

Article

Gravitoelectromagnetic Knot Fields

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Abstract: We construct a class of knot solutions of the time-dependent gravitoelectromagnetic (GEM) equations in vacuum in the linearized gravity approximation by analogy with the Rañada–Hopf fields. For these solutions, the dual metric tensors of the bi-metric geometry of the gravitational vacuum with knot perturbations are given and the geodesic equation as a function of two complex parameters of the time-dependent GEM knots are calculated. By taking stationary potentials, which formally amount to particularizing to time-independent GEM equations, we obtain a set of stationary fields subjected to constraints from the time-dependent GEM knots. Finally, the Landau–Lifshitz pseudo-tensor and a scalar invariant of the static fields are computed.

Keywords: classical theories of gravity; gravitational knots; gravitational Hopfions; gravitoelectromagnetism



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1. Introduction

The gravitoelectromagnetism (GEM) formulation of gravity has proved to be a practical approach to studying gravitating systems and physical processes in the gravitational field. The GEM formalism based on the analogy at the classical level between gravity and electromagnetism has been formulated from different points of view: linearized gravity [1,2], Weyl tensor approach [3], tidal force tensor construction [4–6], irreducible tensor correspondence [7], Lagrangian formulation [8,9], teleparallel gravitoelectromagnetism [10–12].

A very large range of fundamental problems have been addressed in the GEM framework. The generalization of the GEM concepts to the non-linear electromagnetism was studied in [13], to the non-commutative geometry in [14], to higher dimensions in [15], to the teleparallel gravity in [10,11] and to the non-local gravity in [16]. By analogy with the quantization of the electromagnetic systems, several quantum aspects of the GEM were investigated in [17–20]. The gauge properties and the non-abelian extension of GEM formalism were discussed in [9,21]. The thermalization of the GEM systems and the research of several thermal quantum processes was performed in [22–26]. (For excellent reviews on the topic of the GEM formalism and on its applications see, e.g., [27–33]).

In most of these works, the GEM equations are postulated by analogy with Maxwell's equations in flat space-time, like in [1,2,8–12] and in many applications. Other works derive the GEM equations from the analogy between gravitational and electromagnetic phenomena, like the covariant forces exerted on particles and charges, e.g., [4]. The main difference between the two approaches is that the postulated GEM equations allow time-dependent gravitational fields while the derived GEM equations from covariant interactions restrict the GEM analogy to static fields (see also [34–36]). At present, there is no complete resolution of the tension between the formulations of GEM with time-dependent and stationary GEM potentials.

In this paper, we do not aim at contributing to the above discussion. Our purpose is to look for knot solutions in gravity similar to the ones in electromagnetism. To this end, the

linearized gravity is the best starting point since knot solutions are known for Maxwell's equations. Therefore, for the discussion of GEM knot fields the most productive analogy between electromagnetism and GEM is in terms of time-dependent potentials for which the GEM equations are postulated analogously to Maxwell's equations. Formally, the GEM equations that describe the stationary settings in the linearized limit are a particular case of the GEM equations for time-dependent potentials. Thus, we can obtain particular solutions for static potentials from the more general time-dependent solutions by reducing to static fields.

By following the above procedure, we propose in this paper a class of knot solutions of the time-dependent GEM equations in vacuum, in the linearized gravity approach. These are new solutions parameterized by a pair of complex scalar maps ϕ and θ from $\mathbb{R} \times \mathbb{R}^3$ to $\mathbb{C}$ which describe the knot structure of the corresponding gravitomagnetic and gravitoelectric fields. The existence of GEM knot fields guaranteed by the formal equivalence between the GEM equations and the Maxwell equations for which knot solutions are known to exist for some time [37–39]. In the context of GEM, the knot fields bring novelty to the body of the known solutions of the linearly perturbed Einstein equations, in the sense that they describe a topologically non-trivial gravitational vacuum with the knot topology controlled by a set of complex scalar maps.

A general major motivation to study the topological abelian fields is to understand the relationship between their physical and mathematical properties. This is an active area of research in classical electromagnetism with significant progress made recently in this direction by the discovery and the analysis of a large set of new solutions of the field equations that generalize the electromagnetic Rañada fields in the linear [40–59] and non-linear electrodynamics [60–63]. From the mathematical point of view, several mathematical structures related to the topologically non-trivial fields have been explored in the recent literature among which are the twistors [47], rational functions [56,59], fibre bundles [64], space-time foliations [65] and generalized Finsler geometries [66]. The interaction of electromagnetic knots with the matter for classical and quantum particles is discussed. Refs. [66–71] (for recent reviews on many of these results, see, e.g., [72–75]). By generalizing the knot solutions to the gravitating system, one could largely explore the knowledge obtained so far in field theory to better understand the gravitational field's structure at least in the weak field approximation.

This work's primary goal is to give results of the generalization of the knot fields to gravity. Here, we focus on the linearized gravity away from sources in the time-dependent GEM formulation, which originated in the works of Thirring and Lens [67,68]. The existence of the knot fields for gravitational and gravitating electromagnetic fields has been discussed in other research papers [36,76–83]. In particular, the existence of the Hopf–Rañada solutions of the spin 2 field theory and their properties in the GEM formulation in terms of Weyl tensor were analyzed in [76–78,82,84]. In the time-dependent GEM linearized gravity, the discussion of the Hopfions in the gravity/fluid correspondence was recently approached in [36]. For another interesting approach to the knot structure of vacuum see [83].

Since the GEM potentials are identified with some components of the linearized classical perturbation field in the harmonic gauge, the gravitational knots are interesting from mathematical and physical points of view¹. The analogy between gravity and electromagnetism is used to construct the knotted field. However, the GEM fields' gravitational nature imposes typical constraints on the structure of these knots that do not have an electromagnetic equivalent. The existence of knots in the fluctuating gravitational vacuum induces a linearly perturbed space-time bi-metric geometrical structure. The two metrics are related to each other by the GEM equations' duality symmetry under the swap of the gravitoelectric and gravitomagnetic fields and represent a redundant description of the space-time geometry. While this is a general property of the GEM induced geometry away

¹ For a critical discussion of the time-dependent GEM equations in the linearized limit see [4,36].

from sources, in the knot fields, the relationship between the two metrics imposes a set of non-linear constraints on scalar complex maps that parametrize the gravitational knot field. This paper constructs the GEM knot fields and starts the analysis of their geometrical and physical properties.

The paper is organized as follows. In Section 2, we give a concise review of the GEM formulation's fundamental relations in the linearized gravity following the reviews cited above, mainly [29,31,32]. In Section 3, we construct the time-dependent GEM knot fields in vacuum by analogy with the corresponding solutions to the Maxwell equations. A novelty here is the analysis of the time-dependent GEM potentials in the polar representation of the complex scalars. Furthermore, we give the explicit form of the dual metric tensors of the perturbed space-time in terms of the scalar fields and calculate the geodesic equations. Since all the formulas are symmetric under the duality transformation of fields, we present our relation only for one of the scalars to avoid cluttering the formulas. By particularizing the time-dependent knot solutions to stationary potentials, which correspond to the time-independent GEM equations, we obtain a class of stationary magnetic fields for the simplest GEM knot parametrization compatible with the static field condition. The electric fields vanish in this case. More generally, electric fields are also allowed as long as their parameter field obeys a constraint obtained from static field condition. Also, the electric-magnetic duality is broken in this case. The static field condition introduces a constraint on the modulus and phase of the scalar field. The description in terms of a second scalar field, used in the time-dependent settings, is no longer available. Next, we calculate the Landau–Lifshitz pseudo-tensor for the static GEM fields GEM knots and compute the scalar invariant of these fields. In the last section, we discuss our results and give some details on the GEM constraint for the static potentials. Furthermore, we present a shortlist of interesting topics on this subject. Throughout this paper, the Minkowski metric with signature $(+, -, -, -)$ and the natural units $8\pi G = c = 1$ are used.

2. GEM Formalism of Linearized Gravity

In this section, we briefly review all the necessary relations of the GEM formalism following mainly [29,31,32] and we establish our notations.

The GEM is a formulation of the General Relativity in the weak field approximation in which an analogy between the Maxwell equations and the gauge fixed linearized Einstein equation is established. The main equations of the GEM are established by starting with small perturbations of the flat space-time metric defined in the standard way

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \quad (1)$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $|h_{\mu\nu}| \ll 1$. In the decomposition given by Equation (1), the gravitational field is represented by a spin two tensor field $h_{\mu\nu}$ (classical graviton field) in the flat space-time background. The dynamics of the graviton field are obtained from the linearized Einstein tensor. With the purpose to establish our notation, let us briefly review the relevant relations. The linearized Christoffel symbols, the Ricci tensor and the Ricci scalar have the following form [85]

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\eta^{\alpha\rho}(\partial_\nu h_{\mu\rho} + \partial_\mu h_{\nu\rho} - \partial_\rho h_{\mu\nu}), \quad (2)$$

$$R_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}(\partial_\rho\partial_\nu h_{\sigma\mu} + \partial_\rho\partial_\mu h_{\sigma\nu} - \partial_\rho\partial_\sigma h_{\mu\nu} - \partial_\mu\partial_\nu h_{\rho\sigma}), \quad (3)$$

$$R = \eta^{\mu\rho}\eta^{\nu\sigma}(\partial_\nu\partial_\mu h_{\rho\sigma} - \partial_\mu\partial_\rho h_{\nu\sigma}). \quad (4)$$

The linearized Einstein tensor results from combining the equations above and takes the following standard form

$$G_{\mu\nu} = \frac{1}{2}(\partial_\alpha\partial_\nu h^\alpha_\mu + \partial_\alpha\partial_\mu h^\alpha_\nu - \partial^2 h_{\mu\nu} - \partial_\nu\partial_\mu h - \eta_{\mu\nu}\partial_\beta\partial_\alpha h^{\alpha\beta} + \eta_{\mu\nu}\partial^2 h), \quad (5)$$

where $\partial^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu$ and $h = \eta^{\mu\nu} h_{\mu\nu}$. The linearized Einstein equations are simplified by introducing the following trace-reversed metric perturbation

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h. \quad (6)$$

Then the linearized Einstein equations take the following form

$$\partial_\alpha \partial_\nu \tilde{h}^\alpha{}_\mu + \partial_\alpha \partial_\mu \tilde{h}^\alpha{}_\nu - \partial^2 \tilde{h}_{\mu\nu} - \eta_{\mu\nu} \partial_\beta \partial_\alpha \tilde{h}^{\alpha\beta} = T_{\mu\nu}, \quad (7)$$

where $T_{\mu\nu}$ is the matter energy-momentum tensor. As it is well known, the free graviton has a gauge symmetry given by the infinitesimal transformations

$$\delta_\xi h_\mu = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad (8)$$

where ξ is an infinitesimal arbitrary smooth vector field on flat space-time. By using this symmetry and fixing the gauge, the Einstein equations are simplified further. For example, in the trace-reversed harmonic gauge defined by the following equation

$$\tilde{h}^{\mu\nu}{}_{,\nu} = 0, \quad (9)$$

the Einstein equations take the following simpler form

$$\partial^2 \tilde{h}_{\mu\nu} = -2T_{\mu\nu}. \quad (10)$$

The analogy with the Maxwell equations which is at the core of the GEM approach to the linearized gravity is made apparent by the following ansatz for the trace-reversed field $\tilde{h}_{\mu\nu}$ [2,69]

$$\tilde{h}_{0\mu} = 4A_\mu, \quad \tilde{h}_{ij} = 0. \quad (11)$$

The approximation above of the spatial components of $\tilde{h}_{\mu\nu}$ is justified by the dependence of $\tilde{h}_{ij}$ on c at order $O(c^{-4})$ while $\tilde{h}_{0\mu}$ runs as $O(c^{-2})$ ². The GEM potentials are written in the three dimensional form by identifying $A^\mu = (\Phi, \mathbf{A}/2)$ ³. The harmonic gauge condition implies the following relations among the components of the GEM potential

$$A^\mu{}_{,\mu} = 0, \quad \tilde{h}^{ij}{}_{,j} + A^i{}_{,0} = 0. \quad (12)$$

The first equation above corresponds to the Lorentz gauge in the covariant form. The second equation is a typical GEM constraint and it has its origin in the gravitational structure that underlies the potentials. It implies that the spatial components of the potential $\mathbf{A}$ are independent of time for the ansatz given by Equations (11) above. Due to this limitation it imposes on the type of solutions of the GEM equations and to the fact that it contains terms of order $O(c^{-4})$, the second constraint from (12) is sometimes ignored [2,69]. This point of view is adopted here and the properties of the general solutions are analyzed. However, the consequences of the GEM constraints are briefly discussed in the last section.

By analogy with the classical electromagnetism, the GEM electric and magnetic fields are introduced by the following combinations of derivatives of potentials

$$\mathbf{E} = -\nabla\Phi - \partial_0 \left(\frac{\mathbf{A}}{2} \right) \quad (13)$$

$$\mathbf{B} = \nabla \times \left(\frac{\mathbf{A}}{2} \right). \quad (14)$$

² In the references [4,32], it was assumed that $\tilde{h}_{ij}$ take a different form from the one considered here, but this we will not discuss this hypothesis.

³ The numerical factors result from the identification of the GEM potentials with the corresponding components of the tensor $\tilde{h}_{\mu\nu}$ such that the equations obtained for A^μ have the form of Maxwell's equations. For different numerical factors, the analogy between the electric permittivity of vacuum and $\epsilon_G = 1/4\pi G$ is lost.

It follows that the gravitomagnetic field and the gravitoelectric field satisfy the Gauss and the Faraday–Lenz laws, respectively,

$$\partial_i B_i = 0, \quad \varepsilon_{ijk} \partial_j E_k = -\partial_0 B_i. \quad (15)$$

As is the case of the electromagnetism, the above equations are constraints on structure of the fields $\mathbf{E}$ and $\mathbf{B}$ that must be obeyed at all times. The dynamics of $\mathbf{E}$ and $\mathbf{B}$ can be read off of the linearized Einstein equations (10). In the three dimensional formulation, these equations have the following decomposition

$$\partial^2 A^0 = \partial^2 \Phi = \frac{1}{2} T^{00}, \quad (16)$$

$$\partial^2 A^i = \frac{1}{2} \partial^2 A^i = \frac{1}{2} T^{0i}. \quad (17)$$

Equations (16) and (17) are the Gauss law for the gravitoelectric field and the Maxwell–Ampère law, respectively,

$$\partial_i E_i = \frac{1}{2} T_{00}, \quad (18)$$

$$\varepsilon_{ijk} \partial_j B_k = \partial_0 E_i + \frac{1}{2} T_{0i}. \quad (19)$$

Using the electrodynamics tools to study the GEM fields’ background, we gain valuable insight into various gravitating physical systems. The background has a pseudo-Riemann geometry, given by the line element of the form

$$ds^2 = (1 + 2\Phi)(dx^0)^2 - 4\delta_{ij} A^i dx^0 dx^j - (1 - 2\Phi)\delta_{ij} dx^i dx^j. \quad (20)$$

The dynamics of the massive particles in the GEM background results from the classical geodesic equation

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} = 0. \quad (21)$$

Note that the linearized Christoffel symbols calculated from the metric (20) should be employed in the equation in the (21) above. For further details, we refer the reader to [31].

3. General Knot Fields in Linearized GEM

In this section, are constructed a class of knot type solutions of the in-vacuum GEM equations and the space-time geometry generated by the GEM knots is also discussed. These form a class of GEM fields parameterized by two scalar fields that satisfy some differentiability and regularity conditions, introduced by Rañada in the context of the electromagnetism in [38,39]. A different set of knotted GEM fields was obtained in the GEM formulation in terms of Weyl tensor in [76–78].

3.1. Knot Fields in Linearized GEM

In order to explore the knot like solutions in the GEM formalism in the linearized gravity reviewed in the previous section, we need to consider the GEM field away from its sources for which the field equations are (15), (18) and (19) in vacuum. In this case, the knot-like solutions are confirmed by the analogy with the corresponding Maxwell equations that admit topologically non-trivial solutions with knot structure, which were given for the first time in [37–39].

In the absence of the matter fields, the set of GEM Equations (15), (18) and (19) is invariant under the gravitoelectromagnetic duality transformation $E_i \rightarrow B_i$ and $B_i \rightarrow -E_i$. That allows one to describe the fields $\mathbf{E}$ and $\mathbf{B}$ symmetrically in terms of two potentials A and C which help visualize the line structure of each field. Since this is a redundant description of the GEM field, there is a relationship between A and C . Let us construct the general knot solutions of the GEM equations.

Let us start by taking two smooth complex scalar maps $\phi, \theta : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$ whose level equations $\phi(x) = \phi_0 = \text{constant}$ and $\theta(x) = \theta_0 = \text{constant}$ are associated to gravitomagnetic and gravitoelectric field lines ⁴. The functions ϕ and θ are assumed to be regular at spatial infinity and they define the following pair of potentials [79]

$$A_\mu = \frac{\sqrt{a}}{4\pi i} \left(\frac{\bar{\phi} \partial_\mu \phi - \phi \partial_\mu \bar{\phi}}{1 + |\phi|^2} \right), \quad (22)$$

$$C_\mu = \frac{\sqrt{a}}{4\pi i} \left(\frac{\bar{\theta} \partial_\mu \theta - \theta \partial_\mu \bar{\theta}}{1 + |\theta|^2} \right), \quad (23)$$

where a is a dimensionful constant. Since the two potentials must describe the same fields $\mathbf{E}$ and $\mathbf{B}$, there is the relationship between A and C that has the following form in terms of scalar fields

$$(1 + |\phi|^2)^{-2} \varepsilon_{ijk} \partial_j \phi \partial_k \bar{\phi} = (1 + |\theta|^2)^{-2} (\partial_0 \bar{\theta} \partial_i \theta - \partial_0 \theta \partial_i \bar{\theta}), \quad (24)$$

$$(1 + |\theta|^2)^{-2} \varepsilon_{ijk} \partial_j \theta \partial_k \bar{\theta} = (1 + |\phi|^2)^{-2} (\partial_0 \bar{\phi} \partial_i \phi - \partial_0 \phi \partial_i \bar{\phi}). \quad (25)$$

The potentials defined by Equations (22) and (23) generate the following knotted fields that is in vacuum solutions to the GEM equations

$$E_i = \frac{\sqrt{a}}{2\pi i} \frac{\varepsilon_{ijk} \partial_j \theta \partial_k \bar{\theta}}{(1 + |\theta|^2)^2}, \quad (26)$$

$$B_i = \frac{\sqrt{a}}{2\pi i} \frac{\varepsilon_{ijk} \partial_j \phi \partial_k \bar{\phi}}{(1 + |\phi|^2)^2}. \quad (27)$$

The relations (24) and (25) show that a single potential is sufficient to describe the gravitomagnetic and the gravitoelectric field. For example, by employing only ϕ , we have the following representation of $\mathbf{E}$ and $\mathbf{B}$

$$E_i = \frac{\sqrt{a}}{2\pi i} \frac{\partial_0 \bar{\phi} \partial_i \phi - \partial_0 \phi \partial_i \bar{\phi}}{(1 + |\phi|^2)^2}, \quad (28)$$

$$B_i = \frac{\sqrt{a}}{2\pi i} \frac{\varepsilon_{ijk} \partial_j \phi \partial_k \bar{\phi}}{(1 + |\phi|^2)^2}. \quad (29)$$

The two potentials help to visualize the field lines of each field and display the duality symmetry of the GEM equations. The above solutions form a class of knot type GEM fields because they are defined in rather general complex functions. By taking ϕ and θ of particular forms, sets of solutions with specific features are constructed (see for a recent review [72]).

In order to analyze the properties of the GEM knot fields and the induced space-time geometry, it is convenient to pass to the following polar representation of the scalar fields

$$\phi(x) = \rho(x) e^{i\alpha(x)}, \quad \bar{\phi}(x) = \rho(x) e^{-i\alpha(x)}, \quad (30)$$

$$\theta(x) = \lambda(x) e^{i\beta(x)}, \quad \bar{\theta}(x) = \lambda(x) e^{-i\beta(x)}, \quad (31)$$

⁴ The Hopf knots are given by maps from S^3 to S^2 . We will comment on the particularities of this case in the last section.

where ρ and λ are positively defined functions everywhere and obey regularity conditions at infinity. In this representation, the GEM potentials take the following form

$$A_\mu = \frac{\sqrt{a}}{2\pi} \frac{\rho^2 \partial_\mu \alpha}{(1 + \rho^2)}, \quad (32)$$

$$C_\mu = \frac{\sqrt{a}}{2\pi} \frac{\lambda^2 \partial_\mu \beta}{(1 + \lambda^2)}. \quad (33)$$

From the relations (32) and (33) we derive the gravitoelectric and gravitomagnetic fields in the following form

$$E_i = -\frac{\sqrt{a}}{\pi} \left[\frac{\rho(\partial_0 \rho \partial_i \alpha + \partial_i \rho \partial_0 \alpha)}{(1 + \rho^2)^2} + \frac{\rho^2 \partial_0 \rho \partial_i \alpha}{1 + \rho^2} \right], \quad (34)$$

$$B_i = \frac{\sqrt{a}}{\pi} \varepsilon_{ijk} \frac{\rho \partial_j \rho \partial_k \alpha}{(1 + \rho^2)^2}. \quad (35)$$

The relations (24) and (25) between the complex fields take the following form in polar representation

$$\frac{\rho}{(1 + \rho^2)^2} \varepsilon_{ijk} (\partial_j \rho \partial_k \alpha - \partial_j \alpha \partial_k \rho) = \frac{2\lambda}{(1 + \lambda^2)^2} (\partial_i \lambda \partial_0 \beta - \partial_0 \lambda \partial_i \beta), \quad (36)$$

$$\frac{\lambda}{(1 + \lambda^2)^2} \varepsilon_{ijk} (\partial_j \lambda \partial_k \beta - \partial_j \beta \partial_k \lambda) = \frac{2\rho}{(1 + \rho^2)^2} (\partial_i \rho \partial_0 \alpha - \partial_0 \rho \partial_i \alpha). \quad (37)$$

Next, a few comments are required. As mentioned above, the fact that the fields defined by Equations (26) and (27) are solutions of the in-vacuum GEM equations is a consequence of the formal equivalence between the Maxwell and the GEM equations, respectively. However, there is an important particularity of the gravitoelectric and gravitomagnetic fields which is a result of their gravitational origin, namely, the constraints on $\mathbf{E}$ and $\mathbf{B}$ given by the equations from (12) above. Of particular interest here is the second GEM constraint on which we will comment in the last section. Furthermore, it is known that in the electromagnetism, new solutions are generated from a given knot field by using the invariance of the Maxwell equations under Lorentz and conformal symmetries or by working on specific space-time manifolds. These methods might not be applicable to GEM knots due to the underlying graviton structure [56,59,72].

3.2. Space-Time Geometry from GEM Knots

Interesting information about the space-time geometry generated by GEM knots away from the matter distribution is obtained by calculating the relevant tensorial objects. Since the geometry is of pseudo-Riemann type, its fundamental tensors are obtained from the metric tensor given by the line element from Equation (20) above.

One important characteristic of the knot solutions is that they generate a bi-metric geometry on space-time with the two metrics depending on each other according to the relations (24) and (25). As we have seen before, this is a consequence of the gravitoelectromagnetic duality of the vacuum equations. The bi-metric geometry has one metric generated by the field associated with the potential A and the second (dual) metric associ-

ated with the potential C. The dual metric is obtained by replacing A by C in Equation (20). Explicitly, we have the following metric tensors

$$g_{\mu\nu} = \begin{bmatrix} 1 + \frac{\sqrt{a}}{\pi} \frac{\rho^2 \partial_0 \alpha}{1+\rho^2} & -\frac{4\sqrt{a}}{\pi} \frac{\rho^2 \partial_1 \alpha}{1+\rho^2} & -\frac{4\sqrt{a}}{\pi} \frac{\rho^2 \partial_2 \alpha}{1+\rho^2} & -\frac{4\sqrt{a}}{\pi} \frac{\rho^2 \partial_3 \alpha}{1+\rho^2} \\ -\frac{4\sqrt{a}}{\pi} \frac{\rho^2 \partial_1 \alpha}{1+\rho^2} & -1 + \frac{\sqrt{a}}{\pi} \frac{\rho^2 \partial_0 \alpha}{1+\rho^2} & 0 & 0 \\ -\frac{4\sqrt{a}}{\pi} \frac{\rho^2 \partial_2 \alpha}{1+\rho^2} & 0 & -1 + \frac{\sqrt{a}}{\pi} \frac{\rho^2 \partial_0 \alpha}{1+\rho^2} & 0 \\ -\frac{4\sqrt{a}}{\pi} \frac{\rho^2 \partial_3 \alpha}{1+\rho^2} & 0 & 0 & -1 + \frac{\sqrt{a}}{\pi} \frac{\rho^2 \partial_0 \alpha}{1+\rho^2} \end{bmatrix}, \quad (38)$$

$$f_{\mu\nu} = \begin{bmatrix} 1 + \frac{\sqrt{a}}{\pi} \frac{\lambda^2 \partial_0 \beta}{1+\lambda^2} & -\frac{4\sqrt{a}}{\pi} \frac{\lambda^2 \partial_1 \beta}{1+\lambda^2} & -\frac{4\sqrt{a}}{\pi} \frac{\lambda^2 \partial_2 \beta}{1+\lambda^2} & -\frac{4\sqrt{a}}{\pi} \frac{\lambda^2 \partial_3 \beta}{1+\lambda^2} \\ -\frac{4\sqrt{a}}{\pi} \frac{\lambda^2 \partial_1 \beta}{1+\lambda^2} & -1 + \frac{\sqrt{a}}{\pi} \frac{\lambda^2 \partial_0 \beta}{1+\lambda^2} & 0 & 0 \\ -\frac{4\sqrt{a}}{\pi} \frac{\lambda^2 \partial_2 \beta}{1+\lambda^2} & 0 & -1 + \frac{\sqrt{a}}{\pi} \frac{\lambda^2 \partial_0 \beta}{1+\lambda^2} & 0 \\ -\frac{4\sqrt{a}}{\pi} \frac{\lambda^2 \partial_3 \beta}{1+\lambda^2} & 0 & 0 & -1 + \frac{\sqrt{a}}{\pi} \frac{\lambda^2 \partial_0 \beta}{1+\lambda^2} \end{bmatrix}. \quad (39)$$

The relationship between the two dual geometries is an open problem. The main difficulty in determining the explicit relations between geometrical objects, e.g., the two metric tensors, resides in the non-linearity of Equations (24) and (25) that makes it hard from the analytical point of view to define a duality map between the two geometries. Indeed, as Equations (38) and (39) show, the scalar functions enter non-linearly in the metric tensors $g_{\mu\nu}$ and $f_{\mu\nu}$. The main difficulty is to solve Equations (24) and (25) and to find out an explicit relationship between $g_{\mu\nu}$ and $f_{\mu\nu}$. The same problem occurs in the Rañada construction.

Let us focus on the geodesic equation in the geometry determined by the metric tensor $g_{\mu\nu}$. The dual geometry calculations follow precisely the same line, and avoid cluttering formulas will not be displayed here. In determining the geodesic equation in the GEM knot background, are needed the Christoffel symbols. They result from Equations (2) and (32) above. The following expressions result after applying some straightforward algebra

$$\Gamma^0_{00} = \frac{\sqrt{a}}{2\pi} \left[\frac{2\rho \partial_0 \rho \partial_0 \alpha}{(1+\rho^2)^2} + \frac{\rho^2 \partial_0^2 \alpha}{1+\rho^2} \right], \quad (40)$$

$$\Gamma^0_{0i} = \frac{\sqrt{a}}{2\pi} \left[\frac{2\rho \partial_0 \rho \partial_i \alpha}{(1+\rho^2)^2} + \frac{\rho^2 \partial_0 \partial_i \alpha}{1+\rho^2} \right], \quad (41)$$

$$\Gamma^0_{ij} = \frac{\sqrt{a}}{\pi} \left[\frac{\rho (\partial_i \rho \partial_j \alpha + \partial_j \rho \partial_i \alpha)}{(1+\rho^2)^2} + \frac{\rho^2 \partial_i \partial_j \alpha}{1+\rho^2} \right], \quad (42)$$

$$\Gamma^i_{00} = \frac{\sqrt{a}}{2\pi} \delta^{ik} \left[\frac{2\rho (\partial_k \rho \partial_0 \alpha - 4\partial_0 \rho \partial_k \alpha)}{(1+\rho^2)^2} - \frac{3\rho^2 \partial_k \partial_0 \alpha}{1+\rho^2} \right], \quad (43)$$

$$\Gamma^i_{0j} = -\frac{\sqrt{a}}{\pi} \delta^{ik} \left[\frac{2\rho (\partial_k \rho \partial_j \alpha - \partial_j \rho \partial_k \alpha)}{(1+\rho^2)^2} \right] - \frac{\sqrt{a}}{2\pi} \delta^i_j \left[\frac{2\rho \partial_0 \rho \partial_0 \alpha}{(1+\rho^2)^2} + \frac{\rho^2 \partial_0^2 \alpha}{1+\rho^2} \right], \quad (44)$$

$$\Gamma^i_{jk} = \frac{\sqrt{a}}{2\pi} \delta^{im} \left[\frac{2\rho (\delta_{jk} \partial_m \rho \partial_0 \alpha - \delta_{jm} \partial_k \rho \partial_0 \alpha)}{(1+\rho^2)^2} + \frac{\rho^2 (\delta_{jk} \partial_m \partial_0 \alpha - \delta_{jm} \partial_k \partial_0 \alpha)}{1+\rho^2} \right] - \frac{\sqrt{a}}{2\pi} \delta^{im} \delta_{km} \left[\frac{2\rho \partial_j \rho \partial_0 \alpha}{(1+\rho^2)^2} + \frac{\rho^2 \partial_j \partial_0 \alpha}{1+\rho^2} \right]. \quad (45)$$

The geodesic equation can be easily obtained from the above equations. For completeness, we give it here in the component form

$$\ddot{x}^0 + \frac{\sqrt{a}}{2\pi} \left[\frac{2\rho\partial_0\rho\partial_0\alpha}{(1+\rho^2)^2} + \frac{\rho^2\partial_0\partial_0\alpha}{1+\rho^2} \right] \dot{x}^0\dot{x}^0 + \frac{\sqrt{a}}{\pi} \left[\frac{2\rho\partial_0\rho\partial_i\alpha}{(1+\rho^2)^2} + \frac{\rho^2\partial_0\partial_i\alpha}{1+\rho^2} \right] \dot{x}^0\dot{x}^i + \frac{\sqrt{a}}{2\pi} \left[\frac{2\rho(\partial_i\rho\partial_j\alpha + \partial_j\rho\partial_i\alpha)}{(1+\rho^2)^2} + \frac{\rho^2(\partial_i\partial_j\alpha + \partial_j\partial_i\alpha)}{1+\rho^2} \right] \dot{x}^i\dot{x}^j = 0, \quad (46)$$

$$\begin{aligned} \ddot{x}^i + \frac{\sqrt{a}}{2\pi} \delta^{ik} \left[\frac{2\rho(\partial_k\rho\partial_0\alpha - 4\partial_0\rho\partial_k\alpha)}{(1+\rho^2)^2} + \frac{\rho^2(\partial_k\partial_0\alpha - 4\partial_0\partial_k\alpha)}{1+\rho^2} \right] \dot{x}^0\dot{x}^0 \\ - 2\frac{\sqrt{a}}{\pi} \delta^{ik} \left[\frac{2\rho(\partial_k\rho\partial_j\alpha - \partial_j\rho\partial_k\alpha)}{(1+\rho^2)^2} + \frac{\rho^2(\partial_k\partial_j\alpha - \partial_j\partial_k\alpha)}{1+\rho^2} \right] \dot{x}^0\dot{x}^j \\ - \frac{\sqrt{a}}{\pi} \delta^i_j \left[\frac{2\rho\partial_0\rho\partial_0\alpha}{(1+\rho^2)^2} + \frac{\rho^2\partial_0\partial_0\alpha}{1+\rho^2} \right] \dot{x}^0\dot{x}^j \\ + \frac{\sqrt{a}}{2\pi} \delta^{im} \left[\frac{2\rho(\delta_{jk}\partial_m\rho\partial_0\alpha - \delta_{jm}\partial_k\rho\partial_0\alpha)}{(1+\rho^2)^2} + \frac{\rho^2(\delta_{jk}\partial_m\partial_0\alpha - \delta_{jm}\partial_k\partial_0\alpha)}{1+\rho^2} \right] \dot{x}^j\dot{x}^k \\ - \frac{\sqrt{a}}{2\pi} \delta^{im} \delta_{km} \left[\frac{2\rho\partial_j\rho\partial_0\alpha}{(1+\rho^2)^2} + \frac{\rho^2\partial_j\partial_0\alpha}{1+\rho^2} \right] \dot{x}^j\dot{x}^k = 0. \end{aligned} \quad (47)$$

Equations (46) and (47) describe the dynamics of a test particle of given mass in the GEM knot background.

3.3. Knot Solutions in Static Gravitational Field

As mentioned in the introduction, there are several approaches to GEM in which the potentials are considered static from the very beginning. In the linearized limit, the GEM equations for static potentials can be formally obtained from the time-dependent equations by setting $\partial_0 A_i = 0$. Then solutions of the stationary GEM equations can be obtained from the time-dependent knot fields by substituting the stationary potentials.

Let us discuss the properties of the stationary solutions for the most general case $\partial_0 A_\mu = 0$. (The particular case of this which is studied in the literature is $\partial_0 A_i = 0$.) By inspecting Equations (22) and (23), we see that a scalar field with the property $\partial_0 \phi = 0$ generates a stationary potential A_μ . On the other hand, the electric-magnetic duality of the Equations (15), (18) and (19) is broken. As a consequence, the description of the gravitoelectric and gravitomagnetic fields in terms of two potentials A and C is no longer valid, since the fields are no longer mapped into each other. This implies that one should use a single scalar field which we chose to be ϕ for both E and B fields. Thus, the static solution is described by the Equations (28) and (29) written for static $\phi(\mathbf{x})$, that is

$$\mathbf{E}(\mathbf{x}) = 0, \quad \mathbf{B}(\mathbf{x}) = \frac{\sqrt{a}}{2\pi i} \frac{\nabla\phi(\mathbf{x}) \times \nabla\bar{\phi}(\mathbf{x})}{(1+|\phi(\mathbf{x})|^2)^2}. \quad (48)$$

Note that the static fields obtained from the knots solution are purely gravitomagnetic if the static potentials A_μ have a trivial structure.

In the most general case when $\partial_0 A_\mu = 0$ and $\partial_0 \phi \neq 0$, there are static gravitoelectric and gravitomagnetic solutions of the form given by the Equations (28) and (29) if the modulus and the phase of ϕ satisfy the following equation

$$2\partial_0\rho\partial_\mu\rho + \rho\partial_\mu\partial_0\rho + \rho^3\partial_\mu\partial_0\alpha = 0. \quad (49)$$

Then it is easy to see that the geodesic equation takes the following form

$$\ddot{x}^0 + \frac{\sqrt{a}}{2\pi} \left[\frac{2\rho(\partial_i \rho \partial_j \alpha + \partial_j \rho \partial_i \alpha)}{(1 + \rho^2)^2} + \frac{\rho^2(\partial_i \partial_j \alpha + \partial_j \partial_i \alpha)}{1 + \rho^2} \right] \dot{x}^i \dot{x}^j = 0, \quad (50)$$

$$\ddot{x}^i - 2 \frac{\sqrt{a}}{\pi} \delta^{ik} \left[\frac{2\rho(\partial_k \rho \partial_j \alpha - \partial_j \rho \partial_k \alpha)}{(1 + \rho^2)^2} + \frac{\rho^2(\partial_k \partial_j \alpha - \partial_j \partial_k \alpha)}{1 + \rho^2} \right] \dot{x}^0 \dot{x}^j = 0. \quad (51)$$

In the case $\partial_0 A_i = 0$, only the spatial derivatives $\partial_i \alpha$ enter the Equations (49). The fields **E** and **B** and the geodesics have the general form given in the previous section but with the real functions ρ and $\partial_i \alpha$ related by (49).

It is important to note that while particular solutions for static potential backgrounds can be formally derived from the knot fields, the interpretation of these solutions in terms of knots is not obvious due to the constraints (49) above.

3.4. Energy-Momentum Pseudo-Tensor and Scalar Invariant of Static Fields

The physical and geometrical properties of the GEM knot fields are described in terms of several tensor and scalar objects. One important quantity that contains information similar to the Maxwell tensor and the Poynting vector of the electromagnetic field is the Landau–Lifshitz pseudo-tensor. The general form of it in the GEM formalism was given in [29]. Here, we will discuss its form for static GEM knots with $\partial_0 A_\mu = 0$. After some algebra, we obtain the following relations for the Landau–Lifshitz tensor

$$t_{00} = -\frac{a}{(4\pi)^2(1 + \rho^2)^4} \left[\frac{4\rho^2}{(1 + \rho^2)^4} (\partial_{(i} \rho \partial_{j)} \alpha) (\partial^{(i} \rho \partial^{j)} \alpha) + \frac{8\rho^3}{(1 + \rho^2)^3} \partial_{(i} \rho \partial_{j)} \alpha \partial^i \partial^j \alpha \right. \\ \left. + \frac{4\rho^2}{(1 + \rho^2)^2} \partial_i \partial_j \alpha \partial^i \partial^j \alpha \right], \quad (52)$$

$$t_{0i} = 0, \quad (53)$$

$$t_{ij} = \frac{a\rho^4}{\pi^2(1 + \rho^2)^6} \left[\varepsilon_{imn} \varepsilon_{jrs} \partial^m \rho \partial^r \rho \partial^n \alpha \partial^s \alpha + \delta_{ij} (\partial^m \rho \partial_m \rho) (\partial^n \alpha \partial_n \alpha) - (\partial^m \rho \partial_m \alpha)^2 \right]. \quad (54)$$

From the above equations, we see that the GEM Poynting vector $S_i = \epsilon_{ijk} E_j B_k$ vanishes since for static GEM fields $t_{0i} = 4\epsilon_{ijk} E_j B_k$. Thus, only the gravitomagnetic field contributes to the spatial components of the Landau–Lifshitz pseudo-tensor. However, in the dual picture, only the gravitoelectric field enters the Maxwell tensor analog while the GEM Poynting vectors continue to be zero.

The analogy between linearized gravity and electromagnetism provides more information about the scalar invariants' GEM fields. Let us calculate the invariant F^2 that depends on the first derivative of the $\tilde{h}_{\mu\nu}$. This invariant was discussed for more general configurations in [86]. The general definition of F^2 is

$$F^2 = F^{\alpha\mu\nu} F_{\alpha\mu\nu}, \quad (55)$$

where $F_{\mu\nu\rho}$ is obtained from the linearized Einstein tensor and has the following form

$$F_{\alpha\mu\nu} = \partial_\nu \tilde{h}_{\alpha\mu} + \partial_\mu \tilde{h}_{\alpha\nu} - \partial_\alpha \tilde{h}_{\mu\nu} - \eta_{\mu\nu} \partial^\beta \tilde{h}_{\alpha\beta}. \quad (56)$$

For the GEM fields discussed here, the scalar F^2 takes the following form

$$\begin{aligned} \frac{1}{4}F^2 &= -\frac{16a\rho^2}{\pi^2(1+\rho^2)^4} \left[(\partial^m \rho \partial_m \rho) (\partial^n \alpha \partial_n \alpha) - (\partial^m \rho \partial_m \alpha)^2 \right] \\ &\quad - \frac{8a}{\pi^2} \left[\frac{\rho(\partial_0 \rho \partial_m \alpha + \partial_i \rho \partial_0 \alpha)}{(1+\rho^2)^2} + \frac{\rho^2 \partial_0 \rho \partial_m \alpha}{1+\rho^2} \right]^2 \\ &\quad - \frac{a}{4\pi^2(1+\rho^2)^4} \left\{ \left[\frac{2\rho \partial_m \rho \partial_0 \alpha}{(1+\rho^2)^2} + \frac{\rho^2 \partial_m \partial_0 \alpha}{1+\rho^2} \right]^2 + \left[\frac{2\rho \partial^m \rho \partial_m \alpha}{(1+\rho^2)^2} + \frac{\rho^2 \partial^m \partial_m \alpha}{1+\rho^2} \right]^2 \right\} \\ &\quad + \frac{a}{4\pi^2(1+\rho^2)^4} \left[\frac{2\rho(\partial_m \rho \partial_n \alpha + \partial_n \rho \partial_m \alpha)}{(1+\rho^2)^2} + \frac{\rho^2(\partial_m \partial_n \alpha + \partial_n \partial_m \alpha)}{1+\rho^2} \right]^2, \end{aligned} \quad (57)$$

where $v^2 = v^m v_m = \delta^{mn} v_m v_n$ is the spatial norm.

The results obtained in this section show that the general GEM knot fields' relevant quantities are calculable, but the formulas obtained are in general non-linear and non-polynomial in terms of field line maps. Therefore, in concrete applications, computer-assisted methods and approximations are necessary to extract the physical information from these relations. A complete analysis of the energy-momentum tensor along the line of [87] could be made here, but this is out of the present study's scope.

4. Discussions

This paper has investigated a class of knot solutions of the in-vacuum time-dependent GEM equations and discuss some geometrical and physical properties. We have shown that the time-dependent knot fields induce a bi-metric geometry of the perturbed space-time with the two metric tensors $g_{\mu\nu}$ and $f_{\mu\nu}$ related to each other in a complicated way. The derivation of an explicit relation between the two metric tensors is challenging due to the non-linear equations that connect the two scalar functions that determine these tensors according to Equations (38) and (39). This is just a dual description of the space-time geometry due to the electric-magnetic duality of the GEM equations in the absence of matter. In one of these descriptions, the field lines of the gravitoelectric component of the linearized gravitational tensor $\tilde{h}_{\mu\nu}$ are given by the level lines of the scalar field θ , while in the dual geometry the field lines of the gravitomagnetic components are identified with the level lines of the field ϕ . All this suggests that away from matter sources, one describes the GEM field in two complementary ways. Nevertheless, since only one GEM potential is necessary to solve the motion equations, the two descriptions based on the two potentials A and C should be physically equivalent.

Other results obtained here are the derivation of the geodesic equation that describes a test particle's motion in a time-dependent GEM knot field in one of the geometries. Also, by taking time-independent potentials in the GEM knot solutions, we have obtain stationary fields with the following characteristics: i) they are described in terms of a single complex scalar field ϕ ; ii) the components of ϕ are constrained; and iii) the static fields are purely gravitomagnetic for stationary ϕ . These solutions are given in the Section 3.3. For the static fields, we have calculated the Landau–Lifschitz tensor and a scalar invariant.

A critical feature of the time-dependent GEM knot fields obtained here is that they are parametrized by two complex scalar maps ϕ and θ . However, an essential subtlety arises when the domain of definition and values of the functions ϕ and θ is analyzed. For example, if the domains are $\mathbb{R} \times \mathbb{R}^3$ and $\mathbb{C}$, the maps act upon the arbitrary distant point relative to the origin, which implies that regularity conditions on the type $|\phi(x^0, \mathbf{x})| \rightarrow 0$ and $|\theta(x^0, \mathbf{x})| \rightarrow 0$ at $|\mathbf{x}| \rightarrow \infty$ must be imposed. Another consequence is that in the calculations, the limits $|\phi| \gg 1$ and $|\theta| \gg 1$ are accessible. On the other hand, if the maps have compact subdomains such as $\mathbb{R} \times S^3$ and S^2 which is the case of the Hopf knots, the regularity conditions and the limits are different.

Another important property of the time-dependent GEM knot fields discussed here is the existence of the constraints that must be imposed on the scalar maps due to the gravitational nature of the GEM fields, i.e., Equations (12) above. While the Lorentz gauge

is a natural relation among the potential vector components in electromagnetism and, therefore, leads to known properties of the knot fields, the pure GEM constraint is new and emerges in several places in GEM literature. These equations establish conditions that the knot maps must satisfy, which are new in the knot fields study. To exemplify, let us consider the second equation from (12) with the second equation for the ansatz (11) in the polar representation. It is easy to see that the following set of equations should be satisfied

$$(\rho^2 + 1)\rho\partial_0\partial^i\alpha + 2\partial_0\rho\partial^i\alpha = 0, \quad (58)$$

$$(\lambda^2 + 1)\lambda\partial_0\partial^i\beta + 2\partial_0\lambda\partial^i\beta = 0. \quad (59)$$

Thus, we conclude that given one of the two real scalars that determines either the potential A or C, one can determine the second scalar. To understand these equations' content, consider the variation in x^0 at a fixed point in space x^i . Then a simple calculation shows that (58) and (59) imposes the following constraint among the functions (ρ, α) and (λ, β)

$$\delta_{ij}\partial^i\alpha\partial^j\alpha = \frac{3(1+\rho^2)}{\rho^4}, \quad \delta_{ij}\partial^i\beta\partial^j\beta = \frac{3(1+\lambda^2)}{\lambda^4}. \quad (60)$$

The above equations place the spatial derivatives of the phases of the maps ϕ and θ on spheres whose radii results from calculating the modulus of these functions at any instant of time.

The study of the analogy between classical electromagnetism and the GEM equations in the linearized gravity is an exciting research field. Independently on the final resolution of the tension between the different formulations of GEM, the time-dependent linear GEM equations are an important tool to investigate properties of the GEM fields since, formally, the stationary GEM equations can be viewed as a particular case of the time-dependent equations. Another advantage of the time-dependent approach is that problems like breaking the electric-magnetic duality, conservation laws and breaking of the symmetries of solutions, e.g., Lorentz symmetry, can be easily studied in this set up.

The investigation of the time-dependent GEM knot fields started here unfolds significant and noteworthy problems. One important aspect worth being studied further is the stability of the knots fields in the presence of matter and under perturbations. The knots fields' stability in the presence of matter and under perturbations is an open problem even in classical electromagnetism.

Another interesting topic is the existence of static knots that satisfy Equation (49). Non-trivial static solutions can lead to new types of mathematical knots. By generalizing this problem, one can look for knot solutions of the exact GEM equations, e.g., in the tidal tensor approach of [4]. Furthermore, it is essential to investigate other mathematical and physical properties of the general time-dependent knots such as their topological structure, the energy-momentum tensor (along the line of [87]), the types of fields limited by constraints, the generations of new solutions, the discussion of concrete examples and the analysis of the GEM knot fields in different limits, to mention just a few. We hope to report on some of these topics soon.

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