e}^{\eta\,X_{\vartheta_1}})_* j_1 \\ &= (\mathrm{e}^{\tau\,X_{\vartheta_1}})_* (j_1 - \eta) = (\mathrm{e}^{\tau\,X_{\vartheta_1}})_* (j_1) - \eta,\end{aligned}$$

because η is constant. Moreover,

$$\int_0^1 (\mathrm{e}^{\tau\,X_{\vartheta_1}})_* j_1) \mathrm{d}\big(\mathrm{e}^{\tau\,X_{\vartheta_1}})_* \vartheta_1\big) = \int_0^1 (j_1 - \tau)\,\mathrm{d}\vartheta_1 = \int_0^1 j_1\,\mathrm{d}\vartheta_1 - \tau = j_1 - \tau.$$

Similarly,

$$\begin{aligned}\int_0^1 (\mathrm{e}^{\mathrm{h}\,X_{\vartheta_1}})_* j_1) \mathrm{d}\big(\mathrm{e}^{\mathrm{h}\,X_{\vartheta_1}})_* \vartheta_1\big) &= \int_0^1 (\mathrm{e}^{\tau\,X_{\vartheta_1}})_* j_1 - \eta) \mathrm{d}\big(\mathrm{e}^{\tau\,X_{\vartheta_1}})_* \vartheta_1\big) \\ &= \int_0^1 (\mathrm{e}^{\tau\,X_{\vartheta_1}})_* j_1 \mathrm{d}\big(\mathrm{e}^{\tau\,X_{\vartheta_1}})_* \vartheta_1\big) - \eta \int_0^1 \mathrm{d}\big(\mathrm{e}^{\tau\,X_{\vartheta_1}})_* \vartheta_1\big) \\ &= \int_0^1 p_1\,\mathrm{d}\vartheta_1 - \tau - \eta = \int_0^1 p_1\,\mathrm{d}\vartheta_1 - \mathrm{h} = (n_1 - 1)\mathrm{h},\end{aligned}$$

because T is a Bohr–Sommerfeld torus $T_{\mathbf{n}}$ with quantum numbers $(n_1, \ldots, n_k)$. Thus, $\mathrm{e}^{h\,X_{\vartheta_1}}(T)$ is a Bohr–Sommerfeld torus corresponding to the quantum numbers $(n_1 - 1, n_2, \ldots, n_k)$. This argument may be extended to cover the case where $\hbar = k\tau + \eta$ for any positive integer k and $\eta \in [0, \tau)$. □

3.4. Singularity of Toral Polarization in Completely Integrable Hamiltonian Systems

A completely integrable Hamiltonian system on a symplectic manifold (P, ω) of dimension $2k$ is given by k functions $H_1, ..., H_k \in C^\infty(P)$, which Poisson commute with each other, and are independent on the open dense subset P_0 of P. We assume that, for every $i = 1, ..., k$, and each $x \in P_0$, the maximal integral curve of X_{H_i} through x is periodic with period $T_i(x) > 0$. The complement $P \backslash P_0$ of P_0 in P is the set of singular points of the real polarization $D = \operatorname{span}\{X_{H_1}, ..., X_{H_k}\}$ of (P, ω).

Applying the arguments of Section 3.1 and the beginning of Section 3.2, we obtain the set $\mathcal{S} = \{M\}$ of Bohr–Sommerfeld tori M in P. Each M is an integral manifold of D, which admits a covariantly constant section $\sigma_M : M \to L_{|M}$. The section σ_M is determined up to a non-zero constant. The direct sum

$$\mathfrak{S} = \bigoplus_{M \in \mathcal{S}} \{\mathbb{C}\sigma_M\}$$

is the space of quantum states of the Bohr–Sommerfeld theory. Each Bohr–Sommerfeld torus M represents a one-dimensional subspace of quantum states. The collection $\{\sigma_M\}$ is a basis of $\mathfrak{S}$, and

$$\mathcal{Q}_{H_i}\sigma_M = H_{i|M}\sigma_M.$$

Let $\mathcal{S}_0 = \{M \in \mathcal{S} \mid M \subset P_0\}$ be the set of the Bohr–Sommerfeld tori in P_0. Then,

$$\mathfrak{S}_0 = \bigoplus_{M \in \mathcal{S}_0} \{\mathbb{C}\sigma_M\}$$

is the space of quantum states of the system, which are described by the Bohr–Sommerfeld quantization of P_0. The collection $\{\sigma_M \mid M \subseteq P_0\}$ is a basis of $\mathfrak{S}_0$, and

$$\mathcal{Q}_{H_i}\sigma_M = H_{i|M}\sigma_M$$

for every $M \subseteq P_0$.

The restriction $D_{|P_0}$ of D to P_0 is a toral polarization of $(P_0, \omega_{|P_0})$ discussed earlier. The functions $H_0, \ldots, H_{k-1} \in C^\infty(P)$, which define the system, give rise to action–angle coordinates $(\mathbf{j}, \boldsymbol{\vartheta})$ on P_0, where for each $i = 0, \ldots, k-1$, $j_i = H_{i|P_0}|T_{i|P_0}$ and ϑ_i is the multivalued angle coordinate corresponding to j_i. Since we deal with the single set of action–angle coordinates, most of the analysis of Section 3.3.1 applies to this problem. As in Section 3.3.2, Equation (54), for $i = 1, \ldots, k$ we introduce the notation

$$X_i = -\frac{\partial}{\partial j_i} = X_{\vartheta_i}.$$

Each X_i is a locally Hamiltonian vector field on P_0. However, since $P_0 \neq T^*\mathbb{T}^k$, we cannot assume that the vector field X_i is complete.

In terms of action–angle coordinates $(\mathbf{j}, \boldsymbol{\vartheta})$ on P_0, the Bohr–Sommerfeld tori in P_0 are given by equation

$$\boldsymbol{j} = (j_1, \ldots, j_k) = (n_1\mathrm{h}, \ldots, n_k\mathrm{h}) = \boldsymbol{n}\mathrm{h},$$

where $\boldsymbol{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$. For $\boldsymbol{n} \in \mathbb{Z}^k$,

$$M_{\boldsymbol{n}} = M_{(n_1,\ldots,n_k)} = \{x \in P_0 \mid \mathbf{j}(x) = \boldsymbol{n}\mathrm{h}\} \tag{79}$$

denotes the Bohr–Sommerfeld torus in P_0 corresponding to the eigenvalue $\mathbf{n}$ of $\mathbf{j}$. If $\boldsymbol{n}h$ is not in the spectrum of $\mathbf{j}$, then $M_{\mathbf{n}} = \varnothing$. In a trivialization $L_{|P_0} = \mathbb{C} \times P_0$ of the complex line bundle L restricted to P_0, for each $M_{\mathbf{n}} \neq \varnothing$, we can choose

$$\sigma_{\boldsymbol{n}} : M_{\boldsymbol{n}} \to L : (\vartheta_1, \ldots, \vartheta_k) \mapsto \big((\vartheta_1, \ldots, \vartheta_k), \mathrm{e}^{-2\pi i(n_1\vartheta_1 + \ldots + n_k\vartheta_k)}\big), \tag{80}$$

form a basis in the space $\mathfrak{S}_0$ of quantum states in P_0.

Claim 5 implies the following

Corollary 1. *If, for every $x \in P_0$ and each $i = 1, ..., k$, Planck constant* h *is in the domain of the maximal integral curve $\mathrm{e}^{tX_i}(x)$ of X_i starting at x, then $\mathrm{e}^{\mathrm{h}Z_{X_i}} = \mathrm{e}^{-2\pi i\vartheta_i}\mathrm{e}^{\mathrm{h}\,\mathrm{lift}\,X_i}$ is well defined.*

Under the assumptions of Corollary 1, we may follow the arguments of Section 3.3.1 leading to Equation (70). Applied to the case under consideration, it may be rewritten as follows. For every $\boldsymbol{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$, such that $M_{\boldsymbol{n}} \subseteq P_0$, one has

$$\boldsymbol{a}_{X_i}\sigma_{(n_1,\ldots,n_k)} = \sigma_{(n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots n_k)},$$

if $M_{(n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots n_k)} \subset P_0$, and

$$\boldsymbol{b}_{X_i}\sigma_{(n_1,\ldots,n_k)} = \sigma_{(n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots n_k)},$$

if $M_{(n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots n_k)} \subseteq P_0$.

It remains to extend the action of $\boldsymbol{a}_{X_i}$ and $\boldsymbol{b}_{X_i}$ given above to all states in $\mathfrak{S}$. This involves a study of the integral curves of X_i on P, which originate or end at points in the singular set $P \backslash P_0$.

Suppose we manage to extend the action of the shifting operators to all states in $\mathfrak{S}$. Monodromy occurs when, there exist loops in the local lattice of Bohr–Sommerfeld tori such that for some $\alpha_1, \ldots, \alpha_m \in \{1, \ldots, k\}$ the mapping

$$\left(e^{hX_{\alpha_m}} \circ \cdots \circ e^{hX_{\alpha_1}}\right)_{M_{\boldsymbol{n}}} : M_{\boldsymbol{n}} \to M_{\boldsymbol{n}}$$

need not be the identity on $M_{\boldsymbol{n}}$. In this case shifting operators are multivalued, and there exists a phase factor $e^{i\varphi}$ such that

$$(\mathbf{a}_{X_{\alpha_m}} \circ \cdots \circ \mathbf{a}_{X_{\alpha_1}})\sigma_{\boldsymbol{n}} = e^{i\varphi}\sigma_{\boldsymbol{n}}.$$

Given a non-zero section $\sigma \in \mathfrak{S}$ supported on a Bohr–Sommerfeld torus M. Any maximal family

$$B = \{(\boldsymbol{a}_{X_k}^{n_k} \cdots \boldsymbol{a}_{X_1}^{n_1}\, \sigma) \in \mathfrak{S} \big|\, n_1, \ldots n_k \in \mathbb{Z}\} \tag{81}$$

of sections in $\mathfrak{S}$, such that no two sections in B are supported on the same Bohr–Sommerfeld torus, is a linear base of $\mathfrak{S}$. We can define a scalar product $\langle \cdot \mid \cdot \rangle$ on $\mathfrak{S}$ as follows. First, assume that basic sections supported on different Bohr–Sommerfeld tori are perpendicular to each other. Then, assume that for every $\boldsymbol{a}_{X_k}^{n_k} \cdots \boldsymbol{a}_{X_1}^{n_1}\, \sigma \in B$,

$$\left\langle \boldsymbol{a}_{X_k}^{n_k} \cdots \boldsymbol{a}_{X_1}^{n_1}\, \sigma \mid \boldsymbol{a}_{X_k}^{n_k} \cdots \boldsymbol{a}_{X_1}^{n_1}\, \sigma \right\rangle = \langle \sigma \mid \sigma \rangle\,. \tag{82}$$

This definition works even in the presence of monodromy. The completion of $\mathfrak{S}$ with respect to this scalar product yields a Hilbert space $\mathfrak{H}$ of quantum states in the Bohr–Sommerfeld quantization of the completely integrable Hamiltonian system under consideration.

Example: The 2-d Harmonic Oscillator

We consider the harmonic oscillator with 2 degrees of freedom, see [9]. Its configuration space is $\mathbb{R}^2$ with coordinates $\boldsymbol{q} = (q_1, q_2)$. Its phase space $P = T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ has coordinates $(\boldsymbol{p}, \boldsymbol{q}) = (q_1, q_2, p_1, p_2)$ with symplectic form $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. The 2-d harmonic oscillator is completely integrable with integrals the Hamiltonian $\widetilde{H}$ with $\widetilde{H}(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{2}\,(p_1^2 + p_2^2 + q_1^2 + q_2^2)$ and the angular momentum $\widetilde{J}$ with $\widetilde{J}(\boldsymbol{p}, \boldsymbol{q}) = q_1 p_2 - q_2 p_1$.

The change of variables

$$\psi : \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \longmapsto \begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}$$

is symplectic, that is, $\omega = d\eta_1 \wedge d\xi_1 + d\eta_2 \wedge d\xi_2$, preserves the diagonal form of the Hamiltonian $H = \frac{1}{2}\,(\xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2) = \psi_* \widetilde{H}$, and diagonalizes the angular momentum $J = \frac{1}{2}\,(\xi_1^2 + \eta_1^2 - \xi_2^2 - \eta_2^2) = \psi_* \widetilde{J}$. The functions

$$\begin{aligned} A_1 &= \tfrac{1}{2}\,(\xi_1^2 + \eta_1^2) = H + J, \\ A_2 &= \tfrac{1}{2}\,(\xi_2^2 + \eta_2^2) = H - J \end{aligned} \tag{83}$$

are action variables for the two-dimensional harmonic oscillator. The corresponding angle variables are θ_1 and θ_2, respectively. In the variables $(\boldsymbol{A}, \boldsymbol{\theta}) = (A_1, A_2, \theta_1, \theta_2)$, the symplectic form ω is $\mathrm{d}A_1 \wedge \mathrm{d}\theta_1 + \mathrm{d}A_2 \wedge \mathrm{d}\theta_2$. The rescaled action angle coordinates $(\boldsymbol{j}, \boldsymbol{\vartheta}) = (j_1, j_2, \vartheta_1, \vartheta_2)$, used previously, are given by

$$j_i = 2\pi A_i \ \text{ and } \ \vartheta_i = \theta_i / 2\pi \ \text{ for } \ i = 1, 2.$$

The Bohr–Sommerfeld tori $M_{n,m} = \{x \in T^*\mathbb{R}^2 \mid j_1 = nh, j_2 = mh\}$ are parameterized by two integers n, m. The corresponding basic sections are

$$\sigma_{n,m} : M_{n,m} \to L : (\vartheta_1, \vartheta_2) \mapsto e^{-2\pi i (n\vartheta_1 + m\vartheta_2)},$$

see Equation (80). Equations (83) yield

$$\begin{aligned} H &= \tfrac{1}{2}\,(A_1 + A_2) = \tfrac{1}{2}\,(j_1/2\pi + j_2/2\pi), \\ J &= \tfrac{1}{2}\,(A_1 - A_2) = \tfrac{1}{2}\,(j_1/2\pi - j_2/2\pi). \end{aligned}$$

Hence, the quantum operators $\mathcal{Q}_H$ and $\mathcal{Q}_J$ act on $\sigma_{n,m}$ as follows.

$$\begin{aligned} \mathcal{Q}_H\,\sigma_{n,m} &= H_{|M_{n,m}}\sigma_{n,m} = \tfrac{1}{2}\,(j_1/2\pi + j_2/2\pi)_{|M_{n,m}}\sigma_{n,m} \\ &= \tfrac{1}{2}\,(nh/2\pi + m/2\pi)\sigma_{n,m} = \tfrac{1}{2}\,(n+m)\hbar\sigma_{n,m}, \end{aligned}$$

where $\hbar = h/2\pi$ and

$$\mathcal{Q}_J\,\sigma_{n,m} = J_{|M_{n,m}}\sigma_{n,m} = \tfrac{1}{2}\,(n-m)\hbar\,\sigma_{n,m}.$$

The regular part of $P = T^*\mathbb{R}^2$ is

$$P_0 = \{x \in T^*\mathbb{R}^2 \mid j_1(x) > 0 \text{ and } j_2(x) > 0\}.$$

The singular part of $P = T^*\mathbb{R}^2$ consists of three strata

$$\begin{aligned} S_1 &= \{x \in T^*\mathbb{R}^2 \mid j_1(x) > 0 \text{ and } j_2(x) = 0\}, \\ S_2 &= \{x \in T^*\mathbb{R}^2 \mid j_1(x) = 0 \text{ and } j_2(x) > 0\}, \\ S_0 &= \{x \in T^*\mathbb{R}^2 \mid j_1(x) = 0 \text{ and } j_2(x) = 0\}. \end{aligned}$$

S_0 is the origin of $T^*\mathbb{R}^2$, while S_1 and S_2 are cylinders parameterized by (j_1, ϑ_1) and (j_2, ϑ_2), respectively.

As before, for $i = 1, 2$, we consider the locally Hamiltonian vector fields

$$X_i = -\frac{\partial}{\partial j_i} = X_{\vartheta_i}.$$

The conditions of Corollary 1 are satisfied. Hence, in P_0, we get shifting operators

$$\begin{aligned} \boldsymbol{a}_{X_1}\sigma_{n,m} &= \sigma_{n-1,m}, \text{ provided } n > 1 \text{ and } m > 0, \\ \boldsymbol{b}_{X_1}\sigma_{n,m} &= \sigma_{n+1,m}, \text{ provided } n > 0 \text{ and } m > 0, \\ \boldsymbol{a}_{X_2}\sigma_{n,m} &= \sigma_{n,m-1}, \text{ provided } n > 0 \text{ and } m > 1, \\ \boldsymbol{b}_{X_2}\sigma_{n,m} &= \sigma_{n,m+1}, \text{ provided } n > 0 \text{ and } m > 0. \end{aligned}$$

Next, we have to consider limits as integral curves of X_1. Note that the integral curve $\mathrm{e}^{tX_1}(x_0)$ of X_1, originating at $x \in M_{1,m}$, after time $t = h$ reaches $x_1 = \mathrm{e}^{hX_1}(x_0) \in M_{0,m}$. Moreover, the integral curve $\mathrm{e}^{tX_1}(x_0)$ of X_1 originating at $x \in M_{n,0}$, for $n > 0$, after time $t = h$ reaches $M_{n-1,0}$ and after time $t = nh$ reaches the origin $M_{0,0}$. Similarly, the integral curve $\mathrm{e}^{-tX_1}(x)$ of $-X_1$ originating at $x \in M_{n,0}$ after time $t = h$ reaches $M_{n+1,0}$ and after time $t = kh$ it reaches M_{n+k} for every $k > 0$. This argument also applies to X_2. It enlarges the above table of shifting operators as follows.

$$\begin{aligned} \boldsymbol{a}_{X_1}\sigma_{1,m} &= \sigma_{0,m}, \text{ provided } m \geq 0, \\ \boldsymbol{a}_{X_2}\sigma_{n,1} &= \sigma_{n,0}, \text{ provided } n \geq 0. \end{aligned}$$

Since $X_1(x)$ is unbounded as $j_1 \to 0^+$, it is not possible to discuss integral curves of X_1 starting at points in $M_{0,n}$. However, for $n > 0$,

$$\boldsymbol{b}_{X_1}\sigma_{n,m} = \sigma_{n+1,m} \;\text{ and }\; \boldsymbol{a}_{X_1}\sigma_{n+1,m} = \sigma_{n,m}.$$

Thus, $\boldsymbol{b}_{X_1}$ shifts in the opposite direction to $\boldsymbol{a}_{X_1}$. Similarly, $\boldsymbol{b}_{X_2}$ shifts in the opposite direction to $\boldsymbol{a}_{X_2}$. It is natural to extend these relations to the boundary and assume that

$$\begin{aligned} \boldsymbol{b}_{X_1}\sigma_{0,m} &= \sigma_{1,m}, \text{ provided } m \geq 0, \\ \boldsymbol{b}_{X_2}\sigma_{n,0} &= \sigma_{n,1}, \text{ provided } n \geq 0. \end{aligned}$$

The actions of the lowering operators a_{X_1} on states σ_{0m} and a_{X_2} on states σ_{m0} not defined, but they never occur in the theory. Therefore, we may assume that

$$\boldsymbol{a}_{X_1}\sigma_{0,m} = 0, \text{ and } \ \boldsymbol{a}_{X_2}\sigma_{n,0} = 0.$$

3.5. Monodromy

Suppose that $\mathcal{U} = \{U_i\}_{i \in I}$ is a good covering of P such that for every $i \in I$ the chart (U_i, ψ_i) is the domain of a local trivialization of the toral bundle $\rho : P \to B$, associated to the fibrating toral polarization D of P, given by the local action angle coordinates

$$\rho_{|U_i} : U_i \to V_i \times \mathbb{T}^k : p \mapsto \psi_i(p) = (j^i, \vartheta^i) = (j^i_1, \ldots, j^i_k, \vartheta^i_1, \ldots, \vartheta^i_k)$$

with $(\rho_{|U_i})_*(\omega_{|U_i}) = \sum_{\ell=1}^k \mathrm{d}j^i_\ell \wedge \mathrm{d}\vartheta^i_\ell$. We suppose that the set $\mathcal{S}$ of Bohr–Sommerfeld tori on P has the local lattice structure $\{\mathbb{L}_{U_i}\}_{i \in I}$ of Section 3.3.

Let p and $p' \in P$ and let $\gamma : [0,1] \to P$ be a smooth curve joining p to p'. We can choose a finite good subcovering $\{U_k\}_{k=1}^N$ of $\mathcal{U}$ such that $\gamma([0,1]) \subseteq \cup_{k=1}^N U_k$, where $\gamma(0) \subseteq U_1$ and $\gamma(1) \in U_N$. Using the fact that the local lattices $\{\mathbb{L}_{U_k}\}_{k=1}^N$ are compatible, we can extend the local action functions j^1 on $V_1 = \psi_1(U_1) \subseteq B$ to a local action function j^N on $V_N \subseteq B$. Thus, using the connection $\mathcal{E}$ (see Corollary A1), we may parallel transport a Bohr–Sommerfeld torus $T_{\mathbf{n}} \subseteq U_1$ along the curve γ to a Bohr–Sommerfeld torus $T_{\mathbf{n}'} \subseteq U_N$ (see Claim 7). The action function at p', in general depends on the path γ. If the holonomy group of the connection $\mathcal{E}$ on the bundle $\rho : P \to B$ consists only of the identity element in $\mathrm{Gl}(k, \mathbb{Z})$, then this extension process does not depend on the path γ. Thus, we have shown

Claim 9. *If D is a fibrating toral polarization of (P, ω) with fibration $\rho : P \to B$ and B is simply connected, then there are global action angle coordinates on P and the Bohr–Sommerfeld tori $T_{\mathbf{n}} \in \mathcal{S}$ have a unique quantum number $\mathbf{n}$. Thus, the local lattice structure of $\mathcal{S}$ is the lattice $\mathbb{Z}^k$.*

If the holonomy of the connection $\mathcal{E}$ on P is not the identity element, then the set $\mathcal{S}$ of Bohr–Sommerfeld tori is not a lattice and it is not possible to assign a global labeling by quantum numbers to all the tori in $\mathcal{S}$. This difficulty in assigning quantum numbers to Bohr–Sommerfeld tori has been known to chemists since the early 1920s. Modern papers illustrating it can be found in [18,19]. We give a concrete example where the connection $\mathcal{E}$ has nontrivial holonomy, namely, the spherical pendulum.

Example: Spherical Pendulum

The spherical pendulum is a completely integrable Hamiltonian system $(H, J, T^*S^2, \mathrm{d}\alpha_{T^*S^2})$, where $T^*S^2 = \{(q,p) \in T^*\mathbb{R}^3 \,|\, \langle q, q\rangle = 1 \ \& \ \langle q, p\rangle = 0\}$ is the cotangent bundle of the 2-sphere S^2 with $\langle \ , \ \rangle$ the Euclidean inner product on $\mathbb{R}^3$, see [20]. The Hamiltonian is

$$H : T^*S^2 \to \mathbb{R} : (q,p) \mapsto \tfrac{1}{2}\langle p, p\rangle + \langle q, e_3\rangle,$$

where $e_3^T = (0,0,1) \in \mathbb{R}^3$ and the e_3-component of angular momentum is

$$J : T^*S^2 \to \mathbb{R} : (q,p) \mapsto q_1 p_2 - q_2 p_1.$$

The energy momentum map of the spherical pendulum is

$$\mathcal{EM} : T^*S^2 \to \overline{R} \subseteq \mathbb{R}^2 : (q,p) \mapsto \big(H(q,p), J(q,p)\big).$$

Here, $\overline{R}$ is the closure in $\mathbb{R}^2$ of the set R of regular values of the integral map $\mathcal{EM}$. The point $(1,0) \in \overline{R}$ is an isolated critical value of $\mathcal{EM}$. Thus, the set R has the homotopy type of S^1 and is *not* simply connected. Every fiber of

$$\mathcal{EM}_{|\mathcal{EM}^{-1}(R)} : \mathcal{EM}^{-1}(R) \to R \subseteq \mathbb{R}^2$$

over a point $(h,j) \in R$ is a smooth 2-torus $T^2_{h,j}$, see chapter V of [21]. At every point of $T^*S^2 \setminus (\mathcal{EM}^{-1}(1,0) \cup \mathcal{EM}^{-1}\partial\overline{R})$ there are local action angle coordinates $(A_1, A_2, \vartheta_1, \vartheta_2)$. The actions are $A_1 = \mathcal{A}_1 \circ \mathcal{EM}$ and $A_2 = \mathcal{A}_2 \circ \mathcal{EM}$. Here,

$$\mathcal{A}_1(h,j) = \frac{1}{\pi}\int_{\pi_1^-}^{\pi_1^+} \frac{\sqrt{2(h-\pi_1)(1-\pi_1^2)-j^2}}{1-\pi_1^2}\,\mathrm{d}\pi_1,$$

where $\pi_1^{\pm} \in [-1,1]$ and $2(h-\pi_1^{\pm})(1-(\pi_1^{\pm})^2) - j^2 = 0$, and

$$\mathcal{A}_2(h,j) = j;$$

while the angles are $\vartheta_1 = \widetilde{\vartheta}_1 \circ \mathcal{EM}$ and $\vartheta_2 = \widetilde{\vartheta}_2 \circ \mathcal{EM}$, where

$$\widetilde{\vartheta}_1(h,j) = \frac{j}{\pi}\int_{\pi_1^-}^{\pi_1^+} \frac{1}{(1-\pi_1^2)\sqrt{2(h-\pi_1)(1-\pi_1^2)-j^2}}\,\mathrm{d}\pi_1$$

and

$$\widetilde{\vartheta}_2(h,j) = \frac{t}{2\pi},$$

where t is the time parameter of the integral curves of the vector field X_J on the 2-torus $T^2_{h,j}$, which are periodic of period 2π, see Section 2.4 of [20]. The action map

$$\mathcal{A} : \overline{R} \subseteq \mathbb{R}^2 \to \mathbb{R}_{\geq 0} \times \mathbb{R} : (h,j) \mapsto \big(\mathcal{A}_1(h,j), \mathcal{A}_2(h,j)\big)$$

is a homeomorphism of $\overline{R} \setminus \{(1,0)\}$ onto $(\mathbb{R}_{\geq 0} \times \mathbb{R}) \setminus \{(\frac{4}{\pi}, 0)\}$, which is a real analytic diffeomorphism of $\overline{R} \setminus \{j = 0\}$ onto $\big(\mathbb{R}_{>0} \times (\mathbb{R} \setminus \{0\}\big)$, see Fact 2.4 in [20].

For every $(n,m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$, the Bohr–Sommerfeld tori are

$$T^2_{m,n} = \{(q,p) \in T^*S^2 \,\big|\, A_1(q,p) = n\,\mathrm{h} \ \&\ A_2(q,p) = m\,\mathrm{h}\}.$$

The fibers of $\mathcal{EM}$ corresponding to the dark points in Figure 1 are the Bohr–Sommerfeld tori.

The basic sections of the quantum line bundle $\pi : L \to T^*S^2$ are

$$\sigma_{n,m} : T^2_{n,m} \subseteq T^*S^2 \to L : (\vartheta_1, \vartheta_2) \mapsto \big(\vartheta_1, \vartheta_2, \mathrm{e}^{2\pi i(n\vartheta_1 + m\vartheta_2)}\big).$$

The family of sections $\mathfrak{B} = \{\sigma_{n,m} \big| (n,m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}\}$ forms a basis of quantum states of the Bohr–Sommerfeld theory of the spherical pendulum. Let $\mathfrak{H}$ be the Hilbert space of quantum states for

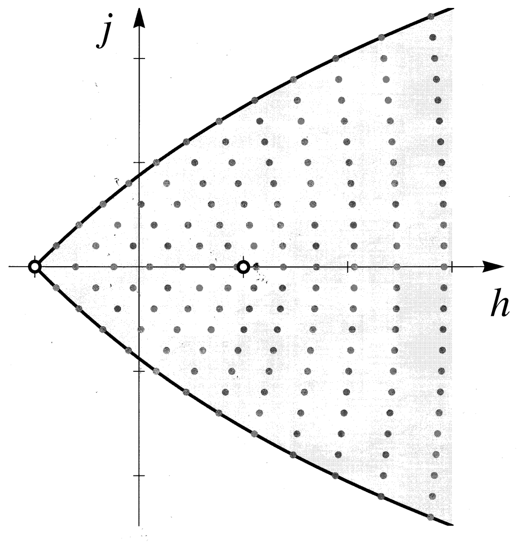

Figure 1. The Bohr–Sommerfeld quantum states of the spherical pendulum in $\overline{R}$.

which $\mathfrak{B}$ is an orthogonal basis. The Bohr–Sommerfeld energy momentum spectrum $\mathfrak{S}$ of the spherical pendulum is the range of the map

$$\begin{aligned} \mathcal{A}^{-1}_{|\mathbb{Z}_{\geq 0}\times\mathbb{Z}} : \mathbb{Z}_{\geq 0} \times \mathbb{Z} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R} \to \overline{R} : \\ (n,m) \mapsto \big(h(n\,\mathrm{h}, m\,\mathrm{h}), j(n\,\mathrm{h}, m\,\mathrm{h})\big) = \big(h_m(n)\,\hbar, m\,\hbar\big). \end{aligned}$$

$(n,m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ are the quantum numbers of the spherical pendulum.

In terms of actions A_1 and A_2, we may write $H = H(A_1, A_2)$. Hence, the quantum operators Q_H and Q_J act on the basic sections $\sigma_{m.n}$ as follows

$$Q_H \sigma_{n,m} = H_{|T^2_{n,m}} \sigma_{n,m} = h_m(n)\,\hbar \sigma_{n,m}$$

and

$$Q_J \sigma_{n,m} = J_{|T^2_{n,m}} \sigma_{n,m} = m\,\hbar \sigma_{n,m}.$$

The regular part of T^*S^2 is

$$S_0 = \mathcal{EM}^{-1}(R) = \mathcal{EM}^{-1}\big(\mathcal{A}^{-1}(\{\mathcal{A}_1 > 0 \;\&\; \mathcal{A}_2 \neq 0\})\big).$$

The singular part of T^*S^2 consists of six strata:

$$\begin{aligned} S_1 &= \mathcal{EM}^{-1}\big(\mathcal{A}^{-1}(\{(0,0)\}\big), \\ S_2 &= \big(\mathcal{A}^{-1}(\{(4/\pi,0)\}\big), \\ S_3 &= \big(\mathcal{A}^{-1}(\{(\mathcal{A}_1,0)\,\big|\, 0 < \mathcal{A}_1 < 4/\pi\}\big), \\ S_4 &= \big(\mathcal{A}^{-1}(\{(\mathcal{A}_1,0)\,\big|\, 4/\pi < \mathcal{A}_1\}\big), \\ S_5 &= \big(\mathcal{A}^{-1}(\{(0,\mathcal{A}_2)\,\big|\, \mathcal{A}_2 > 0\}\big), \\ S_6 &= \big(\mathcal{A}^{-1}(\{(0,\mathcal{A}_2)\,\big|\, \mathcal{A}_2 < 0\}\big). \end{aligned}$$

The stratum S_1 is the point $(0,0,-1,0,0,0) \in T^*S^2$; while the stratum S_2 is the point $(0,0,1,0,0,0)$. The stratum S_3 is the subset of T^*S^2, where $A_1 \in (0, 4/\pi)$ and $A_2 = 0$, which is a cylinder parameterized by (A_1, ϑ_1); while S_4 is the subset where $A_1 \in (4/\pi, \infty)$ and $A_2 = 0$, which is a cylinder parameterized by (A_1, ϑ_1). The stratum S_5 is the subset of T^*S^2 where $A_1 = 0$ and $A_2 > 0$, which is a cylinder parameterized by (A_2, ϑ_2); while S_6 is the subset where $A_1 = 0$ and $A_2 < 0$, which is a cylinder parameterized by (A_2, ϑ_2).

The conditions of Corollary 1 are satisfied. For $i = 1, 2$ let $X_i = X_{\vartheta_i}$. In the regular stratum S_0 we get the shifting operators

$$\begin{aligned}
\mathbf{a}_{X_1}\sigma_{n,m} &= \sigma_{n-1,m}, \text{ provided } n > 1 \text{ and } m \neq 0 \\
\mathbf{b}_{X_1}\sigma_{n,m} &= \sigma_{n+1,m}, \text{ provided } n > 0 \text{ and } m \neq 0 \\
\mathbf{a}_{X_2}\sigma_{n,m} &= \sigma_{n,m-1}, \text{ provided } n > 0 \text{ and } m \neq 1 \\
\mathbf{b}_{X_2}\sigma_{n,m} &= \sigma_{n,m+1}, \text{ provided } n > 0 \text{ and } m \neq 0
\end{aligned}$$

Arguing as in the example of the 2-d harmonic oscillator, we can extend the above relations to

$$\begin{aligned}
\mathbf{b}_{X_1}\sigma_{0,m} &= \sigma_{1,m}, \text{ provided } m \neq 0 \\
\mathbf{b}_{X_2}\sigma_{n,0} &= \sigma_{n,1}, \text{provided } m \neq 0.
\end{aligned}$$

In addition, we may assume that

$$\mathbf{a}_{X_1}\sigma_{0,m} = 0 \text{ and } \mathbf{a}_{X_2}\sigma_{n,0} = 0.$$

Since the are no global action angle coordinates, the action function $\mathcal{A}_1$ on R is multi-valued. After encircling the point $(1,0)$, the quantum number of the Bohr–Sommerfeld torus represented by the upper right hand vertex of the rectangle on the h-axis, see Figure 2, becomes the quantum number of the upper right hand vertex of the parallelogram formed by applying $M^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to the original rectangle, which is the transpose of the monodromy matrix M of the spherical pendulum.

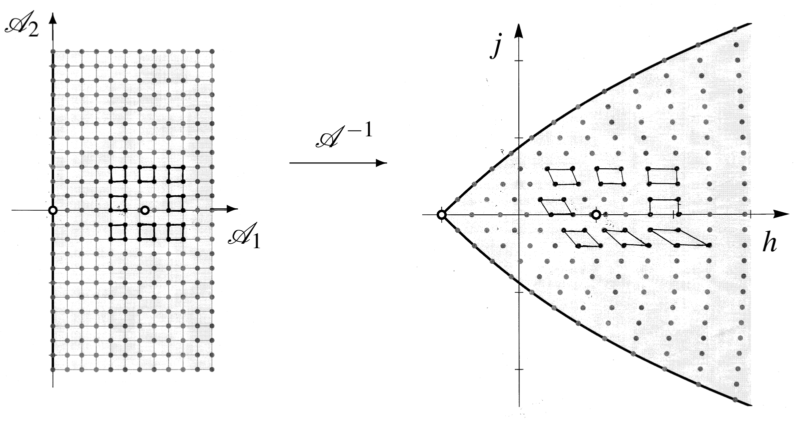

Figure 2. Using the shifting operators to show that the quantized spherical pendulum has monodromy.

The holonomy of the connection $\mathcal{E}$ is called the *monodromy* of the fibrating toral polarization D on (P, ω) with fibration $\rho : P \to B$.

Corollary 2. *Let $\widetilde{B}$ be the universal covering space of B with covering map $\Pi : \widetilde{B} \to B$. The monodromy map M, which is a nonidentity element holonomy group of the connection $\mathcal{E}$ on the bundle ρ sends one sheet of the universal covering space to another sheet.*

Proof. Since the universal covering space $\widetilde{B}$ of B is simply connected and we can pull back the symplectic manifold (P, ω) and the fibrating toral distribution D by the universal covering map to a symplectic manifold $(\widetilde{P}, \widetilde{\omega})$ and a fibrating toral distribution $\widetilde{D}$ with associated fibration $\widetilde{\rho} : \widetilde{P} \to \widetilde{B}$. The connection $\mathcal{E}$ on the bundle ρ pulls back to a connection $\widetilde{\mathcal{E}}$ on the bundle $\widetilde{\rho}$. Let γ be a closed curve on B and let M be the holonomy of the connection $\mathcal{E}$ on B along γ. Then, γ lifts to a curve $\widetilde{\gamma}$ on $\widetilde{B}$, which covers γ, that is, $\widetilde{\rho} \circ \widetilde{\gamma} = \gamma$. Thus, parallel transport of a k-torus $T = \mathbb{R}^k/\mathbb{Z}^k$, which is an integral manifold of the distribution $\widetilde{D}$, along the curve $\widetilde{\gamma}$ gives a linear map M of the lattice $\mathbb{Z}^k$ defining the k-torus $M(\widetilde{T})$. The map M is the same as the linear map M of $\mathbb{Z}^k$ into itself given by parallel transporting T, using the connection $\mathcal{E}$, along the closed γ on B because the connection $\widetilde{\mathcal{E}}$ is the pull back of the connection $\mathcal{E}$ by the covering map ρ. The closed curve γ in B represents an element of the fundamental group of B, which acts as a covering transformation on the universal covering space $\widetilde{B}$ that permutes the sheets (= fibers) of the universal covering map $\widetilde{\Pi}$. □

In the spherical pendulum, the universal covering space $\widetilde{R}$ of $R \setminus \{(1,0)\}$ is $\mathbb{R}^2$. If we cut R by the line segment $\ell = \{(h,0) \in R \big| h > 1\}$, then $R^\times = R \setminus \ell$ is simply connected and hence represents one sheet of the universal covering map of R. For more details on the universal covering map, see [22]. The curve chosen in the example has holonomy $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It gives a map of $\widetilde{R}$ into itself, which sends $R^\times$ to the adjacent sheet of the covering map. Thus, we have a rule how the labelling of the Bohr–Sommerfeld torus $T_{(n_1,n_2)}$, corresponding to $(h,j) \in R^\times$, changes when we go to an adjacent sheet, which covers $R^\times$, namely, we apply the matrix M to the integer vector $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$. Since our chosen curve generates the fundamental group of $R \setminus \{(1,0)\}$, we know what the quantum numbers of Bohr–Sommerfeld are for any closed curve in $R \setminus \{(1,0)\}$, which encircles the origin.

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Appendix A

We return to study the symplectic geometry of a fibrating toral polarization D of the symplectic manifold (P, ω) in order to explain what we mean by its local integral affine structure, see [23].

We assume that the integral manifolds $\{M_p\}_{p \in P}$ of the Lagrangian distribution D on P form a smooth manifold B such that the map

$$\rho : P \to B : p \mapsto M_p$$

is a proper surjective submersion. If the distribution D has these properties we refer to it as a *fibrating polarization* of (P, ω) with *associated fibration* $\rho : P \to B$.

Lemma A1. *Suppose that D is a fibrating polarization of (P, ω). Then, the associated fibration $\rho : P \to B$ has an Ehresmann connection $\mathcal{E}$ with parallel translation. Thus, the fibration $\rho : P \to B$ is locally trivial bundle.*

Proof. We construct the Ehresmann connection as follows. For each $p \in P$ let (U, ψ) be a *Darboux chart* for (P, ω). In other words, $(\psi^{-1})^*(\omega_{|U})$ is the standard symplectic form ω_{2k} on TV, where $V = \psi(U) \subseteq \mathbb{R}^{2k}$ with $\psi(p) = 0$. In more detail, for every $u \in U$ there is a frame $\varepsilon(u)$ of

P at u, whose image under $T_u\psi$ is the frame $\varepsilon(v) = \{\frac{\partial}{\partial x_1}\big|_v, \ldots, \frac{\partial}{\partial x_k}\big|_v, \frac{\partial}{\partial y_1}\big|_v, \ldots, \frac{\partial}{\partial y_k}\big|_v\}$ of $T_vV = \mathbb{R}^{2k}$, where $v = \psi(u)$, such that

$$\omega_{2k}(v)\big(\frac{\partial}{\partial x_i}\Big|_v, \frac{\partial}{\partial x_j}\Big|_v\big) = \omega_{2k}(v)\big(\frac{\partial}{\partial y_i}\Big|_v, \frac{\partial}{\partial y_j}\Big|_v\big) = 0$$

and

$$\omega_{2k}(v)\big(\frac{\partial}{\partial x_i}\Big|_v, \frac{\partial}{\partial y_j}\Big|_v\big) = \delta_{ij}.$$

For $u \in M_p \cap U$, we see that $\lambda_v = T_u\psi(T_uM_p)$ is a Lagrangian subspace of the symplectic vector space $\big(T_vV, \omega_{2k}(v)\big)$. Let $\{\frac{\partial}{\partial z_j}\big|_v\}_{j=1}^k$ be a basis of λ_v with $\{\mathrm{d}z_j(v)\}_{j=1}^k$ the corresponding dual basis of λ_v^*. Extend each covector $\mathrm{d}z_j(v)$ by zero to a covector $\mathrm{d}Z_j(v)$ in T_v^*V, that is, extend the basis $\{\mathrm{d}z_j(v)\}_{j=1}^k$ of λ_v^* to a basis $\{\mathrm{d}Z_j(v)\}_{j=1}^k$ of T_v^*V, where $\begin{cases} \mathrm{d}Z_j(v)_{|\lambda_v} = \mathrm{d}z_j(v), \text{ for } j = 1,\ldots,k \\ \mathrm{d}Z_j(v)_{|\lambda_v} = 0, \text{ for } j = k+1,\ldots,2k. \end{cases}$ Since $\omega_{2k}^\sharp(v) : T_vV \to T_v^*V$ is a linear isomorphism with inverse $\omega_{2k}^\flat(v)$ for every $v \in V$, we see that the collection

$$\{\frac{\partial}{\partial w_j}\Big|_v = \omega_{2k}^\flat(v)(\mathrm{d}Z_j(v))\}_{j=1}^k$$

of vectors in T_vV spans an k-dimensional subspace μ_v. We now show that μ_v is a Lagrangian subspace of $\big(T_vV, \omega_{2k}(v)\big)$. By definition

$$\omega_{2k}(v)\big(\frac{\partial}{\partial w_i}\Big|_v, \frac{\partial}{\partial w_j}\Big|_v\big) = \omega_{2k}^\sharp(v)\big(\frac{\partial}{\partial w_i}\Big|_v\big)\frac{\partial}{\partial w_j}\Big|_v = \mathrm{d}Z_i(v)\frac{\partial}{\partial w_j}\Big|_v = 0.$$

The last equality above follows because $\frac{\partial}{\partial w_j}\big|_v \notin \lambda_v$. To see this we note that

$$\omega_{2k}(v)\big(\frac{\partial}{\partial w_j}\Big|_v, \frac{\partial}{\partial z_j}\Big|_v\big) = \mathrm{d}Z_j(v)\frac{\partial}{\partial z_j}\Big|_v = \mathrm{d}z_j(v)\frac{\partial}{\partial z_j}\Big|_v = 1.$$

The Lagrangian subspace μ_v is complementary to the Lagrangian subspace λ_v, that is, $T_vV = \lambda_v \oplus \mu_v$ for every $v \in V$.

Consequently, $\mathrm{hor}_u = T_v\psi^{-1}\mu_v$ is a Lagrangian subspace of $\big(T_uU, \omega(u)\big)$, which is complementary to the Lagrangian subspace T_uM_p. Since the mapping $\mathrm{hor}_{|U} : U \to TU : u \mapsto \mathrm{hor}_u$ is smooth and has constant rank, it defines a Lagrangian distribution $\mathrm{hor}_{|U}$ on U. Hence, we have a Lagrangian distribution hor on (P, ω). Since T_uM_p is the tangent space to the fiber $\rho^{-1}\big(\rho(p)\big) = M_p$, the distribution $\mathrm{ver}_{|U} : U \to TU : u \mapsto \mathrm{ver}_u = T_uM_p = \lambda_v$ defines the vertical Lagrangian distribution ver on P. Because $\mathrm{ver}_u = \ker T_u\rho$, it follows that $T_u\rho(\mathrm{hor}_u) = T_{\rho(u)}B$. Hence, the linear mapping $T_u\rho_{|\mathrm{hor}_u} : \mathrm{hor}_u \to T_{\rho(u)}B$ is an isomorphism. Since $T_pP = \mathrm{hor}_p \oplus \mathrm{ver}_p$ for every $p \in P$ and the mapping $T_p\rho_{|\mathrm{hor}_p} : \mathrm{hor}_p \to T_{\rho(p)}B$ is an isomorphism for every $p \in P$, the distributions hor and ver on P define an *Ehresmann connection* $\mathcal{E}$ for the Lagrangian fibration $\rho : P \to B$. $\square$

Let X be a smooth complete vector field on B with flow e^{tX}. Because the linear mapping $T_p\rho_{|\mathrm{hor}_p}$: $\mathrm{hor}_p \to T_{\rho(p)}B$ is bijective, there is a unique smooth vector field lift$X$ on P, called the *horizontal lift* of X, which is ρ-related to X, that is, $T_p\rho\,\mathrm{lift}X(p) = X\big(\rho(p)\big)$ for every $p \in P$. Let $\mathrm{e}^{t\,\mathrm{lift}X}$ be the flow of lift$X$. Then, $\rho(\mathrm{e}^{t\,\mathrm{lift}X}) = \mathrm{e}^{tX}(\rho(p))$. Let $\sigma : W \subseteq B \to P$ be a smooth local section of the bundle $\rho : P \to B$. Define the *covariant derivative* $\nabla_X\sigma$ of σ with respect to the vector field X by

$$(\nabla_X\sigma)(w) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{e}^{-t\,\mathrm{lift}\,X}\big(\sigma(\mathrm{e}^{t\,X}(w))\big)$$

for all $w \in W$. Because the bundle projection map ρ is proper, *parallel transport* of each fiber of the bundle $\rho : P \rightarrow B$ by the flow of lift$X$ is defined as long as the flow of X is defined. Because the Ehresmann connection $\mathcal{E}$ has parallel transport, the bundle presented by ρ is locally trivial, see pp. 378–379, [21].

Claim A1. *If D is a fibrating polarization of the symplectic manifold (P, ω), then for every $p \in P$ the integral manifold of D through p is a smooth Lagrangian submanifold of P, which is an k-torus T. In fact T is the fiber over $\rho(p)$ of the associated fibration $\rho : P \rightarrow B$.*

We say that D is a *fibrating toral polarization* of (P, ω) if it satisfies the hypotheses of Claim A1. The proof of Claim A1 requires several preparatory arguments.

Let $f \in C^\infty(B)$. Then, $\rho^* f \in C^\infty(P)$. Let $X_{\rho^* f}$ be the Hamiltonian vector field on (P, ω) with Hamiltonian $\rho^* f$. We have

Lemma A2. *Every fiber of the locally trivial bundle $\rho : P \rightarrow B$ is an invariant manifold of the Hamiltonian vector field $X_{\rho^* f}$.*

Proof. We need only show that for every $p \in P$ and every $q \in M_p$, we have $X_{\rho^* f}(q) \in T_q M_p$. Let Y be a smooth vector field on the integral manifold M_p with flow e^{tY}. Then,

$$\rho^* f\big(\mathrm{e}^{tY}(q)\big) = f\big(\rho(\mathrm{e}^{tY}(q))\big) = f\big(\rho(p)\big),$$

since e^{tY} maps M_p into itself. So

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho^* f\big(\mathrm{e}^{tY}(q)\big) = L_Y(\rho^* f)(q) = \mathrm{d}\big(\rho^* f\big)(q) Y(q) \\ &= -\omega(q)\big(X_{\rho^* f}(q), Y(q)\big). \end{aligned}$$

However, $T_q M_p$ is a Lagrangian subspace of the symplectic vector space $(T_q P, \omega(q))$. Consequently, $X_{\rho^* f}(q) \in T_q M_p$. □

Since the mapping $\rho : P \rightarrow B$ is surjective and proper, for every $b \in B$ the fiber $\rho^{-1}(b)$ is a smooth compact submanifold of P. Hence, the flow $\mathrm{e}^{t\, X_{\rho^* f}}$ of the vector field $X_{\rho^* f}$ is defined for all $t \in \mathbb{R}$.

Lemma A3. *Let $f, g \in C^\infty(B)$. Then, $\{\rho^* f, \rho^* g\} = 0$.*

Proof. For every $p \in P$ and every $q \in M_p$ from Lemma A2 it follows that $X_{\rho^* f}(q)$ and $X_{\rho^* g}(q)$ lie in $T_q M_p$. Because M_p is a Lagrangian submanifold of (P, ω), we get

$$0 = \omega(q)\big(X_{\rho^* g}(q), X_{\rho^* f}(q)\big) = \{\rho^* f, \rho^* g\}(q). \tag{A1}$$

Since $P = \amalg_{p \in P} M_p$, we see that (A1) holds for every $p \in P$. □

Proof of Claim A1. From Lemma A3 it follows that $\big(\rho^*(C^\infty(B)), \{\ ,\ \}, \cdot\big)$ is an abelian subalgebra $\mathfrak{t}$ of the Poisson algebra $(C^\infty(P), \{\ ,\ \}, \cdot)$. Since the bundle projection mapping $\rho : P \rightarrow B$ is surjective and $\dim B = k$, the algebra $\mathfrak{t}$ has k generators, say, $\{\rho^* f_i\}_{i=1}^k$, whose differentials at q span $T_q(\rho^{-1}(b))$ for every $b \in B$ and every $q \in \rho^{-1}(b)$. Using the flow $\mathrm{e}^{t\, X_{\rho^* f_i}}$ of the Hamiltonian vector field $X_{\rho^* f_i}$ on (P, ω) define the $\mathbb{R}^k$-action

$$\Phi : \mathbb{R}^k \times P \rightarrow P; \big(\mathbf{t} = (t_1, \ldots, t_k), p\big) \mapsto \big(\mathrm{e}^{t_1 X_{\rho^* f_1}}(p), \ldots, \mathrm{e}^{t_k X_{\rho^* f_k}}(p)\big) \tag{A2}$$

Since $\mathrm{span}_{1 \le i \le k}\{X_{\rho^* f_i}(q)\} = T_q(\rho^{-1}(b))$ and each fiber is connected, being an integral manifold of the distribution D, it follows that the $\mathbb{R}^k$-action Φ is transitive on each fiber $\rho^{-1}(b)$ of the bundle $\rho : P \rightarrow B$. Thus, $\rho^{-1}(b)$ is diffeomorphic to $\mathbb{R}^k / P_q$, where $P_q = \{\mathbf{t} \in \mathbb{R}^k \,\big|\, \Phi_{\mathbf{t}}(q) = q\}$ is the isotropy group at q.

If $P_q = \{0\}$ for some $q \in P$, then the fiber $\rho^{-1}\big(\rho(q)\big)$ would be diffeomorphic to $\mathbb{R}^k/P_q = \mathbb{R}^k$. However, this contradicts the fact the every fiber of the bundle $\rho : P \to B$ is compact. Hence, $P_q \neq \{0\}$ for every $q \in P$. Since $\mathbb{R}^k/P_q$ is diffeomorphic to $\rho^{-1}(b)$, they have the same dimension, namely, k. Hence, P_q is a zero dimensional Lie subgroup of $\mathbb{R}^k$. Thus, P_q is a rank k lattice $\mathbb{Z}^k$. Thus, the fiber $\rho^{-1}(b)$ is $\mathbb{R}^k/\mathbb{Z}^k$, which is an *affine* k-torus $\mathbb{T}^k$. □

We now apply the action angle theorem, see chapter IX of [21], to the fibrating toral Lagrangian polarization D of the symplectic manifold (P, ω) with associated toral bundle $\rho : P \to B$ to obtain a more precise description of the Ehresmann connection $\mathcal{E}$ constructed in Claim A1. For every $p \in P$ there is an open neighborhood U of the fiber $\rho^{-1}\big(\rho(p)\big)$ in P and a symplectic diffeomorphism

$$\begin{aligned} \psi : U = \rho^{-1}(V) \subseteq P \to V \times \mathbb{T}^k \subseteq \mathbb{R}^k \times \mathbb{T}^k :\\ u \mapsto (j, \vartheta) = (j_1, \ldots, j_k, \vartheta_1, \ldots, \vartheta_k) \end{aligned}$$

such that

$$\rho_{|U} : U \subseteq P \to V \subseteq \mathbb{R}^k : u \mapsto (\pi_1 \circ \psi)(u) = j,$$

is the momentum mapping of the Hamiltonian $\mathbb{T}^k$-action on $(U, \omega_{|U})$. Here, $\pi_1 : V \times \mathbb{T}^k \to V : (j, \vartheta) \to j$. Thus, the bundle $\rho : P \to B$ is locally a principal $\mathbb{T}^k$-bundle. Moreover, we have $(\psi^{-1})^* \omega_{|U} = \sum_{i=1}^{k} \mathrm{d}j_i \wedge \mathrm{d}\vartheta_i$.

Corollary A1. *Using the chart (U, ψ) for action angle coordinates (j, ϕ), the Ehresmann connection $\mathcal{E}_{|U}$ gives an Ehresmann connection $\mathcal{E}_{|V \times \mathbb{T}^n}$ on the bundle $\pi_1 : V \times \mathbb{T}^k \to V$ defined by*

$$\mathrm{ver}_v = \mathrm{span}_{1 \le i \le k}\{\frac{\partial}{\partial \vartheta_i}\Big|_{v=\psi(u)}\} \ \text{ and } \ \mathrm{hor}_v = \mathrm{span}_{1 \le i \le k}\{\frac{\partial}{\partial j_i}\Big|_{v=\psi(u)}\}.$$

Proof. This follows because

$$T_u\psi(\mathrm{ver}_u) = \mathrm{span}_{1 \le i \le k}\{\frac{\partial}{\partial \vartheta_i}\Big|_{v=\psi(u)}\} \ \text{ and } \ T_p\psi(\mathrm{hor}_u) = \mathrm{span}_{1 \le i \le k}\{\frac{\partial}{\partial j_i}\Big|_{v=\psi(u)}\},$$

for every $u \in U$. From the preceding equations for every $u \in U$ we have $\mathrm{ver}_u = \mathrm{span}_{1 \le i \le k}\{X_{\rho^*(j_i)}(u)\}$ and $\mathrm{hor}_u = \mathrm{span}_{1 \le i \le k}\{X_{(\pi_2 \circ \psi)^*(-\vartheta_i)}(u)\}$. Here, $\pi_2 : V \times \mathbb{T}^k \to \mathbb{T}^k : (j, \varphi) \mapsto \varphi$. □

Corollary A2. *The Ehresmann connection $\mathcal{E}$ on the locally trivial toral Lagrangian bundle $\rho : P \to B$ is flat, that is, $\nabla_X \sigma = 0$ for every smooth vector field X on B and every local section σ of $\rho : P \to B$.*

Proof. In action angle coordinates a local section section σ of the bundle $\rho : P \to B$ is given by $\sigma : V \to V \times \mathbb{T}^k : j \mapsto \big(j, \sigma(j)\big)$. Let $X = \frac{\partial}{\partial j_\ell}$ for some $1 \le \ell \le k$ with flow e^{tX}. Let lift$X$ be the horizontal lift of X with respect to the Ehresmann connection $\mathcal{E}_{V \times \mathbb{T}^k}$ on the bundle $\pi_1 : V \times \mathbb{T}^k \to V$. Thus, for every $j \in V$ we have

$$\begin{aligned} (\nabla_X \sigma)(j) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{e}^{t\,\mathrm{lift}X}\big(\sigma(\mathrm{e}^{-tX}(j))\big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{e}^{t\,\mathrm{lift}X}\big(\sigma(j(-t))\big), \quad \text{where } \mathrm{e}^{tX}(j) = j(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{e}^{t\,\mathrm{lift}X}\big(j, \sigma(j)\big), \quad \text{since } j_i \text{ for } 1 \le i \le n \text{ are integrals of } X \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(j(t), \sigma(j(t))\big), \quad \text{since } \pi_1\big(\mathrm{e}^{t\,\mathrm{lift}X}(j, \sigma(j))\big) = \mathrm{e}^{tX}(j) \\ &= 0. \end{aligned}$$

This proves the corollary, since every vector field X on $W \subseteq B$ may be written as $\sum_{i=1}^{k} c_i(j) \frac{\partial}{\partial j_i}$ for some $c_i \in C^\infty(W)$ and the flow $\{\varphi_t^{j_i}\}_{i=1}^{k}$ of $\{\frac{\partial}{\partial j_i}\}_{i=1}^{k}$ on V pairwise commute. $\square$

Claim A2. *Let $\rho : P \to B$ be a locally trivial toral Lagrangian bundle, where (P, ω) is a smooth symplectic manifold. Then, the smooth manifold B has an integral affine structure. In other words, there is a good open covering $\{W_i\}_{i \in I}$ of B such that the overlap maps of the coordinate charts (W_i, φ_i) given by*

$$\varphi_{i\ell} = \varphi_\ell \circ \varphi_i^{-1} : V_i \cap V_\ell \subseteq \mathbb{R}^k \to V_i \cap V_\ell \subseteq \mathbb{R}^k,$$

where $\varphi_i(W_i) = V_i$, have derivative $D\varphi_{i\ell}(v) \in \mathrm{Gl}(k, \mathbb{Z})$, which does not depend on $v \in V_i \cap V_\ell$.

Proof. Cover P by $\mathcal{U} = \{U_i\}_{i \in I}$, where (U_i, ψ_i) is an action angle coordinate chart. Since every open covering of P has a good refinement, we may assume that $\mathcal{U}$ is a good covering. Let $W_i = \rho(U_i)$. Then, $\mathcal{W} = \{W_i\}_{i \in I}$ is a good open covering of B and $(W_i, \varphi_i = \pi_1 \circ \psi_i)$ is a coordinate chart for B. By construction of action angle coordinates, in $V_i \cap V_\ell$ the overlap map $\varphi_{i\ell}$ sends the action coordinates j^i in $V_i \cap V_\ell$ to the action coordinates j^ℓ in $V_i \cap V_\ell$. The period lattices $P_{\psi_i^{-1}(j^i)}$ and $P_{\psi_\ell^{-1}(j^\ell)}$ are equal since for some $p \in W_i \cap W_\ell$ we have $\psi_i(p) = j^i$ and $\psi_\ell(p) = j^\ell$. Moreover, these lattices do not depend on the point p. Thus, the derivative $D\varphi_{i\ell}(j)$ sends the lattice $\mathbb{Z}^k$ spanned by $\{\frac{\partial}{\partial j^i}\big|_j\}_{i=1}^{k}$ into itself. Hence, for every $j \in W_i \cap W_\ell$ the matrix of $D\varphi_{i\ell}(j)$ has integer entries, that is, it lies in $\mathrm{Gl}(k, \mathbb{Z})$ and the map $j \mapsto D\varphi_{i\ell}(j)$ is continuous. However, $\mathrm{Gl}(k, \mathbb{Z})$ is a discrete subgroup of the Lie group $\mathrm{Gl}(k, \mathbb{R})$ and $W_i \cap W_\ell$ is connected, since $\mathcal{W}$ is a good covering. Thus, $D\varphi_{i\ell}(j)$ does not depend on $j \in W_i \cap W_\ell$. $\square$

Corollary A3. *Let $\gamma : [0, 1] \to B$ be a smooth closed curve in B. Let $P_\gamma : [0, 1] \to P$ be parallel translation along γ using the Ehresmann connection $\mathcal{E}$ on the bundle $\rho : P \to B$. Then, the holonomy group of the k-toral fiber $T_{\gamma(0)} = \mathbb{T}^k$ is induced by the group $\mathrm{Gl}(k, \mathbb{Z}) \ltimes \mathbb{Z}^k$ of affine $\mathbb{Z}$-linear maps of $\mathbb{Z}^k$ into itself.*

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