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Dombi Aggregation Operators of Linguistic Cubic Variables for Multiple Attribute Decision Making

Xueping Lu * and Jun Ye

Department of Electrical and Information Engineering, Shaoxing University, 508 Huancheng West Road, Shaoxing 312000, China; yejun@usx.edu.cn

* Correspondence: luxueping@usx.edu.cn

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Abstract: A linguistic cubic variable (LCV) is comprised of interval linguistic variable and single-valued linguistic variable. An LCV contains decision-makers' uncertain and certain linguistic judgments simultaneously. The advantage of the Dombi operators contains flexibility due to its changeable operational parameter. Although the Dombi operations have been extended to many studies to solve decision-making problems; the Dombi operations are not used for linguistic cubic variables (LCVs) so far. Hence, the Dombi operations of LCVs are firstly presented in this paper. A linguistic cubic variable Dombi weighted arithmetic average (LCVDWAA) operator and a linguistic cubic variable Dombi weighted geometric average (LCVDWGA) operator are proposed to aggregate LCVs. Then a multiple attribute decision making (MADM) method is developed in LCV setting on the basis of LCVDWAA and LCVDWGA operators. Finally, two illustrative examples about the optimal choice problems demonstrate the validity and the application of this method.

Keywords: multiple attribute decision making; linguistic cubic variable; Dombi operations; linguistic cubic variable Dombi weighted arithmetic average (LCVDWAA) operator; linguistic cubic variable Dombi weighted geometric average (LCVDWGA) operator

1. Introduction

With the development of society and science, decision-making problems become more and more complex, and they involve more and more fields, such as manufacturing domain [1,2], hospital service quality management [3], evaluation of the supplier criterions [4], and disaster assessment [5]. Since Zadeh [6] firstly proposed that linguistic variable (LV) could evaluate the assessment for objects, many scholars put forward various linguistic aggregation operators and developed corresponding methods to handle decision-making problems with linguistic information in diversified fields [7–12]. So far language variables have come in many forms, which are classified into two types: certain linguistic evaluations and uncertain linguistic evaluations. One of the forms can be used in one decision-making problem and one linguistic variable can represent the evaluation of a decision maker. With respect to an attribute over an alternative, one of the decision makers could give an uncertain evaluation, but another could give a certain evaluation. The pre-proposed LV forms could not express uncertain evaluation and certain evaluation simultaneously. In this study we will use a new linguistic evaluation form which was defined as linguistic cubic variable (LCV) in Ye [13]. An LCV is composed of a certain linguistic variable and an uncertain linguistic variable. An LCV can represent a group of linguistic evaluations over an attribute. Thus, the multiple attribute group decision making (MAGDM) process will be much simpler. We will introduce the development of LV below.

Since single-value linguistic variable was proposed, decision makers thought only one linguistic variable could not accurately provide judgments in some uncertain environments. Xu [14] proposed that decision makers could express their opinions with a linguistic interval. The interval linguistic variable was defined as an uncertain linguistic variable (ULV). They introduced a ULV hybrid aggregation (ULHA) operator and a ULV ordered weighted averaging operator (ULOWA) for MADM under an uncertain environment. Further, some other UL operators were introduced, such as UL hybrid geometric mean operator (ULHGM) [15], UL ordered weighted averaging operator (IULOWA) [16], UL Bonferroni mean operator (ULBM) [17], UL harmonic mean operator (ULHM) [18], and UL power geometric operator (ULPG) [19]. Later, intuitionistic fuzzy sets and linguistic variables were integrated. The concept of linguistic intuitionistic fuzzy numbers was proposed. Liu [20,21] and Li et al. [22] introduced several linguistic intuitionistic MADM methods. Recently, a changeable uncertain linguistic number was defined as a neutrosophic linguistic number (NLN). Some aggregation operators of NLNs were developed to handle MADM problems with NLN information in References [23–25].

Linguistic variable forms, as introduced above, are either certain linguistic evaluations or uncertain linguistic evaluations used to describe evaluation information in the same decision-making problem simultaneously. However, in reality, uncertain linguistic evaluations and certain linguistic evaluations may exist simultaneously. Thus, Ye [13] combined a linguistic variable with a cubic set [26] and proposed a hybrid linguistic form. The hybrid linguistic form was defined as a linguistic cubic variable (LCV). An uncertain linguistic variable and a certain linguistic variable composed an LCV. Meanwhile Ye [13] developed an LCV weighted geometric averaging (LCVWGA) operator and an LCV weighted arithmetic averaging (LCVWAA) operator and further developed a MADM method on the basis of the LCVWGA operator or LCVWAA operator.

Information aggregation operators are effective and powerful tools to handle decision-making problems. Researchers have developed various operators to aggregate evaluation information. Dombi [27] firstly proposed Dombi T-conorm and T-norm operations in 1982. The operations are developed into many information aggregations to deal with various application problems; for instance, Dombi hesitant fuzzy information aggregation operators [5] for disaster assessment, or intuitionistic fuzzy set Dombi Bonferroni mean operators [28] for MADM problems. Then, the advantage of the Dombi operators contains flexibility due to its changeable operational parameter. Up to now, the Dombi operations have not been extended to LCVs. Hence, aggregation operators based the Dombi operations will be developed to handle LCV decision-making problems. So Dombi operational laws of LCVs are proposed in this study. Then an LCV Dombi weighted arithmetic average (LCVDWAA) operator and an LCV Dombi weighted geometric average (LCVDWGA) operator are presented. Further the decision-making approach on basis of the LCVDWAA or LCVDWGA operator is developed for LCV MADM problems.

The remainder of this paper is organized by following six sections. Some concepts of LCVs are introduced in Section 2, and Section 3 defines several Dombi operations of LCVs. LCVDWAA and LCVDWGA operators and some of their properties are presented in Section 4. The MADM approach based on the LCVDWAA or LCVDWGA operator is introduced in Section 5. In Section 6, two application examples are illustrated, and we discuss the validity, the influence of the operational parameter, and the sensitivity of weights. Section 7 gives the conclusions and expectations of the research.

2. Several Concepts of LCVs

Definition 1 [13]. *Set $L = \{L_0, L_1, L_2, \ldots, L_T\}$ as a linguistic term set, in which T is even. A linguistic cubic variable V is constructed by $V = (L, L_M)$, where $L = [L_G, L_H]$ is a ULV and L_M is an LV for $H \geq G$ and L_G, L_H, $L_M \in L$. If $G \leq M \leq H$, $V = ([L_G, L_H], L_M)$ is an internal LCV. If $M < G$ or $M > H$, $V = ([L_G, L_H], L_M)$ is an external LCV.*

Definition 2 [13]. *Set $V = ([L_G, L_H], L_M)$ as an LCV in $L = \{L_0, L_1, L_2, \ldots, L_T\}$ for L_G, L_H, $L_M \in L$. Then the expected value of the LCV is calculated as below:*

$$E(V) = (G + H + M)/3T \qquad for \;\; E(V) \in [0,1] \quad . \tag{1}$$

Definition 3 [13]. *Set $V_1 = ([L_{G1}, L_{H1}], L_{M1})$ and $V_2 = ([L_{G2}, L_{H2}], L_{M2})$ as two LCVs, their expected values are $E(V_1)$ and $E(V_2)$, then their relations are as follows:*

(a) If $E(V_1) \succ E(V_2)$, then $V_1 \succ V_2$;
(b) If $E(V_1) \prec E(V_2)$, then $V_1 \prec V_2$;
(c) If $E(V_1) = E(V_2)$, then $V_1 = V_2$.

3. Some Dombi Operations of LCVs

Dombi T-conorm operation and T-norm operation between two real numbers will be introduced in this section. Then some Dombi operations of LCVs will be proposed.

Definition 4 [27]. *Let Y and X be any two real numbers. If (Y, X) ∈ [0, 1] × [0, 1], the Dombi T-norm and Dombi T-conorm between them are defined as Equations (2) and (3):*

$$D(Y, X) = \frac{1}{1 + \left\{ \left(\frac{1-Y}{Y}\right)^{\rho} + \left(\frac{1-X}{X}\right)^{\rho} \right\}^{\frac{1}{\rho}}}, \tag{2}$$

$$D^{C}(Y, X) = 1 - \frac{1}{1 + \left\{ \left(\frac{Y}{1-Y}\right)^{\rho} + \left(\frac{X}{1-X}\right)^{\rho} \right\}^{\frac{1}{\rho}}}. \tag{3}$$

If $\rho > 0$, the above equations satisfy $D(Y, X) \in [0, 1]$ and $D^c(Y, X) \in [0, 1]$.

According to the above Dombi operations, the following Dombi operational laws of LCVs are defined.

Definition 5. *Let $V_1 = ([L_{G1}, L_{H1}], L_{M1})$ and $V_2 = ([L_{G2}, L_{H2}], L_{M2})$ be two LCVs, where (G_1, H_1, M_1, G_2, H_2, M_2) ∈ [0, T], $\rho > 0$, then their Dombi operations are proposed as follows:*

$$\begin{aligned} &V_1 \oplus V_2 = ([L_{G1}, L_{H1}], L_{M1}) \oplus ([L_{G2}, L_{H2}], L_{M2}) \\ &= \left(\left[L_{T\times(1-\frac{1}{1+\{(\frac{G1/T}{1-G1/T})^{\rho}+(\frac{G2/T}{1-G2/T})^{\rho}\}^{\frac{1}{\rho}}})}, L_{T\times(1-\frac{1}{1+\{(\frac{H1/T}{1-H1/T})^{\rho}+(\frac{H2/T}{1-H2/T})^{\rho}\}^{\frac{1}{\rho}}})} \right], L_{T\times(1-\frac{1}{1+\{(\frac{M1/T}{1-M1/T})^{\rho}+(\frac{M2/T}{1-M2/T})^{\rho}\}^{\frac{1}{\rho}}})} \right) \\ &= \left(\left[L_{T-\frac{T}{1+\{(\frac{G1}{T-G1})^{\rho}+(\frac{G2}{T-G2})^{\rho}\}^{\frac{1}{\rho}}}}, L_{T-\frac{T}{1+\{(\frac{H1}{T-H1})^{\rho}+(\frac{H2}{T-H2})^{\rho}\}^{\frac{1}{\rho}}}} \right], L_{T-\frac{T}{1+\{(\frac{M1}{T-M1})^{\rho}+(\frac{M2}{1-T-M2})^{\rho}\}^{\frac{1}{\rho}}}} \right); \end{aligned} \tag{4}$$

$$\begin{aligned} &V_1 \otimes V_2 = ([L_{G1}, L_{H1}], L_{M1}) \otimes ([L_{G2}, L_{H2}], L_{M2}) \\ &= \left(\left[L_{\frac{T}{1+\{(\frac{1-G1/T}{G1/T})^{\rho}+(\frac{1-G2/T}{G2/T})^{\rho}\}^{\frac{1}{\rho}}}}, L_{\frac{T}{1+\{(\frac{1-H1/T}{H1/T})^{\rho}+(\frac{1-H2/T}{H2/T})^{\rho}\}^{\frac{1}{\rho}}}} \right], L_{\frac{T}{1+\{(\frac{1-M1/T}{M1/T})^{\rho}+(\frac{1-M2/T}{M2/T})^{\rho}\}^{\frac{1}{\rho}}}} \right) \\ &= \left(\left[L_{\frac{T}{1+\{(\frac{T-G1}{G1})^{\rho}+(\frac{T-G2}{G2})^{\rho}\}^{\frac{1}{\rho}}}}, L_{\frac{T}{1+\{(\frac{T-H1}{H1})^{\rho}+(\frac{T-H2}{H2})^{\rho}\}^{\frac{1}{\rho}}}} \right], L_{\frac{T}{1+\{(\frac{T-M1}{M1})^{\rho}+(\frac{T-M2}{M2})^{\rho}\}^{\frac{1}{\rho}}}} \right); \end{aligned} \tag{5}$$

$$
\begin{aligned}
KV_1 &= K([L_{G1}, L_{H1}], L_{M1}) \\
&= \left(\left[L_{T-\frac{T}{1+\{K(\frac{\frac{G1}{T}}{1-\frac{G1}{T}})^{\rho}\}^{\frac{1}{\rho}}}}, L_{T-\frac{T}{1+\{K(\frac{\frac{H1}{T}}{1-\frac{H1}{T}})^{\rho}\}^{\frac{1}{\rho}}}} \right], L_{T-\frac{T}{1+\{K(\frac{\frac{M1}{T}}{1-\frac{M1}{T}})^{\rho}\}^{\frac{1}{\rho}}}} \right) \\
&= \left(\left[L_{T-\frac{T}{1+\{K(\frac{G1}{T-G1})^{\rho}\}^{\frac{1}{\rho}}}}, L_{T-\frac{T}{1+\{K(\frac{H1}{T-H1})^{\rho}\}^{\frac{1}{\rho}}}} \right], L_{T-\frac{T}{1+\{K(\frac{M1}{T-M1})^{\rho}\}^{\frac{1}{\rho}}}} \right);
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
{V_1}^K &= \left(\left[L^K_{G1}, L^K_{H1} \right], L^K_{M1} \right) \\
&= \left(\left[L_{\frac{T}{1+\{K(\frac{1-G1/T}{G1/T})^{\rho}\}^{1/\rho}}}, L_{\frac{T}{1+\{K(\frac{1-H1/T}{H1/T})^{\rho}\}^{1/\rho}}} \right], L_{\frac{T}{1+\{K(\frac{1-M1/T}{M1/T})^{\rho}\}^{1/\rho}}} \right) \\
&= \left(\left[L_{\frac{T}{1+\{K(\frac{T-G1}{G1})^{\rho}\}^{1/\rho}}}, L_{\frac{T}{1+\{K(\frac{T-H1}{H1})^{\rho}\}^{1/\rho}}} \right], L_{\frac{T}{1+\{K(\frac{T-M1}{M1})^{\rho}\}^{1/\rho}}} \right).
\end{aligned}
\tag{7}
$$

Due to functions $(G_1/T, H_1/T, M_1/T, G_2/T, H_2/T, M_2/T) \in [0, 1]$, they satisfy the parameter requirements of Dombi operations. *If $p > 0$, $(Y, X) \in [0, 1] \times [0, 1]$, then $D(Y, X) \in [0, 1]$ and $D^c(Y, X) \in [0, 1]$.* Thus we can get $T^*D^c(Y, X) \in [0, T]$ and $T^*D(Y, X) \in [0, T]$. Obviously the results of Equations (4)–(7) are also LCVs according to the Dombi operations as in Equations (2) and (3). In Equations (4)–(7), we presented Dombi operations of LCVs in the first step and simplified the equations in the second step.

Example 1. *Let V_1 and V_2 be two LCVs in the linguistic term set $L = \{L_i \mid i \in [0, 8]\}$. Assume that $V_1 = ([L_4, L_6], L_5)$, $V_2 = ([L_2, L_7], L_2)$, $k = 0.5$, and $\rho = 1$. According to Equations (4)–(7), the results are calculated respectively as follows:*

$$
\begin{aligned}
V_1 \oplus V_2 &= ([L_4, L_6], L_5) \oplus ([L_2, L_7], L_2) \\
&= \left(\left[L_{8-\frac{8}{1+\{(\frac{4}{8-4})^1+(\frac{2}{8-2})^1\}^{\frac{1}{1}}}}, L_{8-\frac{8}{1+\{(\frac{6}{8-6})^1+(\frac{7}{8-7})^1\}^{\frac{1}{1}}}} \right], L_{8-\frac{8}{1+\{(\frac{5}{8-5})^1+(\frac{2}{8-2})^1\}^{\frac{1}{1}}}} \right) \\
&= ([L_{4.5714}, L_{7.2727}], L_{5.3333});
\end{aligned}
$$

$$
\begin{aligned}
V_1 \otimes V_2 &= ([L_4, L_6], L_5) \otimes ([L_2, L_7], L_2) \\
&= \left(\left[L_{\frac{8}{1+\{(\frac{8-4}{4})^1+(\frac{8-2}{2})^1\}^{\frac{1}{1}}}}, L_{\frac{8}{1+\{(\frac{8-6}{6})^1+(\frac{8-7}{7})^{\rho}\}^{\frac{1}{1}}}} \right], L_{\frac{8}{1+\{(\frac{8-5}{5})^1+(\frac{8-2}{2})^1\}^{\frac{1}{1}}}} \right) \\
&= ([L_{1.6}, L_{5.4193}], L_{1.7391});
\end{aligned}
$$

$$
\begin{aligned}
KV_1 &= K([L_4, L_6], L_5) \\
&= \left(\left[L_{8-\frac{8}{1+\{0.5(\frac{4}{8-4})^1\}^{\frac{1}{1}}}}, L_{8-\frac{8}{1+\{0.5(\frac{6}{8-6})^1\}^{\frac{1}{1}}}} \right], L_{8-\frac{8}{1+\{0.5(\frac{5}{8-5})^1\}^{\frac{1}{1}}}} \right) \\
&= ([L_{2.6667}, L_{4.8}], L_{3.6363});
\end{aligned}
$$

$$
\begin{aligned}
V_1{}^K &= \left([L_4^K, L_6^K], L_5^K\right) \\
&= \left(\left[L_{\frac{8}{1+\{0.5(\frac{8-4}{4})^1\}^{1/1}}}, L_{\frac{8}{1+\{0.5(\frac{8-6}{6})^1\}^{1/1}}}\right], L_{\frac{8}{1+\{0.5(\frac{8-5}{5})^1\}^{1/1}}}\right) \\
&= ([L_{5.3333}, L_{6.8571}], L_{6.1538}).
\end{aligned}
$$

4. Dombi Weighted Aggregation Operators of LCVs

4.1. Dombi Weighted Arithmetic Average Operator of LCVs

Definition 6. *Let $V = \{V_1, V_2, V_3, \ldots, V_n\}$ be an LCV set, then the Dombi weighted arithmetic average operator of LCV_S can be defined as follows:*

$$LCVDWAA(V_1, V_2, \ldots, V_n) = \overset{n}{\underset{i=1}{\oplus}} w_i V_i \tag{8}$$

where the weight vector w_i satisfies $\sum_{i=1}^{i=n} w_i = 1$ and $w_i \in [0, 1]$.

The following Theorem 1 can be induced and proved according to Definitions 4 and 6.

Theorem 1. *Let $V_i = ([L_{Gi}, L_{Hi}], L_{Mi})$ (i = 1, 2, ... , n) be a set of LCVs and the corresponding weight vector is $w = (w_1, w_2, \ldots, w_n)$, where $\sum_{i=1}^{i=n} w_i = 1$ and $w_i \in [0, 1]$, then we can calculate Equation (8) on basis of the predefined operational laws and get the following formula:*

$$
\begin{aligned}
&LCVDWAA(V_1, V_2, \ldots, V_n) \\
&= \left(\left[L_{T-\frac{T}{1+\{\sum_{i=1}^{i=n} w_i(\frac{Gi}{T-Gi})^\rho\}^{1/\rho}}}, L_{T-\frac{T}{1+\{\sum_{i=1}^{i=n} w_i(\frac{Hi}{T-Hi})^\rho\}^{1/\rho}}}\right], L_{T-\frac{T}{1+\{\sum_{i=1}^{i=n} w_i(\frac{Mi}{T-Mi})^\rho\}^{1/\rho}}}\right).
\end{aligned} \tag{9}
$$

Proof:

(1) If n = 2, by the Equations (4) and (6) we can get:

$$
\begin{aligned}
&LCVDWAA(V_1, V_2) = w_1 V_1 \oplus w_2 V_2 \\
&= \left(\left[L_{T-\frac{T}{1+\{w_1(\frac{G1}{T-G1})^\rho\}^{\frac{1}{\rho}}}}, L_{T-\frac{T}{1+\{w_1(\frac{H1}{T-H1})^\rho\}^{\frac{1}{\rho}}}}\right], L_{T-\frac{T}{1+\{w_1(\frac{M1}{T-M1})^\rho\}^{\frac{1}{\rho}}}}\right) \oplus \\
&\quad \left(\left[L_{T-\frac{T}{1+\{w_2(\frac{G2}{T-G2})^\rho\}^{\frac{1}{\rho}}}}, L_{T-\frac{T}{1+\{w_2(\frac{H2}{T-H2})^\rho\}^{\frac{1}{\rho}}}}\right], L_{T-\frac{T}{1+\{w2(\frac{M2}{T-M2})^\rho\}^{\frac{1}{\rho}}}}\right)
\end{aligned}
$$

$$= \left(\begin{bmatrix} L_{T-\frac{T}{1+\left\{\left(\frac{T-\frac{T}{1+\{w_1(\frac{G1}{T-G1})^\rho\}^{\frac{1}{\rho}}}}{T-\left(T-\frac{T}{1+\{w_1(\frac{G1}{T-G1})^\rho\}^{\frac{1}{\rho}}}\right)}\right)^\rho+\left(\frac{T-\frac{T}{1+\{w_2(\frac{G2}{T-G2})^\rho\}^{\frac{1}{\rho}}}}{T-\left(T-\frac{T}{1+\{w_2(\frac{G2}{T-G2})^\rho\}^{\frac{1}{\rho}}}\right)}\right)^\rho\right\}^{\frac{1}{\rho}}}}, \\ L_{T-\frac{T}{1+\left\{\left(\frac{T-\frac{T}{1+\{w_1(\frac{H1}{T-H1})^\rho\}^{\frac{1}{\rho}}}}{T-\left(T-\frac{T}{1+\{w_1(\frac{H1}{T-H1})^\rho\}^{\frac{1}{\rho}}}\right)}\right)^\rho+\left(\frac{T-\frac{T}{1+\{w_2(\frac{H2}{T-H2})^\rho\}^{\frac{1}{\rho}}}}{T-\left(T-\frac{T}{1+\{w_2(\frac{H2}{T-H2})^\rho\}^{\frac{1}{\rho}}}\right)}\right)^\rho\right\}^{\frac{1}{\rho}}}} \end{bmatrix}, \\ L_{T-\frac{T}{1+\left\{\left(\frac{T-\frac{T}{1+\{w_1(\frac{M1}{T-M1})^\rho\}^{\frac{1}{\rho}}}}{T-\left(T-\frac{T}{1+\{w_1(\frac{M1}{T-M1})^\rho\}^{\frac{1}{\rho}}}\right)}\right)^\rho+\left(\frac{T-\frac{T}{1+\{w_2(\frac{M2}{T-M2})^\rho\}^{\frac{1}{\rho}}}}{T-\left(T-\frac{T}{1+\{w_2(\frac{M2}{T-M2})^\rho\}^{\frac{1}{\rho}}}\right)}\right)^\rho\right\}^{\frac{1}{\rho}}}} \right)$$

$$= \left(\left[L_{T-\frac{T}{1+\{w_1(\frac{G1}{T-G1})^\rho+w_2(\frac{G2}{T-G2})^\rho\}^{1/\rho}}}, L_{T-\frac{T}{1+\{w_1(\frac{H1}{T-H1})^\rho+w_2(\frac{H2}{T-H2})^\rho\}^{1/\rho}}} \right], L_{T-\frac{T}{1+\{w_1(\frac{M1}{T-M1})^\rho+w_2(\frac{M2}{T-M2})^\rho\}^{1/\rho}}} \right)$$

$$= \left(\left[L_{T-\frac{T}{1+\{\sum_{i=1}^{i=2} w_i(\frac{Gi}{T-Gi})^\rho\}^{1/\rho}}}, L_{T-\frac{T}{1+\{\sum_{i=1}^{i=2} w_i(\frac{Hi}{T-Hi})^\rho\}^{1/\rho}}} \right], L_{T-\frac{T}{1+\{\sum_{i=1}^{i=2} w_i(\frac{Mi}{T-Mi})^\rho\}^{1/\rho}}} \right).$$

(2) Assume $n = k$, the result is as follows:

$$LCVDWAA(V_1, V_2, \ldots, V_k) = \bigoplus_{i=1}^{k} w_i V_i$$
$$= \left(\left[L_{T-\frac{T}{1+\{\sum_{i=1}^{i=k} w_i(\frac{Gi}{T-Gi})^\rho\}^{1/\rho}}}, L_{T-\frac{T}{1+\{\sum_{i=1}^{i=k} w_i(\frac{Hi}{T-Hi})^\rho\}^{1/\rho}}} \right], L_{T-\frac{T}{1+\{\sum_{i=1}^{i=k} w_i(\frac{Mi}{T-Mi})^\rho\}^{1/\rho}}} \right).$$

(3) If $n = k + 1$, we have:

$$LCVDWAA(V_1, V_2, \ldots, V_k, V_{k+1}) = LCVDWAA(V_1, V_2, \ldots, V_k) \oplus w_{k+1} V_{k+1}$$
$$= \left(\left[L_{T-\frac{T}{1+\{\sum_{i=1}^{i=k} w_i(\frac{Gi}{T-Gi})^\rho\}^{1/\rho}}}, L_{T-\frac{T}{1+\{\sum_{i=1}^{i=k} w_i(\frac{Hi}{T-Hi})^\rho\}^{1/\rho}}} \right], L_{T-\frac{T}{1+\{\sum_{i=1}^{i=k} w_i(\frac{Mi}{T-Mi})^\rho\}^{1/\rho}}} \right)$$
$$\oplus \left(\left[L_{T-\frac{T}{1+\{w_{k+1}(\frac{G_{k+1}}{T-G_{k+1}})^\rho\}^{\frac{1}{\rho}}}}, L_{T-\frac{T}{1+\{w_{k+1}(\frac{H_{k+1}}{T-H_{k+1}})^\rho\}^{\frac{1}{\rho}}}} \right], L_{T-\frac{T}{1+\{w_{k+1}(\frac{M_{k+1}}{T-M_{k+1}})^\rho\}^{\frac{1}{\rho}}}} \right)$$
$$= \left(\left[L_{T-\frac{T}{1+\{\sum_{i=1}^{i=k+1} w_i(\frac{Gi}{T-Gi})^\rho\}^{1/\rho}}}, L_{T-\frac{T}{1+\{\sum_{i=1}^{i=k+1} w_i(\frac{Hi}{T-Hi})^\rho\}^{1/\rho}}} \right], L_{T-\frac{T}{1+\{\sum_{i=1}^{i=k+1} w_i(\frac{Mi}{T-Mi})^\rho\}^{1/\rho}}} \right).$$

Thus, we have proved that Equation (9) is correct for any n. The properties of the LCVDWAA operator are as follows:

(1) Idempotency: If there is LCVs collection $V_i = ([L_{Gi}, L_{Hi}], L_{Mi})$ for $V_i = V$ ($i = 1, 2, \ldots, n$) then LCVDWAA ($V_1, V_2, \ldots, V_n$) = V.

(2) Commutativity: Assume that the LCV set ($V'_1, V'_2, V'_3, \ldots, V'_n$) is any permutation of ($V_1, V_2, \ldots, V_n$). Then, there is LCVDWAA ($V'_1, V'_2, \ldots, V'_n$) = LCVDWAA ($V_1, V_2, \ldots, V_n$).

(3) Boundedness: If there is LCVs collection $V_i = ([L_{Gi}, L_{Hi}], L_{Mi})$ $(i = 1, 2, \ldots, n)$ $V_{\min} = ([L_{\min\limits_i(Gi)}, L_{\min\limits_i(Hi)}], L_{\min\limits_i(Mi)})$, $V_{\max} = ([L_{\max\limits_i(Gi)}, L_{\max\limits_i(Hi)}], L_{\max\limits_i(Mi)})$. Then, $V_{\min} \le LCVDWAA(V_1, V_2, \ldots, V_n) \le V_{\max}$. □

Proof:

(1) Let $V_i = ([L_{Gi}, L_{Hi}], L_{Mi}) = ([L_G, L_H], L_M)$, then we can get the result:

$$\begin{aligned}
&LCVDWAA(V_1, V_2, \ldots, V_n) \\
&= \left(\left[L_{T-\frac{T}{1+\left\{\sum\limits_{i=1}^{i=n} w_i\left(\frac{Gi}{T-Gi}\right)^{\rho}\right\}^{1/\rho}}}, L_{T-\frac{T}{1+\left\{\sum\limits_{i=1}^{i=n} w_i\left(\frac{Hi}{T-Hi}\right)^{\rho}\right\}^{1/\rho}}} \right], L_{T-\frac{T}{1+\left\{\sum\limits_{i=1}^{i=n} w_i\left(\frac{Mi}{T-Mi}\right)^{\rho}\right\}^{1/\rho}}} \right) \\
&= \left(\left[L_{T-\frac{T}{1+\left\{\left(\frac{G}{T-G}\right)^{\rho}\sum\limits_{i=1}^{i=n} w_i\right\}^{1/\rho}}}, L_{T-\frac{T}{1+\left\{\left(\frac{H}{T-H}\right)^{\rho}\sum\limits_{i=1}^{i=n} w_i\right\}^{1/\rho}}} \right], L_{T-\frac{T}{1+\left\{\left(\frac{M}{T-M}\right)^{\rho}\sum\limits_{i=1}^{i=n} w_i\right\}^{1/\rho}}} \right) \\
&= \left(\left[L_{T-\frac{T}{1+\left\{\left(\frac{G}{T-G}\right)^{\rho}\right\}^{1/\rho}}}, L_{T-\frac{T}{1+\left\{\left(\frac{H}{T-H}\right)^{\rho}\right\}^{1/\rho}}} \right], L_{T-\frac{T}{1+\left\{\left(\frac{M}{T-M}\right)^{\rho}\right\}^{1/\rho}}} \right) \\
&= ([L_G, L_H], L_M) = V.
\end{aligned}$$

(2) The proof is obvious.

(3) Since $\min\limits_i(Gi) \le Gi \le \max\limits_i(Gi), \min\limits_i(Hi) \le Hi \le \max\limits_i(Hi), \min\limits_i(Mi) \le Mi \le \max\limits_i(Mi)$. Then the following inequalities can be induced as:

$$\begin{aligned}
T - \frac{T}{1+\left\{\sum\limits_{i=1}^{i=n} w_i\left(\frac{\min\limits_i(Gi)}{T-\min\limits_i(Gi)}\right)^{\rho}\right\}^{1/\rho}} = \min_i(Gi) \le T - \frac{T}{1+\left\{\sum\limits_{i=1}^{i=n} w_i\left(\frac{Gi}{T-Gi}\right)^{\rho}\right\}^{1/\rho}} \le \max_i(Gi) = T - \frac{T}{1+\left\{\sum\limits_{i=1}^{i=n} w_i\left(\frac{\max\limits_i(Gi)}{T-\max\limits_i(Gi)}\right)^{\rho}\right\}^{1/\rho}} \\
T - \frac{T}{1+\left\{\sum\limits_{i=1}^{i=n} w_i\left(\frac{\min\limits_i(Hi)}{T-\min\limits_i(Hi)}\right)^{\rho}\right\}^{1/\rho}} = \min_i(Hi) \le T - \frac{T}{1+\left\{\sum\limits_{i=1}^{i=n} w_i\left(\frac{Hi}{T-Hi}\right)^{\rho}\right\}^{1/\rho}} \le \max_i(Hi) = T - \frac{T}{1+\left\{\sum\limits_{i=1}^{i=n} w_i\left(\frac{\max\limits_i(Hi)}{T-\max\limits_i(Hi)}\right)^{\rho}\right\}^{1/\rho}} \\
T - \frac{T}{1+\left\{\sum\limits_{i=1}^{i=n} w_i\left(\frac{\min\limits_i(Mi)}{T-\min\limits_i(Mi)}\right)^{\rho}\right\}^{1/\rho}} = \min_i(Mi) \le T - \frac{T}{1+\left\{\sum\limits_{i=1}^{i=n} w_i\left(\frac{Mi}{T-Mi}\right)^{\rho}\right\}^{1/\rho}} \le \max_i(Mi) = T - \frac{T}{1+\left\{\sum\limits_{i=1}^{i=n} w_i\left(\frac{\max\limits_i(Mi)}{T-\max\limits_i(Mi)}\right)^{\rho}\right\}^{1/\rho}}
\end{aligned}$$

Hence, $V_{\min} \le LCVDWAA(V_1, V_2, \ldots, V_n) \le V_{\max}$ holds. □

4.2. Dombi Weighted Geometric Average Operator of LCVs

Definition 7. *Let $V = \{V_1, V_2, \ldots, V_n\}$ be an LCV set, then the Dombi weighted geometric average operator of the LCV_S can be defined as:*

$$LCVDWGA(V_1, V_2, \ldots, V_n) = \mathop{\otimes}\limits_{i=1}^{n} w_i V_i \tag{10}$$

where the weight vector w_i satisfies $\sum\limits_{i=1}^{i=n} w_i = 1$ and $w_i \in [0, 1]$.

According to Definitions 5 and 7, the following theorem can be induced and proved.

Theorem 2. *Let $V_i = ([L_{Gi}, L_{Hi}], L_{Mi})$ (i = 1, 2, ... , n) be a set of LCVs and the corresponding weight vector is $w = (w_1, w_2, \ldots, w_n)$, where $\sum_{i=1}^{i=n} w_i = 1$ and $w_i \in [0, 1]$, we can calculate Equation (10) on basis of the predefined operational laws and have:*

$$LCVDWGA(V_1, V_2, \ldots, V_n) = \left(\left[L_{\frac{T}{1+\{\sum_{i=1}^{i=n} w_i (\frac{T-Gi}{Gi})^{\rho}\}^{1/\rho}}}, L_{\frac{T}{1+\{\sum_{i=1}^{i=n} w_i (\frac{T-Hi}{Hi})^{\rho}\}^{1/\rho}}} \right] \right), L_{\frac{T}{1+\{\sum_{i=1}^{i=n} w_i (\frac{T-Mi}{Mi})^{\rho}\}^{1/\rho}}} \right). \quad (11)$$

Theorem 2 is the same proof as Theorem 1. Hence, we do not prove it repeatedly.

The LCVDWGA operator also has Properties (1)–(3) as follows:

(1) Idempotency: If there is LCVs collection $V_i = ([L_{Gi}, L_{Hi}], L_{Mi})$ for $V_i = V$ (i = 1, 2, ... , n). Then LCVDWGA (V_1, V_2, ... , V_n) = V.
(2) Commutativity: If the LCV set (V'_1, V'_2, ... , V'_n) is any permutation of (V_1, V_2, ... , V_n). Then, there is LCVDWGA (V'_1, V'_2, ... , V'_n) = LCVDWGA (V_1, V_2, ... , V_n).
(3) Boundedness: If there is LCVs collection $V_i = ([L_{Gi}, L_{Hi}], L_{Mi})$ (i = 1,2, ... , n) $V_{\min} = ([L_{\min_i (Gi)}, L_{\min_i (Hi)}], L_{\min_i (Mi)})$, $V_{\max} = ([L_{\max_i (Gi)}, L_{\max_i (Hi)}], L_{\max_i (Mi)})$. Then, $V_{\min} \leq LCVDWGA(V_1, V_2, \ldots, V_n) \leq V_{\max}$.

The proofs of the above properties are omitted which are similar with the properties of the LCVDWAA operator.

5. MADM Method on Basis of the LCVDWAA or LCVDWGA Operator

If a MADM problem is described by LCV information, $V = \{V_1, V_2, \ldots, V_m\}$ and $P = \{P_1, P_2, \ldots, P_n\}$ are the sets of alternatives and attributes, respectively. $w = \{w_1, w_2, \ldots, w_n\}$ is the set of weight, where w_j is corresponding to the importance of attribute P_j with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. The LCV V_{ij} is the evaluation of the alternatives $V_i (i = 1, 2, \ldots, m)$ over the attributes $P_j (j = 1, 2, \ldots, n)$. Each LCV includes uncertain linguistic argument and certain linguistic argument. Thus, all the LCVs given by decision makers are constructed as an LCV decision matrix $V = (V_{ij})_{m \times n}$, where $V_{ij} = ([L_{Gij}, L_{Hij}], L_{Mij})$ is an LCV (*i* =*1, 2,* ... *, m; j* = 1, 2, ... , *n*) and $L_{Gij}, L_{Hij}, L_{Mij}$ is from the linguistic term set $L = \{L_k | k \in [0, T]\}$ with even number T.

On basis of the LCVDWAA or LCVWDGA operator, the steps of MADM method are as follows.

Step 1. According to Equation (9) or Equation (11), we can get the collective LCV of each alterative $V_i = LCVDWAA(V_{i1}, V_{i2}, \ldots, V_{in})$ or $V_i = LCVDWGA(V_{i1}, V_{i2}, \ldots, V_{in})$ (i = 1, 2, ... , m).

Step 2. The expected values $E(V_i) (i = 1, 2, \ldots, m)$ of each collective LCV $V_i (i = 1, 2, \ldots, m)$ are calculated according to Equation (1).

Step 3. According to the expected values of $E(V_i) (i = 1, 2, \ldots, m)$, we give the rank order of all the alternatives. The best alternative $V_i (i = 1, 2, \ldots, m)$ is with the greatest value of $E(V_i)$.

6. Illustrative Examples and Discussions

Two application examples are illustrated below, then we discuss the validity of this proposed MADM approach and the influence of the operational parameter.

6.1. Illustrative Examples

Example 2 [13]. *A company needs to hire a soft engineer. There are four candidates (alternatives) V_1, V_2, V_3, and V_4. The decision makers will further evaluate them over four attributes. The four attributes are soft skills, past experience, personality, and self-confidence, in order. The corresponding weight vector of the attributes is w = (0.35, 0.25, 0.2, 0.2). The decision makers evaluate the four candidates by using the linguistic cubic values,* *which are obtained from the linguistic term set L = {L_i | i ∈ [0, 8]}, where L = {L_0 = extremely poor, L_1 = very poor, L_2 = poor, L_3 = slightly poor, L_4 = fair, L_5 = slightly good, L_6 = good, L_7 = very good, L_8 = extremely good}. The linguistic cubic decision matrix V is described as follows:*

$$V = (V_{ij})_{4\times 4} = \begin{bmatrix} ([L_4, L_6], L_5) & ([L_4, L_6], L_4) & ([L_4, L_7], L_6) & ([L_5, L_6], L_6) \\ ([L_3, L_5], L_4) & ([L_5, L_7], L_6) & ([L_4, L_6], L_4) & ([L_6, L_7], L_6) \\ ([L_4, L_7], L_5) & ([L_6, L_7], L_7) & ([L_5, L_7], L_5) & ([L_5, L_7], L_7) \\ ([L_6, L_7], L_7) & ([L_5, L_7], L_6) & ([L_4, L_6], L_5) & ([L_5, L_6], L_5) \end{bmatrix}$$

Now we employ the LCVDWAA operator to solve this MADM problem.

Step 1. According to Equation (9) for ρ = 1 and T = 8, we can get the following collective LCVs for four alternatives:

$$\begin{aligned} V_1 &= LCVDWAA(V_{11}, V_{12}, \ldots, V_{14}) \\ &= \left(\left[L_{T-\frac{T}{1+\{\sum_{i=1}^{i=4} w_i (\frac{G_{1i}}{T-G_{1i}})^{\rho}\}^{1/\rho}}}, L_{T-\frac{T}{1+\{\sum_{i=1}^{i=4} w_i (\frac{H_{1i}}{T-H_{1i}})^{\rho}\}^{1/\rho}}} \right], L_{T-\frac{T}{1+\{\sum_{i=1}^{i=4} w_i (\frac{M_{1i}}{T-M_{1i}})^{\rho}\}^{1/\rho}}} \right) \\ &= \left(\left[L_{8-\frac{8}{1+\sum_{i=1}^{i=4} w_i (\frac{G_{1i}}{T-G_{1i}})}}, L_{8-\frac{8}{1+\sum_{i=1}^{i=4} w_i (\frac{H_{1i}}{T-H_{1i}})}} \right], L_{8-\frac{8}{1+\sum_{i=1}^{i=4} w_i (\frac{M_{1i}}{T-M_{1i}})}} \right) \\ &= ([L_{4.2500}, L_{6.3333}], L_{5.3626}) \\ &= ([L_{G1}, L_{H1}], L_{M1}), \end{aligned}$$

$V_2 = LCVDWAA(V_{21}, V_{22}, \ldots, V_{24}) = ([L_{4.7033}, L_{6.5000}], L_{5.2414})$,
$V_3 = LCVDWAA(V_{31}, V_{32}, \ldots, V_{34}) = ([L_{5.1084}, L_{7.0000}], L_{6.4211})$, and
$V_4 = LCVDWAA(V_{41}, V_{42}, \ldots, V_{44}) = ([L_{5.3333}, L_{6.7500}], L_{6.3562})$.

Step 2. The expected values $E(V_i)(i = 1, 2, \ldots, m)$ of each collective LCV $V_i(i = 1, 2, 3, 4)$ are calculated according to Equation (1). The results are as follows:

$$E(V_1) = (G_1 + H_1 + M_1)/3T = 0.6644, E(V_2) = (G_2 + H_2 + M_2)/3T = 0.6852,$$
$$E(V_3) = (G_3 + H_3 + M_3)/3T = 0.7721, E(V_4) = (G_4 + H_4 + M_4)/3T = 0.7683.$$

Step 3. According to the above expected values and the rank principle, the rank order of the four candidates is $V_3 \succ V_4 \succ V_2 \succ V_1$.

Alternatively, we use LCVDWGA operator for this MADM problem with the same decision steps.

Step 1. We aggregate the LCVs for four candidates according to Equation (11) for ρ = 1 and T = 8.

$$\begin{aligned} V_1 &= LCVDWGA(V_{11}, V_{12}, \ldots, V_{14}) \\ &= \left(\left[L_{\frac{T}{1+\{\sum_{i=1}^{i=4} w_i (\frac{T-G1i}{G1i})^{\rho}\}^{1/\rho}}}, L_{\frac{T}{1+\{\sum_{i=1}^{i=4} w_i (\frac{T-H1i}{H1i})^{\rho}\}^{1/\rho}}} \right] \right), L_{\frac{T}{1+\{\sum_{i=1}^{i=4} w_i (\frac{T-M1i}{M1i})^{\rho}\}^{1/\rho}}} \Bigg) \\ &= \left(\left[L_{\frac{8}{1+\sum_{i=1}^{i=4} w_i (\frac{T-G1i}{G1i})}}, L_{\frac{8}{1+\sum_{i=1}^{i=4} w_i (\frac{T-H1i}{H1i})}} \right] \right), L_{\frac{8}{1+\sum_{i=1}^{i=4} w_i (\frac{T-M1i}{M1i})}} \Bigg) \\ &= ([L_{4.1677}, L_{6.1765}], L_{5.0209}) \end{aligned}$$

$V_2 = LCVDWGA(V_{21}, V_{22}, \ldots, V_{24}) = ([L_{4.0000}, L_{5.9659}], L_{4.7059})$,
$V_3 = LCVDWGA(V_{31}, V_{32}, \ldots, V_{34}) = ([L_{4.7809}, L_{7.0000}], L_{5.7377})$, and
$V_4 = LCVDWGA(V_{41}, V_{42}, \ldots, V_{44}) = ([L_{5.0420}, L_{6.5625}], L_{5.8252})$.

Step 2. The expected values $E(V_i)(i = 1, 2, \ldots, m)$ of each collective LCV $V_i(i = 1, 2, 3, 4)$ are calculated according to Equation (1). The results are as follows:

$E(V_1) = 0.6402, E(V_2) = 0.6113, E(V_3) = 0.7299, E(V_4) = 0.7262.$

Step 3. According to the above expected values and the rank principle, the rank order of the four candidates is $V_3 \succ V_4 \succ V_1 \succ V_2$.

By following the same steps above, we apply the LCVDWAA operator and LCVDWGA operator to Example 2 with parameter ρ from 1 to 100, the ranking results are shown as following Tables 1 and 2.

Table 1. Ranking orders of the LCVDWAA [1] operator, $\rho \in$ [1–5,10,15,20,30,50,100].

ρ	E(V$_1$) [2], E(V$_2$) [3], E(V$_3$) [4], E(V$_4$) [5]	Ranking Order	The Best Candidate
1	0.6644, 0.6852, 0.7721, 0.7683	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3
2	0.6766, 0.7139, 0.7875, 0.7843	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3
3	0.6876, 0.7332, 0.7978, 0.7954	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3
4	0.6969, 0.7459, 0.8049, 0.8031	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3
5	0.7045, 0.7545, 0.8099, 0.8085	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3
10	0.7253, 0.7731, 0.8214, 0.8207	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3
15	0.7336, 0.7794, 0.8254, 0.8250	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3
20	0.7377, 0.7825, 0.8274, 0.8271	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3
30	0.7419, 0.7856, 0.8294, 0.8292	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3
50	0.7451, 0.7881, 0.8310, 0.8309	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3
100	0.7476, 0.7899, 0.8322, 0.8321	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3

[1] LCVDWAA = linguistic cubic variable Dombi weighted arithmetic average; [2] E(V$_1$) = expected value of V$_1$; [3] E(V$_2$) = expected value of V$_2$; [4] E(V$_3$) = expected value of V$_3$; [5] E(V$_4$) = expected value of V$_4$.

Table 2. Ranking orders of the LCVDWGA [1] operator, $\rho \in$ [1–5,10,15,20,30,50,100].

ρ	E(V$_1$), E(V$_2$), E(V$_3$), E(V$_4$)	Ranking Order	The Best Candidate
1	0.6402, 0.6113, 0.7299, 0.7262	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3
2	0.6300, 0.5827, 0.7143, 0.7076	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3
3	0.6219, 0.5633, 0.7039, 0.6929	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3
4	0.6155, 0.5503, 0.6968, 0.6816	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3
5	0.6106, 0.5414, 0.6918, 0.6730	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3
10	0.5980, 0.5213, 0.6800, 0.6509	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3
15	0.5932, 0.5143, 0.6756, 0.6424	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3
20	0.5907, 0.5107, 0.6734, 0.6381	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3
30	0.5883, 0.5071, 0.6711, 0.6337	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3
50	0.5863, 0.5043, 0.6693, 0.6303	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3
100	0.5848, 0.5021, 0.6680, 0.6276	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3

[1] LCVDWGA = linguistic cubic variable Dombi weighted geometric average.

Example 3. *Customers want to buy an air-conditioner; they choose three brands as alternatives V_1, V_2, V_3. Further, they need to evaluate the three alternatives from three attributes which are as follows: (i) P_1 is cooling effect; (ii) P_2 is heating effect; and (iii) P_3 is appearance design. Their importance lies in the weight vector w = (1/2,1/3,1/6). The customers give their evaluations over the three attributes by the linguistic cubic values V_{ij} based on the uniform linguistic term set L as Example 2. The LCVs provided by the customers constitute the decision matrix V.*

$$V = (V_{ij})_{3\times 3} = \begin{bmatrix} ([L_2, L_7], L_3) & ([L_4, L_7], L_2) & ([L_2, L_7], L_1) \\ ([L_2, L_7], L_5) & ([L_2, L_7], L_3) & ([L_2, L_7], L_3) \\ ([L_2, L_5], L_5) & ([L_1, L_6], L_4) & ([L_2, L_5], L_2) \end{bmatrix}$$

By using the same steps, we apply the LCVDWAA operator or LCVDWGA operator to this MADM problem. The ranking results based on the LCVDWAA operator with parameters ρ from 1 to 5 are shown in the Table 3. Similarly, the ranking orders on basis of the LCVDWGA operator are shown in Table 4.

Table 3. Ranking orders of the LCVDWAA operator, $\rho \in$ [1–5,10,15,20,30,50,100].

ρ	E(V1), E(V2), E(V3)	Ranking Order	The Best Alterative
1	0.5117, 0.5795, 0.4804	$V_2 \succ V_1 \succ V_3$	V_2
2	0.5280, 0.5932, 0.4927	$V_2 \succ V_1 \succ V_3$	V_2
3	0.5404, 6005, 0.5016	$V_2 \succ V_1 \succ V_3$	V_2
4	0.5490, 0.6052, 0.5084	$V_2 \succ V_1 \succ V_3$	V_2
5	0.5551, 0.6085, 0.5135	$V_2 \succ V_1 \succ V_3$	V_2
10	0.5688, 0.6165, 0.5267	$V_2 \succ V_1 \succ V_3$	V_2
15	0.5736, 0.6193, 0.5317	$V_2 \succ V_1 \succ V_3$	V_2
20	0.5761, 0.6208, 0.5342	$V_2 \succ V_1 \succ V_3$	V_2
30	0.5785, 0.6222, 0.5367	$V_2 \succ V_1 \succ V_3$	V_2
50	0.5804, 0.6233, 0.5387	$V_2 \succ V_1 \succ V_3$	V_2
100	0.5819, 0.6242, 0.5402	$V_2 \succ V_1 \succ V_3$	V_2

Table 4. Ranking orders of the LCVDWGA operator, $\rho \in$ [1–5,10,15,20,30,50,100].

ρ	E(V1), E(V2), E(V3)	Ranking Order	The Best Alterative
1	0.4750, 0.4826, 0.4393	$V_2 \succ V_1 \succ V_3$	V_2
2	0.4598, 0.4353, 0.4144	$V_1 \succ V_2 \succ V_3$	V_1
3	0.4488, 0.4078, 0.3946	$V_1 \succ V_2 \succ V_3$	V_1
4	0.4414, 0.3907, 0.3810	$V_1 \succ V_2 \succ V_3$	V_1
5	0.4364, 0.3796, 0.3718	$V_1 \succ V_2 \succ V_3$	V_1
10	0.4262, 0.3563, 0.3523	$V_1 \succ V_2 \succ V_3$	V_1
15	0.4229, 0.3486, 0.3459	$V_1 \succ V_2 \succ V_3$	V_1
20	0.4213, 0.3447, 0.3427	$V_1 \succ V_2 \succ V_3$	V_1
30	0.4197, 0.3409, 0.3395	$V_1 \succ V_2 \succ V_3$	V_1
50	0.4185, 0.3379, 0.3370	$V_1 \succ V_2 \succ V_3$	V_1
100	0.4176, 0.3356, 0.3352	$V_1 \succ V_2 \succ V_3$	V_1

6.2. Discussion

6.2.1. Validity of the Method

Ye [13] firstly proposed the concept of LCVs, and then used the LCVWAA operator and LCVWGA operator to handle the MADM problem of Example 2. As shown in Table 5, the ranking orders using the LCVDWAA operator and LCVDWGA operator with parameters ρ from 1 to 100 are the same as those using the LCVWAA operator [13] and LCVWGA operator [13], respectively. In Example 3, the ranking results based on the LCVDWAA operator are the same as those based on the LCVWAA operator [13] when parameter ρ ranges from 1 to 100. Then, the ranking orders based on the LCVDWGA operator are the same as those based on the LCVWGA operator [13] when parameter ρ is equal to 1.

Table 5. Ranking results of different aggregation operators with different parameters.

Example	MADM [1] Method	Ranking Order	The Best Alterative
2	LCVDWAA (ρ = 1 to 100)	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3
	LCVWAA [2] [13]	$V_3 \succ V_4 \succ V_2 \succ V_1$	V_3
	LCVDWGA (ρ = 1 to 100)	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3
	LCVWGA [3] [13]	$V_3 \succ V_4 \succ V_1 \succ V_2$	V_3
3	LCVDWAA (ρ = 1 to 100)	$V_2 \succ V_1 \succ V_3$	V_2
	LCVWAA [13]	$V_2 \succ V_1 \succ V_3$	V_2
	LCVDWGA (ρ = 1)	$V_2 \succ V_1 \succ V_3$	V_2
	LCVDWGA (ρ = 2 to 100)	$V_1 \succ V_2 \succ V_3$	V_1
	LCVWGA [13]	$V_2 \succ V_1 \succ V_3$	V_2

[1] MADM = multiple attribute decision making; [2] LCVWAA = linguistic cubic variable weighted arithmetic average; [3] LCVWGA = linguistic cubic variable weighted geometric average.

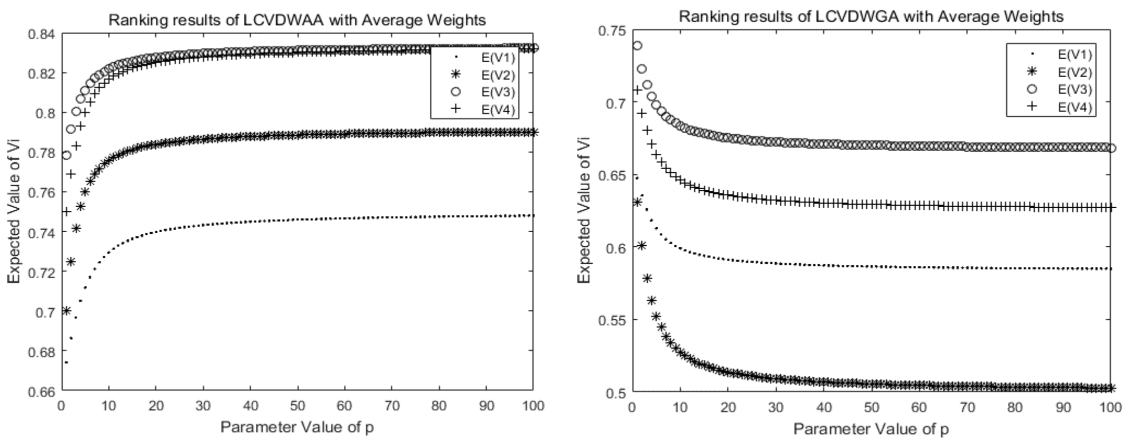
6.2.2. The Influence of the Parameter ρ

As shown in Table 5, parameter value of ρ has no effect on the ranking results in Example 2. In Example 3, the ranking results are not sensitive to parameters ρ for the LCVDWAA operator. However, corresponding to the LCVDWGA operator, the ranking results are more sensitive to parameter ρ. When $\rho = 1$, the ranking orders of the LCVDWGA operator are the same as those of the LCVWAA [13], LCVWGA [13], and LCVDWAA operators ($\rho = 1$ to 100), and the best alternative is V_2. While the ranking orders of the LCVDWGA operator are obviously changed when ρ is from 2 to 100, the best alternative is V_1. From Tables 1–4, we can see that parameter value of ρ is greater and the expected values of $E(V_i)$ are greater in the LCVDWAA operator. While in the LCVDWGA operator the value of parameter ρ is greater and the expected values of $E(V_i)$ are smaller.

In any case, by using LCVDWAA or LCVDWGA operator to aggregate decision-making information, the presented approach is valid to handle MADM problems with LCV information. Especially the LCVDWGA operator is more flexible in actual applications.

6.2.3. The Sensitivity Analysis of Weights

In order to demonstrate the sensitivity of weights, we change the weights of the attributes in Examples 2 and 3. $W = (0.25, 0.25, 0.25, 0.25)$ and $w = (1/3, 1/3, 1/3)$ are used as the weight vectors in Examples 2 and 3, respectively. Then we apply LCVDWAA operator and LCVDWGA operator to the two applications again and change the parameter value of ρ from 1 to 100. The ranking results of Example 2 are shown in Figure 1 and the ranking results of Example 3 are shown in Figure 2. The curves of collective expected values $E(V_{i)}$ were shown in Figures 1 and 2. The curves clearly show that LCVDWAA and LCVDWGA have different effects on the expected value. Additionally, we find that the ranking results are identical with Table 5 when the weights are changed. Especially as Figure 2b shows, the best alternative is V_2 when ρ is equal to 1, while the best alternative is V_1 when ρ ranges from 2 to 100. It fits perfectly with Table 5. Thus, we can think that the LCVDWAA and LCVDWGA are not sensitive to the changes of weights.



(**a**) (**b**)

Figure 1. Ranking results with average weights in Example 2. LCVDWAA = linguistic cubic variable Dombi weighted arithmetic average; LCVDWGA = linguistic cubic variable Dombi weighted geometric average. (**a**) LCVDWAA operator; (**b**) LCVDWGA operator.

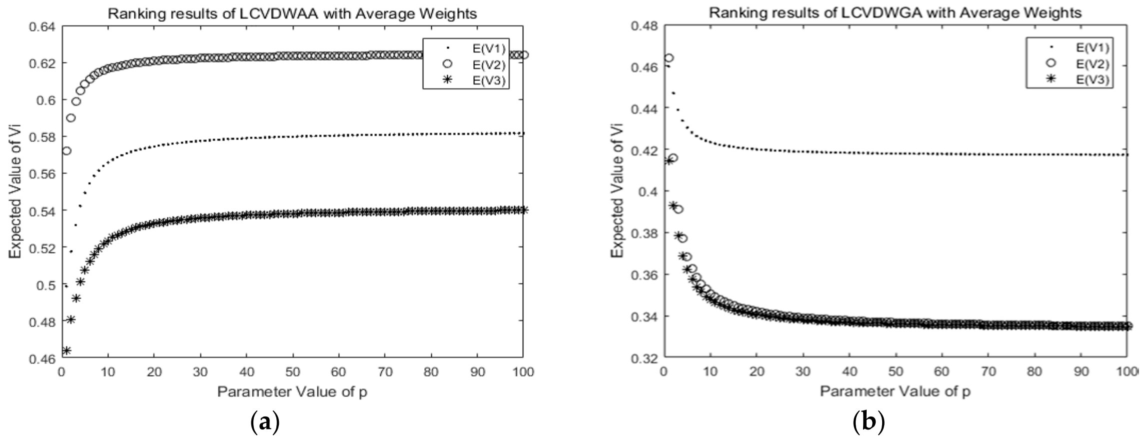


Figure 2. Ranking results with average weights in Example 3. (**a**) LCVDWAA operator; (**b**) LCVDWGA operator.

7. Conclusions

The concept of LCV was proposed by Ye [13] recently. There are few studies on LCV information aggregation operators and MADM methods about LCV information problems. In this paper, the Dombi operations were extended to an LCV environment. We proposed an LCVDWAA operator and an LCVDWGA operator, then discussed their properties. Further, based on the LCVDWAA or LCVDWGA operator, a MADM method was developed. Finally, the proposed approach was applied to two application examples. This MADM method is very simple. There is only one decision-making matrix with LCVs information in a MAGDM problem. The results demonstrated this approach is feasible and valid as the method proposed in Ye [13]. Compared with the method proposed in Ye [13], this approach not only can handle decision-making problems effectively, but also can affect the ranking order based on the LCVDWAA or LCVDWGA operator by the changeable parameter ρ. In an actual decision-making process, we can specify various parameter values based on the decision makers' preferences and requirements. However, the flexibility of the LCVDWAA or LCVDWGA operator was not fully reflected in the two examples. In order to observe the sensitivity of weights, we changed the weight vectors of the two examples and changed parameter values from 1 to 100. We found that the results were not changed when the weight was averaged. Although the operators were not sensitive to the changes of weights, there were some changes in the ranking results when we changed the weight vectors to extreme cases in the study. Thus, the ranking results are determined by weights and parameter values together for the same decision-making matrix. In future work, we can continue to develop more flexible aggregation operators of LCVs and use them to solve MADM problems in various fields.

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