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# On Correlated Equilibria in Marinatto-Weber Type Quantum Games 

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Received: 2 November 2020; Accepted: 11 December 2020; Published: 16 December 2020


#### Abstract

Players' choices in quantum game schemes are often correlated by a quantum state. This enables players to obtain payoffs that may not be achievable when classical pure or mixed strategies are used. On the other hand, players' choices can be correlated due to a classical probability distribution, and if no player benefits by a unilateral deviation from the vector of recommended strategies, the probability distribution is a correlated equilibrium. The aim of this paper is to investigate relation between correlated equilibria and Nash equilibria in the MW-type schemes for quantum games.


Keywords: quantum game; Nash equilibrium; correlated equilibrium

## 1. Introduction

Technological progress that occurred in the last few years made controllable manipulations of single quantum objects possible. This resulted in the emergence and intensive development of a new branch of science placed on the borders of mathematics, computer science, and quantum physics-quantum information theory. The playing of a game is connected to a transfer of information between players and possibly an arbiter. If a carrier of this information is a quantum object, we deal with so called quantum games, the theory of which has been intensively developed during the last twenty years.

Quantum game theory begun with considering a simple extensive form game in [1]. D. Meyer showed that a player equipped with unitary strategies has a winning strategy. Other fundamental papers on quantum games include [2]. The scheme defined by J. Eisert, M. Wilkens, and M. Lewenstein was the first formal protocol of playing quantum games. This scheme uses quantum computing formalism in describing $2 \times 2$ bimatrix games. According to [2], players' strategies are unitary operators that depend on two parameters and act on maximally entangled two-qubit states. The scheme gives the possiblity of obtaining more efficient results in comparison with the results that may be obtained in games played classically. This feature is well illustrated by the prisoner's dilemma game that in a classical version has a unique, inefficient Nash equilibrium. The Eisert-Wilkens-Lewenstein (EWL) scheme enables players to obtain a Pareto optimal Nash equilibrium. Marinatto and Weber [3] introduced an alternative model of playing a quantum game by applying quantum formalism to classical game theory in a more straightforward way. In the general case of $m \times n$ bimatrix games, players' strategies are identified with permutation matrices which are performed on a mn-level quantum system [4], and then measurements are done. This simple model has found applications in many branches of game theory: from evolutionary games [5,6] to games in extensive form [7] and duopoly problems [8,9].

In general, a large part of noncooperative quantum game theory is devoted to studying results of a quantum game by simply applying nonclassical moves, seeking rational strategy profiles among quantum strategies, and pointing out differences between classical and nonclassical solutions [10-15].

This paper presents a completely different approach. Our goal is to identify elements of a quantum scheme that can be described by classical terms. This new approach may provide for further developments of quantum game theory. By identifying solution concepts from classical game theory in quantum games, we can modify existing quantum schemes or construct new ones. We found that there is a strict connection between correlated equilibria of a game and Nash equilibria in the corresponding quantum game. On this basis, we formulated a new scheme for bimatrix games.

## 2. Preliminaries for Game Theory

In this section, we review relevant notion from classical game theory that is needed to follow our work. A reader who is not familiar with that topic is encouraged to see, for example [16].

The basic model of games studied in game theory is a game in strategic form.
Definition 1. [16] A game in strategic form (or in normal form) is an ordered triple $\left(N,\left(S_{i}\right)_{i \in N},\left(u_{i}\right)_{i \in N}\right)$, in which

- $\quad N=\{1,2, \ldots, r\}$ is a finite set of players;
- $S_{i}$ is the set of strategies of player $i$, for every player $i \in N$;
- $u_{i}: S_{1} \times S_{2} \times \cdots \times S_{r} \rightarrow \mathbb{R}$ is a function associating each vector of strategies $s=\left(s_{i}\right)_{i \in N}$ with the payoff $u_{i}(s)$ to player $i$, for every player $i \in N$.

In the case of a finite two-person game, i.e., $N=\{1,2\}, S_{1}=\{0,1, \ldots, m-1\}$, $S_{2}=\{0,1, \ldots, n-1\}$, the game can be written as a bimatrix with entries $\left(u_{1}(s), u_{2}(s)\right)$,

$$
\begin{align*}
&  \tag{1}\\
& 0 \\
& 1 \\
& \vdots \\
& m-1
\end{align*}\left(\begin{array}{cccc}
0 & 1 & \cdots & n-1 \\
\left(a_{00}, b_{00}\right) & \left(a_{01}, b_{01}\right) & \cdots & \left(a_{0, n-1}, b_{0, n-1}\right) \\
\left(a_{10}, b_{10}\right) & \left(a_{11}, b_{11}\right) & \cdots & \left(a_{1, n-1}, b_{1, n-1}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\left(a_{m-1,0}, b_{m-1,0}\right) & \left(a_{m-1,1}, b_{m-1,1}\right) & \cdots & \left(a_{m-1, n-1}, b_{m-1, n-1}\right)
\end{array}\right) .
$$

The notion of Nash equilibrium is one of the most important solution concepts in noncooperative game theory. It defines a strategy vector at which each strategy is a best reply to the strategies of the other players.

Definition 2. [16] A strategy vector $s^{*}=\left(s_{1}^{*}, s_{2}^{*}, \ldots, s_{r}^{*}\right)$ is a Nash equilibrium iffor each player $i \in N$ and each strategy $s_{i} \in S_{i}$ the following is satisfied:

$$
\begin{equation*}
u_{i}\left(s^{*}\right) \geq u_{i}\left(s_{i}, s_{-i}^{*}\right) \tag{2}
\end{equation*}
$$

where $s_{-i}^{*}=\left(s_{1}^{*}, \ldots, s_{i-1}^{*}, s_{i+1}^{*}, \ldots, s_{r}^{*}\right)$.
In particular, if a strategic form game is described in bimatrix form, the Nash equilibrium can be defined as follows:

Definition 3. A position $(i, j)$ in a bimatrix game (1) is a Nash equilibrium if

$$
\begin{equation*}
a_{i j} \geq a_{k j} \text { for all } k \in\{0,1, \ldots, m-1\} \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
b_{i j} \geq b_{i l} \text { for all } l \in\{0,1, \ldots, n-1\} \tag{4}
\end{equation*}
$$

In a Nash equilibrium, the players make their choices independently of one another. A more general solution concept is a correlated equilibrium. It covers situations in which the players can choose their strategies on the basis of the recommended strategy profiles.

Definition 4. [17] A correlated equilibrium in the bimatrix game (1) is a probability distribution $P=\left(p_{i j}\right)$ satisfying

$$
\begin{align*}
& \sum_{j=0}^{n-1}\left(a_{i j}-a_{k j}\right) p_{i j} \geq 0 \quad \text { for all } \quad i, k=0, \ldots, m-1  \tag{5}\\
& \sum_{i=0}^{m-1}\left(b_{i j}-b_{i l}\right) p_{i j} \geq 0 \quad \text { for all } \quad j, l=0, \ldots, n-1 \tag{6}
\end{align*}
$$

Games with incomplete information concern problems in which players may not be informed about certain elements of the game, for example, about payoff functions of other players.

Definition 5. [16] A Harasanyi game with incomplete information is a quintuple

$$
\begin{equation*}
\left(N,\left(T_{i}\right)_{i \in N}, p, S,\left(s_{t}\right)_{t \in \times_{i \in N} T_{i}}\right) \tag{7}
\end{equation*}
$$

where:

- $\quad N$ is a finite set of players.
- $\quad T_{i}$ is a finite set of types for player $i$, for each $i \in N$. The set of type vectors is denoted by $T=\times_{i \in N} T_{i}$.
- $\quad p \in \Delta(T)$ is a probability distribution over the set of type vectors that satisfies $p\left(t_{i}\right)=$ $\sum_{t_{-i} \in T_{-i}} p\left(t_{i}, t_{-i}\right)>0$ for every player $i \in N$ and every type $t_{i} \in T_{i}$.
- $\quad S$ is a set of states of nature. Every state of nature $s \in S$ is a triple $s=\left(N,\left(A_{i}\right)_{i \in N},\left(u_{i}\right)_{i \in N}\right)$, where $A_{i}$ is a nonempty set of actions of player $i$ and $u_{i}: \times_{i \in N} A_{i} \rightarrow \mathbb{R}$ is the payoff function of player $i$.
- $s_{t}=\left(N,\left(A_{i}\left(t_{i}\right)\right)_{i \in N},\left(u_{i}(t)\right)_{i \in N}\right) \in S$ is the state game for the type vector $t$, for every $t \in T$. Thus, player $i^{\prime}$ s action set in the state game $s_{t}$ depends on his type $t_{i}$ only, and is independent of the types of the other players.

A Harasanyi game with incomplete information proceeds in the following way [16]:

- A chance mover chooses a type vector $t=\left(t_{1}, t_{2}, \ldots, t_{n}\right) \in T$ according to the probability distribution $p$.
- Each player $i$ knows his type $t_{i}$, but does not know the types $t_{-i}=\left(t_{j}\right)_{j \neq i}$ of the other players.
- Each player $i$ chooses an action $a_{i} \in A_{i}\left(t_{i}\right)$.
- Each player $i$ obtains the payoff $u_{i}(t, a)$, where $a=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ is the profile of actions chosen by all the players.

Definition 6. [16] A pure strategy of player $i$ in a game with incomplete information is a function $s_{i}: T_{i} \rightarrow \bigcup_{t_{i} \in T_{i}} A_{i}\left(t_{i}\right)$ that satisfies

$$
\begin{equation*}
s_{i}\left(t_{i}\right) \in A_{i}\left(t_{i}\right) \tag{8}
\end{equation*}
$$

for each $t_{i} \in T_{i}$.
Let $s=\left(s_{i}\right)_{i \in N}$ be a strategy profile. The expected payoff in a game with incomplete information is

$$
\begin{equation*}
U_{i}(s)=\sum_{t \in T} p(t) u_{i}(t, s) \tag{9}
\end{equation*}
$$

where $u_{i}(t, s)$ is a payoff resulting from playing the strategy profile $s$ in the game associated with the type vector $t$.

## 3. The Generalized Marinatto-Weber Scheme

In this section, we recall the Marinatto-Weber (MW) scheme for bimatrix games and then we present the generalized model that was introduced in [4].

The (MW) scheme was originally designed for a $2 \times 2$ game:

$$
\left(\begin{array}{ll}
\left(a_{00}, b_{00}\right) & \left(a_{01}, b_{01}\right)  \tag{10}\\
\left(a_{10}, b_{10}\right) & \left(a_{11}, b_{11}\right)
\end{array}\right), a_{i j} \in \mathbb{R} .
$$

According to the model, each of the two players acts with the identity matrix $I$ of size 2 and the Pauli matrix $\sigma_{x}$ on his own qubit of some fixed two-qubit state $|\Psi\rangle$

$$
\begin{align*}
& \left|\Psi_{\mathrm{f}}\right\rangle\left\langle\Psi_{\mathrm{f}}\right|=p q I \otimes I|\Psi\rangle\langle\Psi| I \otimes I+p(1-q) I \otimes \sigma_{x}|\Psi\rangle\langle\Psi| I \otimes \sigma_{x} \\
& \quad+(1-p) q \sigma_{x} \otimes I|\Psi\rangle\langle\Psi| \sigma_{x} \otimes I+(1-p)(1-q) \sigma_{x} \otimes \sigma_{x}|\Psi\rangle\langle\Psi| \sigma_{x} \otimes \sigma_{x} \tag{11}
\end{align*}
$$

where $p$ and $q$ are the probabilities of choosing the identity I by player 1 and player 2 , respectively. Player $i$ 's payoff $u_{i}$ depends on $p$ and $q$, and through the measurement operators

$$
\begin{equation*}
M_{1}=\sum_{i=0}^{1} \sum_{j=0}^{1} a_{i j}, \quad M_{2}=\sum_{i=0}^{1} \sum_{j=0}^{1} b_{i j} \tag{12}
\end{equation*}
$$

it is given by the following formula:

$$
\begin{equation*}
u_{i}(p, q)=\operatorname{tr}\left(\left|\Psi_{\mathrm{f}}\right\rangle\left\langle\Psi_{\mathrm{f}}\right| M_{i}\right) . \tag{13}
\end{equation*}
$$

The MW scheme imples the classical $2 \times 2$ game by putting $|\Psi\rangle=|i j\rangle, i, j \in\{0,1\}$. The strategy sets in the generalization of the MW scheme for $m \times n$ bimatrix games are sets of permutation matrices. It enables one to obtain a classical $m \times n$ bimatrix game when $|\Psi\rangle=|i j\rangle, i \in\{0,1, \ldots, m-1\}$ and $j \in\{0,1, \ldots, n-1\}$.

Let us consider $m \times n$ bimatrix game (1). The generalized Marinatto-Weber (gMW) scheme is defined by a triple

$$
\begin{equation*}
\Gamma_{g M W}=\left(|\Psi\rangle,\left(S_{1}, S_{2}\right),\left(u_{1}, u_{2}\right)\right) \tag{14}
\end{equation*}
$$

where

- $\quad|\Psi\rangle$ is a joint state of $m$-dimensional and $n$-dimensional quantum systems:

$$
\begin{equation*}
|\Psi\rangle=\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \alpha_{i j}|i j\rangle, \quad \alpha_{i j} \in \mathbb{C} ; \tag{15}
\end{equation*}
$$

- $\quad S_{i}$ is a set of strategies of player $i \in\{1,2\}$ :

$$
\begin{equation*}
S_{1}=\left\{V_{k}: k=0, \ldots, m-1\right\}, \quad S_{2}=\left\{V_{k}: k=0, \ldots, n-1\right\} \tag{16}
\end{equation*}
$$

and $V_{k}$ for $k=0,1, \ldots, r-1$ acts on states of the computational basis $\{|0\rangle,|1\rangle, \ldots,|r-1\rangle\}$ as follows:

$$
\begin{equation*}
V_{k}|i\rangle=\left|i+_{r} k\right\rangle, k=0,1, \ldots, r-1, \tag{17}
\end{equation*}
$$

where the symbol $+r$ denotes addition modulo $r$;

- $u_{i}: S_{1} \times S_{2} \rightarrow \mathbb{R}$ is the payoff of player $i \in\{1,2\}$ defined as the expected value of the measurement $M_{i}$,

$$
\begin{equation*}
M_{1}=\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{i j}|i j\rangle\langle i j|, \quad M_{2}=\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} b_{i j}|i j\rangle\langle i j| \tag{18}
\end{equation*}
$$

on the final state

$$
\begin{equation*}
\left|\Psi_{\mathrm{f}}\right\rangle=\left(V_{k} \otimes V_{l}\right)|\Psi\rangle \tag{19}
\end{equation*}
$$

i.e.,

$$
\begin{equation*}
u_{i}\left(V_{k}, V_{l}\right)=\operatorname{tr}\left(\left|\Psi_{\mathrm{f}}\right\rangle\left\langle\Psi_{\mathrm{f}}\right| M_{i}\right) \tag{20}
\end{equation*}
$$

for $k=0,1, \ldots, m-1$ and $l=0,1, \ldots, n-1$.
This scheme can certainly reproduce a classical $m \times n$ bimatrix game. If $|\Psi\rangle=|00\rangle$, the form of the final state $\left|\Psi_{\mathrm{f}}\right\rangle$ is determined by each pair of $\left(V_{k_{i}}, V_{l_{j}}\right)$ and is given in the following matrix:

$$
\begin{align*}
&  \tag{21}\\
& V_{0} \\
& V_{1} \\
& \vdots \\
& V_{m-1} \\
& \hline
\end{align*}\left(\begin{array}{cccc}
V_{0} & V_{1} & \cdots & V_{n-1} \\
|00\rangle & |01\rangle & \cdots & |0, n-1\rangle \\
|10\rangle & |11\rangle & \cdots & |1, n-1\rangle \\
\vdots & \vdots & \ddots & \vdots \\
|m-1,0\rangle & |m-1,1\rangle & \vdots & |m-1, n-1\rangle
\end{array}\right)
$$

Formula (20) applied to each entry of (21) results in the classical game (1). The same bimatrix up to the order of players' strategies could also be obtained if one uses any $|\Psi\rangle=|i j\rangle$. Note also that the original MW scheme for a $2 \times 2$ game introduced in [3] can be obtained by taking $m=n=2$.

## 4. Relationship between the Correlated Equilibrium and the Nash Equilibrium in the MW Scheme

It is intuitively clear from the construction of the MW scheme that the initial quantum state correlates payoff vectors. After the players choose their own strategies $V_{k}$, the payoff vectors are distributed according to the measurement on the resulting quantum state. On the other hand, a correlated equilibrium plays a similar role in the coordination of strategy profiles. An interesting question arises whether there is a connection between the notion of correlated equilibrium and the MW scheme. The following example confirms this fact.

Example 1. Let us consider the following $2 \times 3$ game

$$
\left(\begin{array}{lll}
(3,2) & (2,0) & (1,3)  \tag{22}\\
(4,2) & (1,3) & (2,1)
\end{array}\right) .
$$

Conditions (5) and (6) for $m=2$ and $n=3$ are of the form:

$$
\begin{align*}
& \left(a_{00}-a_{10}\right) p_{00}+\left(a_{01}-a_{11}\right) p_{01}+\left(a_{02}-a_{12}\right) p_{02} \geq 0, \\
& \left(a_{10}-a_{00}\right) p_{10}+\left(a_{11}-a_{01}\right) p_{11}+\left(a_{12}-a_{02}\right) p_{12} \geq 0 \\
& \left(b_{00}-b_{01}\right) p_{00}+\left(b_{10}-b_{11}\right) p_{10} \geq 0 \\
& \left(b_{00}-b_{02}\right) p_{00}+\left(b_{10}-b_{12}\right) p_{10} \geq 0 \\
& \left(b_{01}-b_{00}\right) p_{01}+\left(b_{11}-b_{10}\right) p_{11} \geq 0  \tag{23}\\
& \left(b_{01}-b_{02}\right) p_{01}+\left(b_{11}-b_{12}\right) p_{11} \geq 0 \\
& \left(b_{02}-b_{00}\right) p_{02}+\left(b_{12}-b_{10}\right) p_{12} \geq 0 \\
& \left(b_{02}-b_{01}\right) p_{02}+\left(b_{12}-b_{11}\right) p_{12} \geq 0 .
\end{align*}
$$

Substituting payoffs of (22) to (23) implies the system of inequalities

$$
\begin{align*}
& -p_{00}+p_{01}-p_{02} \geq 0, \\
& p_{10}-p_{11}+p_{12} \geq 0, \\
& 2 p_{00}-p_{10} \geq 0, \\
& -p_{00}+p_{10} \geq 0,  \tag{24}\\
& -2 p_{01}+p_{11} \geq 0, \\
& -3 p_{01}+2 p_{11} \geq 0, \\
& p_{02}-p_{12} \geq 0, \\
& 3 p_{02}-2 p_{12} \geq 0 .
\end{align*}
$$

Inequalities (24) together with $i=1,2$ and $j=1,2,3$ reduce to

$$
\begin{equation*}
p_{10}>0, p_{00}=\frac{p_{10}}{2}, p_{12}=p_{02}=0, p_{11}=p_{10}, p_{01}=\frac{p_{11}}{2} \tag{25}
\end{equation*}
$$

Combining (25) with $\sum_{i j} p_{i j}=1$ yields the unique correlated equilibrium

$$
\begin{equation*}
\left(p_{00}, p_{01}, p_{02}, p_{10}, p_{11}, p_{12}\right)=\left(\frac{1}{6}, \frac{1}{6}, 0, \frac{1}{3}, \frac{1}{3}, 0\right) . \tag{26}
\end{equation*}
$$

To see how correlated equilibrium (26) in game (22) implies a Nash equilibrium in the MW approach to that game, consider the initial state $|\Psi\rangle$ in (14) such that $\alpha_{i j}=\sqrt{p_{i j}}$ for $i=0,1$ and $j=0,1,2$. Then

$$
\begin{equation*}
|\Psi\rangle=\sqrt{\frac{1}{6}}|00\rangle+\sqrt{\frac{1}{6}}|01\rangle+\sqrt{\frac{2}{6}}|10\rangle+\sqrt{\frac{2}{6}}|11\rangle . \tag{27}
\end{equation*}
$$

Taking into account the strategy sets and the payoffs of (22), the other components of (14) are

$$
\begin{equation*}
S_{1}=\left\{V_{0}, V_{1}\right\}, \quad S_{2}=\left\{V_{0}, V_{1}, V_{2}\right\}, \tag{28}
\end{equation*}
$$

and

$$
\begin{align*}
& M_{1}=3|00\rangle\langle 00|+2|01\rangle\langle 01|+|02\rangle\langle 02|+4|10\rangle\langle 10|+|11\rangle\langle 11|+2|12\rangle\langle 12|,  \tag{29}\\
& M_{2}=2|00\rangle\langle 00|+3|02\rangle\langle 02|+2|10\rangle\langle 10|+3|11\rangle\langle 11|+|12\rangle\langle 12|
\end{align*}
$$

Now, consider the strategy profile $\left(V_{0}, V_{0}\right)$. Payoffs of player 1 and 2 corresponding to that profile are

$$
\begin{equation*}
u_{1}\left(\left(V_{0}, V_{0}\right)\right)=\operatorname{tr}\left(|\Psi\rangle\langle\Psi| M_{1}\right)=\frac{5}{2}, \quad u_{2}\left(\left(V_{0}, V_{0}\right)\right)=\operatorname{tr}\left(|\Psi\rangle\langle\Psi| M_{2}\right)=2 \tag{30}
\end{equation*}
$$

Interestingly, no player has a profitable unilateral deviation from $\left(V_{0}, V_{0}\right)$. In the case of player 1 , if he deviates to $V_{1}$, then his payoff is still $5 / 2$; i.e.,

$$
\begin{equation*}
u_{1}\left(\left(V_{1}, V_{0}\right)\right)=\operatorname{tr}\left(\left(V_{1} \otimes V_{0}\right)|\Psi\rangle\langle\Psi|\left(V_{1} \otimes V_{0}\right)^{\dagger} M_{1}\right)=\frac{5}{2} \tag{31}
\end{equation*}
$$

Similarly, player 2 will not benefit from playing the strategies $V_{1}$ and $V_{2}$,

$$
\begin{equation*}
u_{2}\left(V_{0}, V_{j}\right)=\operatorname{tr}\left(\left(V_{0} \otimes V_{j}\right)|\Psi\rangle\langle\Psi|\left(V_{0} \otimes V_{j}\right)^{\dagger} M_{2}\right)=\frac{11}{6} \text { for } j=1,2 \tag{32}
\end{equation*}
$$

Since

$$
\begin{align*}
& u_{1}\left(\left(V_{0}, V_{0}\right)\right) \geq u_{1}\left(\left(V_{1}, V_{0}\right)\right) \\
& u_{2}\left(\left(V_{0}, V_{0}\right)\right) \geq u_{2}\left(\left(V_{0}, V_{j}\right)\right) \text { for } j=1,2 \tag{33}
\end{align*}
$$

the strategy profile $\left(V_{0}, V_{0}\right)$ is a Nash equilibrium (see Definition 2 ).

Example 1 shows that the MW approach to (22) in which the initial state is based on correlated equilibrium (26) results in the payoff-equivalent Nash equilibrium. As the following proposition states, this property is true in general. The MW approach in which the amplitudes of the initial state are the square roots of the respective probabilities of the correlated equilibrium always implies the Nash equilibrium $\left(V_{0}, V_{0}\right)$.

Proposition 1. Let $\left(p_{i j}\right)$ be a correlated equilibrium in (1), and let $\left(|\Psi\rangle,\left(S_{1}, S_{2}\right), M\right)$ be the MW approach associated with (1). If $|\Psi\rangle=\sum_{i j} \sqrt{p_{i j}} \mathrm{e}^{\mathrm{i} \theta_{i j}}|i j\rangle$ then $\left(V_{0}, V_{0}\right)$ is a Nash equilibrium in $\left(|\Psi\rangle,\left(S_{1}, S_{2}\right), M\right)$ with the equilibrium outcome $\sum_{i j} p_{i j}\left(a_{i j}, b_{i j}\right)$.

Proof. First note that indicies $k$ and $l$ in (5) and (6) go from 0 to $m-1$ and from 0 to $n-1$, respectively. Thus, (5) and (6) can be written as

$$
\begin{align*}
& \sum_{j=0}^{n-1}\left(a_{i j}-a_{i+m} k, j\right) p_{i j} \geq 0 \quad \text { for all } \quad i, k=0, \ldots, m-1  \tag{34}\\
& \sum_{i=0}^{m-1}\left(b_{i j}-b_{i, j+n}\right) p_{i j} \geq 0 \quad \text { for all } \quad j, l=0, \ldots, n-1 \tag{35}
\end{align*}
$$

Let us determine $u_{1}\left(V_{k}, V_{0}\right)$. First we obtain

$$
\begin{equation*}
V_{k} \otimes V_{0}|\Psi\rangle=\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \alpha_{i j}\left|i+_{m} k, j\right\rangle \tag{36}
\end{equation*}
$$

Then

$$
\begin{equation*}
u_{1}\left(V_{k}, V_{0}\right)=\operatorname{tr}\left(V_{k} \otimes V_{0}|\Psi\rangle\langle\Psi| V_{k}^{\dagger} \otimes V_{0}^{\dagger} M_{1}\right)=\sum_{i=0}^{m-1} \sum_{j=0}^{n-1}\left|\alpha_{i j}\right|^{2} a_{i+m} k, j \tag{37}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
u_{1}\left(V_{0}, V_{0}\right) \geq u_{1}\left(V_{k}, V_{0}\right) \Leftrightarrow \sum_{i=0}^{m-1} \sum_{j=0}^{n-1}\left|\alpha_{i j}\right|^{2}\left(a_{i j}-a_{i+m} k, j\right) \geq 0 \tag{38}
\end{equation*}
$$

for each $k=0,1, \ldots, m$. Using the fact that $\alpha_{i j}=\sqrt{p_{i j}} \mathrm{e}^{\mathrm{i} \theta_{i j}}$ for all $i, j$ we can rewrite (38) as

$$
\begin{equation*}
\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} p_{i j}\left(a_{i j}-a_{i+{ }_{m} k, j}\right) \geq 0 \tag{39}
\end{equation*}
$$

Inequality (39) is a consequence of the correlated equilibrium conditions (34).
In a similar manner we can prove that

$$
\begin{equation*}
u_{2}\left(V_{0}, V_{0}\right) \geq u_{2}\left(V_{0}, V_{l}\right) \Leftrightarrow \sum_{i=0}^{m-1} \sum_{j=0}^{n-1}\left|\alpha_{i j}\right|^{2}\left(b_{i j}-b_{i, j+n}\right) \geq 0 \tag{40}
\end{equation*}
$$

It is worth noting that the converse of Proposition 1 may not hold. Given the Nash equilibrium $\left(V_{0}, V_{0}\right)$ in the MW game, the corresponding initial state may not determine a correlated equilibrium in the classical game. The following example illustrates this point.

Example 2. Consider again the game given by bimatrix (22) and the MW approach to that game with the initial state

$$
\begin{equation*}
|\Psi\rangle=\sqrt{\frac{1}{6}} \sum_{i=0}^{1} \sum_{j=0}^{2}|i j\rangle \tag{41}
\end{equation*}
$$

The state $|\Psi\rangle$ in the equal superposition means that the players cannot change the state $|\Psi\rangle$; i.e.,

$$
\begin{equation*}
\left(V_{k} \otimes V_{l}\right)|\Psi\rangle=|\Psi\rangle \text { for all } k=0,1 \text { all } l=0,1,2 \tag{42}
\end{equation*}
$$

As a result, each strategy profile $\left(V_{k} \otimes V_{l}\right)$ implies the same payoff vector,

$$
\begin{align*}
& u_{1}\left(\left(V_{k}, V_{l}\right)\right)=\operatorname{tr}\left(|\Psi\rangle\langle\Psi| M_{1}\right)=\frac{13}{6}  \tag{43}\\
& u_{2}\left(\left(V_{k}, V_{l}\right)\right)=\operatorname{tr}\left(|\Psi\rangle\langle\Psi| M_{2}\right)=\frac{11}{6} \tag{44}
\end{align*}
$$

for $k=0,1$ all $l=0,1,2$. The same payoff vectors for each strategy profile imply that $\left(V_{0}, V_{0}\right)$ satisfies the Nash equilibrium conditions (2). However, a probability distribution $P=\left(p_{i j}\right)$ such that $p_{i j}=1 / 6$ for each $i=0,1$ and $j=0,1,2$ is not a correlated equilibrium in (22)—the left hand side of the first condition of (23) is equal to $-1 / 6$.

## 5. Bimatrix Representation of the MW Scheme

The MW scheme can be described in terms of classical game theory. The model

$$
\begin{equation*}
\left(\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \alpha_{i j}|i j\rangle,\left(S_{1}, S_{2}\right),\left(u_{1}, u_{2}\right)\right) \tag{45}
\end{equation*}
$$

can be viewed as a family of bimatrix games in which the rows and columns of (1) are permuted according to a probability distribution $P=\left(\left|\alpha_{i j}\right|^{2}\right)$. In the case of the MW approach to a $2 \times 2$ game, the players play one of the following games

$$
\begin{align*}
& G_{I_{0}, I I_{0}}: \begin{array}{cc}
V_{0} & V_{1} \\
V_{0}\left(\begin{array}{cc}
\left(a_{00}, b_{00}\right) & \left(a_{01}, b_{01}\right) \\
V_{1} \\
\left(a_{10}, b_{10}\right) & \left(a_{11}, b_{11}\right)
\end{array}\right),
\end{array}, \quad G_{I_{0}, I I_{1}}: \begin{array}{cc}
V_{0} & V_{1} \\
V_{0} \\
V_{1}\left(\begin{array}{cc}
\left(a_{01}, b_{01}\right) & \left(a_{00}, b_{00}\right) \\
\left(a_{11}, b_{11}\right) & \left(a_{10}, b_{10}\right)
\end{array}\right), \\
V_{0} & V_{1}
\end{array}  \tag{46}\\
& G_{I_{1}, I I_{0}}: \begin{array}{ll}
V_{0} \\
V_{1}
\end{array}\left(\begin{array}{ll}
\left(a_{10}, b_{10}\right) & \left(a_{11}, b_{11}\right) \\
\left(a_{00}, b_{00}\right) & \left(a_{01}, b_{01}\right)
\end{array}\right), \quad G_{I_{1}, I I_{1}}: \begin{array}{ll}
V_{0} \\
V_{1}
\end{array}\left(\begin{array}{ll}
\left(a_{11}, b_{11}\right) & \left(a_{10}, b_{10}\right) \\
\left(a_{01}, b_{01}\right) & \left(a_{00}, b_{00}\right)
\end{array}\right) .
\end{align*}
$$

These games occur with probabilities

$$
\begin{array}{c|cccc}
G & G_{I_{0}, I I_{0}} & G_{I_{0}, I I_{1}} & G_{I_{1}, I I_{0}} & G_{I_{1}, I I_{1}}  \tag{47}\\
\hline p(G) & \left|\alpha_{00}\right|^{2} & \left|\alpha_{01}\right|^{2} & \left|\alpha_{10}\right|^{2} & \left|\alpha_{11}\right|^{2}
\end{array}
$$

One can check that the expected payoffs in the game defined by (46) and (47) are

$$
\begin{align*}
u_{1(2)}\left(\left(V_{k}, V_{l}\right)\right)=a_{k, l}\left(b_{k, l}\right)\left|\alpha_{00}\right|^{2}+a_{k, l+{ }_{2} 1} & \left(b_{k, l+{ }_{2} 1}\right)\left|\alpha_{01}\right|^{2} \\
& +a_{k+2_{2} 1, l}\left(b_{k+2_{2} 1, l}\right)\left|\alpha_{10}\right|^{2}+a_{k+2_{2} 1, l+{ }_{2} 1}\left(b_{k+2_{2} 1, l+{ }_{2} 1}\right)\left|\alpha_{11}\right|^{2} . \tag{48}
\end{align*}
$$

The same expected payoffs are obtained if we use Formula (20) for the MW approach to $2 \times 2$ game; i.e.,

$$
\begin{equation*}
u_{1(2)}\left(\left(V_{k}, V_{l}\right)\right)=\operatorname{tr}\left(\left(V_{k} \otimes V_{l}\right)|\Psi\rangle\langle\Psi|\left(V_{k} \otimes V_{l}\right)^{\dagger} M_{1(2)}\right) \tag{49}
\end{equation*}
$$

with

$$
\begin{equation*}
|\Psi\rangle=\sum_{i, j=0}^{1} \alpha_{i j}|i j\rangle \text { and } M_{1(2)}=\sum_{i, j=0}^{1} a_{i j}\left(b_{i j}\right)|i j\rangle\langle i j| . \tag{50}
\end{equation*}
$$

In what follows, we modify the MW scheme. The idea behind our scheme can also be explained with the use of the game given by (46) and (47). Let us assume in addition to the fact that probability distribution (47) is a common knowledge among the players that player 1 is informed that either the set of games $\left\{G_{I_{0}, I I_{0}}, G_{I_{0}, I I_{1}}\right\}$ or the set of games $\left\{G_{I_{1}, I I_{0}}, G_{I_{1}, I I_{1}}\right\}$ has to be taken into consideration. Player 2 knows that either $\left\{G_{I_{0}, I I_{0}}, G_{I_{1}, I I_{0}}\right\}$ or $\left\{G_{I_{0}, I I_{1}}, G_{I_{1}, I I_{1}}\right\}$ is actually played. To be more precise, we assume that the players play a game with incomplete information (see Definition (5)) defined as follows:

- The sets of types are

$$
\begin{equation*}
T_{1}=\left\{I_{0}, I_{1}\right\}, \quad T_{2}=\left\{I I_{0}, I I_{1}\right\} \tag{51}
\end{equation*}
$$

- The probability distribution $p$ over $T=\left(\left(I_{0}, I I_{0}\right),\left(I_{0}, I I_{1}\right),\left(I_{1}, I I_{0}\right),\left(I_{1}, I I_{1}\right)\right)$

$$
\begin{equation*}
p=\left(\left|\alpha_{00}\right|^{2},\left|\alpha_{01}\right|^{2},\left|\alpha_{10}\right|^{2},\left|\alpha_{11}\right|^{2}\right) \tag{52}
\end{equation*}
$$

such that

$$
\begin{align*}
& p\left(I_{0}\right)=p\left(I_{0}, I I_{0}\right)+p\left(I_{0}, I I_{1}\right)=\left|\alpha_{00}\right|^{2}+\left|\alpha_{01}\right|^{2} \\
& p\left(I_{1}\right)=p\left(I_{1}, I I_{0}\right)+p\left(I_{1}, I I_{1}\right)=\left|\alpha_{10}\right|^{2}+\left|\alpha_{11}\right|^{2} \\
& p\left(I I_{0}\right)=p\left(I_{0}, I I_{0}\right)+p\left(I_{1}, I I_{0}\right)=\left|\alpha_{00}\right|^{2}+\left|\alpha_{10}\right|^{2}  \tag{53}\\
& p\left(I I_{1}\right)=p\left(I_{0}, I I_{1}\right)+p\left(I_{1}, I I_{1}\right)=\left|\alpha_{01}\right|^{2}+\left|\alpha_{11}\right|^{2}
\end{align*}
$$

- States of nature are:

$$
\begin{equation*}
G_{I_{0}, I I_{0}}, G_{I_{0}, I I_{1}}, G_{I_{1}, I I_{0}}, G_{I_{1}, I I_{1}} \tag{54}
\end{equation*}
$$

Now, from Definition 6, the players' strategies $\tau_{1}$ and $\tau_{2}$ are the functions

$$
\begin{equation*}
\tau_{1}:\left\{I_{0}, I_{1}\right\} \rightarrow\left\{V_{0}, V_{1}\right\}, \quad \tau_{2}:\left\{I I_{0}, I I_{1}\right\} \rightarrow\left\{V_{0}, V_{1}\right\} \tag{55}
\end{equation*}
$$

Strategy $s_{i}$ specifies an action $V_{0}$ or $V_{1}$ for player $i$ depending on his type. Denote by $\left(V_{k_{0}}, V_{k_{1}}\right)$ the strategy of player 1. The first element of the pair is the action for player 1 of type $I_{0}$, the second one is the action of type $I_{1}$. Similarly, denote by $\left(V_{l_{0}}, V_{l_{1}}\right)$ the strategy of player 2.

To find the expected payoff in the game (51)-(55) we use Formula (9). For example, if the strategy profile is $\left(\left(V_{0}, V_{1}\right),\left(V_{1}, V_{1}\right)\right)$, the action profile $\left(V_{0}, V_{1}\right)$ is played in $G_{I_{0}, I I_{0}}$ and $G_{I_{0}, I I_{1}}$, and $\left(V_{1}, V_{1}\right)$ in games $G_{I_{1}, I I_{0}}$ and $G_{I_{1}, I I_{1}}$. Since games $G_{I_{0}, I I_{0}}, G_{I_{0}, I I_{1}}, G_{I_{1}, I I_{0}}$ and $G_{I_{1}, I I_{1}}$ occur with the probability $\left|\alpha_{00}\right|^{2},\left|\alpha_{01}\right|^{2},\left|\alpha_{10}\right|^{2}$, and $\left|\alpha_{11}\right|^{2}$, respectively, the payoffs resulting from playing $\left(\left(V_{0}, V_{1}\right),\left(V_{1}, V_{1}\right)\right)$ are

$$
\begin{align*}
& U_{1}\left(\left(V_{0}, V_{1}\right),\left(V_{1}, V_{1}\right)\right)=\left|\alpha_{00}\right|^{2} a_{01}+\left|\alpha_{01}\right|^{2} a_{00}+\left|\alpha_{10}\right|^{2} a_{01}+\left|\alpha_{11}\right|^{2} a_{00} \\
& U_{2}\left(\left(V_{0}, V_{1}\right),\left(V_{1}, V_{1}\right)\right)=\left|\alpha_{00}\right|^{2} b_{01}+\left|\alpha_{01}\right|^{2} b_{00}+\left|\alpha_{10}\right|^{2} b_{01}+\left|\alpha_{11}\right|^{2} b_{00} \tag{56}
\end{align*}
$$

In general, the expected payoffs in the game (51)-(55) can be given by the following formula:

$$
\begin{align*}
\left(U_{1}, U_{2}\right)\left(\left(V_{k_{0}}, V_{k_{1}}\right),\left(V_{l_{0}}, V_{l_{1}}\right)\right)= & \left|\alpha_{00}\right|^{2}\left(a_{k_{0}, l_{0}}, b_{k_{0}, l_{0}}\right)+\left|\alpha_{01}\right|^{2}\left(a_{k_{0}, l_{1}+2}, b_{k_{0}, l_{1}+21}\right) \\
& +\left|\alpha_{10}\right|^{2}\left(a_{k_{1}+2_{2} 1, l_{0}}, b_{k_{1}+{ }_{2} 1, l_{0}}\right)+\left|\alpha_{11}\right|^{2}\left(a_{k_{1}+{ }_{2} 1, l_{1}+2} 1, b_{k_{1}+{ }_{2} 1, l_{1}+21}\right) \tag{57}
\end{align*}
$$

As we can see, (57) implies a broader range of possible payoffs compared to (48).

## 6. The MW-Type Scheme for Correlated Equilibria

Now, we construct the MW-type scheme that is equivalent to the game defined by the payoff functions (57). Then we formulate the proposition analogous to Proposition 1, where in contrast to
that statement, the correlated equilibrium conditions are necessary and sufficient for the existence of some Nash equilibria.

Let us modify the components of $\Gamma_{g M W}$ to obtain

$$
\begin{equation*}
\Gamma_{c M W}=\left(|\Psi\rangle,\left(B_{1}, B_{2}\right),\left(U_{1}, U_{2}\right)\right) \tag{58}
\end{equation*}
$$

where

- $\quad|\Psi\rangle$ is defined as in (15);
- $\quad B_{i}$ is the set of strategies of player $i \in\{1,2\}$ :

$$
\begin{equation*}
B_{1}=\left\{V_{0}, V_{1}, \ldots, V_{m-1}\right\}^{m}, \quad B_{2}=\left\{V_{0}, V_{1}, \ldots, V_{n-1}\right\}^{n} \tag{59}
\end{equation*}
$$

with typical elements $\left(V_{k_{0}}, V_{k_{1}}, \ldots, V_{k_{m-1}}\right)$ and $\left(V_{l_{0}}, V_{l_{1}}, \ldots, V_{l_{n-1}}\right)$, respectively;

- $\quad U_{i}: B_{1} \times B_{2} \rightarrow \mathbb{R}$ is the payoff of player $i \in\{1,2\}$ defined as the expected value of the measurements (18) on the final state

$$
\begin{equation*}
\rho_{\mathrm{f}}=\sum_{i=0}^{m-1} \sum_{j=0}^{n-1}\left(V_{k_{i}} \otimes V_{l_{j}}\right) P_{i j}|\Psi\rangle\langle\Psi|\left(\left(V_{k_{i}} \otimes V_{l_{j}}\right) P_{i j}\right)^{\dagger}, \quad P_{i j}=|i j\rangle\langle i j| ; \tag{60}
\end{equation*}
$$

i.e.,

$$
\begin{equation*}
U_{t}\left(\left(V_{k_{0}}, V_{k_{1}}, \ldots, V_{k_{m-1}}\right),\left(V_{l_{0}}, V_{l_{1}}, \ldots, V_{l_{n-1}}\right)\right)=\operatorname{tr}\left(\rho_{\mathrm{f}} M_{t}\right), \quad t \in\{1,2\} \tag{61}
\end{equation*}
$$

We now verify that (61) coincides with (57) for $m, n=2$. First note that $|\Psi\rangle\langle\Psi|$ can be written as

$$
\begin{equation*}
|\Psi\rangle\langle\Psi|=\sum_{i, j, i^{\prime}, j^{\prime}=0}^{1} \alpha_{i j} \alpha_{i^{\prime} j^{\prime}}^{\prime}|i j\rangle\left\langle i^{\prime} j^{\prime}\right| . \tag{62}
\end{equation*}
$$

Let us determine each term of

$$
\begin{equation*}
\rho_{\mathrm{f}}=\sum_{i, j=0}^{1}\left(V_{k_{i}} \otimes V_{l_{j}}\right) P_{i j}|\Psi\rangle\langle\Psi| P_{i j}^{\dagger}\left(V_{k_{i}} \otimes V_{l_{j}}\right)^{\dagger} \tag{63}
\end{equation*}
$$

For

- $\quad i=0, j=0$

$$
\begin{equation*}
\left(V_{k_{0}} \otimes V_{l_{0}}\right) P_{00}|\Psi\rangle\langle\Psi| P_{00}^{\dagger}\left(V_{k_{0}} \otimes V_{l_{0}}\right)^{\dagger}=\left|\alpha_{00}\right|^{2}\left|k_{0}, l_{0}\right\rangle\left\langle k_{0}, l_{0}\right| \tag{64}
\end{equation*}
$$

- $\quad i=0, j=1$

$$
\begin{equation*}
\left(V_{k_{0}} \otimes V_{l_{1}}\right) P_{01}|\Psi\rangle\langle\Psi| P_{01}^{\dagger}\left(V_{k_{0}} \otimes V_{l_{1}}\right)^{\dagger}=\left|\alpha_{01}\right|^{2}\left|k_{0}, l_{1}+{ }_{2} 1\right\rangle\left\langle k_{0}, l_{1}+{ }_{2} 1\right| ; \tag{65}
\end{equation*}
$$

- $\quad i=1, j=0$

$$
\begin{equation*}
\left(V_{k_{1}} \otimes V_{l_{0}}\right) P_{10}|\Psi\rangle\langle\Psi| P_{10}^{\dagger}\left(V_{k_{1}} \otimes V_{l_{0}}\right)^{\dagger}=\left|\alpha_{10}\right|^{2}\left|k_{1}+21, l_{0}\right\rangle\left\langle k_{1}+21, l_{0}\right| \tag{66}
\end{equation*}
$$

- $\quad i=1, j=1$

$$
\begin{equation*}
\left(V_{k_{1}} \otimes V_{l_{1}}\right) P_{11}|\Psi\rangle\langle\Psi| P_{11}^{\dagger}\left(V_{k_{1}} \otimes V_{l_{1}}\right)^{\dagger}=\left|\alpha_{11}\right|^{2}\left|k_{1}+21, l_{1}+21\right\rangle\left\langle k_{1}+{ }_{2} 1, l_{1}+21\right| . \tag{67}
\end{equation*}
$$

Measuring $\rho_{\mathrm{f}}$ with respect to $M_{1}$ and $M_{2}$ given by (50) yields the same expected payoff as (57).
It is commonly required in quantum game theory that a given scheme must coincide with the classical game under specific settings. The classical game in the MW game is obtained by taking
$|00 \ldots 0\rangle$ in (15). Another way of deriving the classical game can be carried out by a restriction of the players' strategies. In the seminal Eisert-Wilkens-Lewenstein scheme [2], restricting strategies to some type of one-parameter unitary operators yields the classical game. Similarly, a proper limitation of the sets $B_{1}$ and $B_{2}$ in $\Gamma_{c M W}$ leads to the game that is equivalent to (1). That is the content of the following proposition:

Proposition 2. If

$$
\begin{align*}
& B_{1}^{\prime}=\left\{\left(V_{i}, V_{(m-1)+{ }_{m} i}, V_{(m-2)+_{m} i} \ldots, V_{1+m}\right) ; i \in\{0,1, \ldots, m-1\}\right\},  \tag{68}\\
& B_{2}^{\prime}=\left\{\left(V_{j}, V_{(n-1)+_{n} j}, V_{(n-2)+{ }_{n} j} \ldots, V_{1+n} j\right) ; j \in\{0,1, \ldots, n-1\}\right\}
\end{align*}
$$

then the game determined by $\Gamma_{c M W}=\left(|\Psi\rangle,\left(B_{1}^{\prime}, B_{2}^{\prime}\right),\left(U_{1}, U_{2}\right)\right)$ can be identified with a bimatrix game $m \times n$.
Proof. Let us consider first a strategy profile consisting of a strategy ( $V_{0}, V_{m-1}, V_{m-2}, \ldots, V_{1}$ ) of player 1 and $\left(V_{0}, V_{n-1}, V_{n-2}, \ldots, V_{1}\right)$ of player 2 . Keeping in mind that $a \equiv 0(\bmod a)$, we can write final state (60) in the form of

$$
\begin{align*}
\rho_{\mathrm{f}} & =\sum_{i=0}^{m-1} \sum_{j=0}^{n-1}\left|\alpha_{i j}\right|^{2}\left(V_{m-i} \otimes V_{n-j}\right)|i j\rangle\langle i j|\left(V_{m-i} \otimes V_{n-j}\right)^{\dagger} \\
& =\sum_{i=0}^{m-1} \sum_{j=0}^{n-1}\left|\alpha_{i j}\right|^{2}|i+m(m-i), j+n(n-j)\rangle\left\langle i+_{m}(m-i), j+_{n}(n-j)\right|  \tag{69}\\
& =\sum_{i=0}^{m-1} \sum_{j=0}^{n-1}\left|\alpha_{i j}\right|^{2}|00\rangle\langle 00| .
\end{align*}
$$

In general, adding $i$ modulo $m$ and $j$ modulo $n$ to the subscripts of $\left(V_{0}, V_{m-1}, V_{m-2}, \ldots, V_{1}\right)$ and $\left(V_{0}, V_{n-1}, V_{n-2}, \ldots, V_{1}\right)$ respectively, yields

$$
\begin{equation*}
\rho_{\mathrm{f}}=\sum_{i=0}^{m-1} \sum_{j=0}^{n-1}\left|\alpha_{i j}\right|^{2}|i j\rangle\langle i j| \tag{70}
\end{equation*}
$$

This implies pure payoff profiles of the bimatrix game:

$$
\begin{equation*}
\left(U_{1}, U_{2}\right)\left(\left(V_{i}, V_{(m-1)+{ }_{m} i}, V_{(m-2)+{ }_{m} i} \ldots, V_{1+{ }_{m} i}\right),\left(V_{j}, V_{(n-1)+_{n} j}, V_{(n-2)+_{n} j} \ldots, V_{1+{ }_{n} j}\right)\right)=\left(a_{i j}, b_{i j}\right) \tag{71}
\end{equation*}
$$

Note that $\left|B_{1}^{\prime}\right|=m$ and $\left|B_{2}^{\prime}\right|=n$, which completes the proof.
It was shown in Example 2 that a Nash equilibrium in the MW scheme may not necessarily mean that the initial state generates a correlated equilibrium in the classical game. The strategy profile $\left(V_{0}, V_{0}\right)$ is a Nash equilibrium in the MW scheme, even though the probability distribution based on squared amplitudes of the initial state is not a correlated equilibrium. In the case of scheme (58), the correlated equilibrium and the existence of some specific Nash equilibria are equivalent.

Example 3. Consider the scheme $\Gamma_{c M W}$ associated with (22). Let the initial state $|\Psi\rangle$ be given by (41). Since $m=2$ and $n=3$ for bimatrix game (22), a typical strategy of player 1 is $\left(V_{k_{0}}, V_{k_{1}}\right)$, and a typical strategy of player 2 is of the form $\left(V_{l_{0}}, V_{l_{1}}, V_{l_{2}}\right)$. The final state $\rho_{\mathrm{f}}$ resulting from playing a general strategy profile is

$$
\begin{align*}
& \rho_{\mathrm{f}}=\frac{1}{6} \sum_{i=0}^{1} \sum_{j=0}^{2}\left(V_{k_{i}} \otimes\right.\left.V_{l_{j}}\right)|i j\rangle\langle i j|\left(V_{k_{i}} \otimes V_{l_{j}}\right)^{+} \\
&=\frac{1}{6}\left(\left|k_{0}, l_{0}\right\rangle\left\langle k_{0}, l_{0}\right|+\left|k_{0}, l_{1}+{ }_{3} 1\right\rangle\left\langle k_{0}, l_{1}+{ }_{3} 1\right|+\left|k_{0}, l_{2}+{ }_{3} 2\right\rangle\left\langle k_{0}, l_{1}+{ }_{3} 2\right|\right. \\
&+\left|k_{1}+{ }_{2} 1, l_{0}\right\rangle\left\langle k_{1}+{ }_{2} 1, l_{0}\right|+\left|k_{1}+{ }_{2} 1, l_{1}+{ }_{3} 1\right\rangle\left\langle k_{1}+{ }_{2} 1, l_{1}+{ }_{3} 1\right| \\
&\left.+\left|k_{1}+{ }_{2} 1, l_{2}+{ }_{3} 2\right\rangle\left\langle k_{1}+{ }_{2} 1, l_{2}+{ }_{3} 2\right|\right) . \tag{72}
\end{align*}
$$

Hence

$$
\begin{align*}
& \left(U_{1}, U_{2}\right)\left(\left(V_{k_{0}}, V_{k_{1}}\right),\left(V_{l_{0}}, V_{l_{1}}, V_{l_{2}}\right)\right) \\
& =\frac{1}{6}\left(\left(a_{k_{0}, l_{0}}, b_{k_{0}, l_{0}}\right)+\left(a_{k_{0}, l_{1}+31}, b_{k_{0}, l_{1}+{ }_{3} 1}\right)+\left(a_{k_{0}, l_{2}+3}, b_{k_{0}, l_{2}+3_{2}}\right)+\left(a_{k_{1}+2,1, l_{0}}, b_{k_{1}+2, l_{0}}\right)\right. \\
& \left.+\left(a_{k_{1}+21, l_{1} 3_{3} 1}, b_{k_{1}+21, l_{1}+31}\right)+\left(a_{k_{1}+21, l_{2}+3^{2}}, b_{k_{1}+21, l_{2}+3^{2}}\right)\right) . \tag{73}
\end{align*}
$$

By substituting the payoffs of (22) we deduce that the payoff of player 1 that corresponds to the strategy profile $\left(\left(V_{0}, V_{0}\right),\left(V_{0}, V_{0}, V_{0}\right)\right)$ is

$$
\begin{equation*}
U_{1}\left(\left(\left(V_{0}, V_{0}\right),\left(V_{0}, V_{0}, V_{0}\right)\right)\right)=\frac{13}{6} . \tag{74}
\end{equation*}
$$

If player 1 deviates to $\left(V_{1}, V_{0}\right)$, the resulting payoff is

$$
\begin{equation*}
U_{1}\left(\left(\left(V_{1}, V_{0}\right),\left(V_{0}, V_{0}, V_{0}\right)\right)\right)=\frac{7}{3} . \tag{75}
\end{equation*}
$$

Thus, the strategy profile $\left(\left(V_{0}, V_{0}\right),\left(V_{0}, V_{0}, V_{0}\right)\right)$ is not a Nash equilibrium in the game $\Gamma_{c M W}$.
The next proposition states that the equivalence between the correlated equilibrium and the Nash equilibrium can be generalized to every bimatrix game.

Proposition 3. Let $\left(p_{i j}\right)$ be a probability distribution over the outcomes of bimatrix game (1), and let $\Gamma_{c M W}=\left(\sum_{i j} \sqrt{p_{i j}}{ }^{\mathrm{i}}{ }^{\mathrm{i}} \mathrm{i}_{i j}|i j\rangle,\left(B_{1}, B_{2}\right),\left(U_{1}, U_{2}\right)\right)$ be associated with (1). Then $\left(p_{i j}\right)$ is a correlated equilibrium in (1) if and only if the strategy profile $\left(\tau^{*}, \tau^{*}\right)=\left(\prod_{i=0}^{m-1} V_{0}^{(i)}, \prod_{j=0}^{n-1} V_{0}^{(j)}\right)$ is a Nash equilibrium in $\Gamma_{c M W}$.

Proof. Let us consider the expression

$$
\begin{equation*}
U_{1}\left(\tau^{*}, \tau^{*}\right) \geq U_{1}\left(\left(V_{k_{0}}, V_{0}, V_{0}, \ldots, V_{0}\right), \tau^{*}\right) \tag{76}
\end{equation*}
$$

According to (61),

$$
\begin{align*}
& U_{1}\left(\left(V_{k_{0}}, V_{0}, V_{0}, \ldots, V_{0}\right), \tau^{*}\right) \\
& \quad=\operatorname{tr}\left(\sum_{j=0}^{n-1}\left(V_{k_{0}} \otimes \mathbb{1}\right) P_{0, j}|\Psi\rangle\langle\Psi|\left(\left(V_{k_{0}} \otimes \mathbb{1}\right) P_{0, j}\right)^{\dagger} M_{1}\right)+\operatorname{tr}\left(\sum_{i=1}^{m-1} \sum_{j=0}^{n-1} P_{i j}|\Psi\rangle\langle\Psi| P_{i j}^{\dagger} M_{1}\right) . \tag{77}
\end{align*}
$$

## Hence,

$$
\begin{align*}
U_{1}\left(\tau^{*}, \tau^{*}\right)- & U_{1}\left(\left(V_{k_{0}}, V_{0}, V_{0}, \ldots, V_{0}\right), \tau^{*}\right) \\
& =\operatorname{tr}\left(\sum_{j=0}^{n-1} P_{0, j}|\Psi\rangle\langle\Psi| P_{0, j}^{+} M_{1}\right)-\operatorname{tr}\left(\sum_{j=0}^{n-1}\left(V_{k_{0}} \otimes \mathbb{1}\right) P_{0, j}|\Psi\rangle\langle\Psi| P_{0, j}^{+}\left(V_{k_{0}} \otimes \mathbb{1}\right)^{\dagger} M_{1}\right) . \tag{78}
\end{align*}
$$

Note that

$$
\begin{equation*}
\sum_{j=0}^{n-1}\left(V_{k_{0}} \otimes \mathbb{1}\right) P_{0, j}|\Psi\rangle\langle\Psi| P_{0, j}^{+}\left(V_{k_{0}} \otimes \mathbb{1}\right)^{\dagger}=\sum_{j=0}^{n-1} p_{0, j}\left|k_{0}, j\right\rangle\left\langle k_{0}, j\right| \tag{79}
\end{equation*}
$$

Therefore, Inequality (76) is equivalent to $\sum_{j=0}^{n-1} p_{0, j}\left(a_{0, j}-a_{k_{0}, j}\right)$ for $k_{0} \in\{0,1, \ldots, n-1\}$.
In general,

$$
\begin{align*}
U_{1}\left(\tau^{*}, \tau^{*}\right)- & U_{1}\left(V_{0}, V_{0}, \ldots, V_{0}, V_{k_{i}}, V_{0}, \ldots, V_{0}\right) \\
= & \operatorname{tr}\left(\sum_{j=0}^{n-1} P_{i j}|\Psi\rangle\langle\Psi| P_{i j}^{\dagger} M_{1}\right)-\operatorname{tr}\left(\sum_{j=0}^{n-1}\left(V_{k_{i}} \otimes \mathbb{1}\right) P_{i j}|\Psi\rangle\langle\Psi| P_{i j}^{\dagger}\left(V_{k_{i}} \otimes \mathbb{1}\right)^{\dagger} M_{1}\right) \\
& =\sum_{j=0}^{n-1} p_{i j}\left(a_{i j}-a_{i{ }_{m} k_{i}, j}\right) \geq 0 \tag{80}
\end{align*}
$$

for $i, k_{i} \in\{0,1, \ldots, m-1\}$. Analysis similar to the above one shows that

$$
\begin{align*}
& U_{2}\left(\tau^{*}, \tau^{*}\right)-U_{2}\left(V_{0}, V_{0}, \ldots, V_{0}, V_{l_{j}}, V_{0}, \ldots, V_{0}\right) \\
& =\operatorname{tr}\left(\sum_{i=0}^{m-1} P_{i j}|\Psi\rangle\langle\Psi| P_{i j}^{\dagger} M_{2}\right)-\operatorname{tr}\left(\sum_{i=0}^{m-1}\left(\mathbb{1} \otimes V_{l_{j}}\right) P_{i j}|\Psi\rangle\langle\Psi| P_{i j}^{\dagger}\left(\mathbb{1} \otimes V_{l_{j}}\right)^{\dagger} M_{2}\right) \\
& =\sum_{i=0}^{m-1} p_{i j}\left(b_{i j}-b_{i, j+{ }_{n} l_{j}}\right) \geq 0 \tag{81}
\end{align*}
$$

for $j, l_{j} \in\{0,1, \ldots, n-1\}$.
In this way we have shown that the correlated equilibrium in a bimatrix $m \times n$ game is a necessary condition for the existence of the Nash equilibrium $\left(\tau^{*}, \tau^{*}\right)$ in $\Gamma_{c M W}$.

To prove that these two notions are equivalent, we show that the other inequalities of the definition of Nash equilibrium are consequences of (80) and (81).

Note that $U_{1}\left(\left(V_{k_{0}}, V_{k_{1}}, V_{0}, V_{0} \ldots, V_{0}\right), \tau^{*}\right)$ can be written as

$$
\begin{align*}
& U_{1}\left(\left(V_{k_{0}}, V_{k_{1}}, V_{0}, V_{0} \ldots, V_{0}\right), \tau^{*}\right)=\operatorname{tr}\left(\sum_{j=0}^{n-1}\left(V_{k_{0}} \otimes \mathbb{1}\right) P_{0 j}|\Psi\rangle\langle\Psi|\left(\left(V_{k_{0}} \otimes \mathbb{1}\right) P_{0 j}\right)^{\dagger} M_{1}\right) \\
& \quad+\operatorname{tr}\left(\sum_{j=0}^{n-1}\left(V_{k_{1}} \otimes \mathbb{1}\right) P_{1 j}|\Psi\rangle\langle\Psi|\left(\left(V_{k_{1}} \otimes \mathbb{1}\right) P_{1 j}\right)^{\dagger} M_{1}\right)+\operatorname{tr}\left(\sum_{i=2}^{m-1} \sum_{j=0}^{n-1} P_{i j}|\Psi\rangle\langle\Psi| P_{i j}^{\dagger} M_{1}\right) \tag{82}
\end{align*}
$$

for $V_{k_{0}}, V_{k_{1}} \in\left\{V_{0}, V_{1}, \ldots, V_{m-1}\right\}$. Then inequalities

$$
\begin{equation*}
U_{1}\left(\tau^{*}, \tau^{*}\right) \geq U_{1}\left(\left(V_{k_{0}}, V_{0}, V_{0}, \ldots, V_{0}\right), \tau^{*}\right) \tag{83}
\end{equation*}
$$

and

$$
\begin{equation*}
U_{1}\left(\tau^{*}, \tau^{*}\right) \geq U_{1}\left(\left(V_{0}, V_{k_{1}}, V_{0}, \ldots, V_{0}\right), \tau^{*}\right) \tag{84}
\end{equation*}
$$

imply

$$
\begin{equation*}
\operatorname{tr}\left(\sum_{j=0}^{n-1} P_{0 j}|\Psi\rangle\langle\Psi| P_{0 j}^{\dagger} M_{1}\right) \geq \operatorname{tr}\left(\sum_{j=0}^{n-1}\left(V_{k_{0}} \otimes \mathbb{1}\right) P_{0 j}|\Psi\rangle\langle\Psi|\left(\left(V_{k_{0}} \otimes \mathbb{1}\right) P_{0 j}\right)^{\dagger} M_{1}\right) \tag{85}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{tr}\left(\sum_{j=0}^{n-1} P_{0 j}|\Psi\rangle\langle\Psi| P_{0 j}^{\dagger} M_{1}\right) \geq \operatorname{tr}\left(\sum_{j=0}^{n-1}\left(V_{k_{1}} \otimes \mathbb{1}\right) P_{1 j}|\Psi\rangle\langle\Psi|\left(\left(V_{k_{1}} \otimes \mathbb{1}\right) P_{1 j}\right)^{\dagger} M_{1}\right) . \tag{86}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
U_{1}\left(\tau^{*}, \tau^{*}\right) \geq U_{1}\left(\left(V_{k_{0}}, V_{k_{1}}, V_{0} \ldots, V_{0}\right), \tau^{*}\right) \tag{87}
\end{equation*}
$$

## 7. Summary and Conclusions

In quantum game theory it is required that a given quantum scheme coincides with the classical game under specific settings. This means that a quantum approach is a proper generalization of the classical way of playing a game. The new game so obtained is also defined by the sets of strategies and the payoff functions. Therefore, the quantum model is still a game in terms of game theory

As an example, we examined the notion of correlated equilibrium and its role in the MW-type quantum game. By taking the amplitudes as the square roots of respective probabilities of the correlated equilibrium, we found that the MW game has a pure Nash equilibrium payoff equivalent to the correlated equilibrium. We also pointed out that the MW-type approach to a bimatrix game can be viewed as a game with incomplete information. In particular, the MW model for a $2 \times 2$ game requires four bimatrix games in terms of incomplete information. The MW approach to a $3 \times 3$ game would be represented by eight $3 \times 3$ bimatrix games, and in general, the MW scheme for a $n \times n$ game would require $2^{n}$ bimatrix games. In this sense, the MW model is an effective way of presenting some specific games of incomplete information.

In the later part of the work, we modified the generalized MW scheme so that a Nash equilibrium in the quantum game could determine the correlated equilibrium in the classical game. The construction was based on the Harsanyi model of games with incomplete information, among which each player has two types.

Our work has shown that quantum computing approach to non-cooperative games may have a representation in classical game theory. The examined quantum model for a given classical game turned out to be a more general (classical) game: a Nash equilibrium in the quantum game becomes a correlated equilibrium in the classical bimatrix game; the quantum bimatrix game becomes the bimatrix game with incomplete information. We believe that this feature of quantum games may help to construct new schemes for quantum games.

Funding: This research was funded by the Pomeranian University in Słupsk.
Conflicts of Interest: The author declares no conflict of interest.

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