

## Article

# Particle Production at High Energy: DGLAP, BFKL and Beyond<sup>†</sup>

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**Abstract:** Particle production in high energy hadronic/nuclear collisions in the Bjorken limit  $Q^2, \sqrt{s} \rightarrow \infty$  can be described in the collinear factorization framework of perturbative Quantum Chromodynamics (QCD). On the other hand in the Regge limit, at fixed and not too high  $Q^2$  with  $\sqrt{s} \rightarrow \infty$ , a  $k_\perp$  factorization approach (or a generalization of it) is the appropriate framework. A new effective action approach to QCD in the Regge limit, known as the Color Glass Condensate (CGC) formalism, has been developed which allows one to investigate particle production in high energy collisions in the kinematics where collinear factorization breaks down. Here we give a brief overview of particle production in CGC framework and the evolution equation which governs energy dependence of the observables in this formalism. We show that the new evolution equation reduces to previously known evolution equations in the appropriate limits.

**Keywords:** quantum chromodynamics; small  $x$ ; evolution equations; JIMWLK; BFKL; BJKP

## 1. Introduction

Quantum chromodynamics (QCD) is the fundamental theory of strong interactions describing interactions of colored quarks and gluons and their binding into colorless hadrons. QCD is amazing rich but complex theory; perhaps the most well-understood aspect of it is its high  $Q^2$  limit due to the phenomenon of asymptotic freedom: the coupling constant  $\alpha_s(Q^2) \rightarrow 0$  when  $Q^2 \rightarrow \infty$ . This allows a systematic expansion of physical observables in terms of a small parameter, the coupling constant. Nevertheless the smallness of the coupling constant is often spoiled by appearance of logarithmic divergences arising from the singular nature of correlators of field operators in quantum field theories.

Parton Model [1] in the context of Operator Product Expansion (OPE) and collinear factorization theorems [2] provide a useful platform for calculating particle production in high energy hadronic collisions. Collinear factorization allows one to cleanly separate long distance (low momentum) physics from that of short distance (high momentum). The long distance physics is contained in non-perturbative parton distribution and fragmentation functions while short distance physics containing the hard scattering is calculable in perturbation theory to any order in the coupling constant. The emerging singular logarithms are absorbed into parton distribution and fragmentation functions and lead to their scale ( $Q^2$ ) dependence, usually referred to as evolution of distribution or fragmentation functions. The power of collinear factorization theorems relies on the fact that the non-perturbative distribution and fragmentation functions are process independent i.e., universal. As such they can be measured in one process and used in any other. This formalism has been extremely successful in describing high  $p_t$  particle production in proton-proton collisions as well as structure functions (total

cross section) in Deep Inelastic Scattering (DIS). For example, neutral pion production cross section in high energy proton-proton collisions can be symbolically written as

$$E \frac{d\sigma^{pp \rightarrow \pi^0 X}}{d^3p} = f_1(x_1, Q^2) \otimes f_2(x_2, Q^2) \otimes \frac{d\sigma}{dt} \otimes D(z, Q^2) \quad (1)$$

where  $f_1(x_1, Q^2), f_2(x_2, Q^2)$  are parton distribution functions of the incoming protons and  $D(z, Q^2)$  is the parton-pion fragmentation function. Here  $x$  is the fraction of the energy of a proton carried by a parton while  $z$  is the fraction of the energy of a parton carried by the produced pion. The scale factorization scale  $Q$  is usually taken to be equal or proportional to the produced particle's transverse momentum  $p_t$  in order to avoid the need for resummation of additional logs of the form  $\log Q^2/p_t^2$ . Very roughly the factorization scale can be thought to arise from radiation of extra partons by the incoming or outgoing partons. If the momentum of the radiated parton is less than this scale then this (potentially divergent) contribution is absorbed into the parton distribution or fragmentation functions. Radiated partons with momenta larger than this scale are real and are part of higher order (in  $\alpha_s$ ) corrections to the partonic cross section  $\frac{d\sigma}{dt}$ . It is important to realize that there are corrections to Equation (1) which break this factorization but are suppressed by powers of the scale  $Q^2$  so that this relation becomes exact in the limit  $p_t \rightarrow \infty$ . Furthermore and in the context of collinear factorization, distribution and fragmentation functions are defined as expectation values of bi-local operators of quark (or gluon) fields between the relevant states. These bi-local operators are divergent, renormalization of which leads to the scale dependence of the distribution and fragmentation functions. This scale dependence is governed by the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation [3–5] which can be written as (for one quark flavor)

$$\frac{d}{d \log Q^2} \begin{pmatrix} q(x, Q^2) \\ g(x, Q^2) \end{pmatrix} = \int_x^1 \frac{dz}{z} \begin{pmatrix} P_{qq}(z, \alpha_s) & P_{qg}(z, \alpha_s) \\ P_{gq}(z, \alpha_s) & P_{gg}(z, \alpha_s) \end{pmatrix} \begin{pmatrix} q(x/z, Q^2) \\ g(x/z, Q^2) \end{pmatrix}. \quad (2)$$

The splitting functions  $P_{ab}(z, \alpha_s)$  give the probability for radiation of a parton of type  $a$  by another parton of type  $b$  carrying a fraction  $z$  of its energy. These splitting functions are calculable in perturbation theory and in addition to energy fraction  $z$  also depend on the coupling constant (and hence implicitly on scale  $Q \sim p_t$ ). There is also an explicit dependence on the scale  $Q$  when one goes beyond Leading Order (LO) approximation.

Collinear factorization formalism in pQCD has been extremely successful in describing and predicting transverse momentum and rapidity dependence of particle production in high energy hadronic collisions at high transverse momentum  $p_t$ . On the other hand as one consider production of particles in lower momenta, the power suppressed become important and even dominant at lower  $p_t$ . These corrections are known as higher twist corrections which lead to a break down of collinear factorization in pQCD. Furthermore, it is an experimental fact that parton (specially gluon) distribution functions grow rapidly as  $x$  gets smaller. Given the kinematic relation between center of mass energy  $\sqrt{s}$ , transverse momentum and rapidity of produced particle  $p_t, y$  and energy fraction  $x$

$$x_{1,2} = \frac{p_t}{\sqrt{s}} e^{\pm y} \quad (3)$$

we see that at least one of the energy fractions  $x_{1,2}$  in parton distribution functions becomes very small. Due to the fast growth of parton distribution functions [6] as  $x \rightarrow 0$  this leads to a proton wave function that is densely populated with gluons and sea quarks [7]. Noticing that the essence of collinear factorization is treating partons inside a proton as uncorrelated and non-interacting (with each other) high parton density effects also lead to a breakdown of collinear factorization formalism. Therefore one would need a new formalism for particle production at very high energies as long as the transverse momentum of produced particles is not too high.

## 2. Particle Production in Color Glass Condensate Formalism

The Color Glass Condensate (CGC) formalism [8,9] is an effective action approach to QCD at high energy. Rather than considering individual partons of a proton participating in a high energy collision, it treats the wave function of a high energy proton as a classical color field. This is justified due to the presence of large number of color charges carried by quarks and gluons of a proton at small  $x$  (equivalently high energy at fixed transverse momentum). Therefore, instead of thinking of a high energy collision proton-proton collision as a collision of one parton from one proton on another parton from the other proton one thinks of collision of classical color fields of the colliding protons. This is the so-called classical approximation which is then modified due to quantum corrections. Due to the kinematics of interest (small  $x$ ) one is interested in quantum corrections which result in large logarithms of  $x$  ( $\alpha_s \log 1/x$ ) which then need to be resummed, analogously to the resummation of large logarithms of  $Q^2$  as done by the DGLAP evolution equation. The CGC formalism has been applied to many processes [8,9] including high energy heavy ion collisions where one expects formation of a Quark-Gluon Plasma [10]. In order to illustrate the methods and techniques of CGC formalism here we focus on the simple case of scattering of a quark parton from a dense system of gluons, treated as a classical color field representing a high energy proton (or nucleus).

### 2.1. Scattering in High Energy Limit: Classical Approximation

We consider scattering of a quark from the color field  $A_a^\mu(x)$  of the target proton (or nucleus) [11–17]. This is meant as a sub-process for proton-nucleus scattering at high energy and in particular in the forward rapidity region where the values of target  $x$  are very small and the approximations inherent in the CGC formalism are best satisfied. The classical field  $A_a^\mu(x)$  represents the small  $x$  gluons of the target radiated coherently by the large  $x$  color charges (valence quarks, large  $x$  gluons and sea quarks). This is due to the fact that small  $x$  gluons have small momentum  $p^+$  and equivalently a large longitudinal wave length, much larger than the longitudinal spacing between the large  $x$  color charges. Therefore they can not resolve the individual color charges along the longitudinal direction and couple to them coherently. Furthermore and in the high energy limit and due to the vast difference between the natural time scales between the high  $x$  and low  $x$  degrees of freedom, one treats the color sources as light-cone time ( $x^+$ ) independent. Specifically, for large  $x$  degrees of freedom the natural time scale is  $x^+ \sim \frac{1}{p^-} \sim \frac{p^+}{p_t^2}$  while for the small  $x$  modes we have  $x^+ \sim \frac{1}{p^-} \sim x \frac{p^+}{p_t^2}$  so that the relevant time scale for the small  $x$  gluon modes is much smaller than the corresponding time scale for the large  $x$  modes when  $x \ll 1$ . This means that small  $x$  gluon modes see the large  $x$  color modes as static, i.e., “frozen” (in light cone time  $x^+$ ) [18,19]. With this approximation one can solve the classical equation of motion in the presence of a static color charge for a target proton/nucleus moving in the negative  $z$  direction while the quark is moving in the positive  $z$  direction,

$$D_\mu F^{\mu\nu} = J^\nu \quad (4)$$

where due to the high energy kinematics (large boost in the negative  $z$  direction) the current  $J^\mu$  has only one large component,  $J_a^\mu = \delta^{\mu-} \rho_a$  and is independent of  $x^-$  coordinate (suppressed by the boost factor  $\gamma$ ). Furthermore if we work in the light cone gauge  $A^+ = 0$  (equivalent to covariant gauge in the classical approximation) the component of the equation of motion with the source reduces to

$$\partial_+ F^{+-} = J^-(x^+, x_t) \quad (5)$$

whose solution is independent of  $x^-$  coordinate,

$$A^-(x) = A^-(x^+, x_t) \quad (6)$$

with the other components of the field being zero. Our problem then reduces to calculating the amplitude for multiple scattering of the high energy quark projectile on this background color field. Since the color field  $A^-(x^+, x_t)$  is independent of  $x^-$  one gets a light cone energy conserving  $\delta(p_1^+ - p_2^+)$  for each scattering vertex where  $p_1^+, p_2^+$  are the energies of the incoming and outgoing quark. Furthermore and assuming the incoming quark has positive  $p^+$ , contour integration over the  $i$ th intermediate momentum  $p_i^-$  results in a theta function forcing a path ordering in the longitudinal coordinate  $x^+$  along the direction of propagation of the quark. One also ignores all terms of the form  $\frac{p_t}{p^+}$  which then allows one to perform the integration over the intermediate transverse momenta which results in all the transverse coordinates of the fields being equal. For the  $j$ th scatterings we get

$$i\mathcal{M}_j = 2\pi\delta(p^+ - q^+) \bar{u}(q) \not{n} \int d^2x_t e^{-i(q_t - p_t) \cdot x_t} \left\{ (ig)^j (-i)^j (i)^j \int dx_1^+ dx_2^+ \cdots dx_j^+ \theta(x_j^+ - x_{j-1}^+) \cdots \theta(x_2^+ - x_1^+) \right. \\ \left. \left[ A^-(x_j^+, x_t) A^-(x_{j-1}^+, x_t) \cdots A^-(x_2^+, x_t) A^-(x_1^+, x_t) \right] \right\} u(p) \quad (7)$$

where  $n^\mu$  is a light-like vector pointing in the  $-$  direction so that  $\not{n} = \gamma^+$ . Summing over all  $j$  scatterings ( $i\mathcal{M} = \sum_{j=1}^{\infty} i\mathcal{M}_j$ ) gives

$$i\mathcal{M}(p, q) = 2\pi\delta(p^+ - q^+) \bar{u}(q) \not{n} \int d^2x_t e^{-i(q_t - p_t) \cdot x_t} [V(x_t) - 1] u(p) \quad (8)$$

where the Wilson line  $V$  in the fundamental representation is defined as

$$V(x_t) \equiv \hat{P} \exp \left\{ ig \int_{-\infty}^{+\infty} dx^+ A_a^-(x^+, x_t) t_a \right\} \quad (9)$$

and describes propagating of a high energy quark in the background color field and multiply scattering from it at longitudinal coordinates  $x_i^+$  while staying at the same transverse coordinate  $x_t$ . This is the standard eikonal approximation. To get the scattering cross section one needs to multiply the amplitude  $\sim V$  by the complex conjugate amplitude  $\sim V^\dagger$  which gives the so-called  $T$  matrix, the quark anti-quark pair (called a dipole) scattering total cross section  $T$  (this is the imaginary part of the forward scattering amplitude for the quark anti-quark pair scattering from the background color field),

$$T(x_t, y_t) \equiv \frac{1}{N_c} < \text{Tr} [1 - V(x_t) V^\dagger(y_t)] >_\rho \quad (10)$$

where  $< >_\rho$  means one needs to average over all color charge configuration with some weight for  $\rho$ , usually taken to be a Gaussian. The result so far is a classical result (multiple scattering generalization of a tree level calculation). We now consider quantum corrections in the one-loop approximation.

## 2.2. Scattering in High Energy Limit: Quantum Corrections

In the collinear factorization approach one would look for quantum corrections to a tree level process which results in large logarithms of  $Q^2$ . Since here we are interested in the kinematics where logarithms of  $1/x$  are the largest we need to look for quantum corrections which results in large logarithms of  $1/x$  gluon radiation is ordered in the  $+$  momentum. This is the so-called Regge kinematics where one considers the high energy limit  $\sqrt{s} \rightarrow \infty$  but where the momentum exchanged  $\sqrt{t}$  is small but still large enough to be perturbative,  $s \gg t \gg \Lambda_{QCD}$  such that  $\alpha_s \log s/t \sim \alpha_s \log 1/x \sim 1$ .

One loop correction to the  $T$  matrix results in the so called BK [20,21] evolution equation (this is the large  $N_c$ , mean field approximation to the JIMWLK evolution equation [22–26]),

$$\frac{d}{d \log 1/x} T(x, x_t - y_t) = \frac{N_c \alpha_s}{2\pi^2} \int d^2 z_t \frac{(x_t - z_t) \cdot (y_t - z_t)}{(x_t - z_t)^2 (y_t - z_t)^2} \left[ T(x, x_t - z_t) + T(x, y_t - z_t) - T(x, x_t - y_t) - T(x, x_t - z_t) T(x, z_t - y_t) \right] \quad (11)$$

where  $x_t, y_t$  and  $z_t$  are two-dimensional coordinate vectors in position space. We have also assumed translational invariance on the transverse plane of the target proton/nucleus and (in case of BK equation) made a large  $N_c$  approximation. This equation allows a simple interpretation in terms of multiple scattering of a quark anti-quark system (hence the name dipole) on the target proton/nucleus. Initially the quark is at space coordinate  $x_t$  and the anti-quark is at coordinate  $y_t$ . Both quark and anti-quark are multiply scattering from the target as encoded in the Wilson line. To calculate a one-loop correction to this scattering one consider radiation of one gluon, either from the quark or the anti-quark which can then be either real (crossing the cut line) or virtual (ending on the same side of the cut). The radiated gluon is at transverse coordinate  $z_t$  and in the large  $N_c$  limit can be thought of as a new quark anti-quark (both at  $z_t$ ) system. The first term in (11) corresponds to the case when the original quark with the new anti-quark forming a dipole scatter from the target. The second term corresponds to the original anti-quark forming a new dipole system with the new quark and scattering on the target. The third term is the virtual correction where the radiated gluon does not cross the cut line and is not produced. The last term corresponds to both dipole multiply scattering from the target and is a new term which appears due to high gluon density effects in the target proton/nucleus. In the low gluon density regime one can disregard the last term (non-linear) in (11) which is essential in the high gluon density regime. We are then left with a linear equation known as the BFKL equation [27,28] which has been studied for a long time. It is known that the solution to BFKL equation has a power dependence on energy and grows without bound hence violating perturbative unitarity. The non-linear correction to BFKL in the high gluon regime restores perturbative unitarity such that scattering probability is never larger than one.

The BFKL equation can be understood as describing the energy dependence of cross sections due to exchange of Reggeized gluons in the  $t$  channel. This can be done by expanding the Wilson line to the first non-trivial power in the gluon field. The resulting equation is then for the energy dependence (evolution) of two-point function of gluon fields. This singlet state of two Reggeized gluons is known as a (hard or perturbative) Pomeron. It is also possible to study energy dependence of states of higher number of Reggeized gluons to which we turn now.

### 2.3. Scattering in High Energy Limit: JIMWLK Equation and Higher States of Reggeized Gluons

The BK equation is the large  $N_c$  approximation to the JIMWLK equation, which in its original formulation, describes the evolution of a high energy proton or nucleus wave function as one decreases  $x$ . It can be written as

$$\frac{d}{d \log 1/x} W[\rho] = \mathcal{H} W[\rho] \quad (12)$$

where the weight functional  $W[\rho]$  describes distribution of color charges  $\rho$  in the proton or nucleus wave function and  $\mathcal{H}$  is the Hamiltonian. This is then used to write down the evolution equation for any operator in the CGC effective theory, for example, the two-point function of Wilson lines as in (11). To be specific, for any operator  $O$  we have

$$\frac{d}{d \log 1/x} \langle O \rangle = \frac{1}{2} \left\langle \int d^2 x_t d^2 y_t \frac{\delta}{\delta \alpha_x^b} \eta_{xy}^{bd} \frac{\delta}{\delta \alpha_y^d} O \right\rangle, \quad (13)$$

where  $\alpha_x^a \equiv \int dx^+ A^{-,a}(x^+, x_t)$  and

$$\eta_{xy}^{bd} = \frac{1}{\pi} \int \frac{d^2 z_t}{(2\pi)^2} \frac{(x_t - z_t) \cdot (y_t - z_t)}{(x_t - z_t)^2 (y_t - z_t)^2} \left[ 1 + U_{x_t}^\dagger U_{y_t} - U_{x_t}^\dagger U_{z_t} - U_{z_t}^\dagger U_{y_t} \right]^{bd}. \quad (14)$$

and  $U$  is a Wilson line in the adjoint representation. In general higher correlators of Wilson lines appear in production cross sections, for example, the quadrupole defined as

$$Q(x_t, y_t, z_t, r_t) \equiv \frac{1}{N_c} \langle \text{Tr} V(x_t) V^\dagger(y_t) V(z_t) V^\dagger(r_t) \rangle_\rho, \quad (15)$$

which appears in the production cross section for dijets in Deep Inelastic Scattering as well as in high energy proton-nucleus collisions [11]. Using (13) for the quadrupole operator defined in (15) we get

$$\begin{aligned} \frac{d \langle Q(r, \bar{r}, \bar{s}, s) \rangle}{d \log 1/x} &= \frac{N_c \alpha_s}{(2\pi)^2} \int d^2 z \left\{ \left\langle \begin{aligned} &\left[ \frac{(r - \bar{r})^2}{(r - z)^2 (\bar{r} - z)^2} + \frac{(r - s)^2}{(r - z)^2 (s - z)^2} - \frac{(\bar{r} - s)^2}{(\bar{r} - z)^2 (s - z)^2} \right] Q(z, \bar{r}, \bar{s}, s) S(r, z) \\ &+ \left[ \frac{(r - \bar{r})^2}{(r - z)^2 (\bar{r} - z)^2} + \frac{(\bar{r} - \bar{s})^2}{(\bar{r} - z)^2 (\bar{s} - z)^2} - \frac{(r - \bar{s})^2}{(r - z)^2 (\bar{s} - z)^2} \right] Q(r, z, \bar{s}, s) S(z, \bar{r}) \\ &+ \left[ \frac{(\bar{r} - \bar{s})^2}{(\bar{r} - z)^2 (\bar{s} - z)^2} + \frac{(s - \bar{s})^2}{(s - z)^2 (\bar{s} - z)^2} - \frac{(\bar{r} - s)^2}{(s - z)^2 (\bar{r} - z)^2} \right] Q(r, \bar{r}, z, s) S(\bar{s}, z) \\ &+ \left[ \frac{(r - s)^2}{(r - z)^2 (s - z)^2} + \frac{(s - \bar{s})^2}{(s - z)^2 (\bar{s} - z)^2} - \frac{(r - \bar{s})^2}{(r - z)^2 (\bar{s} - z)^2} \right] Q(r, \bar{r}, \bar{s}, z) S(z, s) \\ &- \left[ \frac{(r - \bar{r})^2}{(r - z)^2 (\bar{r} - z)^2} + \frac{(s - \bar{s})^2}{(s - z)^2 (\bar{s} - z)^2} + \frac{(r - s)^2}{(r - z)^2 (s - z)^2} + \frac{(\bar{r} - \bar{s})^2}{(\bar{r} - z)^2 (\bar{s} - z)^2} \right] Q(r, \bar{r}, \bar{s}, s) \\ &- \left[ \frac{(r - s)^2}{(r - z)^2 (s - z)^2} + \frac{(\bar{r} - \bar{s})^2}{(\bar{r} - z)^2 (\bar{s} - z)^2} - \frac{(\bar{r} - s)^2}{(\bar{r} - z)^2 (s - z)^2} - \frac{(r - \bar{s})^2}{(r - z)^2 (\bar{s} - z)^2} \right] S(r, s) S(\bar{r}, \bar{s}) \\ &- \left[ \frac{(r - \bar{r})^2}{(r - z)^2 (\bar{r} - z)^2} + \frac{(s - \bar{s})^2}{(s - z)^2 (\bar{s} - z)^2} - \frac{(r - \bar{s})^2}{(r - z)^2 (\bar{s} - z)^2} - \frac{(\bar{r} - s)^2}{(\bar{r} - z)^2 (s - z)^2} \right] S(r, \bar{r}) S(\bar{s}, s) \end{aligned} \right\} \end{aligned} \quad (16)$$

where the  $S$  matrix is defined as

$$S(r, \bar{r}) \equiv \frac{1}{N_c} \text{tr} V_r V_{\bar{r}}^\dagger \quad (17)$$

so that  $S = 1 - T$  and all the coordinates appearing above are two-dimensional transverse coordinates. Clearly this is a complicated equation which can only be solved by approximate methods, the most common one being Gaussian approximation after which all operators can be written in terms of the dipole operator  $T$  or  $S$ . One can also study this equation numerically in various kinematics as in cite papers. Of particular interest is the low gluon density limit where all the Wilson lines can be expanded with only the first non-trivial term kept. It can be shown that the JIMWLK evolution equation for the quadrupole (15) reduces [29,30] to the known BJKP equation [31–34] for the energy dependence of a state of four Reggeized gluons given by

$$\begin{aligned} \frac{d}{d \log 1/x} \hat{T}_4(l_1, l_2, l_3, l_4) &= \frac{N_c \alpha_s}{\pi^2} \int d^2 p_t \left[ \frac{p_t^i}{p_t^2} - \frac{(p_t^i - l_1^i)}{(p_t + l_1)^2} \right] \cdot \left[ \frac{p_t^i}{p_t^2} - \frac{(p_t^i - l_2^i)}{(p_t + l_2)^2} \right] \hat{T}_4(p_t + l_1, l_2 - p_t, l_3, l_4) \\ &+ \dots \\ &- \frac{N_c \alpha_s}{(2\pi)^2} \int d^2 p_t \left[ \frac{l_1^2}{p_t^2 (l_1 - p_t)^2} + \{l_1 \rightarrow l_2, l_3, l_4\} \right] \hat{T}_4(l_1, l_2, l_3, l_4) \end{aligned} \quad (18)$$

where  $\hat{T}_4$  is the quadrupole operator with each Wilson line expanded to first order in the gluon field and written in terms of the color charge density  $\rho$  and  $\dots$  denotes a cyclic permutation of the external momenta. This equivalence between the BJKP equation for energy dependence of a state of  $n$



Reggeized gluons and the low gluon density limit of JIMWLK evolution equation is quite general and can be shown to be true for any  $n$ . Finally it also seems possible to go beyond small  $x$  approximation and include large  $x$  contributions which become dominant at high transverse momentum [35,36].

### 3. Summary

The JIMWLK evolution equation describes the energy (or equivalently  $x$ ) dependence of operators appearing in multi-particle production cross sections in high energy hadronic collisions. It reduces to the previously known evolution equations such as BFKL and BJKP in the low gluon density limit, where it can be understood to describe the energy dependence of states containing  $2n$  Reggeized gluons.

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