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Observer-Based Consensus Control for Heterogeneous Multi-Agent Systems with Output Saturations

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Abstract: This paper studies the consensus problem for heterogeneous multi-agent systems with output saturations. We consider the agents to have different dynamics and assume that the agents are neutrally stable and that the communication graph is undirected. The goal of this paper is to achieve the consensus for leaderless and leader-following cases. To solve this problem, we propose the observer-based distributed consensus algorithms, which consists of three parts: the nonlinear observer, the reference generator, and the regulator. Then, we analyze the consensus based on the Lasalle's Invariance Principle and the input-to-state stability. Finally, we provide numerical examples to demonstrate the validity of the proposed algorithms.

Keywords: heterogeneous multi-agent system; output saturation; consensus; observer

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1. Introduction

In the past decade, the consensus problem for multi-agent systems has received a lot of attention since it has wide applications such as formation control [1], and distributed filtering [2], cooperative control [3]. The goal of consensus is to achieve an agreement using local interactions between agents, and the consensus is one of the most studied methods to control the multi-agent systems in a distributed way.

Most pioneering works have considered the consensus problem for homogeneous agents, which have identical dynamics, with single-integrator [1,2,4,5], double-integrator [6–8], high-order linear [9–12], and nonlinear dynamics [13–15]. However, in real applications, it is often unrealistic for all agents to have an identical model. Therefore, in recent years, consensus for heterogeneous agents, which have nonidentical dynamics, has been widely studied [16–19]. Specifically, the necessary and sufficient condition for heterogeneous linear agents was studied in [16]. They showed that all agents must have a common internal model such that they can generate the same trajectory. Then, under this assumption, the observer-based consensus algorithm was constructed via the output regulation approach. In [17], the dynamic consensus protocol was proposed for leaderless and leader-following cases. In [18], the output consensus problem using the state feedback and the output feedback was studied. In [19], the dynamic event-triggered consensus protocol was developed based on input-to-state stability. In [20], heterogeneous oscillator were considered and the adaptive observer was developed.

Most actuators and sensors in real systems have saturation constraints owing to their limited capacity. Since the saturations lead to poor performance of the system, control problems for systems under saturations are an important issue in real applications. Specifically, the consensus problem for agents with input saturations has been widely studied in recent years [21–28]. For homogeneous agents with input saturations, semi-global consensus and global consensus were studied in [21–25]. For heterogeneous agents with input saturations, the semi-global output consensus was studied in [26–28]. Although there are many results dealing with input saturations in the consensus problem, the consensus problem under the output saturations has rarely been studied [29–32]. Since the consensus algorithm uses relative information between two agents, consensus may not be realized when the outputs are saturated. In [29–31], the necessary and sufficient conditions for single-integrator agents with output saturations were investigated. They showed that the weighted average in a group should be bounded such that the consensus trajectory can be measured. In [32], leader-following consensus for homogeneous high-order linear agents with output saturations was studied. The authors developed an observer-based consensus algorithm inspired by the nonlinear observer [33] and analyzed the asymptotic convergence based on the Lyapunov stability theory. However, to the best of our knowledge, the existing works on the consensus problem under the output saturations have considered homogeneous agents.

Motivated by the above observations, this paper studies the consensus problem for the heterogeneous agents with output saturations. We assume that the agents are neutrally stable and the communication graph is undirected. Then, we propose a distributed observer-based consensus algorithm based on the output regulation approach. The main contributions of this paper are summarized as follows. First, the output consensus problem for the heterogeneous agents with output saturations is investigated, and the leaderless and leader-following cases are considered. Therefore, this paper is a generalized version of the previous papers, which considered homogeneous agents [29–32]. Second, we construct the observer-based algorithm considering the output saturations. The output regulation approach has been applied to solve the consensus problem for heterogeneous agents in [16–18]. Then, by solving the linear matrix equations, called the regulation equations, they developed the consensus algorithms, which consist of three parts: the first part is the state observer, the second part is the reference generator, and the third part is the regulator. Then, by choosing control gain matrices such that the error systems are Hurwitz, the consensus is analyzed. However, in the presence of output saturations, the analysis techniques of [16–18] cannot be applied, since the observer contains saturation nonlinearity. Therefore, inspired by the works [32,33], this paper proposes the nonlinear observer and analyzes the consensus based on the Lasalle's Invariance Principle and the input-to-state stability.

The rest of this paper is organized as follows. In Section 2, the mathematical background and problem formulation are presented. In Section 3, the observer-based consensus algorithms for the leaderless and leader-following cases are constructed. In Section 4, numerical examples are provided, and conclusions are made in Section 5.

2. Preliminaries and Problem Formulation

2.1. Notations and Graph Theory

For a vector $x \in \mathbf{R}^n$, $x_{(i)}$ denotes the ith component of x. For a matrix $A \in \mathbf{R}^{n\times n}$, A^T and A^{-1} denote the transpose and the inverse of A, respectively, and $\lambda_i(A)$, $i = 1, 2, ..., n$, are the eigenvalues of A in ascending order, i.e., $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$. We say that $A \in \mathbf{R}^{n\times n}$ is Hurwitz if every eigenvalue of A has strictly negative real part, and is neutrally stable if every eigenvalue of A has non-positive real part with those on the imaginary axis being simple. $A \otimes B$ denotes the Kronecker product of A and B. $I_N \in \mathbf{R}^{N\times N}$ and $\mathbf{1}_N \in \mathbf{R}^N$ denote the identity matrix and the column vector with all entries equal to 1. $\text{blkdiag}(A_i)_{i=1}^N$ represents a block-diagonal matrix with matrices A_i, $i = 1, ..., N$, on its diagonal.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be an undirected graph, which represents the communication between agents, with a set of nodes (or agents) $\mathcal{V} := \{1, 2, ..., N\}$, a set of undirected edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and an weighted adjacency matrix $\mathcal{A} = [\alpha_{ij}] \in \mathbf{R}^{N\times N}$. For the undirected graph, if $(i, j) \in \mathcal{E}$, then $(j, i) \in \mathcal{E}$, which means the agents i and j can communicate with each other. The weights $\alpha_{ij} = \alpha_{ji} > 0$ if and only if $(i, j) \in \mathcal{E}$ and $\alpha_{ij} = \alpha_{ji} = 0$ otherwise.

The Laplacian matrix of the graph $\mathcal{G}$ is denoted by $L = [l_{ij}] \in \mathbf{R}^{N\times N}$, where $l_{ii} = \sum_{j=1, j\neq i}^N \alpha_{ij}$, $l_{ij} = -\alpha_{ij}$, $i \neq j$, and, thus, L is a positive semi-definite matrix with $0 = \lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_N(L)$. The undirected graph is connected if there exists a path between any two distinct nodes. For the connected graph, L has a simple zero eigenvalue that is $0 = \lambda_1(L) < \lambda_2(L)$.

2.2. Problem Formulation

This paper considers a heterogeneous multi-agent system. The dynamics of each agent is described by

$$\begin{aligned} \dot{x}_i =& A_i x_i + B_i u_i \\ y_i =& C_i x_i \\ z_i =& \sigma(y_i), \qquad i \in \mathcal{V} := \{1, 2, ..., N\}, \end{aligned} \tag{1}$$

where $x_i \in \mathbf{R}^{n_i}$, $u_i \in \mathbf{R}^{m_i}$, $y_i \in \mathbf{R}^q$, and $z_i \in \mathbf{R}^q$ represent the state, control input, controlled(or real) output, and measured output of ith agent, respectively, and A_i, B_i, and C_i are real constant matrices with compatible dimensions. The function $\sigma(\cdot)$ is a normalized standard saturation function defined by

$$\begin{aligned} \sigma(y_i) =& \left[\sigma(y_{i(1)}), \sigma(y_{i(2)}), ..., \sigma(y_{i(q)})\right]^T \\ \sigma(y_{i(j)}) =& \operatorname{sign}(y_{i(j)}) \min\left\{\left|y_{i(j)}\right|, 1\right\}. \end{aligned} \tag{2}$$

Then, the goal of this paper is to achieve the output consensus, that is,

$$\lim_{t\to\infty} \left\|y_i(t) - y_j(t)\right\| = 0, \quad \forall i, j \in \mathcal{V}. \tag{3}$$

To achieve the consensus, we require some standard assumptions. We first consider the existence of solution to the output consensus (3). It was shown in [16] that the necessary condition for the output consensus is the existence of a common internal model such that all agents can generate the same trajectory. This condition can be summarized as the following assumption [16]:

Assumption 1. *There exist matrices $S \in \mathbf{R}^{n_0 \times n_0}$, $R \in \mathbf{R}^{q \times n_0}$, $\Pi_i \in \mathbf{R}^{n_i \times n_0}$, and $\Gamma_i \in \mathbf{R}^{m_i \times n_0}$ for the following linear matrix equations:*

$$\begin{aligned} A_i \Pi_i + B_i \Gamma_i =& \Pi_i S, \\ C_i \Pi_i =& R, \qquad \forall i \in \mathcal{V}. \end{aligned} \tag{4}$$

We next consider the following assumptions to control the agents under output saturations.

Assumption 2. *The agents satisfy the following conditions:*

1. *For all $i \in \mathcal{V}$, (A_i, B_i) is stabilizable and (C_i, A_i) is detectable.*
2. *The matrices $A_i, \forall i \in \mathcal{V}$, and S are neutrally stable.*

Note that Condition 1 in Assumption 2 is the standard assumption to construct an observer-based controller. Moreover, Condition 2 requires controling the system in the global sense, since we cannot track the exponentially growing signals in the presence of saturation nonlinearities. Next, we suppose that the communication graph between the agents in (1) is given by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. Then, we further consider the following assumption, which is the necessary condition for the consensus.

Assumption 3. *The communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is undirected and connected.*

Before we construct the consensus algorithm, we consider the dynamics of agents in (1). From Condition 2 of Assumption 2, there exits a non-singular matrix $T_i \in \mathbf{R}^{n_i \times n_i}$ such that [32]

$$\begin{aligned} \bar{A}_i =& T_i A_i T_i^{-1} = \begin{bmatrix} A_{s,i} & 0 \\ 0 & A_{h,i} \end{bmatrix}, \quad \bar{B}_i = T_i B_i = \begin{bmatrix} B_{s,i} \\ B_{h,i} \end{bmatrix}, \\ \bar{C}_i =& C_i T_i^{-1} = \begin{bmatrix} C_{s,i} & C_{h,i} \end{bmatrix}, \quad i \in \mathcal{V}, \end{aligned} \tag{5}$$

where $A_{h,i} \in \mathbf{R}^{n_{h,i} \times n_{h,i}}$, and $A_{s,i} \in \mathbf{R}^{(n_i - n_{h,i}) \times (n_i - n_{h,i})}$ are Hurwitz and skew-symmetric matrix, respectively, the pair $(A_{s,i}, B_{s,i})$ is controllable, and $(C_{s,i}, A_{s,i})$ is observable. Then, we consider the following lemma that will be used to construct the consensus algorithm [32].

Lemma 1. *For the matrices $\bar{A}_i$ and $\bar{C}_i$ in (5), there exists a positive definite matrix $P_i \in \mathbf{R}^{n_i \times n_i}$ given by*

$$P_i = \begin{bmatrix} I_{n_i - n_{h,i}} & 0 \\ 0 & P_{h,i} \end{bmatrix}, \tag{6}$$

with a symmetric positive semi-definite matrix $P_{h,i} \in \mathbf{R}^{n_{h,i} \times n_{h,i}}$ such that $A_{h,i}^T P_{h,i} + P_{h,i} A_{h,i} < 0$. Moreover, for any positive constant $\beta_i > 0$, the matrix $(\bar{A}_i - \beta_i P_i^{-1} \bar{C}_i^T \bar{C}_i)$ is Hurwitz.

We next consider the following lemmas that will play a crucial role to analyze the consensus [33,34].

Lemma 2. *(LaSalle's Invariance Principle) For the system $\dot{x} = f(x)$ with $x \in \mathbf{R}^n$, if there exists a continuously differentiable function $V(x) : \mathbf{R}^n \to \mathbf{R}_+$ such that*

1. *$V(x)$ is a radially unbounded, positive definite function,*
2. *$\dot{V}(x) \leq 0$ for all $x \in \mathbf{R}^n$,*
3. *Let $\mathcal{M} = \{x \in \mathbf{R}^n : \dot{V}(x) = 0\}$, and no solution can stay identical in $\mathcal{M}$ except for $x = 0$;*

then, the origin is globally asymptotically stable.

Lemma 3. *(Input-to-State Stability) If the system $\dot{x} = f_1(x, z)$ is input-to-state stable and the origin of the system $\dot{z} = f_2(z)$ is globally asymptotically stable, then the origin of the cascade connection*

$$\begin{aligned} \dot{x} &= f_1(x, z) \\ \dot{z} &= f_2(z), \end{aligned} \tag{7}$$

is globally asymptotically stable.

3. Main Results

In this section, we construct the consensus algorithms for heterogeneous agents with the output saturations and consider the cases of leaderless and leader-following. The proposed algorithms are composed of three parts, as shown in Figure 1. The first part is the nonlinear observer to measure the state using the measured output, the second part is the distributed reference generator to generate the common trajectory, and the third part is the controller to track the common trajectory based on the output regulation theory.

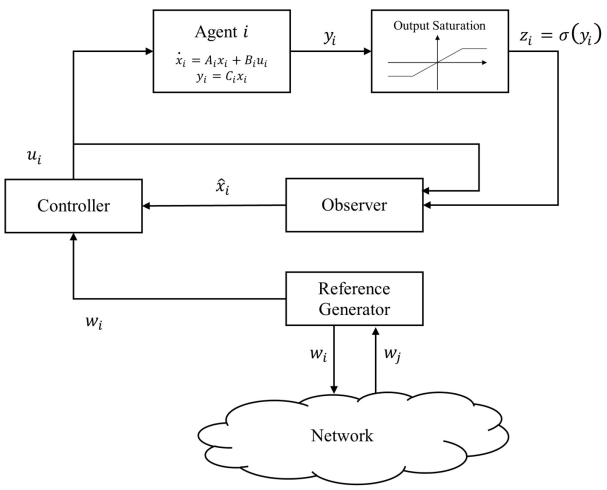

Figure 1. Block diagram of agent i with the observer-based consensus algorithm.

3.1. Leaderless Case

In this subsection, we consider the output consensus without the leader agent. We propose the observer-based consensus algorithm as follows:

$$\dot{\hat{x}}_i = A_i\hat{x}_i + B_iu_i + H_i\hat{z}_i \tag{8a}$$

$$\dot{w}_i = Sw_i + F\left(\sum_{j=1}^{N}\alpha_{ij}(w_j - w_i)\right) \tag{8b}$$

$$u_i = K_i(\hat{x}_i - \Pi_i w_i) + \Gamma_i w_i \tag{8c}$$

$$\hat{z}_i = \sigma(C_ix_i) - \sigma(C_i\hat{x}_i), \quad i \in \mathcal{V}, \tag{8d}$$

where $\hat{x}_i \in \mathbf{R}^{n_i}$ and $w_i \in \mathbf{R}^{n_0}$ are the states of the observer and the reference generator of the ith agent, respectively; H_i, F and K_i are the gain matrices with compatible dimensions that will be determined later; and S, Π_i and Γ_i are the constant matrices satisfying Assumption 1. Before we present our main result, we consider the following lemma [11].

Lemma 4. *Suppose that the pair (S, Q) is stabilizable. Then, there exists a symmetric positive-definite matrix W such that*

$$WS + S^TW - 2WQQ^TW < 0. \tag{9}$$

Then, we can solve the leaderless output consensus problem from the following theorem.

Theorem 1. *Consider a group of N heterogeneous agents (1), and suppose that Assumptions 1–3 hold. Then, we can achieve the output consensus using the observer-based consensus algorithm (8) if the gain matrices satisfy the following conditions:*

1. *$H_i = \beta_iT_i^{-1}P_i^{-1}\bar{C}_i^T$, where β_i is any positive constant, T_i and $\bar{C}_i$ are given in (5), and P_i is given in Lemma 1.*
2. *$F = \tau QQ^TW$, where τ is a positive constant such that $\tau\lambda_2(L) > 1$, and Q and W are the solution of (9) in Lemma 4.*
3. *K_i is a constant matrix such that $A_i + B_iK_i$ is Hurwitz.*
4. *$w_i(0)$ is bounded such that $\|R\varpi(t)\|_\infty \leq 1 - \delta$, for $0 < \delta < 1$, $\forall t \geq 0$, where ϖ is the solution of the following dynamics:*

$$\dot{\varpi} = S\varpi, \quad \varpi(0) = \frac{1}{N}\sum_{i=1}^{N}w_i(0). \tag{10}$$

Proof of Theorem 1. To prove the consensus, we investigate the asymptotic stability of the error dynamics. Then, we first consider the reference generator (8b) and define the state vector $w = [w_1^T, ..., w_N^T]^T$. Then, from (8b), we have

$$\dot{w} = (I_N \otimes S - L \otimes F)w. \tag{11}$$

Since the graph is undirected and connected, there exists an orthogonal matrix $U = \left[\frac{1}{\sqrt{N}}1_N\, U_2\right]$, such that $U^TLU = \Lambda = \text{diag}(0, \lambda_2(L), ..., \lambda_N(L))$ with $U_2 \in \mathbf{R}^{N\times(N-1)}$ and $U_2^TU_2 = I_{N-1}$ [14]. Let $\zeta = (U^T \otimes I_{n_0})w$ with $\zeta = [\zeta_1^T, \zeta_2^T, ..., \zeta_N^T]^T$ and $\bar{\zeta} = (U_2^T \otimes I_{n_0})w$ with $\bar{\zeta} = [\zeta_2^T, \zeta_3^T ..., \zeta_N^T]^T$. In [12], it was shown that $(w_i - w_j) \to 0$, $\forall i, j = 1, ..., N$, if and only if $\bar{\zeta} \to 0$. Moreover, if $(w_i - w_j) \to 0$, $\forall i, j = 1, ..., N$, $w_i(t) \to \varpi(t)$, where $\varpi(t)$ is the solution of (10) [10]. Therefore, in what follows, we derive the asymptotic convergence of $\bar{\zeta}$ to the origin. After the coordinate transformation, we have

$$\dot{\bar{\zeta}} = (I_{N-1} \otimes S - \bar{\Lambda} \otimes F)\bar{\zeta}, \tag{12}$$

where $\bar{\Lambda} = \text{diag}(\lambda_2(L), \lambda_3(L), ..., \lambda_N(L))$.

We next consider the observer (8a) and the regulator (8c) and define the observer error and the regulation error as $e_i = T_i(x_i - \hat{x}_i)$ and $\epsilon_i = \hat{x}_i - \Pi_i w_i$, respectively. Then, from (5), the observer error dynamics can be written as

$$\begin{aligned}\dot{e}_i =& T_i(\dot{x}_i - \dot{\hat{x}}_i) \\ =& T_i A_i(x_i - \hat{x}_i) - T_i H_i \hat{z}_i \\ =& \bar{A}_i e_i - T_i H_i \hat{z}_i,\end{aligned} \tag{13}$$

and, from Assumption 1, the regulation error dynamics is given by

$$\begin{aligned}\dot{\epsilon}_i =& \dot{\hat{x}}_i - \Pi_i \dot{w}_i \\ =& A_i\hat{x}_i + B_i(K_i(\hat{x}_i - \Pi_i w_i) + \Gamma_i w_i) + H_i\hat{z}_i - \Pi_i S w_i - \Pi_i F\left(\sum_{j=1}^{N} \alpha_{ij}(w_j - w_i)\right) \\ =& A_i\hat{x}_i + B_i(K_i\epsilon_i + \Gamma_i w_i) + H_i\hat{z}_i - A_i\Pi_i w_i - B_i\Gamma_i w_i - \Pi_i F\left(\sum_{j=1}^{N} \alpha_{ij}(w_j - w_i)\right) \\ =& (A_i + B_iK_i)\epsilon_i + H_i\hat{z}_i - \Pi_i F\left(\sum_{j=1}^{N} \alpha_{ij}(w_j - w_i)\right).\end{aligned} \tag{14}$$

Let $e = [e_1^T, ..., e_N^T]^T$, $\epsilon = [\epsilon_1^T, ..., \epsilon_N^T]^T$, $\hat{x} = [\hat{x}_1^T, ..., \hat{x}_N^T]^T$, $\hat{z} = [\hat{z}_1^T, ..., \hat{z}_N^T]^T$, $w = [w_1^T, ..., w_N^T]^T$, $T = \text{blkdiag}(T_i)_{i=1}^N$, $A = \text{blkdiag}(A_i)_{i=1}^N$, $B = \text{blkdiag}(B_i)_{i=1}^N$, $K = \text{blkdiag}(K_i)_{i=1}^N$, $H = \text{blkdiag}(H_i)_{i=1}^N$, $\Pi = \text{blkdiag}(\Pi_i)_{i=1}^N$, and $\bar{A} = \text{blkdiag}(\bar{A}_i)_{i=1}^N$. Then, from (12), (13), and (14), the overall error dynamics can be written as

$$\begin{aligned}\dot{\bar{\zeta}} =& (I_{N-1} \otimes S - \bar{\Lambda} \otimes F)\bar{\zeta}, \\ \dot{e} =& \bar{A}e - TH\hat{z} \\ \dot{\epsilon} =& (A + BK)\epsilon + H\hat{z} + \Pi(L \otimes F)w.\end{aligned} \tag{15}$$

It is clear that we can achieve the consensus if the error dynamics (15) is asymptotically stable, i.e., if $(\bar{\zeta}, e, \epsilon) \to (0, 0, 0)$, then $y_i - y_j = 0$, $\forall i, j \in \mathcal{V}$. Then, to prove the consensus, we first analyze the asymptotic convergence of $\bar{\zeta}$ and e to the origin, applying Lemma 2. We define the continuously differentiable function$V$ as follows:

$$V = \bar{\zeta}^T(I_{N-1} \otimes W)\bar{\zeta} + e^T P e, \tag{16}$$

where W is the solution of (9) in Lemma 4 and $P = \text{blkdiag}(P_i)_{i=1}^N$ with P_i given in Lemma 1. Let $\beta = \text{diag}(\beta_1, ..., \beta_N)$ and $\bar{C} = \text{blkdiag}(\bar{C}_i)_{i=1}^N$. Then, the time-derivative of V is given by

$$\begin{aligned}\dot{V} =& 2\bar{\zeta}^T(I_{N-1} \otimes W)\dot{\bar{\zeta}} + 2e^T P\dot{e} \\ =& 2\bar{\zeta}^T(I_{N-1} \otimes W)(I_{N-1} \otimes S - \bar{\Lambda} \otimes F)\bar{\zeta} + 2e^T P(\bar{A}e - TH\hat{z}) \\ =& \bar{\zeta}^T(I_{N-1} \otimes WS + S^TW - 2\bar{\Lambda} \otimes \tau QQ^TW)\bar{\zeta} + e^T(P\bar{A} + \bar{A}^TP)e \\ & - 2e^TPT\beta T^{-1}P^{-1}\bar{C}^T\hat{z} \\ =& \bar{\zeta}^T(I_{N-1} \otimes WS + S^TW - 2\tau\bar{\Lambda} \otimes WQQ^TW)\bar{\zeta} + e^T(P\bar{A} + \bar{A}^TP)e - 2\beta e^T\bar{C}^T\hat{z} \\ =& \sum_{i=2}^{N} \zeta_i^T(WS + S^TW - 2\tau\lambda_i(L)WQQ^TW)\zeta_i + \sum_{i=1}^{N} e_i^T(A_{s,i}^T + A_{s,i} + A_{h,i}^TP_{h,i} + P_{h,i}A_{h,i})e \\ & - 2\sum_{i=1}^{N} \beta_i(C_ix_i - C_i\hat{x}_i)^T(\sigma(C_ix_i) - \sigma(C_i\hat{x}_i)).\end{aligned} \tag{17}$$

Then, from Lemma 1 and 4, and the fact that $\tau\lambda_2(L) > 1$, the time-derivative of V can be rewritten as

$$\begin{aligned}\dot{V} \leq& \sum_{i=2}^{N} \zeta_i^T (WS + S^T W - 2WQQ^T W)\zeta_i - 2\sum_{i=1}^{N} \beta_i (C_i x_i - C_i \hat{x}_i)^T (\sigma(C_i x_i) - \sigma(C_i \hat{x}_i)) \\ \leq& 0,\end{aligned} \tag{18}$$

where we have used the fact that, for any $a, b \in \mathbf{R}$, $\text{sign}(a-b) = \text{sign}(\sigma(a) - \sigma(b))$. Then, we have shown that $V \geq 0$ is radially unbounded and $\dot{V} \leq 0$, which implies that Conditions 1 and 2 in Lemma 2 hold. Then, to prove Condition 3, we define $\mathcal{M} := \{(e, \bar{\zeta}) : \dot{V} = 0\}$. $\dot{V} = 0$ implies $\bar{\zeta} = 0$, i.e., $(w_i - w_j) = 0$, $\forall i, j \in \mathcal{V}$, and $(L \otimes I_{n_0})w = 0$, and $\hat{z}_i = (\sigma(C_i x_i) - \sigma(C_i \hat{x}_i)) = 0$, $\forall i \in \mathcal{V}$. Then, in the set $\mathcal{M}$, the error dynamics in (15) can be rewritten as

$$\begin{aligned}\dot{e} =& \bar{A} \\ \dot{\epsilon} =& (A + BK)\epsilon.\end{aligned} \tag{19}$$

Since $(A_i + B_i K_i)$ is Hurwitz, $\forall i \in \mathcal{V}$, we have

$$\lim_{t\to\infty} \|\epsilon_i\| = \lim_{t\to\infty} \|\hat{x}_i - \Pi_i w_i\| = \lim_{t\to\infty} \|C_i \hat{x}_i - C_i \Pi_i w_i\| = \lim_{t\to\infty} \|C_i \hat{x}_i - R w_i\| = 0. \tag{20}$$

Moreover, $\bar{\zeta} = 0$ implies $w_i = \bar{w}$, and, thus, from Condition 4 in Theorem 1, we have $\|R w_i\|_\infty = \|R \bar{w}\|_\infty < 1$ in $\mathcal{M}$. Therefore, there exists a finite time t_1 such that $\sigma(C_i \hat{x}_i) = C_i \hat{x}_i$, $\forall t \geq t_1$, which gives, for $t \geq t_1$, $\hat{z}_i = (\sigma(C_i x_i) - \sigma(C_i \hat{x}_i)) = (C_i x_i - C_i \hat{x}_i) = \bar{C}_i e_i = 0$ in $\mathcal{M}$. Then, for $t \geq t_1$, the observer error dynamics can be rewritten as

$$\dot{e} = \bar{A} e = (\bar{A} - \beta P^{-1} \bar{C}^T \bar{C}) e. \tag{21}$$

Since $(\bar{A} - \beta P^{-1} \bar{C}^T \bar{C})$ is Hurwitz from Lemma 1, e converges to 0.

In summary, we have shown that $\dot{V} \leq 0$ and $e = \bar{\zeta} = 0$ is a unique, asymptotically stable equilibrium point in $\mathcal{M}$. Therefore, according to Lemma 2, we have $\lim_{t\to\infty} \|e_i\| = 0$ and $\lim_{t\to\infty} \|\bar{\zeta}_i\| = 0$, $\forall i \in \mathcal{V}$. Moreover, since $(A + BK)$ is Hurwitz, the regulation error dynamics is input-to-state stable. Then, from Lemma 3, we have $\lim_{t\to\infty} \|\epsilon_i\| = 0$. Finally, we can conclude that $(\bar{\zeta}, e, \epsilon) \to (0, 0, 0)$ and $\lim_{t\to\infty} \|C_i x_i - C_j x_j\| = 0$, $\forall i, j \in \mathcal{V}$, which completes the proof. □

3.2. Leader-Following Case

In this subsection, we consider the leader agent, which generates the reference trajectory. The dynamics of the leader is given by

$$\begin{aligned}\dot{x}_0 =& S x_0 \\ y_0 =& R x_0,\end{aligned} \tag{22}$$

where $x_0 \in \mathbf{R}^{n_0}$ and $y_0 \in \mathbf{R}^q$ are the state and output, respectively, of the leader. S and R are the constant matrices satisfying Assumption 1. Then, to achieve the consensus, we consider the following assumption to control the follower agents into the leader's trajectory.

Assumption 4. *The leader agent* (22) *satisfies the following conditions:*

1. *(R, S) is detectable.*
2. *For the leader agent, there exists $0 < \delta < 1$, satisfying*

$$x_0(0) \in \mathcal{X}_0 \Rightarrow \|R x_0(t)\|_\infty \leq 1 - \delta, \quad \forall t \geq 0. \tag{23}$$

Then, to achieve the consensus, we propose the following consensus algorithm:

$$\dot{\hat{x}}_i = A_i\hat{x}_i + B_iu_i + H_i\hat{z}_i \tag{24a}$$

$$\dot{w}_i = Sw_i + F\left(\sum_{j=1}^{N}\alpha_{ij}(w_j - w_i) + \alpha_{i0}(x_0 - w_i)\right) \tag{24b}$$

$$u_i = K_i(\hat{x}_i - \Pi_i w_i) + \Gamma_i w_i \tag{24c}$$

$$\hat{z}_i = \sigma(C_i x_i) - \sigma(C_i\hat{x}_i), \quad i \in \mathcal{V}, \tag{24d}$$

where $\hat{x}_i \in \mathbf{R}^{n_i}$ and $w_i \in \mathbf{R}^{n_0}$ are the states of the observer and the reference generator of the ith follower, respectively; H_i, F and K_i are the gain matrices with compatible dimensions that will be determined later; and Π_i and Γ_i are the solution of linear matrix equations in Assumption 1. We next consider the following lemma [10], which is the dual problem of Lemma 4:

Lemma 5. *Suppose that the pair (R, S) is detectable. Then, there exists a symmetric positive-definite matrix W such that*

$$WS + S^T W - 2R^T R < 0. \tag{25}$$

Then, we can solve the leader-following output consensus from the following theorem.

Theorem 2. *Consider a group of N follower agents (1) with the leader (22). Suppose that Assumptions 1–4 hold, and there exists at least one follower that can receive the information from the leader. Then, we can achieve the leader-following output consensus using the observer-based consensus algorithm (24) if the gain matrices satisfy the following conditions:*

1. *$H_i = \beta_i T_i^{-1} P_i^{-1} \bar{C}_i^T$, where β_i is any positive constant, T_i and $\bar{C}_i$ are given in (5), and P_i is given in Lemma 1.*
2. *$F = \tau W^{-1} R^T R$, where τ is a positive constant such that $\tau\lambda_1(\bar{L}) > 1$, where $\bar{L} = L + \mathrm{diag}(\alpha_{10}, \ldots, \alpha_{N0})$, and the positive definite matrix W is the solution of (25) in Lemma 5.*
3. *K_i is a constant matrix such that $A_i + B_iK_i$ is Hurwitz.*

Proof of Theorem 2. We apply the same procedure as in the proof of Theorem 1. We define the tracking error of the reference generator, the observer error and the regulation error as $\zeta_i = w_i - x_0$, $e_i = T_i(x_i - \hat{x}_i)$, and $\epsilon_i = \hat{x}_i - \Pi_i w_i$, respectively. Then, the tracking error dynamics is given by

$$\begin{aligned}\dot{\zeta}_i &= Sw_i + F\left(\sum_{j=1}^{N}\alpha_{ij}(w_j - w_i) + \alpha_{i0}(x_0 - w_i)\right) - Sx_0 \\ &= S\zeta_i + F\left(\sum_{j=1}^{N}\alpha_{ij}(\zeta_j - \zeta_i) - \alpha_{i0}\zeta_i\right),\end{aligned} \tag{26}$$

and, from (13) and (14), the observer error dynamics and the regulation error dynamics are given by

$$\dot{e}_i = \bar{A}_i e_i - T_i H_i \hat{z}_i \tag{27a}$$

$$\dot{\epsilon}_i = (A_i + B_iK_i)\epsilon_i + H_i\hat{z}_i - \Pi_i F\left(\sum_{j=1}^{N}\alpha_{ij}(\zeta_j - \zeta_i) - \alpha_{i0}\zeta_i\right), \tag{27b}$$

where we have used the fact that $(w_j - w_i) = (\zeta_j - \zeta_i)$.

Let $\zeta = [\zeta_1^T, ..., \zeta_N^T]^T$ and $\bar{L} = L + \mathrm{diag}(\alpha_{10}, ..., \alpha_{N0})$, and define $e, \epsilon, T, H, A, B, K$, and Π as in the proof of Theorem 1. Then, the overall error dynamics can be written as

$$\begin{aligned} \zeta =&(I_N \otimes S - \bar{L} \otimes F)\zeta \\ \dot{e} =&\bar{A}e - TH\hat{z} \\ \dot{\epsilon} =&(A + BK)\epsilon + H\hat{z} + \Pi(\bar{L} \otimes F)\zeta. \end{aligned} \tag{28}$$

It is clear that we can achieve the leader-following output consensus if the error dynamics (28) is asymptotically stable; i.e., if $(\zeta, e, \epsilon) \rightarrow (0, 0, 0)$, then $y_i - y_0 = 0$, $\forall i \in \mathcal{V}$. Then, to prove the consensus, we define the continuously differentiable function$V$ as follows:

$$V = \zeta^T(I_N \otimes W)\zeta + e^T Pe, \tag{29}$$

where W is the solution of (25) and $P = \mathrm{blkdiag}(P_i)_{i=1}^N$ with P_i given in Lemma 1. Let $\beta = \mathrm{diag}(\beta_1, ..., \beta_N)$ and $\bar{C} = \mathrm{blkdiag}(\bar{C}_i)_{i=1}^N$. Then, the time-derivative of V is given by

$$\begin{aligned} \dot{V} =&2\zeta^T(I_N \otimes W)\zeta + 2e^T P\dot{e} \\ =&2\zeta^T(I_N \otimes W)(I_N \otimes S - \bar{L} \otimes F)\zeta + 2e^T P(\bar{A}e - TH\hat{z}) \\ =&\zeta^T(I_N \otimes WS + S^T W - 2\bar{L} \otimes \tau R^T R)\zeta + e^T(P\bar{A} + \bar{A}^T P)e - 2\beta e^T \bar{C}^T \hat{z} \\ \leq& \sum_{i=1}^N \zeta_i^T (WS + S^T W - 2\tau\lambda_1(\bar{L})R^T R)\zeta_i + e^T(P\bar{A} + \bar{A}^T P)e \\ &- 2\sum_{i=1}^N \beta_i (\bar{C}_i e_i)^T(\sigma(\bar{C}_i x_i) - \sigma(C_i \hat{x}_i)) \\ \leq& \sum_{i=1}^N \zeta_i^T (WS + S^T W - 2R^T R)\zeta_i - 2\sum_{i=1}^N \beta_i(\bar{C}_i x_i - \bar{C}_i \hat{x}_i)^T(\sigma(\bar{C}_i x_i) - \sigma(C_i \hat{x}_i)) \\ \leq& 0, \end{aligned} \tag{30}$$

where we have used the fact that $\tau\lambda_1(\bar{L}) > 1$. Then, we have shown that $V \geq 0$ is radially unbounded and $\dot{V} \leq 0$, which implies that conditions 1 and 2 in Lemma 2 hold. Then, to prove Condition 3, we define $\mathcal{M} := \{(e, \zeta) : \dot{V} = 0\}$. $\dot{V} = 0$ implies $\zeta = 0$, i.e., $(w_i - x_0) = 0$, $\forall i \in \mathcal{V}$, and $\hat{z}_i = (\sigma(C_i x_i) - \sigma(C_i \hat{x}_i)) = 0$, $\forall i \in \mathcal{V}$. Then, in $\mathcal{M}$, the error dynamics (28) can be rewritten as

$$\begin{aligned} \dot{e} =&\bar{A}e \\ \dot{\epsilon} =&(A + BK)\epsilon. \end{aligned} \tag{31}$$

Then, since $(A + BK)$ is Hurwitz and $\|Rx_0\|_\infty < 1$, we have

$$\lim_{t \to \infty} \|\epsilon_i\| = \lim_{t \to \infty} \|C_i \hat{x}_i - Rx_0\| = 0, \quad \forall i \in \mathcal{V}, \tag{32}$$

and, thus, there exists a finite time t_1 such that $\sigma(C_i \hat{x}_i) = C_i \hat{x}_i$, $\forall t \geq t_1$. Therefore, following the proof of Theorem 1, we can prove that e converges to 0.

In summary, we have shown that $\dot{V} \leq 0$, and $e = \zeta = 0$ is a unique, asymptotically stable equilibrium point in $\mathcal{M}$. Therefore, according to Lemma 2, we have $\lim_{t\to\infty} \|e_i\| = 0$ and $\lim_{t\to\infty} \|\zeta_i\| = 0$, $\forall i \in \mathcal{V}$. Moreover, since $(A + BK)$ is Hurwitz, the regulation error dynamics is input-to-state stable. Then, from Lemma 3, we have $\lim_{t\to\infty} \|\epsilon_i\| = 0$. Finally, we can conclude that $(\zeta, e, \epsilon) \rightarrow (0, 0, 0)$ and $\lim_{t\to\infty} \|C_i x_i - Rx_0\| = 0$, $\forall i \in \mathcal{V}$, which completes the proof. □

Remark 1. *Note that Theorems 1 and 2 give the sufficient conditions to achieve the output consensus. The existence of control gains follows from Assumptions 1–4, which are the standard* *assumptions to achieve the consensus for the heterogeneous agents. Moreover, the control gains can be constructed from the knowledge of the system matrices except τ, which requires the global information, i.e., $\lambda_2(L)$ in Theorem 1 and $\lambda_1(\bar{L})$ in Theorem 2. However, by choosing τ as an arbitrarily large constant, the conditions can be satisfied.*

4. Simulations

In this section, we present two numerical examples to demonstrate the theoretical results.

4.1. Leaderless Case

In this subsection, we consider the leaderless case with 10 agents modeled by harmonic oscillators. The dynamics of each agent is of the form (1) with, for $i \in \mathcal{V} = \{1, 2, ..., 10\}$,

$$A_i = \begin{bmatrix} 0 & 1 \\ -\psi_i & 0 \end{bmatrix}, \quad \psi_i = 0.5 \times i, \quad B_i = B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = C = \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{33}$$

Then, we can choose the non-singular matrix T_i as follows:

$$T_i = \begin{bmatrix} \sqrt{\psi_i} & 0 \\ 0 & 1 \end{bmatrix}, \tag{34}$$

which gives, from (5),

$$\bar{A}_i = \begin{bmatrix} 0 & \sqrt{\psi_i} \\ -\sqrt{\psi_i} & 0 \end{bmatrix}, \quad \bar{B}_i = \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{C}_i = \bar{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad P_i = P = I_2. \tag{35}$$

We next consider the linear matrix equations (4) in Assumption 1. The analytic solution of (4) is given in [20] as follows:

$$\begin{aligned} S &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{N}\sum_{i=1}^{N}\psi_i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2.75 & 0 \end{bmatrix}, \quad \Pi_i = \Pi = I_2, \\ \Gamma_i &= \begin{bmatrix} \frac{1}{N}\sum_{j=1}^{N}(\psi_i - \psi_j) & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 1 \end{bmatrix}. \end{aligned} \tag{36}$$

In this simulation, we assume that the communication between agents is as ring topology given in Figure 2. We consider $\alpha_{ij} = 1$, if $(i, j) \in \mathcal{E}$ and $\alpha_{ij} = 0$ otherwise. Then, the Laplacian matrix is given by

$$L = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}, \tag{37}$$

and $\lambda_2(L) = 0.3820$. We next construct the observer-based consensus algorithm (8) applying Theorem 1. From the condition 1, we choose

$$H_i = \beta_i T_i^{-1} P_i^{-1} \bar{C}_i^T = \beta_i \begin{bmatrix} \frac{1}{\sqrt{\psi_i}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}. \tag{38}$$

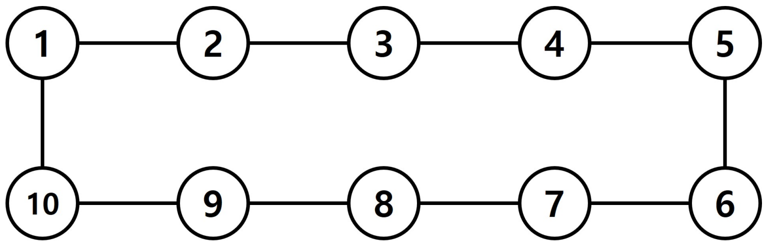

Figure 2. Communication topology between 10 agents.

We next choose $W = Q = I_2$ such that (S, Q) is stabilize and Lemma 4 is satisfied. Then, from the condition 2, we choose

$$F = \tau Q Q^T W = 10 \times I_2, \tag{39}$$

where $\tau = 10$ such that $\tau\lambda_2(L) = 3.82 > 1$. Finally, we choose $K_i = -[100 \ \ 10]$ such that Condition 3 is satisfied, i.e., $A_i + B_i K_i$ is Hurwitz. We conduct the simulation with the initial conditions satisfying Condition 4 as follows:

$$\begin{aligned}
&x_1(0) = \begin{bmatrix} 7 \\ -7 \end{bmatrix}, x_2(0) = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, x_3(0) = \begin{bmatrix} -8 \\ 3 \end{bmatrix}, x_4(0) = \begin{bmatrix} 5 \\ -8 \end{bmatrix}, x_5(0) = \begin{bmatrix} 6 \\ -6 \end{bmatrix}, \\
&x_6(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, x_7(0) = \begin{bmatrix} -4 \\ 5 \end{bmatrix}, x_8(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_9(0) = \begin{bmatrix} -10 \\ -3 \end{bmatrix}, x_{10}(0) = \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \\
&w_1(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, w_2(0) = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}, w_3(0) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, w_4(0) = \begin{bmatrix} 0.1 \\ 2 \end{bmatrix}, w_5(0) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \\
&w_6(0) = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}, w_7(0) = \begin{bmatrix} 1.4 \\ -1 \end{bmatrix}, w_8(0) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, w_9(0) = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}, w_{10}(0) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \\
&\hat{x}_i(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \forall i \in \mathcal{V}.
\end{aligned} \tag{40}$$

The simulation results using the proposed algorithm are shown in Figures 3–6. Figure 3 shows the state trajectories of the reference generators. As we can see from Figure 3, the reference generators converge to the common trajectory. Figure 4 shows the square norms of the observer errors, which converge to zero. Thus, the states of the agents can be measured under the proposed nonlinear observer (8a). Figure 5 depicts the controlled and the measured output trajectories of agents. Although the measured outputs are saturated, the agents achieve the consensus under the proposed algorithm. Moreover, to investigate the effect of the control gain β_i, we conduct the simulation with $\beta_i = \beta = 1, 5, 10, 100$, and the square norms of the output errors between agents are shown in Figure 6. The simulation result shows that, as the control gain β_i increases, the agents achieve a fast convergence speed.

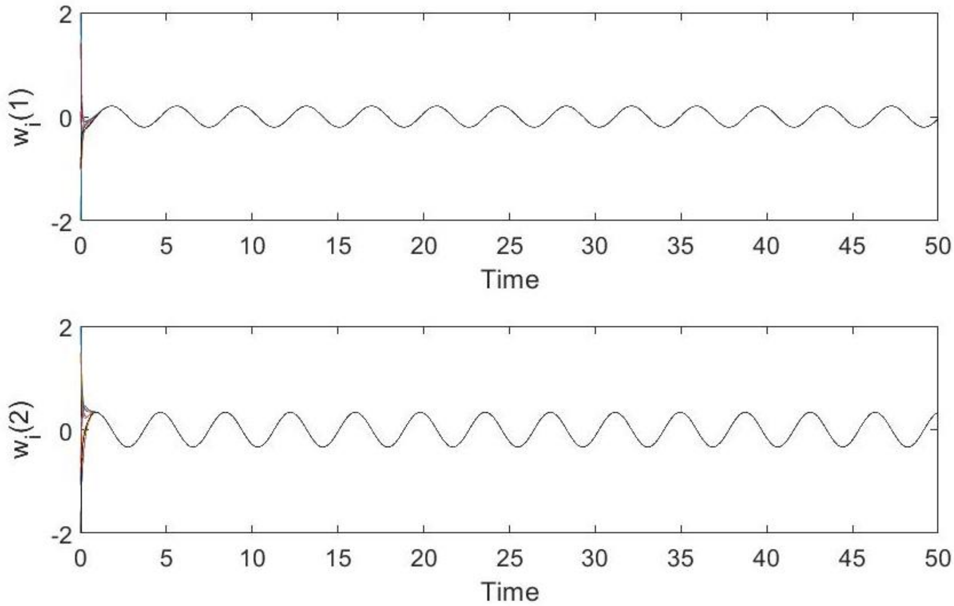

Figure 3. The state trajectories of the reference generators using the proposed algorithm (8).

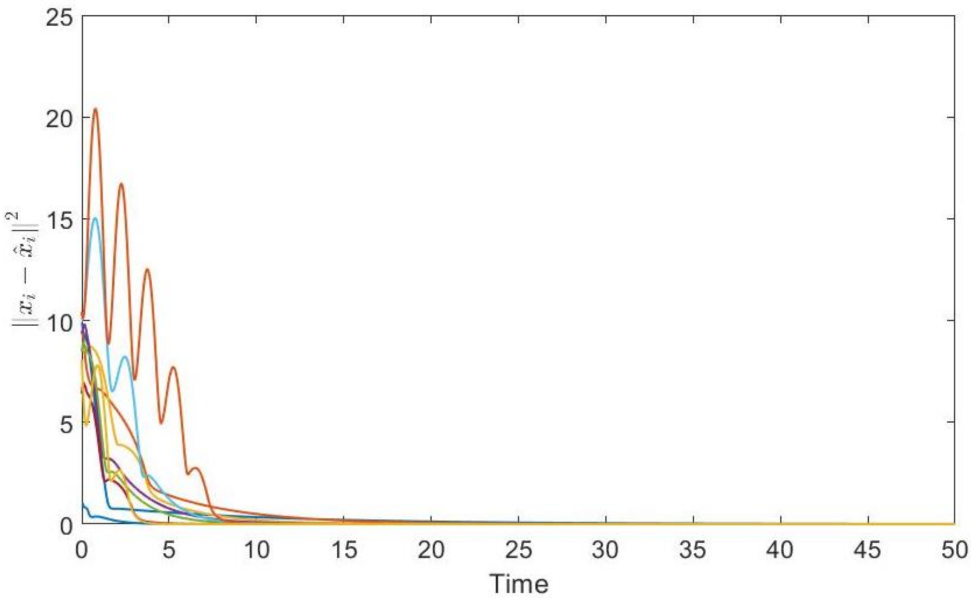

Figure 4. The observer errors using the proposed algorithm (8).

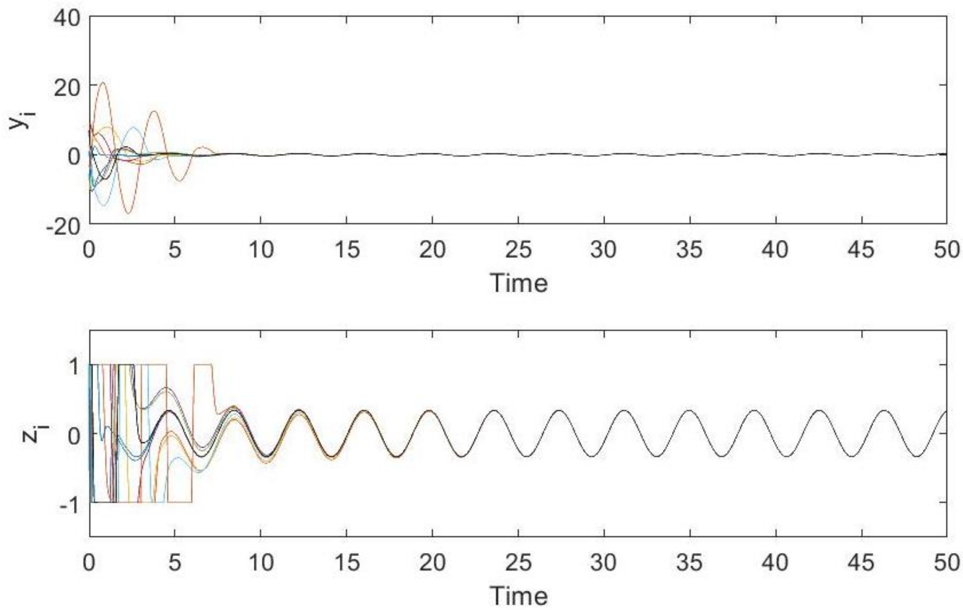

Figure 5. The trajectories of the controlled outputs and the measured outputs using the proposed algorithm (8).

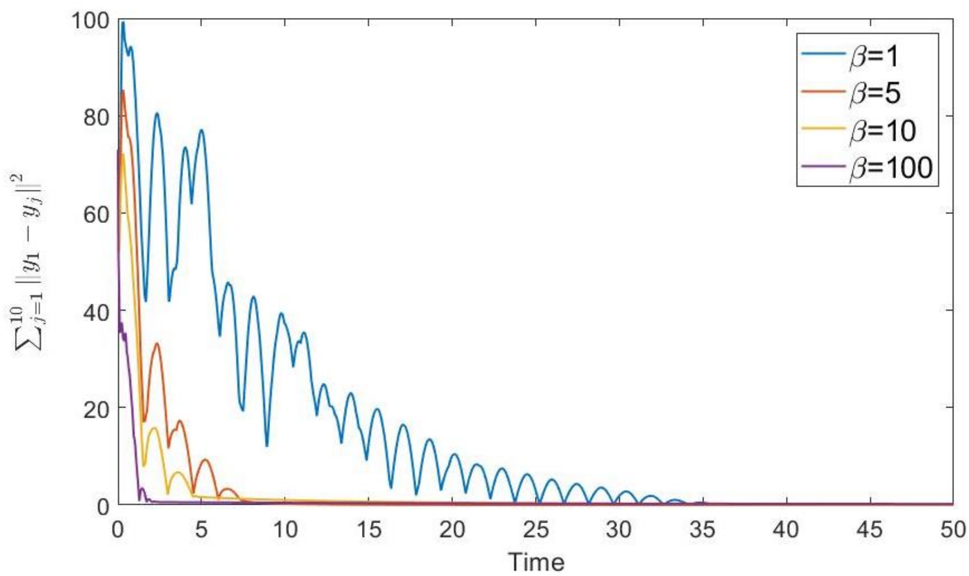

Figure 6. The output errors between agents using the proposed algorithm (8) with $\beta = 1, 5, 10, 100$.

4.2. Leader-Following Case

In this subsection, we consider a group of four agents (1), i.e., $\mathcal{V} = \{1, 2, 3, 4\}$, and a leader agent (22) with [18]

$$\begin{aligned}
&A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -0.8 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -1.5 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ -1 & -1.2 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 1 \\ -0.5 & -1.4 \end{bmatrix}, \\
&B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix}, B_4 = \begin{bmatrix} 0 \\ 1.4 \end{bmatrix}, C_i = C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \\
&S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \end{bmatrix},
\end{aligned} \tag{41}$$

which satisfies Assumption 2 and Condition 1 in Assumption 4. Then, we can choose the non-singular matrix $T_i = I_2, \forall i \in \mathcal{V}$ and the positive semi-definite matrix P_i satisfying Lemma 1 as follows:

$$P_1 = \begin{bmatrix} 1.00 & 0.27 \\ 0.27 & 0.60 \end{bmatrix}, P_2 = \begin{bmatrix} 1.03 & 0.41 \\ 0.41 & 0.97 \end{bmatrix}, P_3 = \begin{bmatrix} 0.88 & 0.53 \\ 0.53 & 1.52 \end{bmatrix}, P_4 = \begin{bmatrix} 0.50 & 0.47 \\ 0.47 & 2.30 \end{bmatrix}. \quad (42)$$

Moreover, by solving the linear equations in Assumption 1, we have

$$\Pi_i = \Pi = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \Gamma_1^T = \begin{bmatrix} 0.8 \\ -1 \end{bmatrix}, \Gamma_2^T = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, \Gamma_3^T = \begin{bmatrix} 1.2 \\ 0 \end{bmatrix}, \Gamma_4^T = \begin{bmatrix} 1.4 \\ 0.5 \end{bmatrix}. \quad (43)$$

In this simulation, we consider the communication topology between the agents given in Figure 7. We consider $\alpha_{ij} = 1$, if $(i,j) \in \mathcal{E}$ and $\alpha_{ij} = 0$ otherwise. Then, the Laplacian matrix is given by

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}, \alpha_{10} = \alpha_{20} = 1, \alpha_{30} = \alpha_{40} = 0, \quad (44)$$

and $\lambda(L) = \{0, 1, 3, 4\}$ and $\lambda(\bar{L}) = \{0.4, 1.2, 3.6, 4.9\}$. We next construct the observer-based consensus algorithm (24) applying Theorem 2. From Condition 1, we choose $H_i = \beta_i T_i^{-1} P_i^{-1} \bar{C}_i^T$ with $\beta_i = \beta = 20$, $T_i = I_2$, and P_i given in (42). We next choose W satisfying Lemma 5 as

$$W = \begin{bmatrix} 6.49 & -0.50 \\ -0.50 & 6.49 \end{bmatrix}. \quad (45)$$

Then, from Condition 2, we choose $F = \tau W^{-1} R^T R$ with $\tau = 3$ such that $\tau\lambda_1(\bar{L}) = 1.2 > 1$. Finally, we choose the gain matrix $K_i = -[100 \;\; 100], \forall i \in \mathcal{V}$ such that $A_i + B_i K_i$ is Hurwitz

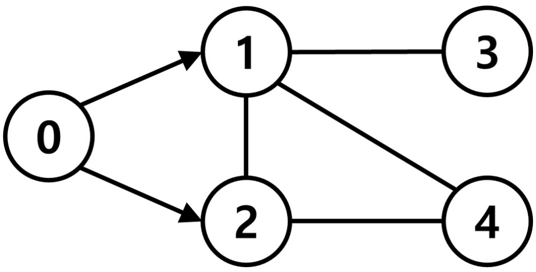

Figure 7. Communication topology between four agents, labeled by 1, 2, 3, 4, and a leader, labeled by 0.

In this simulation, we choose the initial conditions of the followers as follows:

$$\begin{aligned} x_1(0) &= \begin{bmatrix} 7 \\ -7 \end{bmatrix}, x_2(0) = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, x_3(0) = \begin{bmatrix} -8 \\ 3 \end{bmatrix}, x_4(0) = \begin{bmatrix} 5 \\ -8 \end{bmatrix}, \hat{x}_i(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \forall i \in \mathcal{V}, \\ w_1(0) &= \begin{bmatrix} 2 \\ -1 \end{bmatrix}, w_2(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, w_3(0) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, w_4(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \end{aligned} \quad (46)$$

The initial condition of the leader is chosen as $x_0(0) = [0.5, -0.5]^T$ such that Assumption 4 is satisfied. The simulation results are given in Figures 8–10. In Figure 8, the solid line shows the trajectories of the reference generators, while the dashed line shows the trajectory of the leader. It can be observed from Figure 8 that the reference generators track the leader's trajectory. In Figure 9, we can see that the observer errors converge to zero, which means the proposed nonlinear observer (24a) performs well. Moreover, the controlled and measured output trajectories of the followers in solid line are given in

Figure 10. It is clear that the followers track the leader's trajectory in the dashed line, and, thus, the proposed algorithm solves the leader-following consensus problem.

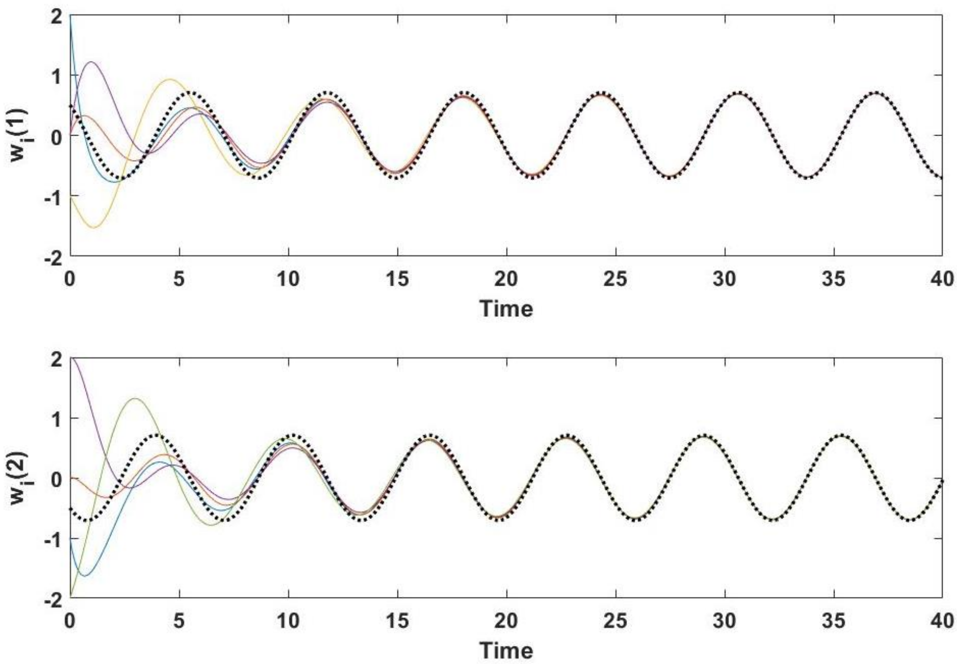

Figure 8. The state trajectories of the reference generators (solid line) and the leader (dashed line) using the proposed algorithm (24).

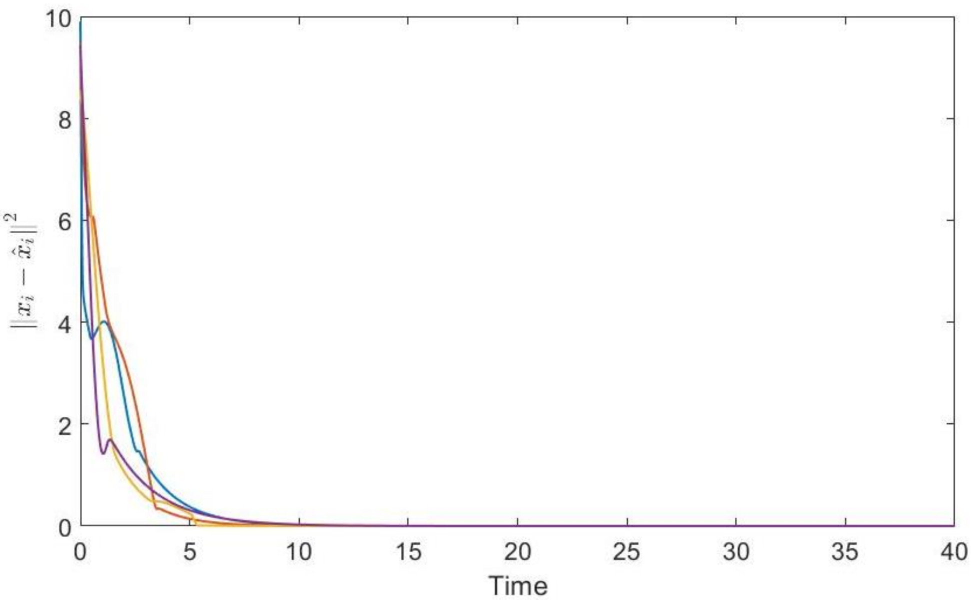

Figure 9. The observer errors using the proposed algorithm (24).

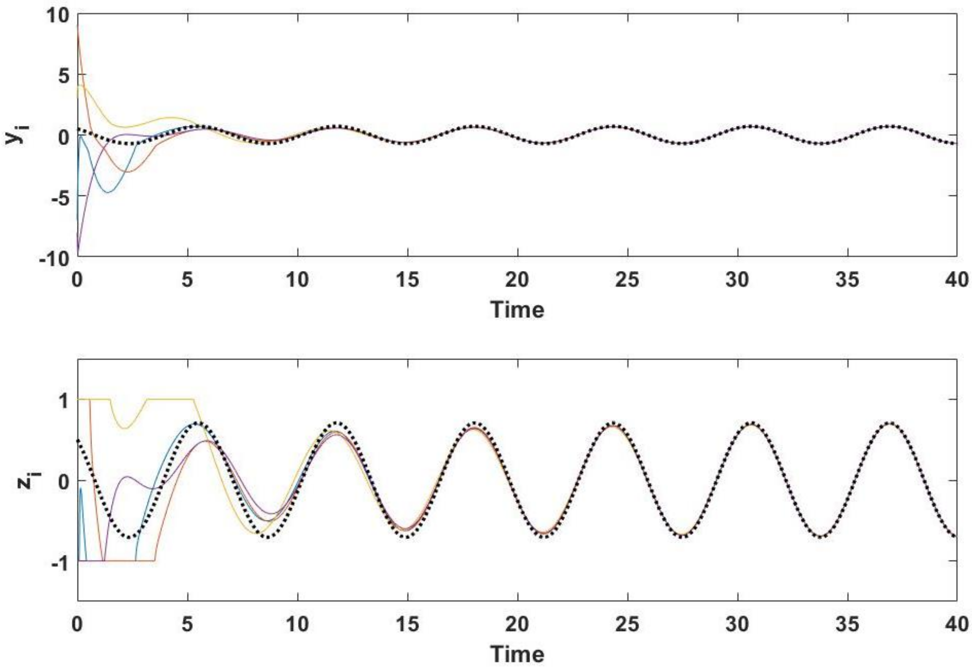

Figure 10. The trajectories of the controlled outputs (solid line), the measured outputs (solid line), and the leader's output (dashed line) using the proposed algorithm (24).

5. Conclusions

In this paper, the output consensus problems for heterogeneous agents were studied under the output saturations. Applying the output regulation approach to solve the consensus problem, we have proposed the observer-based consensus algorithms considering leaderless and leader-following cases. Specifically, the proposed algorithm consists of three parts: the nonlinear observer, the reference generator, and the regulator. By defining the error dynamics, we have transformed the consensus problem into the stability problem of the error dynamics. Then, based on the Lasalle's Invariance Principle and the input-to-state stability, the stability of error dynamics and the existence of the control gains have been derived under the standard assumptions for the consensus. Finally, two numerical examples have been given to demonstrate the theoretical results. Although the effect of the control gain β_i has been investigated by simulation, the performance in a group has not been addressed. Thus, the consensus control with the performance analysis would be worthwhile for a further study. Moreover, as mentioned in Remark 1, the proposed algorithm requires global information, i.e., $\lambda_2(L)$ and $\lambda_1(\bar{L})$. To solve this problem, the fully distributed algorithm has been widely used [14]. By using the state dependent control gain, the consensus can be solved without global information. Therefore, it would be interesting to extend the results of this paper to fully distributed consensus.

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