axioms

MDPI

Article

On Some Coupled Fixed Points of Generalized T-Contraction Mappings in a $b_v(s)$-Metric Space and Its Application

Reny George [1,2,*], Zoran D. Mitrović [3,*] and Stojan Radenović [4]

1 Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia
2 Department of Mathematics and Computer Science, St. Thomas College, Bhilai, Chhattisgarh 491022, India
3 Faculty of Electrical Emgineering, University of Banja Luka, Patre 5, 78000 Banja Luka, Bosnia and Herzegovina
4 Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11000 Beograd, Serbia; radens@beotel.net
* Correspondence: r.kunnelchacko@psau.edu.sa (R.G.); zoran.mitrovic@etf.unibl.org (Z.D.M.)

Received: 16 October 2020; Accepted: 7 November 2020; Published: 9 November 2020

Abstract: Common coupled fixed point theorems for generalized T-contractions are proved for a pair of mappings $S : X \times X \to X$ and $g : X \to X$ in a $b_v(s)$-metric space, which generalize, extend, and improve some recent results on coupled fixed points. As an application, we prove an existence and uniqueness theorem for the solution of a system of nonlinear integral equations under some weaker conditions and given a convergence criteria for the unique solution, which has been properly verified by using suitable example.

Keywords: common coupled fixed point; $b_v(s)$-metric space; T-contraction; weakly compatible mapping

1. Introduction

In the last three decades, the definition of a metric space has been altered by many authors to give new and generalized forms of a metric space. In 1989, Bakhtin [1] introduced one such generalization in the form of a b-metric space and in the year 2000 Branciari [2] gave another generalization in the form a rectangular metric space and generalized metric space. Thereafter, using the above two concepts, many generalizations of a metric space appeared in the form of rectangular b-metric space [3], hexagonal b-metric space [4], pentagonal b-metric space [5], etc. The latest such generalization was given by Mitrović and Radenović [6] in which the authors defined a $b_v(s)$-metric space which is a generalization of all the concepts told above. Some recent fixed point theorems in such generalized metric spaces can be found in [6–9]. In [10–12], one can find some interesting coupled fixed point theorems and their applications proved in some generalized forms of a metric space. In the present note, we have given coupled fixed point results for a pair of generalized T-contraction mappings in a $b_v(s)$-metric space. Our results are new and it extends, generalize, and improve some of the coupled fixed point theorems recently dealt with in [10–12].

In recent years, fixed point theory has been successfully applied in establishing the existence of solution of nonlinear integral equations (see [11–15]). We have applied one of our results to prove the existence and convergence of a unique solution of a system of nonlinear integral equations using some weaker conditions as compared to those existing in literature.

2. Preliminaries

Definition 1. *[6] Let X be a nonempty set. Assume that, for all $x, y, \in X$ and distinct $u_1, \cdots, u_v \in X - \{x, y\}$, $d_v : X \times X \to R$ satisfies :*

1. $d_v(x, y) \geq 0$ *and* $d_v(x, y) = 0$ *if and only if* $x = y$,
2. $d_v(x, y) = d_v(y, x)$,
3. $d_v(x, y) \leq s[d_v(x, u_1) + d_v(u_1, u_2) + \cdots + d_v(u_{v-1}, u_v) + d_v(u_v, y)]$, *for some* $s \geq 1$.

Then, (X, d_v) *is a* $b_v(s)$*-metric space.*

Definition 2. *[6] In the* $b_v(s)$*-metric space* (X, d_v)*, the sequence* $< u_n >$

(a) *converges to* $u \in X$ *if* $d_v(u_n, u) \to 0$ *as* $n \to \infty$;
(b) *is a Cauchy sequence if* $d_v(u_n, u_m) \to 0$ *as* $n, m \to +\infty$.

Clearly, $b_1(1)$-metric space is the usual metric space, whereas $b_1(s)$, $b_2(1)$, $b_2(s)$, and $b_v(1)$-metric spaces are, respectively, the b-metric space ([1]), rectangular metric space ([2]), rectangular b-metric space ([3]), and v-generalized metric space ([2]).

Lemma 1. *[6] If* (X, d_v) *is a* $b_v(s)$*-metric space, then* (X, d_v) *is a* $b_{2v}(s^2)$*-metric space.*

Definition 3. *An element* $(u, v) \in X \times X$ *is called a coupled coincidence point of* $S : X \times X \to X$ *and* $g : X \to X$ *if* $g(u) = S(u, v)$ *and* $g(v) = S(v, u)$*. In this case, we also say that* $(g(u), g(v))$ *is the point of coupled coincidence of* S *and* g*. If* $u = g(u) = S(u, v)$ *and* $v = g(v) = S(v, u)$*, then we say that* (u, v) *is a common coupled fixed point of* S *and* g.

We will denote by $COCP\{S, g\}$ and $CCOFP\{S, g\}$ respectively the set of all coupled coincidence points and the set of all common coupled fixed points of S and g.

Definition 4. $S : X \times X \to X$ *and* $g : X \to X$ *are said to be weakly compatible if and only if* $S(g(u), g(v)) = g(S(u, v))$ *for all* $(u, v) \in COCP\{S, g\}$.

3. Main Results

We will start this section by proving the following lemma which is an extension of Lemma 1.12 of [6] to two sequences:

Lemma 2. *Let* (X, d_v) *be a* $b_v(s)$*-metric space and let* $< u_n >$ *and* $< v_n >$ *be two sequences in* X *such that* $u_n \neq u_{n+1}, v_n \neq v_{n+1}$ $(n \geq 0)$*. Suppose that* $\lambda \in [0, 1)$ *and* c_1, c_2 *are real nonnegative numbers such that*

$$K_{m,n} \leq \lambda K_{m-1,n-1} + c_1\lambda^m + c_2\lambda^n, \text{ for all } m, n \in \mathbb{N}, \tag{1}$$

where $K_{m,n} = \max\{d_v(u_m, u_n), d_v(v_m, v_n)\}$ *or* $K_{m,n} = d_v(u_m, u_n) + d_v(v_m, v_n)$*. Then,* $< u_n >$ *and* $< v_n >$ *are Cauchy sequences.*

Proof. From (1), we have

$$\begin{aligned} K_{n,n+1} &\leq \lambda K_{n-1,n} + c_1\lambda^n + c_2\lambda^{n+1} \\ &\leq \cdots \\ &\leq \lambda^n K_{0,1} + c_1 n\lambda^n + c_2 n\lambda^{n+1} \\ &\leq \lambda^n K_{0,1} + C_0 n\lambda^n. \end{aligned} \tag{2}$$

For $m, n, k \in N$, by (1), we have

$$
\begin{aligned}
K_{m+k,n+k} \quad \leq \quad & \lambda \max\{K_{m+k-1,n+k-1}, c_1\lambda^{m+k-1} + c_2\lambda^{n+k-1})\} \\
& \leq \lambda K_{m+k-1,n+k-1} + c_1\lambda^{m+k} + c_2\lambda^{n+k}) \\
& \cdots \\
& \leq \lambda^k K_{m,n} + kC_1\lambda^k(\lambda^m + \lambda^n).
\end{aligned} \tag{3}
$$

Since $0 < \lambda < 1$, we can find a positive integer q_k such that $0 < \lambda^{q_k} < \frac{1}{s}$. Now, suppose $v \geq 2$. Then, by using condition 3. of a $b_v(s)$-metric and inequalities (2) and (3), we have

$$
\begin{aligned}
K_{m,n} \quad \leq \quad & s[K_{m,m+1} + K_{m+1,m+2} + \cdots + K_{m+v-3,m+v-2} + K_{m+v-2,m+q_k} + K_{m+q_k,n+q_k} + K_{n+q_k,n}] \\
& \leq s[\lambda^m + \lambda^{m+1} + \cdots + \lambda^{m+v-3}]K_0 + sC_0[m\lambda^m + (m+1)\lambda^{m+1} + \cdots + (m+v-3)\lambda^{m+v-2}] \\
& + s[\lambda^m K_{v-2,q_k} + m\lambda^m(\lambda^{v-2} + \lambda^{q_k})K_0] \\
& + s[\lambda^{q_k} K_{m,n} + q_k\lambda^{q_k}(\lambda^m + \lambda^n)K_0] + s[\lambda^n K_{q_k,0} + n\lambda^n(\lambda^{q_k} + 1)K_0].
\end{aligned}
$$

Then,

$$
\begin{aligned}
K_{m,n} \quad \leq \quad & \frac{s\lambda^m}{(1 - s\lambda^{q_k})(1-\lambda)}K_{0,1} + \frac{s(m+v-3)\lambda^m}{(1-\lambda)(1-s\lambda^{q_k})} \\
& + \frac{s}{1 - s\lambda^{q_k}}[\lambda^m K_{v-2,q_k} + m\lambda^m(\lambda^{v-2} + \lambda^{q_k})K_{0,1}] \\
& + \frac{s}{1 - s\lambda^{q_k}}[q_k\lambda^{q_k}(\lambda^m + \lambda^n)K_{0,1}] + \frac{s}{1 - s\lambda^{q_k}}[\lambda^n K_{q_k,0} + n\lambda^n(\lambda^{q_k} + 1)K_{0,1}].
\end{aligned}
$$

Thus, from the definition of $K_{m,n}$, we see that, as $m, n \to +\infty$, $d_v(u_m, u_n) \to 0$ and $d_v(v_m, v_n) \to 0$ and thus $< u_n >$ and $< v_n >$ are Cauchy sequences. $\square$

3.1. Coupled Fixed Point Theorems

We now present our main theorems as follows:

Theorem 1. *Let (X, d_v) be a $b_v(s)$-metric space , $T \colon X \to X$ be a one to one mapping, $S \colon X \times X \to X$ and $g \colon X \to X$ be mappings such that $S(X \times X) \subset g(X)$, $Tg(X)$ is complete. If there exist real numbers λ, μ, ν with $0 \leq \lambda < 1$, $0 \leq \mu, \nu \leq 1$, $\min\{\lambda\mu, \lambda\nu\} < \frac{1}{s}$ such that, for all $u, v, w, z \in X$*

$$
\begin{aligned}
d_v(TS(u,v), TS(w,z)) \quad \leq \quad & \lambda \max\{d_v(Tgu, Tgw), d_v(Tgv, Tgz), \mu d_v(Tgu, TS(u,v)), \mu d_v(Tgv, TS(v,u)), \\
& \nu d_v(Tgw, TS(w,z)), \nu d_v(Tgz, TS(z,w))\}
\end{aligned} \tag{4}
$$

then the following holds :

1. *There exist w_{x_0}, w_{y_0} in X, such that sequences $< Tgu_n >$ and $< Tgv_n >$ converge to Tgw_{x_0} and Tgw_{y_0} respectively, where the iterative sequences $< gu_n >$ and $< gv_n >$ are defined by $gu_n = S(u_{n-1}, v_{n-1})$ and $gv_n = S(v_{n-1}, u_{n-1})$ for some arbitrary $(u_0, v_0) \in X \times X$.*
2. *$(w_{x_0}, w_{y_0}) \in COCP\{S, g\}$.*
3. *If S and g are weakly compatible, then S and g have a unique common coupled fixed point.*

Proof. 1. We shall start the proof by showing that the sequences $< Tgu_n >$ and $< Tgv_n >$ are Cauchy sequences, where $< gu_n >$ and $< gv_n >$ are as mentioned in the hypothesis.

By (4), we have

$$
\begin{aligned}
d_v(Tgu_n, Tgu_{n+1}) &= d_v(TS(u_{n-1}, v_{n-1}), TS(u_n, v_n)) \\
&\leq \lambda \max\{d_v(Tgu_{n-1}, Tgu_n), d_v(Tgv_{n-1}, Tgv_n), \mu d_v(Tgu_{n-1}, TS(u_{n-1}, v_{n-1})), \\
&\quad \mu d_v(Tgv_{n-1}, TS(v_{n-1}, u_{n-1})), \nu d_v(Tgu_n, TS(u_n, v_n)), \nu d_v(Tgv_n, TS(v_n, u_n))\} \\
&\leq \lambda \max\{d_v(Tgu_{n-1}, Tgu_n), d_v(Tgv_{n-1}, Tgv_n), d_v(Tgu_{n-1}, Tgu_n), \\
&\quad d_v(Tgv_{n-1}, Tgv_n), d_v(Tgu_n, Tgu_{n+1}), d_v(Tgv_n, Tgv_{n+1})\}.
\end{aligned} \tag{5}
$$

Similarly, we get

$$
\begin{aligned}
d_v(Tgv_n, Tgv_{n+1}) \leq\ & \lambda \max\{d_v(Tgv_{n-1}, Tgv_n), d_v(Tgu_{n-1}, Tgu_n), d_v(Tgv_{n-1}, Tgv_n), \\
& d_v(Tgu_{n-1}, Tgu_n), d_v(Tgv_n, Tgv_{n+1}), d_v(Tgu_n, Tgu_{n+1})\}.
\end{aligned} \tag{6}
$$

Let $K_n = \max\{d_v(Tgu_n, Tgu_{n+1}), d_v(Tgv_n, Tgv_{n+1})\}$. By (5) and (6), we get

$$
K_n \leq \lambda \max\{d_v(Tgv_{n-1}, Tgv_n), d_v(Tgu_{n-1}, Tgu_n), d_v(Tgv_n, Tgv_{n+1}), d_v(Tgu_n, Tgu_{n+1})\}. \tag{7}
$$

If

$$
\begin{aligned}
&\max\{d_v(Tgv_{n-1}, Tgv_n), d_v(Tgu_{n-1}, Tgu_n), d_v(Tgv_n, Tgv_{n+1}), d_v(Tgu_n, Tgu_{n+1})\} \\
&\qquad = d_v(Tgv_n, Tgv_{n+1}) \text{ or } d_v(Tgu_n, Tgu_{n+1}),
\end{aligned}
$$

then (7) will yield a contradiction. Thus, we have

$$
\begin{aligned}
&\max\{d_v(Tgv_{n-1}, Tgv_n), d_v(Tgu_{n-1}, Tgu_n), d_v(Tgv_n, Tgv_{n+1}), d_v(Tgu_n, Tgu_{n+1})\} \\
&\qquad = \max\{d_v(Tgv_{n-1}, Tgv_n), d_v(Tgu_{n-1}, Tgu_n)\},
\end{aligned}
$$

and then (7) gives

$$
K_n \leq \lambda \max\{d_v(Tgv_{n-1}, Tgv_n), d_v(Tgu_{n-1}, Tgu_n)\} = \lambda K_{n-1} \preceq \lambda^2 K_{n-2} \preceq \cdots \preceq \lambda^n K_0. \tag{8}
$$

For any $m, n \in N$, we have

$$
\begin{aligned}
d_v(Tgu_m, Tgu_n) &= d_v(TS(u_{m-1}, v_{m-1}), TS(u_{n-1}, v_{n-1}) \\
&\leq \lambda \max\{d_v(Tgu_{m-1}, Tgu_{n-1}), d_v(Tgv_{m-1}, Tgv_{n-1}), \\
&\quad \mu d_v(Tgu_{m-1}, TS(u_{m-1}, v_{m-1})), \mu d_v(Tgv_{m-1}, TS(v_{m-1}, u_{m-1})), \\
&\quad \nu d_v(Tgu_{n-1}, TS(u_{n-1}, v_{n-1})), \nu d_v(Tgv_{n-1}, TS(v_{n-1}, u_{n-1}))\} \\
&\leq \lambda \max\{d_v(Tgu_{m-1}, Tgu_{n-1}), d_v(Tgv_{m-1}, Tgv_{n-1}), d_v(Tgu_{m-1}, Tgu_m), \\
&\quad d_v(Tgv_{m-1}, Tgv_m), d_v(Tgu_{n-1}, Tgu_n), d_v(Tgv_{n-1}, Tgv_n)\}.
\end{aligned}
$$

Then, by using (8), we get

$$
\begin{aligned}
d_v(Tgu_m, Tgu_n) \leq\ & \lambda \max\{d_v(Tgu_{m-1}, Tgu_{n-1}), d_v(Tgv_{m-1}, Tgv_{n-1})\} \\
& +(\lambda^m + \lambda^n)K_0\}.
\end{aligned} \tag{9}
$$

Similarly, we have

$$\begin{aligned} d_v(Tgv_m, Tgv_n) \quad &\leq \quad \lambda \max\{d_v(Tgu_{m-1}, Tgu_{n-1}), d_v(Tgv_{m-1}, Tgv_{n-1})\} \\ &\quad + (\lambda^m + \lambda^n)K_0\}. \end{aligned} \tag{10}$$

Let $K_{m,n} = \max\{d_v(Tgu_m, Tgu_n), d_v(Tgv_m, Tgv_n)\}$. By (9) and (10), we get

$$K_{m,n} \quad \leq \quad \lambda K_{m-1,n-1} + (\lambda^m + \lambda^n)K_0.$$

Thus, we see that inequality (1) is satisfied with $c_1 = c_2 = K_0$. Hence, by Lemma 2, $< Tgu_n >$ and $< Tgv_n >$ are Cauchy sequences. For $v = 1$, the same follows from Lemma 1.
Since $(Tg(X), d)$ is complete, we can find $w_{x_0}, w_{y_0} \in X$ such that

$$\lim_{n\to\infty} Tgu_n = Tgw_{x_0} and \lim_{n\to\infty} Tgv_n = Tgw_{y_0}.$$

2. Now,

$$\begin{aligned} & d_v(TS(w_{x_0}, w_{y_0}), Tgw_{x_0}) \leq s[d_v(TS(w_{x_0}, w_{y_0}), TS(u_n, v_n)) + d_v(TS(u_n, v_n), TS(u_{n+1}, v_{n+1})) \\ & + \cdots + d_v(TS(u_{n+v-2}, v_{n+v-2}), TS(u_{n+v-1}, v_{n+v-1}) + d_v(TS(u_{n+v-1}, v_{n+v-1}), Tgw_{x_0}) \\ \leq\; & s[\lambda max\{d_v(Tgw_{x_0}, Tgu_n), d_v(Tgw_{y_0}, Tgv_n), \mu d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})), \\ & \mu d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0}), \nu d_v(Tgu_n, TS(u_n, v_n)), \nu d_v(Tgv_n, TS(v_n, u_n))\} \\ & + d_v(Tgu_{n+1}, Tgu_{n+2}) + \cdots + d_v(Tgu_{n+v-1}, Tgu_{n+v}) + d_v(Tgu_{n+v}, Tgw_{x_0}) \\ \leq\; & s[\lambda max\{d_v(Tgw_{x_0}, Tgu_n), d_v(Tgw_{y_0}, Tgv_n), \mu d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})), \\ & \mu d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0}), \nu d_v(Tgu_n, Tgu_{n+1}), \nu d_v(Tgv_n, Tgv_{n+1})\} \\ & + d_v(Tgu_{n+1}, Tgu_{n+2}) + \cdots + d_v(Tgu_{n+v-1}, Tgu_{n+v} + d_v(Tgu_{n+v}, Tgw_{x_0}). \end{aligned} \tag{11}$$

Note that, since $< Tgu_n >$ and $< Tgv_n >$ are Cauchy sequences, by definition, $d_v(Tgu_n, Tgu_{n+1}) \to 0$, $d_v(Tgv_n, Tgv_{n+1}) \to 0$ as $n \to \infty$. Thus, from (11), as $n \to \infty$, we get

$$d_v(TS(w_{x_0}, w_{y_0}), Tgw_{x_0}) \quad \leq \quad s\lambda \max\{\mu d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})), \mu d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0}))\}.$$

Similarly, we get

$$d_v(TS(w_{y_0}, w_{x_0}), Tgw_{y_0}) \quad \leq \quad s\lambda \max\{\mu d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})), \mu d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0})\}.$$

Thus, we have

$$\begin{aligned} & \max\{d_v(TS(w_{x_0}, w_{y_0}), Tgw_{x_0}), d_v(TS(w_{y_0}, w_{x_0}), Tgw_{y_0})\} \\ \leq\; & s\lambda\mu \max\{d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})), d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0})\}. \end{aligned} \tag{12}$$

Proceeding along the same lines as above, we also have

$$\begin{aligned} & \max\{d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})), d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0}))\} \\ \leq\; & s\lambda\nu \max\{d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})), d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0})\}. \end{aligned} \tag{13}$$

Using (12) and (13) along with the condition $\min\{\lambda\mu, \lambda\nu\} < \frac{1}{s}$, we get $TS(w_{x_0}, w_{y_0}) = Tgw_{x_0}$ and $TS(w_{y_0}, w_{x_0}) = Tgw_{y_0}$. As T is one to one, we have $S(w_{x_0}, w_{y_0}) = gw_{x_0}$ and $S(w_{y_0}, w_{x_0}) = gw_{y_0}$. Therefore, $(w_{x_0}, w_{y_0}) \in COCP\{S, g\}$.

3. Suppose S and g are weakly compatible. First, we will show that, if $(w^*_{x_0}, w^*_{y_0}) \in COCP\{S, g\}$, then $gw^*_{x_0} = gw_{x_0}$ and $gw^*_{y_0} = gw_{y_0}$, or in other words the point of coupled coincidence of S and g is unique. By (5), we have

$$\begin{aligned} d_v(Tgw^*_{x_0}, Tgw_{x_0}) &= d_v(TS(w^*_{x_0}, w^*_{y_0}), TS(w_{x_0}, w_{y_0})) \\ &\leq \lambda max\{d_v(Tgw^*_{x_0}, Tgw_{x_0}), d_v(Tgw^*_{y_0}, Tgw_{y_0}), \mu d_v(Tgw^*_{x_0}, TS(w^*_{x_0}, w^*_{y_0})), \\ &\quad \mu d_v(Tgw^*_{y_0}, TS(w^*_{y_0}, w^*_{x_0}), \nu d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})), \nu d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0}))\} \\ &\leq \lambda max\{d_v(Tgw^*_{x_0}, Tgw_{x_0}), d_v(Tgw^*_{y_0}, Tgw_{y_0})\}. \end{aligned}$$

Similarly, we have

$$d_v(Tgw^*_{y_0}, Tgw_{y_0}) \leq \lambda max\{d_v(Tgw^*_{x_0}, Tgw_{x_0}), d_v(Tgw^*_{y_0}, Tgw_{y_0})\}.$$

Thus, from the above two inequalities, we get

$$max\{d_v(Tgw^*_{x_0}, Tgw_{x_0}), d_v(Tgw^*_{y_0}, Tgw_{y_0}) \leq \lambda max\{d_v(Tgw^*_{x_0}, Tgw_{x_0}), d_v(Tgw^*_{y_0}, Tgw_{y_0})\}$$

which implies that $Tgw^*_{x_0} = Tgw_{x_0}$ and $Tgw^*_{y_0} = Tgw_{y_0}$. Since T is one to one, we get $gw^*_{x_0} = gw_{x_0}$ and $gw^*_{y_0} = gw_{y_0}$, which is the point of coupled coincidence of S and g is unique. Since S and g are weakly compatible and, since $(w_{x_0}, w_{y_0}) \in COCP\{S, g\}$, we have

$$ggw_{x_0} = gS(w_{x_0}, w_{y_0}) = S(gw_{x_0}, gw_{y_0})$$

and

$$ggw_{y_0} = gS(w_{y_0}, w_{x_0}) = S(gw_{y_0}, gw_{x_0})$$

which shows that $(gw_{x_0}, gw_{y_0}) \in COCP\{S, g\}$. By the uniqueness of the point of coupled coincidence, we get $ggw_{x_0} = gw_{x_0}$ and $ggw_{y_0} = gw_{y_0}$ and thus $(gw_{x_0}, gw_{y_0}) \in CCOFP\{S, g\}$. Uniqueness of the coupled fixed point follows easily from (4). □

Our next result is a generalized version of Theorem 2.1 of Gu [10].

Theorem 2. *Let (X, d_v), T, S and g be as in Theorem 1 and suppose there exist $\beta_1, \beta_2, \beta_3$ in the interval [0,1), such that $\beta_1 + \beta_2 + \beta_3 < 1$, minimum$\{\beta_2, \beta_3\} < \frac{1}{s}$ and for all $u, v, w, z \in X$*

$$\begin{aligned} &d_v(TS(u,v), TS(w,z) + d_v(TS(v,u), TS(z,w) \leq \beta_1(d_v(Tgu, Tgw) + d_v(Tgv, Tgz)) + \\ &\beta_2(d_v(Tgu, TS(u,v)) + d_v(Tgv, TS(v,u)) + \beta_3(d_v(Tgw, TS(w,z)) + d_v(Tgz, TS(z,w))). \end{aligned} \tag{14}$$

Then, conclusions 1, 2, and 3 of Theorem 1 are true.

Proof. Let $K'_n = d_v(Tgu_n, Tgu_{n+1}) + d_v(Tgv_n, Tgv_{n+1})$ and $K'_{m,n} = d_v(Tgu_m, Tgu_n) + d_v(Tgv_m, Tgv_n)$. From condition (14), we obtain

$$\begin{aligned} &d_v(Tgu_n, Tgu_{n+1}) + d_v(Tgv_n, Tgv_{n+1}) = d_v(TS(u_{n-1}, v_{n-1}), TS(u_n, v_n)) + \\ &d_v(TS(v_{n-1}, u_{n-1}), TS(v_n, u_n)) \\ \leq\ & \beta_1[d_v(Tgu_{n-1}, Tgu_n) + d_v(Tgv_{n-1}, Tgv_n)] + \beta_2[d_v(Tgu_{n-1}, TS(u_{n-1}, v_{n-1})) \\ &+ d_v(Tgv_{n-1}, TS(v_{n-1}, u_{n-1}))] + \beta_3[d_v(Tgu_n, TS(u_n, v_n)) + d_v(Tgv_n, TS(v_n, u_n))] \\ \leq\ & (\beta_1 + \beta_2)[d_v(Tgu_{n-1}, Tgu_n) + d_v(Tgv_{n-1}, Tgv_n)] \\ &+ \beta_3[d_v(Tgu_n, Tgu_{n+1}) + d_v(Tgv_n, Tgv_{n+1})]. \end{aligned}$$

Therefore,

$$d_v(Tgu_n, Tgu_{n+1}) + d_v(Tgv_n, Tgv_{n+1}) \leq \lambda^{'}[d_v(Tgu_{n-1}, Tgu_n) + d_v(Tgv_{n-1}, Tgv_n)],$$

where $\lambda^{'} = \dfrac{\beta_1 + \beta_2}{1 - \beta_3} < 1$. Thus, we get

$$K_n^{'} \leq \lambda^{'} K_{n-1}^{'} \leq \cdots \leq \lambda^{'^n} K_0^{'}. \tag{15}$$

For any $m, n \in \mathbb{N}$, we have

$$\begin{aligned}
d_v(Tgu_m, Tgu_n) \quad + \quad & d_v(Tgv_m, Tgv_n) = d_v(TS(u_{m-1}, v_{m-1}), TS(u_{n-1}, v_{n-1}) + \\
& d_v(TS(v_{m-1}, u_{m-1}), TS(v_{n-1}, u_{n-1}) \\
\leq \quad & \beta_1[d_v(Tgu_{m-1}, Tgu_{n-1}) + d_v(Tgv_{m-1}, Tgv_{n-1})] \\
& + \beta_2[d_v(Tgu_{m-1}, TS(u_{m-1}, v_{m-1})) + d_v(Tgv_{m-1}, TS(v_{m-1}, u_{m-1}))] \\
& + \beta_3[d_v(Tgu_{n-1}, TS(u_{n-1}, v_{n-1})) + d_v(Tgv_{n-1}, TS(v_{n-1}, u_{n-1}))] \\
\leq \quad & \beta_[d_v(Tgu_{m-1}, Tgu_{n-1}) + d_v(Tgv_{m-1}, Tgv_{n-1})] + \beta_2[d_v(Tgu_{m-1}, Tgu_m) \\
& + d_v(Tgv_{m-1}, Tgv_m)] + \beta_3[d_v(Tgu_{n-1}, Tgu_n) + d_v(Tgv_{n-1}, Tgv_n)].
\end{aligned}$$

Then, by using (15), we get

$$\begin{aligned}
d_v(Tgu_m, Tgu_n) + d_v(Tgv_m, Tgv_n) \quad \leq \quad & \beta_1[d_v(Tgu_{m-1}, Tgu_{n-1}) + d_v(Tgv_{m-1}, Tgv_{n-1})] \\
& + (\beta_2 \lambda^{'^m} + \beta_3 \lambda^{'^n}) K_0^{'} \}.
\end{aligned}$$

That is,

$$K_{m,n}^{'} \quad \leq \quad \lambda K_{m-1,n-1}^{'} + (\lambda^m + \lambda^n) K_0^{'}$$

where $\lambda^{'} = \beta_1 + \beta_2 + \beta_3 < 1$. Now for $m, n, r \in N$. Thus, we see that inequality (1) is satisfied with $c_1 = c_2 = K_0$. Hence, by Lemma 2, $< Tgu_n >$ and $< Tgv_n >$ are Cauchy sequences. For $v = 1$, the same follows from Lemma 1.

Since $(Tg(X), d)$ is complete, we can find $w_{x_0}, w_{y_0} \in X$ such that

$$\lim_{n \to \infty} Tgu_n = Tgw_{x_0} and \lim_{n \to \infty} Tgv_n = Tgw_{y_0}.$$

Again, from condition 3 in Definition 1, we have

$$\begin{aligned}
d_v(TS(w_{x_0}, w_{y_0}), Tgw_{x_0})) \quad \leq \quad & s[d_v(TS(w_{x_0}, w_{y_0}), TS(u_n, v_n)) + d_v(TS(u_n, v_n), TS(u_{n+1}, v_{n+1})) + \cdots + \\
& + d_v(TS(u_{n+v-2}, v_{n+v-2}), TS(u_{n+v-1}, v_{n+v-1})) + \\
& d_v(TS(u_{n+v-1}, v_{n+v-1}), Tgw_{x_0}))]
\end{aligned}$$

and

$$\begin{aligned}
d_v(TS(w_{y_0}, w_{x_0}), Tgw_{y_0})) \quad \leq \quad & s[d_v(TS(w_{y_0}, w_{x_0}), TS(v_n, u_n)) + d_v(TS(v_n, u_n), TS(v_{n+1}, u_{n+1})) + \cdots + \\
& d_v(TS(v_{n+v-2}, u_{n+v-2}), TS(v_{n+v-1}, u_{n+v-1})) + \\
& d_v(TS(v_{n+v-1}, u_{n+v-1}), Tgw_{x_0}))].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&d_v(TS(w_{x_0}, w_{y_0}), Tgw_{x_0}) + d_v(TS(w_{y_0}, w_{x_0}), Tgw_{y_0}) \leq s[d_v(TS(w_{x_0}, w_{y_0}), TS(u_n, v_n)\\
&+d_v(TS(w_{y_0}, w_{x_0}), TS(v_n, u_n)\\
&+d_v(TS(u_n, v_n), TS(u_{n+1}, v_{n+1})) + \cdots + d_v(TS(u_{n+v-2}, v_{n+v-2}), TS(u_{n+v-1}, v_{n+v-1}))\\
&+d_v(TS(v_n, u_n), TS(v_{n+1}, u_{n+1})) + \cdots + d_v(TS(v_{n+v-2}, u_{n+v-2}), TS(v_{n+v-1}, u_{n+v-1}))\\
&+d_v(TS(u_{n+v-1}, v_{n+v-1}), Tgw_{x_0}) + d_v(TS(v_{n+v-1}, u_{n+v-1}), Tgw_{y_0})]\\
&\leq s[\beta_1(d_v(Tgw_{x_0}, Tgu_n) + d_v(Tgw_{y_0}, Tgv_n)) + \beta_2(d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})) +\\
&d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0})) + \beta_3(d_v(Tgu_n, TS(u_n, v_n)) + d_v(Tgv_n, TS(v_n, u_n)))\}\\
&+d_v(Tgu_n, Tgu_{n+1}) + \cdots + d_v(Tgu_{n-1}, Tgu_n) + +d_v(Tgv_n, Tgv_{n+1}) + \cdots + d_v(Tgv_{n-1}, Tgv_n)\\
&+d_v(Tgu_{n+v-1}, Tgw_{x_0}) + d_v(Tgv_{n+v-1}, Tgw_{y_0})].
\end{aligned}$$

As $n \to \infty$, we get

$$\begin{aligned}
&d_v(TS(w_{x_0}, w_{y_0}), Tgw_{x_0}) + d_v(TS(w_{y_0}, w_{x_0}), Tgw_{y_0})\\
\leq\ & s\beta_2[d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})) + d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0}))].
\end{aligned} \tag{16}$$

Similarly, we can show that

$$\begin{aligned}
&d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})) + d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0}))\\
\leq\ & s\beta_3[d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})) + d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0})]
\end{aligned} \tag{17}$$

Using (16) and (17) along with the condition $\min\{\beta_2, \beta_3\} < \frac{1}{s}$, we get $d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})) + d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0})) = 0$, i.e., $TS(w_{x_0}, w_{y_0}) = Tgw_{x_0}$ and $TS(w_{y_0}, w_{x_0}) = Tgw_{y_0}$. As T is one to one, we have $S(w_{x_0}, w_{y_0}) = gw_{x_0}$ and $S(w_{y_0}, w_{x_0}) = gw_{y_0}$. Therefore, $(w_{x_0}, w_{y_0}) \in COCP\{S, g\}$.

If $(w^*_{x_0}, w^*_{y_0}) \in COCP\{S, g\}$, then, by (14), we have

$$\begin{aligned}
d_v(Tgw^*_{x_0}, Tgw_{x_0}) + d_v(Tgw^*_{y_0}, Tgw_{y_0}) &= d_v(TS(w^*_{x_0}, w^*_{y_0}), TS(w_{x_0}, w_{y_0})) + d_v(TS(w^*_{y_0}, w^*_{x_0}), TS(w_{y_0}, w_{x_0}))\\
&\leq \beta_1[d_v(Tgw^*_{x_0}, Tgw_{x_0}) + d_v(Tgw^*_{y_0}, Tgw_{y_0})] + \beta_2[d_v(Tgw^*_{x_0}, TS(w^*_{x_0}, w^*_{y_0}))\\
&+d_v(Tgw^*_{y_0}, TS(w^*_{y_0}, w^*_{x_0})] + \beta_3[d_v(Tgw_{x_0}, TS(w_{x_0}, w_{y_0})) + d_v(Tgw_{y_0}, TS(w_{y_0}, w_{x_0}))]\\
&\leq \beta_1[d_v(Tgw^*_{x_0}, Tgw_{x_0}) + d_v(Tgw^*_{y_0}, Tgw_{y_0})].
\end{aligned}$$

Thus, $d_v(Tgw^*_{x_0}, Tgw_{x_0}) + d_v(Tgw^*_{y_0}, Tgw_{y_0}) = 0$, which implies that $Tgw^*_{x_0} = Tgw_{x_0}$ and $Tgw^*_{y_0} = Tgw_{y_0}$. Since T is one to one, we get $gw^*_{x_0} = gw_{x_0}$ and $gw^*_{y_0} = gw_{y_0}$, which is the point of coupled coincidence of S, and g is unique. The remaining part of the proof is the same as in the proof of Theorem 1. □

The next results can be proved as in Theorems 1 and 2 and so we will not give the proof.

Theorem 3. *Theorem 1 holds if we replace condition (4) with the following condition:*
There exist $\beta_i \in [0,1), i \in \{1, \ldots, 6\}$ such that $\sum_{i=1}^{6} \beta_i < 1$, $\min\{\beta_3 + \beta_4, \beta_5 + \beta_6\} < \frac{1}{s}$ and for all $u, v, w, z \in X$,

$$\begin{aligned}
&d_v(TS(u, v), TS(w, z)) \leq \beta_1 d_v(Tgu, Tgw) + \beta_2 d_v(Tgv, Tgz) + \beta_3 d_v(Tgu, TS(u, v))\\
&+\beta_4 d_v(Tgv, TS(v, u) + \beta_5 d_v(Tgw, TS(w, z)) + \beta_6 d_v(Tgz, TS(z, w)).
\end{aligned} \tag{18}$$

Taking T to be the identity mapping in Theorems 1–3, we have the following:

Corollary 1. *Let (X, d_v), S, g, λ, μ and ν be as in Theorem 1 such that, for all $u, v, w, z \in X$, the following holds :*

$$\begin{aligned} d_v(S(u,v), S(w,z)) \leq\ & \lambda max\{d_v(gu, gw), d_v(gv, gz), \mu d_v(gu, S(u,v)), \mu d_v(gv, S(v,u)), \\ & \nu d_v(gw, S(w,z)), \nu d_v(gz, S(z,w))\}. \end{aligned} \tag{19}$$

Then, $COCP\{S, g\} \neq \phi$. Furthermore, if S and g are weakly compatible, then S and g has a unique common coupled fixed point. Moreover, for some arbitrary $(u_0, v_0) \in X \times X$, the iterative sequences $(< gu_n >, < gv_n >)$ defined by $gu_n = S(u_{n-1}, v_{n-1})$ and $gv_n = S(v_{n-1}, u_{n-1})$ converge to the unique common coupled fixed point of S and g.

Corollary 2. *Corollary 1 holds if the condition (19) is replaced with the following condition:*

There exist $\beta_1, \beta_2, \beta_3$ in the interval [0,1), such that $\beta_1 + \beta_2 + \beta_3 < 1$, $\min\{\beta_2, \beta_3\} < \frac{1}{s}$ and for all $u, v, w, z \in X$

$$\begin{aligned} & d_v(S(u,v), S(w,z) + d_v(S(v,u), S(z,w) \leq \beta_1(d_v(gu, gw) + d_v(gv, gz)) + \\ & \beta_2(d_v(gu, S(u,v)) + d_v(gv, S(v,u)) + \beta_3(d_v(gw, S(w,z)) + d_v(gz, S(z,w))). \end{aligned} \tag{20}$$

Corollary 3. *Corollary 1 holds if the condition (19) is replaced with the following condition: There exist $\beta_i \in [0,1), i \in \{1, \dots 6\}$ such that $\sum_{i=1}^{6} \beta_i < 1$, $\min\{\beta_3 + \beta_4, \beta_5 + \beta_6\} < \frac{1}{s}$ and, for all $u, v, w, z \in X$,*

$$\begin{aligned} & d_v(S(u,v), S(w,z)) \leq \beta_1 d_v(gu, gw) + \beta_2 d_v(gv, gz) + \\ & \beta_3 d_v(gu, S(u,v)) + \beta_4 d_v(gv, S(v,u) + \beta_5 d_v(gw, S(w,z)) + \beta_6 d_v(gz, S(z,w)). \end{aligned} \tag{21}$$

Remark 1. *Since every b-metric space is a $b_1(s)$ metric space, we note that Theorem 1 is a substantial generalization of Theorem 2.2 of Ramesh and Pitchamani [11]. In fact, we do not require continuity and sub sequential convergence of the function T.*

Remark 2. *Note that condition (2.1) of Gu [10] implies (20) and hence Corollary 2 gives an improved version of Theorem 2.1 of Gu [10].*

Remark 3. *Condition (3.1) of Hussain et al. [12] implies (18) and hence Theorem 3 is an extended and generalized version of Theorem 3.1 of [12].*

3.2. Application to a System of Integral Equations

In this section, we give an application of Theorem 1 to study the existence and uniqueness of solution of a system of nonlinear integral equations.

Let $X = C[0, A]$ be the space of all continuous real valued functions defined on $[0, A]$, $A > 0$. Our problem is to find $(u(t), v(t)) \in X \times X$, $t \in [0, A]$ such that, for $f : [0, A] \times R \times R \to R$ and $G : [0, A] \times [0, A] \to R$ and $K \in C([0, A]$, the following holds:

$$\begin{aligned} u(t) &= \int_0^A G(t,r) f(t, u(r), v(r)) dr + K(t) \\ v(t) &= \int_0^A G(t,r) f(t, v(r), u(r)) dr + K(t). \end{aligned} \tag{22}$$

Now, suppose $F : X \times X \to X$ is given by

$$F(u(t), v(t)) = \int_0^A G(t,r) f(t, u(r), v(r)) dr + K(t).$$

$$F(v(t),u(t)) = \int_0^A G(t,r)f(t,v(r),u(r))dr + K(t).$$

Then, (22) is equivalent to the coupled fixed point problem $F(u(t),v(t)) = u(t)$, $F(v(t),u(t)) = v(t)$.

Theorem 4. *The system of Equation (22) has a unique solution provided the following holds:*

(i) $G : [0,A] \times [0,A] \to R$ *and* $f : [0,A] \times R \times R \to R$ *are continuous functions.*

(ii) $K \in C([0,A]$.

(iii) *For all* $x,y,u,v \in X$ *and* $t \in [0,A]$, *we can find a function* $g : X \to X$ *and real numbers* $p \geq 1$, λ, μ, ν *with* $0 \leq \lambda < 1$, $0 \leq \mu, \nu \leq 1$, *minimum* $\{\lambda\mu, \lambda\nu\} < \frac{1}{3^{s-1}}$ *satisfying*

$$\begin{aligned}(iii-a):\mid f(t,u(r),v(r))) - f(t,x(r),y(r))) \mid^p \leq\ & \lambda^p max\{\mid g(u(r)) - g(x(r)) \mid^p, \mid g(v(r)) - g(y(r)) \mid^p,\\ & \mu \mid g(u(r)) - F(u(r),v(r)) \mid^p, \mu \mid g(v(r)) - F(v(r),u(r)) \mid^p,\\ & \nu \mid g(x(r)) - F(x(r),y(r)) \mid^p, \nu \mid g(y(r)) - F(y(r),x(r)) \mid^p\}.\end{aligned}$$

(iii-b) $F(g(u(t)),g(v(t))) = g(F(u(t),v(t)))$.

(iv) $sup_{t\in[0,A]} \int_0^A \mid G(t,r) \mid^p dr \leq \frac{1}{\lambda^{p-1}}$.

Moreover, for some arbitrary $u_0(t), v_0(t)$ *in X, the sequence* $(< gu_n(t) >, < gv_n(t) >)$ *defined by*

$$\begin{aligned} gu_n(t) &= \int_0^A G(t,r)f(t,u_{n-1}(r),v_{n-1}(r))dr + K(t)\\ gv_n(t) &= \int_0^A G(t,r)f(t,v_{n-1}(r),u_{n-1}(r))dr + K(t) \end{aligned} \tag{23}$$

converges to the unique solution.

Proof. Define d_v: $X \times X \to R$ such that for all $u, v \in X$,

$$d_v(u,v) = sup_{t\in[0,A]} \mid u(t) - v(t) \mid^s . \tag{24}$$

Clearly, d_v is a $b_v((v+1)^{s-1})$-metric space.

For some $r \in [0,A]$, we have

$$\begin{aligned} \mid F(u(t),v(t)) - & F(x(t),y(t)) \mid^p\\ = & \mid \textstyle\int_0^A G(t,r)f(t,u(r),v(r))dr + g(t) - \int_0^A G(t,r)f(t,x(r),y(r))dr + g(t) \mid^p\\ \leq & \textstyle\int_0^A \mid G(t,r) \mid^p \mid f(t,u(r),v(r)) - f(t,x(r),y(r)) \mid^p dr\\ \leq & (\textstyle\int_0^A \mid G(t,r) \mid^p dr)\lambda^p[max\{\mid g(u(r)) - g(x(r)) \mid^p, \mid g(v(r)) - g(y(r)) \mid^p,\\ & \mu \mid g(u(r)) - F(u(r),v(r)) \mid^p, \mu \mid g(v(r)) - F(v(r),u(r)) \mid^p,\\ & \nu \mid g(x(r)) - F(x(r),y(r)) \mid^p, \nu \mid g(y(r)) - F(y(r),x(r)) \mid^p\}.\\ \leq & (\textstyle\int_0^A \mid G(t,r) \mid^p dr)\lambda^p[max\{d_v(g(u),g(x)), d_v(g(v),g(y)), \mu d_v(g(u),F(u,v)), \mu d_v(g(v),F(v,u)),\\ & \nu d_v(g(x),F(x,y)), \nu d_v(g(y),F(y,x))\}. \end{aligned}$$

Thus, using condition (iv), we have

$$\begin{aligned} d_v(F(u,v),F(x,y)) &= sup_{t\in[0,A]} \mid F(u(t),v(t)) - F(x(t),y(t)) \mid^p\\ &\leq \lambda[max\{d_v(g(u),g(x)), d_v(g(v),g(y)), \mu d_v(g(u),F(u,v)), \mu d_v(g(v),F(v,u)),\\ &\quad \nu d_v(g(x),F(x,y)), \nu d_v(g(y),F(y,x))\}. \end{aligned}$$

Thus, all the conditions of Corollary 1 are satisfied and so F has a unique coupled fixed point $(u', v') \in C([0,A] \times C([0,A]$, which is the unique solution of (22) and the sequence $(< gu_n(t) >, < gv_n(t) >)$ defined by (23) converges to the unique solution of (22). □

Example 1. *Let $X = C[0,1]$ be the space of all continuous real valued functions defined on $[0,1]$ and define $d_3 \colon X \times X \to R$ such that, for all $u, v \in X$,*

$$d_3(u,v) = sup_{t\in[0,1]} \mid u(t) - v(t) \mid^2 . \tag{25}$$

Clearly, d_3 is a $b_2(3)$-metric. Now, consider the functions $f : [0,1] \times R \times R \to R$ given by $f(t,u,v) = t^2 + \frac{9}{20}u + \frac{8}{20}v$, $G : [0,1] \times [0,1] \to R$ given by $G(t,r) = \frac{\sqrt{45}(t+r)}{10}$, $K \in C([0,1]$ given by $K(t) = t$. Then, Equation (22) becomes

$$\begin{aligned} u(t) &= t + \int_0^1 \frac{\sqrt{45}(t+r)}{10}(t^2 + \frac{9}{20}u(r) + \frac{8}{20}v(r))dr \\ v(t) &= t + \int_0^1 \frac{\sqrt{45}(t+r)}{10}(t^2 + \frac{9}{20}v(r) + \frac{8}{20}u(r))dr. \end{aligned} \tag{26}$$

Then,

$$\begin{aligned} \mid f(t,u,v) - f(t,x,y) \mid^2 &= \mid \frac{9}{20}(u-x) + \frac{8}{20}(v-y) \mid^2 \\ &\leq \mid Max\{\frac{9}{10}(u-x), \frac{8}{10}(v-y)\} \mid^2 \\ &\leq \frac{81}{100}Max\{\mid u-x \mid^2, \mid v-y) \mid^2\}. \end{aligned}$$

In addition,

$$sup_{t\in[0,1]} \int_0^1 \mid G(t,r) \mid^2 dr = \int_0^1 \frac{45}{100}(t+r)^2 dr = 1.05.$$

We see that all the conditions of Theorem 4 are satisfied, with $\lambda = \frac{9}{10}, \mu = 0, \nu = 0, p = 2$ and $g = I_X$(Identity mapping). Hence, Theorem 4 ensures a unique solution of (26). Now, for $u_0(t) = 1$ and $v_0(t) = 0$, we construct the sequence $(< u_n(t) >, < v_n(t) >\}$ given by

$$\begin{aligned} u_n(t) &= t + \int_0^1 \frac{\sqrt{45}(t+r)}{10}(t^2 + \frac{9}{20}u_{n-1}(r) + \frac{8}{20}v_{n-1}(r))dr \\ v_n(t) &= t + \int_0^1 \frac{\sqrt{45}(t+r)}{10}(t^2 + \frac{9}{20}v_{n-1}(r) + \frac{8}{20}u_{n-1}(r))dr. \end{aligned} \tag{27}$$

Using MATLAB, we see that above sequence converges to $\{0.6708t^3 + 0.3354t^2 + 2.2339t + 0.7677, 0.6708t^3 + 0.3354t^2 + 2.2339t + 0.7677\}$, and this is the unique solution of the system of nonlinear integral Equation (26). The convergence table is given in Table 1 below.

Table 1. Convergence of sequences $< u_n(t) >$ and $< v_n(t) >$.

n	$u_n(t) = t + \int_0^1 \frac{\sqrt{45}(t+r)}{10}(t^2 + \frac{9}{20}u_{n-1}(r) + \frac{8}{20}v_{n-1}(r))dr$	$v_n(t) = t + \int_0^1 \frac{\sqrt{45}(t+r)}{10}(t^2 + \frac{9}{20}v_{n-1}(r) + \frac{8}{20}u_{n-1}(r))dr$
1	$u_1(t) = t + 0.0167(2t+1)(20t^2+9))$	$v_1(t) = t + .0671(2t+1)(5t^2+2))$
2	$u_2(t) = 0.6708t^3 + 0.3354t^2 + 1.3t + 0.5007$	$v_2(t) = 0.6708t^3 + 0.3354t^2 + 1.29t + 0.5115$
3	$u_3(t) = 0.6708t^3 + 0.3354t^2 + 1.8210t + 0.5174$	$v_3(t) = 0.6708t^3 + 0.3354t^2 + 1.8208t + 0.5171$
4	$u_4(t) = 0.6708t^3 + 0.3354t^2 + 1.9734t + 0.6179$	$v_4(t) = 0.6708t^3 + 0.3354t^2 + 1.9734t + 0.6178$
5	$u_5(t) = 0.6708t^3 + 0.3354t^2 + 2.0743t + 0.6755$	$v_5(t) = 0.6708t^3 + 0.3354t^2 + 2.0743t + 0.6755$
6	$u_6(t) = 0.6708t^3 + 0.3354t^2 + 2.1359t + 0.7111$	$v_6(t) = 0.6708t^3 + 0.3354t^2 + 2.1359t + 0.7111$
7	$u_7(t) = 0.6708t^3 + 0.3354t^2 + 2.1737t + 0.73298$	$v_7(t) = 0.6708t^3 + 0.3354t^2 + 2.1737t + 0.73298$
8	$u_8(t) = 0.6708t^3 + 0.3354t^2 + 2.19699t + 0.7464$	$v_8(t) = 0.6708t^3 + 0.3354t^2 + 2.19699t + 0.7464$
9	$u_9(t) = 0.6708t^3 + 0.3354t^2 + 2.2113t + 0.7547$	$v_9(t) = 0.6708t^3 + 0.3354t^2 + 2.2113t + 0.7547$
10	$u_{10}(t) = 0.6708t^3 + 0.3354t^2 + 2.2200t + 0.7597$	$v_{10}(t) = 0.6708t^3 + 0.3354t^2 + 2.2200t + 0.7597$
11	$u_{11}(t) = 0.6708t^3 + 0.3354t^2 + 2.2254t + 0.7628$	$v_{11}(t) = 0.6708t^3 + 0.3354t^2 + 2.2254t + 0.7628$
12	$u_{12}(t) = 0.6708t^3 + 0.3354t^2 + 2.2287t + 0.7647$	$v_{12}(t) = 0.6708t^3 + 0.3354t^2 + 2.2287t + 0.7647$
13	$u_{13}(t) = 0.6708t^3 + 0.3354t^2 + 2.2308t + 0.7658$	$v_{13}(t) = 0.6708t^3 + 0.3354t^2 + 2.2308t + 0.7658$
14	$u_{14}(t) = 0.6708t^3 + 0.3354t^2 + 2.23199t + 0.7666$	$v_{14}(t) = 0.6708t^3 + 0.3354t^2 + 2.23199t + 0.7666$
15	$u_{15}(t) = 0.6708t^3 + 0.3354t^2 + 2.2328t + 0.7671$	$v_{15}(t) = 0.6708t^3 + 0.3354t^2 + 2.2328t + 0.7671$
16	$u_{16}(t) = 0.6708t^3 + 0.3354t^2 + 2.2333t + 0.7674$	$v_{16}(t) = 0.6708t^3 + 0.3354t^2 + 2.2333t + 0.7674$
17	$u_{17}(t) = 0.6708t^3 + 0.3354t^2 + 2.2336t + 0.7675$	$v_{17}(t) = 0.6708t^3 + 0.3354t^2 + 2.2336t + 0.7675$
18	$u_{18}(t) = 0.6708t^3 + 0.3354t^2 + 2.2338t + 0.7676$	$v_{18}(t) = 0.6708t^3 + 0.3354t^2 + 2.2338t + 0.7676$
19	$u_{19}(t) = 0.6708t^3 + 0.3354t^2 + 2.2339t + 0.7677$	$v_{19}(t) = 0.6708t^3 + 0.3354t^2 + 2.2339t + 0.7677$
20	$u_{20}(t) = 0.6708t^3 + 0.3354t^2 + 2.2339t + 0.7677$	$v_{20}(t) = 0.6708t^3 + 0.3354t^2 + 2.2339t + 0.7677$

Remark 4. *Condition (iv) of Theorem 4 above is weaker than the corresponding conditions used in similar theorems of [11,13,14].*

Remark 5. *In example 1 above, we see that* $sup_{t\in[0,1]} \int_0^1 \mid G(t,r) \mid^2 dr = \int_0^1 \frac{45}{100}(t+r)^2 dr = 1.05 > 1$ *and thus condition (v) of Theorem 3.1 of [11], condition (30) of Theorem 3.1 of [13] and condition (iii) of Theorem 3.1 of [14] are not satisfied.*

Author Contributions: Investigation, R.G., Z.D.M., and S.R.; Methodology, R.G.; Software, Z.D.M.; Supervision, R.G., Z.D.M., and S.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: 1. The authors are thankful to the Deanship of Scientific Research at Prince Sattam bin Abdulaziz University, Al-Kharj, Kingdom of Saudi Arabia, for supporting this research. 2. The authors are thankful to the learned reviewers for their valuable comments which helped in improving this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bakhtin, I.A. The contraction mapping principle in quasimetric spaces. *Funct. Anal. Ulianowsk Gos. Ped. Inst.* **1989**, *30*, 26–37.
2. Branciari, A. A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces. *Publicationes Mathematicae Debrecen* **2000**, *57*, 31–37.

3. George, R.; Radenović, S.; Reshma, K.P.; Shukla, S. Rectangular b-metric spaces and contraction principle. *J. Nonlinear Sci. Appl.* **2015**, *8*, 1005–1013. [CrossRef]
4. Hincal, E.; Auwalu, A. A note on Banach contraction mapping principle in cone hexagonal metric space. *Br. J. Math. Comput. Sci.* **2016**, *16*, 1–12.
5. Auwalul, A.; Hincal, E. Kannan type fixed point theorem in cone pentagonal metric spaces. *Intern. J. Pure Appl. Math.* **2016**, *108*, 29–38.
6. Mitrović, Z.D.; Radenović, S. The Banach and Reich contractions in $b_v(s)$-metric spaces. *J. Fixed Point Theory Appl.* **2017**, *19*, 3087–3095. [CrossRef]
7. Mitrović, Z.D. A fixed point theorem for mappings with a contractive iterate in rectangular b-metric spaces. *Matematicki Vesnik* **2018**, *70*, 204–210.
8. George, R.; Mitrović, Z.D. On Reich contraction principle in rectangular cone b-metric space over Banach algebra. *J. Adv. Math. Stud.* **2018**, *11*, 10–16.
9. George, R.; Nabwey, H.A.; Rajagopalan, R.; Radenović, S.; Reshma, K.P. Rectangular cone b-metric spaces over Banach algebra and contraction principle. *Fixed Point Theory Appl.* **2017**, *2017*, 14. [CrossRef]
10. Gu, F. On some common coupled fixed point results in rectangular b-metric spaces. *J. Nonlinear Sci. Appl.* **2017**, *10*, 4085–4098. [CrossRef]
11. Kumar, D.R.; Pitchaimani, M. New coupled fixed point theorems in cone metric spaces with applications to integral equations and Markov process. *Trans. A. Razmadze Math. Inst.* **2018**. [CrossRef]
12. Hussain, N.; Salimi, P.; Al-Mezel, S. Coupled fixed point results on quasi-Banach spaces with application to a system of integral equations. *Fixed Point Theory Appl.* **2013**, *2013*, 261. [CrossRef]
13. Nashine, H.K.; Sintunavarat, W.; Kumam, P. Cyclic generalized contractions and fixed point results with applications to an integral equation. *Fixed Point Theory Appl.* **2012**, *2012*, 217. [CrossRef]
14. Hussain, N.; Roshan, J.R.; Parvaneh, V.; Abbas, M. Common fixed point results for weak contractive mappings in ordered b-dislocated metric spaces with applications. *J. Ineq. Appl.* **2013**, *2013*, 486. [CrossRef]
15. Garai, H.; Dey, L.K.; Mondal, P.; Radenović, S. Some remarks and fixed point results with an application in $b_v(s)$-metric spaces. *Nonlinear Anal. Model. Control* **2020**, *25*, 1015–1034. [CrossRef]