

Review

Perturbative versus Non-Perturbative Quantum Field Theory: Tao's Method, the Casimir Effect, and Interacting Wightman Theories

Walter Felipe Wreszinski

Instituto de Física, Universidade de São Paulo, São Paulo 05508-090, SP, Brazil; wreszins@gmail.com

Abstract: We dwell upon certain points concerning the meaning of quantum field theory: the problems with the perturbative approach, and the question raised by 't Hooft of the existence of the theory in a well-defined (rigorous) mathematical sense, as well as some of the few existent mathematically precise results on fully quantized field theories. Emphasis is brought on how the mathematical contributions help to elucidate or illuminate certain conceptual aspects of the theory when applied to real physical phenomena, in particular, the singular nature of quantum fields. In a first part, we present a comprehensive review of divergent versus asymptotic series, with qed as background example, as well as a method due to Terence Tao which conveys mathematical sense to divergent series. In a second part, we apply Tao's method to the Casimir effect in its simplest form, consisting of perfectly conducting parallel plates, arguing that the usual theory, which makes use of the Euler-MacLaurin formula, still contains a residual infinity, which is eliminated in our approach. In the third part, we revisit the general theory of nonperturbative quantum fields, in the form of newly proposed (with Christian Jaekel) Wightman axioms for interacting field theories, with applications to "dressed" electrons in a theory with massless particles (such as qed), as well as unstable particles. Various problems (mostly open) are finally discussed in connection with concrete models.

Keywords: quantum fields; divergent series; Casimir effect; Wightman axioms; singularity hypothesis



Citation: Wreszinski, W.F. Perturbative versus Non-Perturbative Quantum Field Theory: Tao's Method, the Casimir Effect, and Interacting Wightman Theories. *Universe* **2021**, *7*, 229. <https://doi.org/10.3390/universe7070229>

Academic Editor: Dmitry Gitman

Received: 14 June 2021

Accepted: 2 July 2021

Published: 5 July 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

One of the most impressive measurements in physics is of the electron anomalous magnetic moment of the electron, which is known with a precision of a few parts in 10^{14} [1]. On the other hand, the theoretical prediction needed to match this result requires the use of perturbation theory containing up to five loop contributions. As is well known, higher order contributions in systems containing an infinite number of degrees of freedom lead to the appearance of divergences which are taken care of by renormalization theory [2]. The *renormalized* perturbation series in enormously successful theories, such as quantum electrodynamics (qed), is, however, strongly conjectured to be a *divergent* series (see Reference [3] for the first result in this direction), and G. 't Hooft [4] has, among others, raised the question whether qed has a meaning in a rigorous mathematical sense.

Several years ago, Arthur S. Wightman, one of the leaders of axiomatic quantum field theory, asked the related question "Should we believe in quantum field theory?", in a still very readable review article [5]. After his program of constructing interacting quantum field theories in four space-time dimensions failed, the question remains in the air, and we should like to suggest in the present review that the answer to it does remain affirmative, although the open problems are very difficult.

One of the problems with quantum field theory (qft) is that it seems to have lost contact with its main object of study, viz., explaining the observed phenomena in the theory of elementary particles. One instance of this fact is that, except for a few of the lightest particles, all the remaining ones are unstable, and there exists up to the present

time no single mathematically rigorous model of an unstable particle (this is reviewed in Reference [6], and we come back to this point in Sections 4 and 5). Alternative theories have not supplied any new experimentally measured numbers in elementary particle physics, such as the impressive one in qed mentioned above.

A recent, important field of applications of qft came from condensed matter physics: the structure of graphene, of which certain (possibly experimentally verifiable) phenomena have been successfully studied by Gavrilov, Gitman et al. [7] with nonperturbative methods from the theory of strong-field qed with unstable vacuum. So far, however, the only experimental consequence of a non-perturbative quantum field theoretic model concerns the Thirring model [8] in its lattice version, the Luttinger model. The latter's first correct solution was due to Lieb and Mattis [9]; the rigorous explanation of the thereby new emergent quasi-particles through fluctuation observables was provided by Verbeure and Zagrebnov in Reference [10]. Luttinger's model yields a well-established picture of conductivity along one-dimensional quantum wires [11].

In Section 2, we present a comprehensive review of divergent versus asymptotic series, with qed as background example, as well as a method due to Terence Tao which conveys mathematical sense to divergent series.

In Section 3, we apply Tao's method to one of the very few nonperturbative effects in qed, the Casimir effect in its simplest form, consisting of perfectly conducting parallel plates, arguing that the usual theory, which makes use of the Euler-MacLaurin formula, still contains a residual infinity, which is eliminated in our approach. The fact that Tao's smoothing of the discrete sums eliminates the divergences is seen to be directly related to the singular nature of the quantum fields.

In Section 4, we revisit the general theory of nonperturbative quantum fields, in the form of newly proposed (together with Christian Jaekel) Wightman axioms for interacting field theories, with applications to "dressed" electrons in a theory with massless particles (such as qed), as well as unstable particles [12]. Here, again, the singular nature of quantum fields appears, in the form of a "singularity hypothesis", characteristic of *interacting* quantum fields, which plays a major role. It clarifies the role of the nonperturbative wavefunction renormalization constant Z , and permits to show that the condition $Z = 0$ is not a *universal* one for interacting theories, but is, rather, related to different phenomena: the "dressing" of particles in charged sectors, in the presence of massless particles, considered by Buchholz [13], as well as the existence of unstable particles. As remarked by Weinberg [2], the characterization of unstable particles by the condition $Z = 0$ is an intrinsically *nonperturbative* phenomenon.

Section 5 discusses the latter problem in connection with some concrete models of the Lee-Friedrichs type (see Reference [6] and references given there), both in the bound state case (atomic resonances), as well as for particles, in which case the problem remains open. The relation to Haag's theorem [14] is pointed out, completing the "singularity picture" of quantum fields developed in the preceding sections.

Section 6 is reserved to the conclusion and brief comments on alternative approaches.

2. Asymptotic versus Divergent Series: Tao's Method

We now come back to the perturbation series for the renormalized gyromagnetic ratio g of the electron (see Reference [15], pp. 79, 109), which is a prototype of divergent series in physics:

$$\frac{g-2}{2} = \frac{1}{2} \left(\frac{\alpha}{\pi} \right) - 0.328 \left(\frac{\alpha}{\pi} \right)^2 + O((\alpha)^3), \quad (1)$$

where $\alpha = \frac{1}{137.036}$ denotes the fine-structure constant. As remarked by Wightman [5], which we now follow, the series is likely to be an *asymptotic series*, defined as below.

In mathematics, divergent series have been studied since the XVIIIth century and many summability methods have been developed; see, for instance, Reference [16]. In particular, Terence Tao [17] suggested a powerful smoothing method which explains the numbers obtained by these summability methods. On the other hand, one of the few nonperturbative

effects in qed, in which a divergent series appears, is the calculation of the force between two infinite conducting planes, also known as the Casimir effect [18]. Here, the zero point energy of the electromagnetic field diverges due to the existence of an infinite number of normal modes. In Section 3, we study the Casimir effect using Tao's summation method.

To set the stage, we now recapitulate the concepts of asymptotic and divergent series, reserving Section 2.2 to the presentation of inconsistencies in the treatment of divergent series by the Euler and Ramanujan summation methods, and Section 2.3 to a brief review of Tao's method of smoothed sums.

2.1. Asymptotic and Divergent Series

Given a function f , defined on an open interval $a < x < b$, one says that a function f_N defined on the interval is *asymptotic to f at order N at a* if

$$\lim_{x \rightarrow a+} \frac{f(x) - f_N(x)}{(x - a)^N} = 0, \quad (2)$$

where $x \rightarrow a+$ means that x tends to a from the right. This general definition applies in particular to the expression

$$f_N(x) = \sum_{n=0}^N a_n (x - a)^n$$

as follows: given a sequence (a_0, a_1, a_2, \dots) , if $\sum_{n=0}^N a_n (x - a)^n$ is asymptotic to f of order N at a for all $N = 0, 1, 2, \dots$, then one says that the series $\sum_{n=0}^{\infty} a_n (x - a)^n$ is *asymptotic to f at a* and writes

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - a)^n. \quad (3)$$

Now, if $\lim_{x \rightarrow a+} (f(x) - f_N(x))(x - a)^{-N} = 0$, then it follows that $\lim_{x \rightarrow a+} (f(x) - f_N(x))(x - a)^{-k} = 0$ for $k = 0, 1, \dots, N - 1$. Therefore, when $\sum_{n=0}^N a_n (x - a)^n$ is asymptotic to f of order N at a ,

$$\begin{aligned} \lim_{x \rightarrow a+} f(x) &= a_0 = f(a+) \\ \lim_{x \rightarrow a+} (f(x) - f(a+))(x - a)^{-1} &= a_1 = f^{(1)}(a+) \\ \lim_{x \rightarrow a+} (f(x) - f(a+) - f'(a+)(x - a))(x - a)^{-2} &= a_2 = \frac{f^{(2)}(a+)}{2!} \dots \\ \lim_{x \rightarrow a+} (f(x) - \sum_{n=0}^{N-1} \frac{f^{(n)}(a+)}{n!} (x - a)^n) &= a_N = \frac{f^{(N)}(a+)}{N!} \end{aligned} \quad (4)$$

Above, and throughout the paper, the superscripts denote the orders of the derivatives. Thus, when $\sum_{n=0}^N a_n (x - a)^n$ is asymptotic to f of order N at a , the function f has necessarily N derivatives from the right at a , and the coefficients a_n are uniquely determined as Taylor coefficients

$$a_n = \frac{f^{(n)}(a+)}{n!} \text{ for } n = 0, 1, \dots, N.$$

Thus, as Wightman observes, the assertion, that the gyromagnetic anomaly $\frac{g-2}{2}$ has the asymptotic series $\sum_{n=1}^{\infty} a_n (\alpha/\pi)^n$ at 0, means "no more and no less" than the fact that $\frac{g-2}{2}$ is defined for α in some interval $0 < \alpha < \alpha_0$, and has derivatives of all orders from the right at 0: then the above formula for the a_n holds for all N with $f = \frac{g-2}{2}$ and $a = 0$. If we, thus, write, following (2)–(4),

$$f(x) = \sum_{n=0}^N \frac{f^{(n)}(a+)}{n!} (x - a)^n + R_N(x), \quad (5)$$

where R_N is the rest in Taylor's series (see, e.g., Reference [19], p. 126):

$$R_N(x) = \int_a^x \frac{f^{(N+1)}(t)}{N!} (x-t)^N dt, \quad (6)$$

and

$$R_N(x) = O((x-a)^{N+1}). \quad (7)$$

Above, $g(x) = O(h(x))$ as $x \rightarrow a+$ means that there exists an open interval (a, a_0) and a positive number A independent of $x \in [a, a_0)$ such that

$$\left| \frac{g(x)}{h(x)} \right| \leq A \text{ for all } x \in [a, a_0). \quad (8)$$

It follows then, by what was said before, that $\sum_{n=0}^N \frac{f^{(n)}(a+)}{n!} (x-a)^n$ is asymptotic to f at $a+$.

The above is an example of a power series: it may, or not, be convergent. We come now to a general asymptotic series, which will be shown to be divergent and will play an important role in Section 3, the *Euler-Maclaurin sum formula* (Reference [16], Chapter XIII, p. 318). Let a function f be given which satisfies certain conditions [16]: we shall assume that f is infinitely differentiable to the right at a . Let

$$g(N) \equiv \sum_{n=1}^N f(n) - \int_a^N f(x) dx - C - \frac{1}{2}f(N). \quad (9)$$

Then,

$$g(N) \sim \sum_{r=1}^{\infty} (-1)^{r-1} \frac{B_r}{2r!} f^{(2r-1)}(N). \quad (10)$$

Above, C is a constant, and the $\{B_r\}$ are the Bernoulli numbers, defined recursively by the formula

$$\sum_{k=0}^{s-1} \frac{s!}{k!(s-k)!} B_k = s, \quad (11)$$

for all $s = 1, 2, \dots$, or equivalently, by the generating function

$$\sum_{k=0}^{\infty} \frac{B_k}{k!} t^k = \frac{t \exp(t)}{(\exp(t) - 1)}. \quad (12)$$

The first few Bernoulli numbers are

$$B_0 = 1; B_1 = \frac{1}{2}; B_2 = \frac{1}{6}; B_3 = 0; B_4 = -\frac{1}{30}; \dots \quad (13)$$

We refer to Reference [17], p. 7, or Reference [16], p. 320).

As in the case of power series, the sign \sim in (10) means

$$g(n) = \sum_{r=1}^N (-1)^{r-1} \frac{B_r}{(2r)!} f^{(2r-1)}(n) + R_N(n), \quad (14)$$

where, for each finite N ,

$$\lim_{n \rightarrow \infty} \frac{R_N(n)}{\sum_{r=1}^N (-1)^{r-1} \frac{B_r}{(2r)!} f^{(2r-1)}(n)} = 0, \quad (15)$$

but, for each finite n ,

$$\lim_{N \rightarrow \infty} R_N(n) = \infty. \quad (16)$$

(16) expresses the fact that, in general, the Euler-Maclaurin series *diverges*, due to the fast increase of the Bernoulli numbers B_r with r . For an example, take $a = 1$ and $f(x) = \log x$ in (9). Then,

$$g(n) = \log(n!) - (n + \frac{1}{2}) \log n + n + \frac{1}{2} \log(2\pi), \quad (17)$$

with $C = \frac{1}{2} \log(2\pi)$ (for a justification, see Reference [16]). In this case, (14) may be written

$$g(n) = \sum_{m=1}^N \frac{B_{2m}}{2m(2m-1)n^{2m-1}} + R_N(n), \quad (18)$$

where the remainder is

$$|R_N(n)| \leq \frac{B_{2N+2}}{(2N+1)(2N+2)n^{2N+1}} \quad (19)$$

(see Reference [20], 6.1.42). Therefore, both (15) and (16) are seen to hold. The zeroth term in (17), which corresponds to set $g(n) = 0$ in (17), is the well-known Stirling approximation, widely used in statistical mechanics.

The divergence of the series (9) is due to the inequalities (Reference [20], 23.1.15):

$$(-1)^{n+1} B_{2n} > \frac{2(2n)!}{(2\pi)^{2n}}, \quad (20)$$

as well as the fact that, even if the function f (as a function on the positive reals) is analytic in a neighbourhood of the origin and, thus, satisfies the bound

$$|f^{(r)}| \leq r! c^r, \quad (21)$$

for some constant c , by Cauchy's integral formula, the r -th term of the series is of order $r!$ by (20). This is explicitly seen in the special case $f(x) = \log x$ and $a = 1$: (19) is obtained.

We now come back to the power series example (1). As remarked in the introduction, the first proof of divergence of perturbation theory for a quantum field theoretic model (scalar field Φ , with interaction term $\lambda \Phi^3$) was given by Thirring [3]. His results are consistent with the assumption (here extrapolated to the unknown case of qed) that the sum

$$f(\alpha) = \frac{g-2}{2} = \sum_{n=0}^{\infty} a_n \alpha^n \quad (22)$$

in (1) is such that the a_n increase no worse than $n!$, that is, the $R_N(\alpha)$, defined by (6), is

$$R_N(\alpha) = O(N! \alpha^N). \quad (23)$$

One may try to find N such that this remainder after N terms is smallest possible. By (23) and Stirling's formula ((17), with $g(n) = 0$), and treating N as continuous parameter for simplicity, we find

$$\log(R_N(\alpha)) \approx N \log(N) - N + N \log(\alpha), \quad (24)$$

from which

$$\frac{d}{dN} \log(R_N(\alpha)) \approx \log(N) + \log(\alpha) = 0 \text{ with solution } \alpha = \frac{1}{N} \quad (25)$$

The above value of α characterizes a minimum because

$$\frac{d^2}{dN^2} (\log(R_N(\alpha)))_{\alpha=\frac{1}{N}} = \frac{1}{N} > 0. \quad (26)$$

We now come to a fundamental question posed by Wightman in Reference [5], p. 994): if a series is asymptotic, is it useful theoretically or experimentally? His answer is:

Observation A: “If one knows nothing about the function to which the series is asymptotic, the answer is no”.

We shall come to Observation A again later, but, for the moment, adopt the “practical” attitude he advocates in Reference [5], p. 995: according to (24)–(26), the description of the function $f(\alpha)$ in (22) should improve with the order of approximation, until one gets to order $N \approx 137$ by (25), and then one should stop; by (26), beyond that order, the approximations will become worse. Richard P. Feynman comments on this issue in the discussion in Reference [21], p. 226: “The question is whether this theory if carried out to the ultimate in all orders will give a satisfactory series (I don’t mean in agreement with experiments, but with logic). Is it unitary, for example, in the 137th order? I do not know, and am not at all convinced that it is”. Gerard ‘t Hooft [4] also observes (p. 11): “Fact is, however, that there is no proof of the existence of such a model beyond its perturbation expansion”. These remarks suggest that there are, indeed, deep issues associated to the (expected) divergence of series, such as (1).

We now come back to Observation A: what can be said about the function f to which a given series is asymptotic? One first important remark concerns *unicity*. The function

$$f_0(z) = \exp\left(-\frac{1}{z^\beta}\right) \text{ with } 0 < \beta < 1 \quad (27)$$

has the property

$$f_0^{(n)}(0+) = 0 \text{ for all } n = 0, 1, 2, \dots \quad (28)$$

There is, thus, in general, no unique function f to which a given series is asymptotic because the function f_0 may be added to f , by (28). There are, however, so-called *summability methods*: operations on infinite series, divergent or convergent, which yield convergent series or functions. For functions, one such powerful method of summability is that of Borel. Applied to a formal power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

and it yields

$$\frac{1}{x} \int_0^{\infty} \exp\left(-\frac{z}{x}\right) \left(\sum_{n=0}^{\infty} \frac{a_n z^n}{n!}\right) dz.$$

The above sum under the integral sign is understood as defined by analytic continuation if necessary. As reviewed by Wightman in [5], there are properties which may be assumed on the function f in (3) which guarantee, for instance, that the Borel summability method yields a unique answer, the right one. They typically exclude functions of the form (27), but are very difficult to prove in concrete situations, such as (1), because, as remarked by ‘t Hooft, in the case of qed, nothing is known beyond the asymptotic series on the right of (3) (or, in a concrete case, (1)).

One may also be concerned with associating a *number* (not a function) to a given divergent series of scalars by a given summability method. A unicity issue of a different kind arises: given one such series, does the possibility exist that different summability methods yield different (finite) numbers? This issue is, of course, not new, and is dealt at length, and with elegance, in G. H. Hardy’s monograph on divergent series [16]. In Reference [16], p. 346, paragraph 13.17, Hardy observes that the summability methods of Euler and Ramanujan “have a narrow range and demand great caution in their application”. Today, it is well known that the usual rules of calculation cannot be used when handling divergent series (see, e.g., the introduction in Reference [22]). We review the subject, however, in Section 2.2, as a useful introduction to Tao’s method in Section 2.3.

2.2. Inconsistencies in the Standard Treatment of Divergent Series

We now consider two examples, the divergent series S_0 and S_1 :

$$S_0 = 1 + 1 + 1 + \cdots, \quad (29)$$

and

$$S_1 = 1 + 2 + 3 + \cdots. \quad (30)$$

Although divergent, the series S_0 and S_1 may be “evaluated” by certain summability methods, two of which we now explain. For the first, consider the Riemann zeta function $\zeta(s)$, defined as the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (31)$$

which converges for $\Re(s) > 1$. As shown in Reference [16], analytic continuation from a certain complex integral representation for $\zeta(s)$ may be used to extend ζ to values of s beyond $\Re(s) > 1$. Let $\zeta_{a.c.}$ denote the analytically continued function, and call *Ramanujan sums*, denoted by the symbol \mathcal{R} , as in Reference [16], the corresponding sums, e.g., for (29) and (30),

$$S_0(\mathcal{R}) = \zeta_{a.c.}(0) = -\frac{1}{2}, \quad (32)$$

and

$$S_1(\mathcal{R}) = \zeta_{a.c.}(-1) = -\frac{1}{12}. \quad (33)$$

For the second method, due to Euler, consider a (possibly divergent) series $S \equiv \sum_{n=1}^{\infty} a_n$, and define the function

$$f_S(t) = \sum_{n=1}^{\infty} t^n a_n \quad (34)$$

for $|t| \leq 1$. Assume that $\sum_{n=1}^{\infty} t^n a_n < \infty$ if $|t| < 1$, and define the Euler sum of the original series S by

$$S(\mathcal{E}) = \lim_{t \rightarrow 1^-} \sum_{n=1}^{\infty} t^n a_n. \quad (35)$$

It is clear that

$$f_{S_0}(t) = \frac{1}{1-t}, \quad (36)$$

and

$$f_{S_1}(t) = \frac{1}{(1-t)^2}; \quad (37)$$

thus, by (35),

$$S_0(\mathcal{E}) = S_1(\mathcal{E}) = +\infty. \quad (38)$$

Thus, the Euler sums of S_0 and S_1 disagree with their Ramanujan counterparts, as seen from (32), (33), and (38). We have, however, for $\Re(s) > 1$,

$$1^{-s} + 3^{-s} + 5^{-s} + \cdots = (1 - 2^{-s})\zeta(s), \quad (39)$$

$$2^{-s} + 4^{-s} + \cdots = 2^{-s}\zeta(s). \quad (40)$$

Subtracting (40) from (39), we find

$$(1 - 2^{1-s})\zeta(s) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots, \quad (41)$$

or

$$\zeta(s) = (1 - 2^{1-s})^{-1}(1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots). \quad (42)$$

(42) expresses $\zeta(s)$ as an alternating series, to which, when Euler summation is applied, indeed yields (as may be easily checked):

$$S_0(\mathcal{E}) = -\frac{1}{2}, \quad (43)$$

as well as

$$S_1(\mathcal{E}) = -\frac{1}{12}. \quad (44)$$

What is, now, the Ramanujan sum associated to ζ , given by the r.h.s. of (42), in the case $s = -1$? For $s = -1$, it corresponds to the divergent alternating series

$$S'_1 \equiv -\frac{1}{3}(1 - 2 + 3 - 4 + \dots). \quad (45)$$

We find, however, that operations of the same type as those leading to (42) (summation term by term, multiplication by a scalar) yield the following Ramanujan sum values:

$$2 + 4 + 6 + \dots = 2(1 + 2 + 3 + \dots) = 2S_1(\mathcal{R}) = -\frac{1}{6}, \quad (46)$$

and further

$$1 + 3 + 5 + \dots = 2 + 4 + 6 + \dots - (1 + 1 + 1 + \dots) = 2S_1(\mathcal{R}) - S_0(\mathcal{R}) = -\frac{1}{6} + \frac{1}{2} = \frac{1}{3} \quad (47)$$

by (32) and (33); thus, by (45),

$$-3S_1(\mathcal{R}) = (1 + 3 + \dots) - (2 + 4 + \dots) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}, \quad (48)$$

and, finally,

$$S'_1(\mathcal{R}) = \frac{-1}{3} \times \frac{1}{2} = -\frac{1}{6} \neq S_1(\mathcal{R}). \quad (49)$$

Therefore, although (42) coincides with (31) throughout the region of convergence, its Ramanujan sum *differs* from the corresponding one for (31). This means that the algebraic operations of scalar multiplication and term-by-term summation are not preserved by analytic continuation, about which we shall shortly say more. Incidentally, (44) and (49) imply.

Observation B: The Euler and Ramanujan summation methods may yield *different* finite values for a given divergent series.

Observation B does not seem, surprisingly, to have been made before, although Hardy (Reference [16], p. 345, paragraph 13.17) remarked the disagreement between (33) and the second of (38).

There exists a way of reconciling (45) with (33), which is pointed out by Hardy (Reference [16], p. 346, paragraph 13.17). Interpret, in (48), $1 + 3 + \dots$ as $1 + 0 + 3 + 0 + \dots$, and $2 + 4 + \dots$ as $0 + 2 + 0 + 4 + \dots$, which are consistent with the r.h.s. of (39) and (40) for $s = -1$. In this way, one obtains for the r.h.s. of (45)

$$S''_1(\mathcal{R}) = -\frac{1}{3}((-1)(-\frac{1}{12}) - 2(\frac{-1}{12})) = -\frac{1}{3} \times \frac{1}{4} = -\frac{1}{12}, \quad (50)$$

which does agree with (33)! The value of $S''_1(\mathcal{R})$ depends, however, on the bizarre properties $(1 + 0 + 3 + 0 + \dots) \neq (1 + 3 + \dots)$, as well as $(0 + 2 + 0 + 4 + \dots) \neq (2 + 4 + \dots)$, i.e., the sums are not invariant by the addition of zeroes, or, alternatively, term-by-term summation is not allowed (or does not hold). Summarizing, analytic continuation does not preserve the fundamental algebraic properties of scalar multiplication and term-by-term summation; as a consequence, Observation B holds; also see Reference [22] about this issue.

Equation (42) motivates the introduction of a third divergent series, called *Grandi's series* G :

$$G = \sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - \dots \quad (51)$$

The partial sums

$$\sum_{n=1}^N (-1)^{n-1} = \frac{1}{2} + \frac{1}{2}(-1)^{N-1} \quad (52)$$

oscillate between 1 and 0, so that the series is neither conditionally nor absolutely convergent. By (42) and (32), its Ramanujan sum is

$$G(\mathcal{R}) = \frac{1}{2}. \quad (53)$$

Parenthetically, define the Cesàro (C) sums by, for $S = \sum_{n=1}^{\infty} a_n$,

$$S(\mathcal{C}) = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{l=0}^n \sum_{k=0}^l a_k. \quad (54)$$

Equation (54) corresponds to the process of averaging the partial sums and is, yet, another popular summability method. Using (52), the reader may verify that

$$G(\mathcal{C}) = \frac{1}{2}, \quad (55)$$

and it is immediate that

$$G(\mathcal{E}) = \frac{1}{2}, \quad (56)$$

where \mathcal{E} means “Euler”. Thus, for Grandi's series, all three summability methods, Ramanujan, Euler, and Cesàro, yield the same sum $\frac{1}{2}$. It might be expected, therefore, that this number $\frac{1}{2}$ is indeed attached to the series G in some way, particularly because the partial sums oscillate between one and zero by (52). We shall see how in the next subsection.

2.3. A New Look at Divergent Series: Terence Tao's Method of Smoothed Sums

The Ramanujan method of analytic continuation, when used in connection with the zeta function for general s in (31), yields a generalization of (43) and (44), namely

$$S_s(\mathcal{R}) = \zeta_{a.c.}(-s) = -\frac{B_{s+1}}{s+1}, \quad (57)$$

where the B_s are the Bernoulli numbers (11), (12); the corresponding formal series are

$$S_s \equiv \sum_{n=1}^{\infty} n^s, \quad (58)$$

with partial sums

$$S_s(N) = \sum_{n=1}^N n^s = \frac{1}{s+1} N^{s+1} + \frac{1}{2} N^s + \frac{s}{12} N^{s-1} + \dots + B_s N \quad (59)$$

(Faulhaber's formula). For $s = 0, 1$, (57) yields the Ramanujan sums

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + \dots = -\frac{1}{2}, \quad (60)$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots = -\frac{1}{12}. \quad (61)$$

We complete the list with Grandi's series (51), with partial sums (52), and Ramanujan sum

$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - \dots = \frac{1}{2}. \quad (62)$$

As remarked by Terence Tao [17], the partial sums $S_s(N)$ do not resemble the r.h.s. of (57), e.g., for $s = 0, 1$,

$$S_0(N) = \sum_{n=1}^N 1 = N, \quad (63)$$

$$S_1(N) = \sum_{n=1}^N n = \frac{1}{2}N^2 + \frac{1}{2}N. \quad (64)$$

Tao's main observation is that, if N is viewed as a real number in (59), $S_s(N)$ has jump discontinuities at each positive integer N , which play a crucial role in its asymptotic behavior with N ; he proposes to replace $S_s(N)$ by *smoothed sums*

$$S_s(\eta_N) \equiv \sum_{n=1}^{\infty} \eta\left(\frac{n}{N}\right) n^s, \quad (65)$$

where

$$\eta_N(x) \equiv \eta\left(\frac{x}{N}\right). \quad (66)$$

Above, $\eta : \mathbf{R}_+ \rightarrow \mathbf{R}$, the *cutoff function*, is a bounded function of compact support, which we take, without loss of generality, to be the interval $[0, 1]$, i.e., $\eta(x) = 0$ if $x \notin [0, 1]$, and such that

$$\eta(0+) = 1. \quad (67)$$

The values of η on the negative real axis are of no concern. We assume, for simplicity, that η is smooth (infinitely differentiable) on the set $(0, 1]$: this means that the function and all its derivatives at $x = 1$ are zero. The latter behavior contrasts with that of the characteristic function χ of $[0, 1]$, which is defined to equal one if $x \in [0, 1]$, and zero otherwise. If η is replaced by χ , a finite (step) discontinuity at $x = 1$ takes place, and, in (65), we recover the traditional partial sums (59).

We now review the simplest case treated by Tao, that of Grandi's series (62), for which the corresponding smoothed sum (65) may be written by regrouping the summands as

$$\begin{aligned} G(\eta_N) &= \sum_{n=1}^{\infty} \eta\left(\frac{n}{N}\right) (-1)^{n-1} = \\ &= \frac{\eta\left(\frac{1}{N}\right)}{2} + \sum_{n=1}^{\infty} \frac{[\eta\left(\frac{(2m-1)}{N}\right) - 2\eta\left(\frac{2m}{N}\right) + \eta\left(\frac{(2m+1)}{N}\right)]}{2}. \end{aligned}$$

If η is twice continuously differentiable, it follows from the Taylor expansion that each summand in $G(\eta_N)$ is $O(\frac{1}{N^2})$ (because it is a double difference). From the compact support of η , the number of terms in the infinite sum in $G(\eta_N)$ is finite and $O(N)$. Thus, the sum occurring in the expression for $G(\eta_N)$ is $O(\frac{1}{N})$, and, due to (67), the first term in the expression of $G(\eta_N)$ is $O(\frac{1}{N})$, too. Hence,

$$G(\eta_N) = \frac{1}{2} + O\left(\frac{1}{N}\right) \quad (68)$$

The common Ramanujan, Euler and Cesàro sum of G , namely one half- appears, therefore, as leading term in the smoothed asymptotics.

The above method may be applied to $S_s(\eta_N)$, defined by (65), yielding [17]

$$S_s(\eta_N) = -\frac{B_{s+1}}{s+1} + C_{\eta_N, s} N^{s+1} + O\left(\frac{1}{N}\right) \quad (69)$$

for any fixed $s = 0, 1, 2, \dots$, where

$$C_{\eta, s} \equiv \int_0^\infty x^s \eta(x) dx \quad (70)$$

is the Mellin transform of η (see, e.g., Reference [23], p. 80, Chapter 3). By (66),

$$C_{\eta_N, s} = C_{\eta, s} N^{s+1}. \quad (71)$$

By (69) we obtain, for $s = 0$ and $s = 1$ the results

$$S_0(\eta_N) = \sum_{n=1}^\infty \eta\left(\frac{n}{N}\right) = -\frac{1}{2} + C_{\eta_N, 0} N + O\left(\frac{1}{N}\right), \quad (72)$$

$$S_1(\eta_N) = \sum_{n=1}^\infty n \eta\left(\frac{n}{N}\right) = -\frac{1}{12} + C_{\eta_N, 1} N^2 + O\left(\frac{1}{N}\right). \quad (73)$$

Equations (72) and (73) should be compared with (32) and (33), respectively. They show that Ramanujan sums $-\frac{1}{2}$ and $-\frac{1}{12}$ appear as the leading terms of asymptotic expansions in N of $S_0(\eta_N)$ and $S_1(\eta_N)$, respectively, for large N , which, as remarked by Tao, do not contradict the positivity of the summands on the l.h.s. of (72) and (73) because the $C_{\eta_N, s}$ terms are, by (71), both strictly positive and dominate for large N . In particular, the inconsistencies pointed out in Section 2.1 connected with term-by-term summation and scalar multiplication, in the conventional treatment, disappear when definition (65) of smoothed partial sums is adopted. The simplest example is that of the scalar multiplication, take the first (innocent-looking) equation of (46):

$$2 + 4 + 6 + \dots = 2(1 + 2 + 3 + \dots).$$

This is no longer true for smoothed sums. For,

$$\sum_{n=1}^\infty 2n \eta\left(\frac{2n}{N}\right) \neq 2 \sum_{n=1}^\infty n \eta\left(\frac{n}{N}\right).$$

Similarly, term-by-term summation yields no contradictions because of the necessary associated shifts in the arguments of the function η .

We now come back to (69). This formula was proved by Tao as a corollary of his very elegant and simple proof of the Euler-Maclaurin formula. Let f satisfy

Assumption A f is a real valued function on $(0, \infty)$, at least continuously differentiable up to order $(s+2)$, vanishing, together with all its derivatives up to order $(s+1)$ at N .

Let

$$\tilde{g}(N) \equiv \int_0^N f(x) dx - \frac{1}{2} f(0) - \sum_{n=1}^N f(n). \quad (74)$$

Then,

$$\tilde{g}(N) = \sum_{k=2}^{s+1} \frac{B_k}{k!} f^{(k-1)}(0) + O(N \|f\|_{C^{s+2}}), \quad (75)$$

where $s = 1, 2, \dots$, and

$$\|f\|_{C^k} \equiv \sup_{x \in \mathbf{R}} |f^{(k)}(x)|. \quad (76)$$

Applying (75) with

$$f(x) = x^s \eta\left(\frac{x}{N}\right), \quad (77)$$

we see, from (74), the conditions on η stated between equations (65) and (67), and (71) that the l.h.s. of (75) equals $C_{\eta,s} N^{s+1}$. It may be easily checked that all terms in the sum in (75) vanish, except for the $k = s + 1$ term, which is $\frac{B_{s+1}}{s+1}$. Applying the product rule and the chain rule to definition (76), and using that η is supported on the interval $[0, 1]$, we also check that

$$\|f\|_{C^{s+2}} = O\left(\frac{1}{N^2}\right), \quad (78)$$

from which (69) follows. From (69),

$$\lim_{N \rightarrow \infty} [S_s(\eta_N) - C_{\eta,s} N^{s+1}] = -\frac{B_{s+1}}{s+1}. \quad (79)$$

One should compare (79) with the result which follows from Faulhaber's formula (59), namely $S_s(N) - \frac{N^{s+1}}{s+1} \rightarrow \infty$. This is because, when η is replaced by the characteristic function of $[0, 1]$, Assumption A does not hold, due to the boundary terms at N (see exercise 5 in Reference [17]).

3. Application of Tao's Method to the Casimir Effect for Perfectly Conducting Parallel Plates

In this section, we demonstrate the applicability of Tao's method to an important effect in nonperturbative qed, the Casimir effect [24–26]. For a review, see Reference [27]; for recent advances, see Reference [28]; and, for a pedagogical treatment, see Reference [29]. We shall follow Reference [30]; also see Reference [25], Section 2.7. Only the very simplest case, that of perfectly conducting parallel plates, will be considered.

Consider an empty cubic box Λ with perfectly conducting boundaries. It has thickness d and lateral sizes of surface L^2 :

$$\Lambda = \{\vec{x} = (x, y, z) | 0 \leq x \leq d, -L/2 \leq y \leq L/2, -L/2 \leq z \leq L/2\}.$$

The electric field in the box is the solution of Maxwell's equations, with proper boundary conditions (see Reference [25,30], Section 2.7). It yields a classical energy, whose quantization provides the expression for the total Hamiltonian H_Λ :

$$H_\Lambda = \sum_{\vec{k}, \lambda'} \hbar \omega_{\vec{k}} a_{\vec{k}, \lambda}^\dagger a_{\vec{k}, \lambda} + \frac{1}{2} \sum_{\vec{k}, \lambda'} \hbar \omega_{\vec{k}}.$$

Above, the modes (\vec{k}, λ) are described by a wave-number $\vec{k} = (k_x, k_y, k_z)$ and a polarization index $\lambda = 1, 2$, $a^\dagger(\vec{k}), a(\vec{k}, \lambda)$ are creation and annihilation operators satisfying commutation relations $[a(\vec{k}, \lambda), a^\dagger(\vec{k}', \lambda')] = \delta_{\vec{k}, \vec{k}'} \delta_{\lambda, \lambda'}$, $\omega_{\vec{k}} = c|\vec{k}|$ is the photon energy. and the prime in (\vec{k}, λ') means that only one polarization is possible when one of the wave-numbers (k_x, k_y, k_z) equals zero. The wave-numbers k_x, k_y, k_z and the eigenfrequencies are

$$k_x = \frac{\pi n_x}{d}, k_y = \frac{\pi n_y}{L}, k_z = \frac{\pi n_z}{L}, n_x, n_y, n_z = 0, 1, 2, \dots ;$$

$$\omega_{\vec{k}} = c|\vec{k}| = c\sqrt{\left(\frac{\pi n_x}{d}\right)^2 + \left(\frac{\pi n_y}{L}\right)^2 + \left(\frac{\pi n_z}{L}\right)^2}.$$

where c denotes the speed of light. The boundary condition on the electric field, i.e., that its tangential components vanish on the boundary of Λ , lead to the property of its y and z components to be proportional to $\sin(k_x x)$, with k_x , as above. The (infinite) last term

in the Hamiltonian represents the zero-point energy of the electromagnetic field. As in Reference [30], we adopt Casimir's point of view that the vacuum energy

$$\begin{aligned}\Sigma_{\Lambda} &= \langle 0 | \frac{1}{8\pi} \int_{\Lambda} dx (|\vec{E}(\vec{x})|^2 + |\vec{B}(\vec{x})|^2) | 0 \rangle = \\ &= \frac{1}{2} \sum_{\vec{k}, \lambda'} \hbar \omega_{\vec{k}}\end{aligned}$$

represents mean square fluctuations of the fields in the box Λ that exist even in the absence of photons but may produce physically observable effects because they depend on the geometry (shape, size) of the spatial domain containing the field. Accordingly, let $f(d) \equiv \frac{F(d)}{L^2}$ denote the force per unit surface induced by these vacuum fluctuations between two faces of the metallic box at distance d . Since a real metal is characterized by a frequency-dependent dielectric function $\epsilon(\omega)$ such that $\epsilon(\omega) \rightarrow \infty$ as $\omega \rightarrow 0$, but which tends to the vacuum value ϵ_0 as $\omega \rightarrow \infty$, namely, when $\omega \gg \omega_a$, with ω_a a characteristic atomic frequency, high enough frequencies should not contribute to the force. For this reason, we introduce a cutoff function $g(\omega/\omega_a)$ in the above formula for the zero point energy, such that

$$g(0) = 1, \quad (80)$$

as well as

$$g\left(\frac{\omega}{\omega_a}\right) \rightarrow 0 \text{ as } \omega \rightarrow \infty. \quad (81)$$

This means to consider Σ_{Λ} above as given by

$$\Sigma_{\Lambda} = \frac{1}{2} \sum_{\vec{k}, \lambda'} \hbar \omega_{\vec{k}} g\left(\frac{\omega_{\vec{k}}}{\omega_a}\right). \quad (82)$$

The cutoff function will be removed at the end by letting $\omega_a \rightarrow \infty$.

Extending the plates to infinity in the y, z directions yields the energy per unit surface

$$\begin{aligned}u_d &= \lim_{L \rightarrow \infty} \frac{\Sigma_{\Lambda}}{L^2} = 2 \frac{1}{\pi^2} \int_0^{\infty} dk_y \int_0^{\infty} dk_z \\ &\left(\frac{1}{2} \sum_0^{\infty'} \hbar \omega_n(\sqrt{k_y^2 + k_z^2}) \right) g\left(\frac{\omega_n(\sqrt{k_y^2 + k_z^2})}{\omega_a}\right) = \\ &= 2 \frac{1}{4\pi} \sum_{n=0}^{\infty'} \int_0^{\infty} dq q \hbar \omega_n(q) g\left(\frac{\omega_n(q)}{\omega_a}\right),\end{aligned}$$

where

$$\omega_n(q) = c \sqrt{\left(\frac{\pi n}{d}\right)^2 + q^2}.$$

Above, \vec{q} is the two-dimensional wave-number vector in the (y, z) -plane, $q = |\vec{q}|$, and we have introduced polar coordinates in the (y, z) plane, with angular sector $\frac{2\pi}{4}$. The prefactor 2 in u_d is due to the two polarization states of the photon and the prime in the sum means that the term $n = 0$ must have an additional factor $\frac{1}{2}$. With the change of variable $q \rightarrow \omega = \omega_n(q)$, $\omega d\omega = c^2 q dq$, u_d may be written, finally,

$$u_d = \frac{1}{4\pi c^2} \sum_{n=0}^{\infty'} \int_{\frac{c\pi n}{d}}^{\infty} d\omega \omega \hbar \omega g\left(\frac{\omega}{\omega_a}\right);$$

therefore,

$$u_d = \frac{\hbar c \pi^2}{2d^3} \sum_{n=0}^{\infty'} F(n), \quad (83)$$

where

$$F(s) \equiv \int_s^\infty dv v^2 g\left(\frac{\pi c v}{d \omega_a}\right). \quad (84)$$

A possibility for a cutoff function g satisfying (80) and (81) is to consider, for each finite integer N (this is no restriction because one may always consider for any given sequence of numbers $\alpha_i, i = 1, \dots$ the largest integers $N_i \leq \alpha_i, i = 1, \dots$), the function $g(x) = g_N(x) = \chi(\frac{x}{N})$, where χ denotes the characteristic function of the interval $[0, 1]$, i.e., $\chi(x) = 1$ if $x \in [0, 1]$, and $\chi(x) = 0$ otherwise. Recalling, however, the problems mentioned after (79), we are led to adopt the following definition for g instead:

$$g_N(x) \equiv \eta_N(x), \quad (85)$$

with η_N defined as in (66), (67). Since $\frac{\omega_n(q)}{\omega_a} \leq N$ implies $\frac{c \pi n}{d} \leq N$, we may write (83) and (84) in the form

$$u_d = \frac{\hbar c \pi^2}{2d^3} \lim_{N \rightarrow \infty} \sum_{n=0}^{N'} F_N(n), \quad (86)$$

where

$$F_N(n) \equiv \int_n^\infty dv v^2 g_N\left(\frac{\pi c v}{d \omega_a}\right). \quad (87)$$

The force between the plates, assuming that the electromagnetic field is entirely enclosed in the cavity, is $f_d = -\frac{\partial}{\partial d} u_d$, with u_d given by (83) and F replaced by F_N . There is, however, also a field in the space external to the cavity: the external face of the plate at d will be subject to a force in the opposite direction due to the vacuum fluctuations in the semi-infinite space to its right, namely

$$f_{ext} = -\left(-\lim_{d \rightarrow \infty} \frac{\partial}{\partial d} u_d\right), \quad (88)$$

which yields

$$u_d^{ext} \approx \frac{d}{2\pi^2 c^3} \int_0^\infty dv \int_v^\infty d\omega \omega (\hbar \omega) g\left(\frac{\omega}{\omega_a}\right). \quad (89)$$

By the formula for u_d just preceding (83), the corresponding energy per unit surface is

$$u_d^{ext} \approx \frac{\hbar c \pi^2}{2d^3} \int_0^\infty ds F_N(s), \quad (90)$$

as $d \rightarrow \infty$, up to negligible corrections which are discarded. Note that this Ansatz amounts to normalizing the force in such a way that a single plate in infinite space feels no resulting force. Define, thus, the total energy u^t per unit surface by

$$u^t = u_d - u_d^{ext}. \quad (91)$$

By (83) and (89), it is given by

$$u^t = \frac{\pi^2 \hbar c}{2d^3} \lim_{N \rightarrow \infty} u_N^t, \quad (92)$$

where

$$u_N^t \equiv \sum_{n=1}^\infty F_N(n) + \frac{1}{2} F_N(0) - \int_0^\infty F_N(s). \quad (93)$$

Our main result is the following theorem, which is an application of Tao's method to a less trivial case than (65), i.e., in which the relevant function f is not a polynomial:

Theorem 1.

$$u^t = \frac{-2\pi^2 \hbar c}{2d^3} \frac{B_4}{4!} = \frac{-\pi^2 \hbar c}{720d^3}, \quad (94)$$

where B_4 is a Bernoulli number (13).

Proof. Let $\lambda \equiv \frac{\pi c}{d\omega_a}$. By (86),

$$\begin{aligned} F_N^{(1)}(s) &= -s^2 g_N(\lambda s) \\ F_N^{(2)}(s) &= -2s g_N(\lambda s) - \lambda s^2 g_N^{(1)}(\lambda s) \\ F_N^{(3)}(s) &= -2\eta(0) - 4s \lambda g_N^{(1)}(\lambda s) - \lambda^2 s^2 g_N^{(2)}(\lambda s) \\ F_N^{(4)}(s) &= -4\lambda g_N^{(1)}(\lambda s) - 6s \lambda^2 g_N^{(2)}(\lambda s) - \lambda^3 s^2 g_N^{(3)}(\lambda s) \\ F_N^{(5)}(s) &= -10\lambda^2 g_N^{(2)}(\lambda s) - 8s \lambda^3 g_N^{(3)}(\lambda s) - \lambda^4 s^2 g_N^{(4)}(\lambda s). \end{aligned}$$

Above, as before, the superscripts denote the order of the derivatives. From the above formulas, we find that $F_N^{(1)}(0) = 0$, $F_N^{(2)}(0) = 0$. Inserting these results into (74)–(76), and taking into account (92) and (93), we obtain

$$u_N^t = -\frac{B_4}{4!} F_N^{(3)}(0) + O(N \|F_N\|_{C^5}), \quad (95)$$

where, from the above explicit formula for $F_N^{(5)}$,

$$\|F_N\|_{C^5} \equiv \sup_{x \in \mathbf{R}_+} |F_N^{(5)}(x)| \leq 10\lambda^2 \frac{\lambda}{N^2} + O\left(\frac{1}{N^3}\right). \quad (96)$$

The explicit formula for $F_N^{(3)}$ yields

$$F_N^3(0) = -2\eta(0+). \quad (97)$$

Inserting (96) and (97) into (95), and taking into account the normalization (67), we obtain

$$u_N^t = \frac{-\pi^2 \hbar c}{720 d^3} + O\left(\frac{1}{N}\right), \quad (98)$$

from which, together with (92), (94) follows. \square

We have shown that Tao's method of smoothed sums yields the correct formula (94) for the total energy density for the simplest Casimir effect, of perfectly conducting parallel plates. Most proofs of this effect use the Euler-MacLaurin formula, and write (94) in the form

$$u^t = -\frac{\pi^2 \hbar c}{720 d^3} + O(d^{-4}) \quad (99)$$

(see, e.g., Reference [31], p. 171, (13.121), for a recent reference). If u^t were a *known* function, the r.h.s of (99) would be its usual asymptotic expansion, which would determine it with any degree of precision. If, however, nothing is known about the l.h.s., as is the case with the Casimir effect, the r.h.s. of (99) must be taken as the *definition* of u^t , and then Wightman's Observation A applies. Indeed, by this definition, $u^t = +\infty$, since the $O(d^{-4})$ term in (99) indicates that one is supposed to sum the series, which, however, diverges, whatever (nonzero) value of the small parameter is filled in. Due to (75) and (80), every finite approximation to u^t is independent of the cutoff g , but the rest diverges as $N \rightarrow \infty$. Taking a sufficiently large number of terms, even the sign of u^t eventually changes from negative to positive, in analogy to (72) and (73).

It is to be remarked that the aforementioned "residual divergence" is not removed by any process of renormalization, and is, in this respect, quite analogous to the situation in perturbative qed (1), which refers to the *renormalized* perturbation series (for the gyromagnetic ratio of the electron). It is present in all approaches which use the Euler-MacLaurin series. This does not, of course, mean that these approaches are "wrong": it means that

they are not *mathematically precise*, the issue being one of striving towards a higher level of understanding.

We would like to expand slightly on this important issue because it touches on the philosophy of science. As Jaffe observes in Reference [32], lesson III, p. 7: “Arthur (Wightman) insisted: A great physical theory is not mature until it has been put in precise mathematical form”. As discussed in Section 2, perturbative qed is also, in a similar way, not mathematically precise, in spite of remaining one of the greatest successes of physics, but, as Lieb observes in Reference [33], “it is as much an enigma as it is a success”. One important point in this connection is that perturbation theory provides a wrong picture of the photon cloud which surrounds an electron; see Reference [33] and references given there, as well as Reference [34,35].

What can be said, in analogy, about the Casimir effect? It is certainly one of the very few *nonperturbative* effects of qed. As remarked in Reference [31], p. 170, together with blackbody radiation, it provides the most direct (experimental) evidence for the quantum nature of the Maxwell field (to which one might add the phenomenon of spontaneous emission; see Reference [15]). On the theoretical side, there are strong *conceptual* arguments which require that the electromagnetic field be quantized [36]. In this same paper, Bohr and Rosenfeld point out that the square of the fields at a single point, such as in the first expression for the vacuum energy Σ_Λ , are ill-defined. In fact, as discussed in Reference [15], p. 33, what we measure by a test body is the field strength averaged over some small region about a point: fields are what is termed operator-valued distributions, a notion which is basic to the axiomatic (or general) theory of quantized fields, according to which only the so-called Wick dots : $(\vec{E}(\vec{x}))^2$: exist [37]. Building on this notion, a few different approaches to the Casimir effect, which do not use the Euler-MacLaurin series (see Reference [38–40]), arrive at the same result independently. Of particular interest is Reference [39], which uses the image method in a field theoretic context and arrives at (94) by summing a convergent series. These different conceptual formulations are free of infinities, even of these mentioned in connection with (99), but they are rather special, in contrast to Tao’s formulation, which is quite general: there, the would-be rest in (99) disappears in the limit $N \rightarrow \infty$ (see Theorem 3.1).

In conclusion, the previously mentioned references, as well as Theorem 1, are mathematically precise statements of the Casimir effect. In different ways, they introduce a “smoothing”, which has its roots in the previously discussed singular nature of the quantum fields. This “smoothing” is familiar from distribution theory, which is one of the basic mathematical pillars of classical mathematical physics [41]. Coming back to sequences and series of functions, as in Section 2, consider the sequence of infinitely differentiable functions $f_j(x) \equiv \sin(jx)$, $j = 1, 2, \dots$. It certainly has no limit in the sense of functions, but let ϕ be an infinitely differentiable function, equal to zero outside the set $[-\pi, \pi]$. A partial integration shows that

$$(T_j, \phi) \equiv \int_{-\pi}^{\pi} \sin(jx)\phi(x)dx = \frac{1}{j} \int_{-\pi}^{\pi} \cos(jx)\phi'(x)dx,$$

where the prime indicates differentiation. Thus, $\lim_{j \rightarrow \infty} (T_j, \phi) = 0$. Similarly (see, e.g., Reference [42], Chapter 3, p. 36), it may be shown that the series

$$\delta_j(x) \equiv \frac{1}{2\pi} \sum_{k=-j}^{k=j} \exp(ikx) = \frac{\sin((j+1/2)x)}{\sin(x/2)} \quad (100)$$

is such that

$$\lim_{j \rightarrow \infty} (\delta_j, \phi) = \lim_{j \rightarrow \infty} \int_{-\pi}^{\pi} dx \delta_j(x) \phi(x) = \phi(0), \quad (101)$$

that is, “ δ_j ” is a “delta-sequence”: the r.h.s of (100) becomes more and more concentrated around the point $x = 0$ as j grows large, but the “distributional limit” (101) does exist.

That is, a smoothing around the singular point $x = 0$ enables the limit to exist. In analogy, with (95) and (97),

$$\lim_{N \rightarrow \infty} u_N^t = \text{const. } \eta(0+). \quad (102)$$

This means that a smoothing of the “steps” at each integer N also enables the limit $N \rightarrow \infty$ in Theorem 3.1 to exist. Since the derivative of the step function is a “delta function” in the sense of distributions (see, e.g., Reference [41], p. 82), the two notions are related.

We now come to more general nonperturbative approaches, in which the singular nature of quantum fields also plays a major role.

4. General Aspects of Nonperturbative Quantum Fields: Wightman Axioms for Interacting Quantum Fields, Dressed Particles in a Charged Sector, and Unstable Particles

We shall try to keep with the general objective of remaining at the theoretical physicist’s mathematical level [2,43]. The theory will be defined by its vacuum expectation values (VEV) or n -point Wightman functions of a (for simplicity) scalar field $A(x)$: $W_n(x_1, \dots, x_n) = (\Omega, A(x_1), \dots, A(x_n)\Omega)$, where Ω denotes the vacuum. Because of the previously mentioned singular nature of the quantum fields $A(x)$, the $A(x)$ are not (operator-valued) functions of the space-time variables $x = (x_0, \vec{x})$, but rather *functionals*, denoted by $A(f)$, which may be heuristically pictured as “smeared” objects $\int dx f(x) A(x)$, with f smooth, fast decreasing at infinity functions, taken to belong to the Schwartz space [41]: one speaks of “operator-valued tempered distributions” on the Schwartz space \mathcal{S} .

In Reference [14], pp. 107–110, it is shown that, if the (n-point) Wightman functions satisfy

- (a.) the relativistic transformation law;
- (b.) the spectral condition;
- (c.) hermiticity;
- (d.) local commutativity;
- (e.) positive-definiteness,

then they are the *vacuum expectation values of a field theory* satisfying the so-called Wightman axioms, except, eventually, the uniqueness of the vacuum state. We refer to Reference [14] for an account of Wightman theory; also see Reference [37].

We shall assume (a.) – (e.) for the n -point functions of the observable fields, with, in addition, the following requirement:

- (f.) interacting fields are assumed to satisfy the singularity hypothesis (the forthcoming Definition).

4.1. The Källén-Lehmann Representation

In a non-perturbative framework, there exists a theory of renormalization of masses and fields, which “has nothing directly to do with the presence of infinities” (Reference [2], p. 441, Section 10.3). We adopt a related proposal, which we formulate as previously mentioned, for simplicity, for a theory of a self-interacting scalar field A of mass m satisfying the Wightman axioms, assumed to be an operator-valued tempered distribution on the Schwartz space \mathcal{S} .

The following result is the spectral representation of the two-point function W_2 (Reference [2], p. 457):

$$W_2^m(x - y) = \langle \Omega, A(x) A(y) \Omega \rangle = \frac{1}{i} \int_0^\infty d\rho(m_\circ^2) \Delta_+^{m_\circ}(x - y), \quad (103)$$

where Ω denotes the vacuum vector, $x = (x_0, \vec{x})$, and

$$\Delta_+^{m_\circ}(x) = \frac{i}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3\vec{k} \frac{e^{-ix_0\sqrt{m_\circ^2 + \vec{k}^2} + i\vec{x}\cdot\vec{k}}}{\sqrt{m_\circ^2 + \vec{k}^2}} \quad (104)$$

is the two-point function of the free scalar field of mass m_\circ . It is further assumed that

$$\langle \Omega, A(f)\Omega \rangle = 0 \quad \forall f \in \mathcal{S}. \quad (105)$$

The spectral measure ρ may be further decomposed in the sum of a discrete and a continuous part

$$d\rho(m_\circ^2) = Z\delta(m_\circ^2 - m^2) + d\sigma(m_\circ^2), \quad (106)$$

where σ is a continuous measure and

$$0 \leq Z < \infty. \quad (107)$$

Of course, if $Z = 0$ in (106), there is no discrete component of mass m in the total mass spectrum of the theory. The positivity of Z is due to the positive-definiteness conditions e.) (or the positive-definite Hilbert space metric). See Reference [43], p. 50, or (Reference [2], p. 461, Equation (10.7.20)), where it is shown that Z is the (non-perturbative) wave function renormalization constant.

4.2. The Singularity Hypothesis

One of the most important features of relativistic quantum field theory is the behavior of the theory at large momenta (or large energies). Renormalization group theory [44] has contributed a significant lore to this issue (even if none of it has been made entirely rigorous): it strongly suggests that the light-cone singularity of the two-point functions of interacting theories is stronger than that of a free theory: this is expected even in asymptotically free quantum chromodynamics, where the critical exponents are anomalous. We refer to this as the “singularity hypothesis”, which will be precisely stated in the next section. This hypothesis is also verified in each order of perturbation theory (if the interaction density has engineering dimension larger than 2).

4.2.1. Steinmann Scaling Degree and A Theorem

In order to formulate the singularity hypothesis in rigorous terms, the Steinmann scaling degree sd of a distribution [45] is a natural concept: for a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$, let u_λ denote the “scaled distribution”, defined by

$$u_\lambda(f) \equiv u(f(\lambda^{-1}\cdot)).$$

As $\lambda \rightarrow 0$, we expect that $u_\lambda \approx \lambda^{-\omega}$ for some ω , the “degree of singularity” of the distribution u . Hence, we set

$$sd(u) \equiv \inf \left\{ \omega \in \mathbb{R} \mid \lim_{\lambda \rightarrow 0} \lambda^\omega u_\lambda = 0 \right\}, \quad (108)$$

with the proviso that, if there is no u satisfying the limiting condition above, we set $sd(u) = \infty$. For the free scalar field of mass $m \geq 0$, it is straightforward to show from the explicit form of the two-point function in terms of modified Bessel functions that

$$sd(\Delta_+) = 2 \quad (109)$$

(see, e.g., Reference [43], (5.15)). In (109), and the forthcoming equations, we omit the mass superscript.

Definition 1. We say that the singularity hypothesis holds for an interacting scalar field if

$$sd(W_+) > 2. \quad (110)$$

In Reference [12], it was proved that:

Theorem 2. If the total spectral mass is finite, i.e.,

$$\int_0^\infty d\rho(a^2) < \infty, \quad (111)$$

then

$$sd(W_+) \leq 2, \quad (112)$$

i.e., the scaling degree of W_+ cannot be strictly greater than that of a free theory; thus, by Definition 1, the singularity hypothesis (110) is not satisfied.

Corollary 1. The singularity hypothesis holds for an interacting scalar field only if $\int_0^\infty d\sigma(m_o^2) = \infty$. This necessary condition is independent of the value of $0 \leq Z < \infty$.

The importance of the above theorem, and especially of its corollary is that it provides a mathematical foundation for the forthcoming interpretation of the condition

$$Z = 0. \quad (113)$$

In this sense, it is a complement to the foundations of quantum field theory. In order to understand why this is so, we have to make a brief interlude.

4.2.2. The ETCR Hypothesis and Its Consequences for the Singularity Hypothesis

For the purposes of identification with Lagrangian field theory, one may equate the $A(\cdot)$ of (103) with the “bare” scalar field ϕ_B (Reference [2], p. 439), whereby

$$A = \sqrt{Z} A_{phys}, \quad (114)$$

under the condition

$$Z > 0. \quad (115)$$

Under the same condition (115), the assumption of equal time commutation relations (ETCR) for the physical fields may be written (in the distributional sense)

$$\left[\frac{\partial A_{phys}(x_0, \vec{x})}{\partial x_0}, A_{phys}(x_0, \vec{y}) \right] = -\frac{i}{Z} \delta(\vec{x} - \vec{y}). \quad (116)$$

Together with (103) and (116), one obtains (Reference [43], (9.19), Reference [2], (10.7.18) (suitably modified by the factor $\frac{1}{Z}$):

$$\frac{1}{Z} = \int_0^\infty d\rho(m_o^2). \quad (117)$$

Formula (117) has been extensively used as a heuristic guide, even, for instance, by the great founders of axiomatic (or general) quantum field theory, Wightman and Haag. Indeed, in Reference [46], p. 201, it is observed that “ $\int_0^\infty d\rho(m_o^2) = \infty$ is what is usually meant by the statement that the field-strength renormalization is infinite”. This follows from (117), with “field-strength renormalization” interpreted as $\frac{1}{Z}$. The connection with the singularity hypothesis comes next (Reference [46], p. 201), with the observation that, by (103), W_2 will have the same singularity, as $(x - y)^2 = 0$, as does $\Delta_+(x - y; m^2)$. As for Haag, he remarks (Reference [47], p. 55): “In the renormalized perturbation expansion one relates formally the true field A_{phys} to the canonical field A (our notation) which satisfies (114), where Z is a

constant (in fact, zero). This means that the fields in an interacting theory are more singular objects than in the free theory, and we do not have the ETCR." Both assertions seems to substantiate the conjecture that $Z = 0$ is expected to be a *general* condition for interacting fields. In this connection, we have the following direct consequence of Corollary 1:

Corollary 2. *If (117) holds, only $Z = 0$ is compatible with the singularity hypothesis.*

It follows from Corollaries 1 and 2 that the two definitions of Z , in (106) and in (117), are *not* equivalent. Since the ETCR, which implies (117), is not generally valid for interacting fields, as briefly reviewed in the forthcoming paragraph, we conclude that the singularity hypothesis opens the possibility of the non-universal validity of (113).

The hypothesis of ETCR has been in serious doubt for a long time, see, e.g., the remarks in Reference [14], p. 101. Its validity has been tested [48] in a large class of models in two-dimensional space-time, where it was definitely proved not to hold in the case of the Thirring model [8] for large coupling. The only case found in [48] in which the ETCR holds was the Schwinger model, which is known to be equivalent to a canonical theory of a massive vector field! A nother interesting case of canonical behavior were the canonical interacting quantum fields on two-dimensional de Sitter space [49], where, however, the canonical character was intrinsically due to the geometry of de Sitter space.

Unfortunately, however, the ETCR (for interacting fields!) is still assumed to hold, without comment, in standard treatises ([2], p. 460; [43], p. 47).

Although, when $0 < Z < \infty$, Z is interpreted as the non-perturbative field strength renormalization, relating "bare" fields to physical fields, as in (114), the remaining case (113) remains to be understood. As stated in Reference [2], p. 461, "the limit $Z = 0$ has an interesting interpretation as a condition for a particle to be composite rather than elementary". This brings us to our next topic.

5. A Proposal for the Meaning of the Condition $Z = 0$: The Presence of Massless and Unstable Particles

In the presence of massless particles (photons), Buchholz [13] used Gauss' law to show that the discrete spectrum of the mass operator

$$P_\sigma P^\sigma = M^2 = P_0^2 - \vec{P}^2 \quad (118)$$

is empty. Above, P^0 is the generator of time translations in the physical representation, i.e., the physical hamiltonian H , and \vec{P} is the physical momentum. This fact is interpreted as a confirmation of the phenomenon that particles carrying an electric charge are accompanied by clouds of soft photons.

Buchholz formulates adequate assumptions which must be valid in order that one may determine the electric charge of a physical state Φ with the help of Gauss' law:

$$\langle \Phi, j_\mu \Phi \rangle = \langle \Phi, \partial^\nu F_{\nu,\mu} \Phi \rangle. \quad (119)$$

$F_{\nu,\mu}$ denotes the electromagnetic field observable, and (119) is assumed to hold in the sense of distributions on $\mathcal{S}(\mathbb{R}^n)$.

When endeavoring to apply Buchholz's theorem to concrete models, such as qed_{1+3} , problems similar to those occurring in connection with the charge superselection rule [50] arise. The most obvious one is that Gauss' law (119) is only expected to be valid (as an operator equation in the distributional sense) in non-covariant gauges, the Coulomb gauge in the case of qed_{1+3} , but not in covariant gauges [50]. If we adopt the present framework, our option is to use the Coulomb gauge and to define the theory in terms of the n -point Wightman functions of observable fields, i.e., gauge-invariant fields, thus maintaining Hilbert-space positivity. The hypotheses of Buchholz's theorem are then in consonance

with the requirements of Wightman's theory [14], and are applicable to qed_{1+3} . In a charged electron sector, denoting spinor indices by α, β , we have

$$\begin{aligned} S_{\alpha,\beta}^+(x-y) &= \langle \Omega, \Psi_\alpha(x) \bar{\Psi}_\beta(y) \Omega \rangle \\ &= \int_0^\infty d\rho_1(m_\circ^2) S_{\alpha,\beta}^+(x-y; m_\circ^2) + \delta_{\alpha,\beta} \int_0^\infty d\rho_2(m_\circ^2) \Delta^+(x-y; m_\circ^2), \end{aligned} \quad (120)$$

with $d\rho_{ph}, d\rho_1, d\rho_2$ positive, measures, and ρ_1 satisfying certain bounds with respect to ρ_2 (Reference [51], p. 350).

$$d\rho_1(m_\circ^2) = Z_2 \delta(m_\circ^2 - m_e^2) + d\sigma_1(m_\circ^2), \quad (121)$$

with m_e the renormalized electron mass, according to conventional notation. Recalling (120), we have the following immediate corollary of Buchholz's theorem:

Corollary 3. *For qed_{1+3} in the Coulomb gauge, assuming it exists in the sense of the framework of this section and satisfies the assumptions of Buchholz's theorem, the following condition holds:*

$$Z_2 = 0. \quad (122)$$

Above, Ψ denote observable fermion fields, which we assume to exist as a generalization of those constructed by Lowenstein and Swieca [52] in qed_{1+1} ; also see Reference [53] for a similar attempt in perturbative qed_{1+3} . In the words of Lieb and Loss [34], who were the first to observe this phenomenon in a relativistic model of qed, “the electron Hilbert space is linked to the photon Hilbert space in an inextricable way”. Thereby, in this way, “dressed photons” and “dressed electrons” arise as new entities.

We now come back to Weinberg's suggestion that the condition $Z = 0$ describes unstable particles.

Turning to scalar fields for simplicity, we consider the case of a scalar particle C , of mass m_C , which may decay into a set of two (for simplicity) stable particles, each of mass m . We have energy conservation in the rest frame of C , i.e.,

$$m_C = \sum_{i=1}^2 \sqrt{\vec{q}_i^2 + m_i^2} \geq \sum_{i=1}^2 m_i,$$

with $m_i = m, i = 1, 2$, and \vec{q}_i the momenta of the two particles in the rest frame of C :

$$m_C > 2m. \quad (123)$$

In order to stress the practical effects of the confusion of the two definitions of Z in the literature, when the ETCR is assumed (including [2]), it should be remarked that the first reference given by Weinberg on a model for unstable particles [54] assumes (what amounts to) $m_C \leq 2m$ instead of (123) (that is, we are in the stability range!), and finally obtains $Z = Z_C = 0$! This is solely due to the “double definition” of the non-perturbative wave function renormalization constant!

In order to check that $Z = Z_C = 0$ when (123) holds, while $0 < Z < \infty$ is valid in the stable case $m_C < 2m$, in a model, we are beset with the difficulty to obtain information on the two-point function.

There exists a quantum model of Lee type of a composite (unstable) particle, satisfying (123), where (113) was indeed found, that of Araki et al. [55]. Unfortunately, however, the (heuristic) results in Reference [55] have one major defect: their model contains “ghosts”. A very good review of the existent (nonrigorous) results on unstable particles is the article by Landsman [56], to which we refer for further references and hints on the intuition behind the criterion (113).

In the next section, we come back to a set of models for atomic resonances and particles, which might support the suggested picture of quantum field theory in terms of “dressed”

and unstable particles, and, in the last section, we discuss the crucial conceptual issues and difficulties associated with this program, comparing it with alternative approaches.

6. Models for Atomic Resonances, Unstable and “Dressed” Particles: What Distinguishes Quantum Field Theory from Many-Body Systems?

In Reference [6] the model below—the Lee-Friedrichs model of atomic resonances—was revisited. Its Hamiltonian may be written

$$H = H_0 + H_I, \quad (124)$$

with

$$H_0 = E_0 \frac{\mathbf{1} + \sigma_z}{2} \otimes \mathbf{1} + \mathbf{1} \otimes \int d^3k |k| a^\dagger(k) a(k), \quad (125)$$

and

$$H_I^1 = \beta [\sigma_- \otimes a^\dagger(g) + \sigma_+ \otimes a(g)]. \quad (126)$$

The operators act on the Hilbert space

$$\mathcal{H} \equiv \mathcal{C}^2 \otimes \mathcal{F}, \quad (127)$$

where \mathcal{F} denotes symmetric (Boson) Fock space on $L^2(\mathbf{R}^3)$ (see, e.g., Reference [57]), which describes the photons. We shall denote by (\cdot, \cdot) the scalar product in \mathcal{H} . Formally, $a(g) \equiv \int d^3k g(k) a(k)$, and k denotes a three-dimensional vector. The \dagger denotes adjoint, $\sigma_\pm = \frac{\sigma_x \pm i\sigma_y}{2}$, and $\sigma_{x,y,z}$ are the usual Pauli matrices. The operator

$$N = \frac{\mathbf{1} + \sigma_z}{2} \otimes \mathbf{1} + \mathbf{1} \otimes \int d^3k k a^\dagger(k) a(k) \quad (128)$$

commutes with H . We write

$$N = \sum_{l=0}^{\infty} l P_l \quad (129)$$

and introduce the notation

$$H_l \equiv P_l H P_l. \quad (130)$$

H_l is the restriction of H to the subspace $P_l \mathcal{H}$. Let, in addition,

$$E_0 > \beta^2 \int d^3k |k|^{-2} |g(k)|^2. \quad (131)$$

Then, the one-dimensional subspace $P_0 \mathcal{H}$ consists of the ground state vector

$$\Phi_0 \equiv |-\rangle \otimes |\Omega\rangle, \quad (132)$$

with energy zero, where

$$\sigma_z |\pm\rangle = \pm |\pm\rangle \quad (133)$$

denote the upper $|+\rangle$ and lower $|-\rangle$ atomic levels, and $|\Omega\rangle$ denotes the zero-photon state in \mathcal{F} . Note that Φ_0 is also eigenstate of the free Hamiltonian H_0 , with energy zero, and we say, therefore, that the model has a persistent zero particle state.

We shall refer to the model described by (124) as *Model 1*. Replace, now, in Model 1, H_I^1 by

$$H_I^2 = \beta [(\sigma_- + \sigma_+) \otimes (a^\dagger(g) + a(g))]. \quad (134)$$

We shall refer to the ensuing model as *Model 2*. Let, now, N, V satisfy *anticommutation relations*, i.e.,

$$\{N, N^+\} = \{V, V^+\} = 1,$$

together with all other anticommutators equal to zero, i.e., $\{N, V^+\} = 0$, etc. By means of Schwinger's representation

$$\begin{aligned}\sigma_+ &= V^+ N, \\ \sigma_- &= N^+ V, \\ \sigma_3 &= \frac{1}{2}(V^+ V - N^+ N),\end{aligned}$$

with the further correspondences

$$\begin{aligned}|+\rangle &= V^+ |0\rangle, \\ |-\rangle &= N^+ |0\rangle,\end{aligned}$$

where $|0\rangle$ denotes the fermion no-particle state, Model 1 becomes the usual Lee model for particles, and Model 2 a “refined” Lee model for particles; we refer to them as *Model 3* and *Model 4*, respectively. There is one big difference, however, between the atomic resonance case and the particle case. In the former case, the function g in (126) or (134) is square-integrable, and, indeed, the physical dipole-moment matrix elements provide natural cutoffs, so that neither infrared nor ultraviolet problems arise. In the latter case, however, one aims at the pointwise limit $g(k) \rightarrow 1$, which should lead to an euclidean-invariant theory. This step is delicate and requires mass and wave function renormalizations. To the particle versions, Model 3 and Model 4, assuming they are well-defined, the forthcoming framework is applicable.

Let a theory of a scalar field of mass $m > 0$ be invariant under the euclidean group, that is, the group of translations and rotations of euclidean space $\vec{x} \rightarrow R\vec{x} + \vec{a}$, where R denotes a rotation. By Haag's theorem (Reference [46], p. 249), in a euclidean field theory which uses the Fock representation, the no-particle state Ψ_0 is euclidean invariant, i.e.,

$$U(\vec{a}, R)\Psi_0 = \Psi_0. \quad (135)$$

We have the following:

Theorem 3. *Let the Hamiltonian be of the form*

$$H = \int \mathcal{H}(\vec{x}) d\vec{x}, \quad (136)$$

where $\mathcal{H}(\vec{x})$ satisfies

$$U(\vec{a}, \mathbf{1})\mathcal{H}(\vec{x})U(\vec{a}, \mathbf{1})^{-1} = \mathcal{H}(\vec{x} + \vec{a}). \quad (137)$$

Then, if Ψ is any state invariant under $U(\vec{a}, \mathbf{1})$, i.e.,

$$U(\vec{a}, \mathbf{1})\Psi = \Psi, \quad (138)$$

then Ψ belongs to the domain of H only if $H\Psi = 0$.

Proof. We have that $(\Psi, \mathcal{H}(\vec{x})\mathcal{H}(\vec{y})\Psi)$ depends only on $\vec{x} - \vec{y}$, so that

$$\|H\Psi\|^2 = \int \int d\vec{x} d\vec{y} (\Psi, \mathcal{H}(\vec{x})\mathcal{H}(\vec{y})\Psi) = 0 \text{ or } \infty, \quad (139)$$

according to whether $H\Psi = 0$ or $\|H\Psi\| = \infty$, that is, H must annihilate any translation-invariant state to which it is applicable. \square

Choosing $\Psi = \Psi_0$ in the Fock representation, it follows from Theorem 3, together with the consequence of Haag's theorem (135), that H can be applied to the no-particle state only if it annihilates it. The above was extracted from Reference [46], p. 250, and, to make it entirely rigorous, (136) should be written as a limit of the “smearing” of \mathcal{H} with smooth

functions. This result may be applied to Model 3 (assuming the renormalizations performed such that the pointwise limit can be taken, leading to an euclidean invariant quantum field theory), with the no-particle state identified to the state given by $\Psi \equiv |\Omega\rangle \otimes |-\rangle$, in and $|\Omega\rangle$ the no-particle photon state: $H\Psi = 0$. We say that there is no *vacuum polarization*. On the other hand, for Model 4, Ψ does not belong to the domain of H due to the term $\sigma_+ a^\dagger(g)$ in (134), which, in terms of fermion operators equals, $V^+ N a^\dagger(g)$, and $V^+ N |-\rangle = |+\rangle$. In the case there is vacuum polarization, Theorem 3 implies that there exists an “infinite energy barrier” between the Fock no-particle state and the true vacuum: non-Fock representations are required, that is, the “physical” Hilbert space is not unitarily equivalent to Fock space.

All this being said, Model 3 turns out, very unexpectedly, to be afflicted by “ghosts”, i.e., states of negative norm (see Reference [43], Chapter 12). We say unexpectedly because the occurrence of “ghosts” in relativistic quantum field theory is known to be a consequence of the use of manifestly covariant gauges (see Reference [44]), and the Lee model is not relativistically invariant. Note, however, that, in the Schwinger representation, the “fermions” V and N are two states of an *infinitely heavy* (spinless) fermion, i.e., *there is no recoil!* Of course, this is a highly unphysical assumption. One may consider, however, the model with recoil, describing the interaction between the photon field and *two* particles V and N , with energies $E_V(p) = (M^2 + p^2)^{1/2}$ and $E_N(p) = (m^2 + p^2)^{1/2}$, and interaction energy $H_I = \int d^3p d^3k g(p, k) [V^\dagger(p) N(p - k) a(k) + \text{h.c.}] \lambda$, with λ proportional to the charge, and $g(p, k) \equiv f(p, k) [8E_V(p) E_N(p - k) \omega(k)]^{-1/2}$, $\omega(k) = (\mu^2 + k^2)^{1/2}$, the latter representing the (eventually massive) “photon” energy. This model is well-defined and free of “ghosts” in the pointwise limit $f \rightarrow 1$, the latter taken in a careful way, according to a renormalization prescription. This was proved by Yndurain [58] in a seldom cited, but very important, paper. Thus, the pathologies associated to the original Lee model just have to do with neglecting recoil!

We henceforth refer to the Yndurain versions of models 3 and 4 as Model 3Y and Model 4Y.

Model 2 (for zero temperature, as a model of atomic resonances) may be the simplest prototype of a model with vacuum polarization. Model 3Y should be suitable to study the criterion $Z = 0$ for unstable particles. The subtlest point in this connection is the fact that the term proportional to the delta measure in (106) has coefficient Z , and, in its absence, due to the condition $Z = 0$, the (renormalized) mass seems to remain undetermined. One must, therefore, be able to determine the renormalized mass uniquely from *alternative* general conditions on the Hamiltonian, such as the requirement that it be self-adjoint and bounded below.

We close this section with some remarks of what distinguishes quantum field theory from many-body systems. The latter are characterized by an interaction Hamiltonian $\int d^3x d^3y \sum_{\sigma, \sigma'} V(x - y) \Psi^\dagger(x, \sigma) \Psi^\dagger(y, \sigma') \Psi(y, \sigma') \Psi(x, \sigma)$, where V denotes the interaction potential, σ and σ' spin indices, and Ψ are boson or fermion quantized operators (see, e.g., Reference [57], p. 110). The interaction term, and consequently the whole Hamiltonian, annihilates, therefore, the no-particle state (Fock vacuum). In contrast, quantum field theories display in general vacuum polarization, as typified by Model 4Y, which is an approximation of the usual trilinear coupling terms which occur in qed and also in quantum chromodynamics (qcd), as a consequence of relativistic invariance and the gauge principle (see Reference [2,44]). Coupled with Haag’s theorem and Theorem 3, this implies non-Fock representations, as previously discussed. This is a further manifestation of the singular nature of quantum fields. Of course, the above reference to many-body systems concerns a finite number of particles. For nonzero density, i.e., an infinite number of degrees of freedom, non Fock representations arise even in the case of free systems, by a well-known mechanism (Reference [59], Section 2.3). What we wish to emphasize is that, in the case of quantum field theories, such representations arise due to a particular reason, namely vacuum polarization.

7. Conclusions

In this review, we focused on the foundations of quantum field theory, which is still believed to be the most fundamental theory, describing in principle all phenomena observed in atomic and particle physics. Unlike quantum mechanics, however, its foundations are still not cleared up. We attempted to describe how some novel approaches lead to a unified picture, in spite of the fact that several difficult open problems remain. The solution of some of them seems to be nontrivial, but feasible. Many other problems demand new ideas, however.

The first issues concern *relativistic* quantum field theories, such as qed or qcd. They have been discussed in Sections 3 and 4, after a pedagogic review of asymptotic and divergent series in Section 2.

The fact that divergent series, even if asymptotic, do not define a theory mathematically, was emphasized in the case of qed; see, in particular, (24)–(26). From the latter, we see that even the assumption that renormalized perturbation series, such as (1), is the asymptotic series of an unknown function, which is probably true and explains the dazzling success of perturbative qed, cannot hold for qcd, where a rough dimensionless measure of the coupling constant is of order $1 - 10$. The improved perturbation theory deriving from renormalization group arguments and asymptotic freedom in qcd (Reference [44], Ch. 18.7) has no comparable experimental consequences.

We, therefore, turned our attention to *nonperturbative* phenomena, one of the very few (in qed) being the Casimir effect, for simplicity in its simplest form, that of perfectly conducting parallel plates in the vacuum. We showed in Section 3 that Tao's method of smoothed sums [17] eliminates the “residual infinity” present in the Euler-maclaurin series for the energy density of the field (Theorem 1). This smoothing of the series is a new method of accounting for the singular nature of quantum fields, which are not pointwise defined.

Going beyond specific models or effects, we reviewed in Section 4 a recent novel criterion to characterize interacting theories in the Wightman framework [12]. Here, a different aspect of the singular nature of quantum fields is touched upon: the nature of the singularity of the two-point Wightman function at small distances. Using the Källén-Lehmann representation of the two-point function W_+ and the Steinmann scaling degree [45] sd , we propose to characterize interacting Wightman theories by the requirement $sd(W_+) > 2$. We were then able to prove a Theorem 2 which states that, in an interacting Wightman theory, the total spectral mass which occurs in the Källén-Lehmann representation is infinite. This allows us, in turn, to state that, contrary to previous belief, the *nonperturbative* wavefunction renormalization constant Z is not universally equal to zero. We then interpret the condition $Z = 0$ in a two-fold way: it is either due to the nonexistence of a pure point part in the mass spectrum in charged sectors in theories, such as qed, due to infraparticles (electrons with their photon clouds), or to the existence of unstable particles, as conjectured by Weinberg (Reference [2], p. 461. We believe that this new approach is connected with the theories of “dressed” photons and electrons in Reference [34,35].

In qcd, the gluons being massless, a similar phenomenon as discussed in the last paragraph should be expected: an interacting theory of “dressed” quarks and gluons. It has been suggested by Casher, Kogut and Susskind [60], and Swieca [61] that massless qed_{1+1} contains what is desired of a theory of quark confinement, in the sense that under short distance probing the theory behaves as if it contained particles which do not manifest themselves as physical states: in that limit one recovers a theory of free (massless) electrons and photons. For large distances, the electrons completely disappear from the picture, giving rise to massive photons, by “Bosonization” (an extreme form of the “dressing” phenomenon). In Reference [12], we formulated a precise criterion for confinement which applies to qed_{1+1} and which, generalized to qcd (under assumptions on the leading infrared singularity of the gluon propagator), yields a surprisingly realistic picture of confinement and asymptotic freedom.

We should like to mention a recent alternative approach, due to Buchholz and Fredenhagen (Reference [62] and references given there). There the assumed existence of a time-arrow yields a novel approach, centered on an interesting, non-commutative structure of dynamical algebras inspired by scattering theory. An early reference which treats unstable particles in the spirit of open systems, associated to a dynamical group (the Poincaré semigroup), is the paper by Alicki, Fannes, and Verbeure [63]. Sewell's book [64] also deals with the emergent macrophysics originating from quantum mechanics, albeit not in the realm of quantum field theory.

More directly aimed at gauge theories is the theory of string-localised fields, due to Mund, Schroer and Yngvason, which has some points of contact with the present approach, specifically the insistence on positivity, i.e., "ghostless" theories. The theory has recently progressed considerably through a deep analysis of Gauss' law in that connection, see [65] and references given there to the previous literature.

A theory of quantum fields in de Sitter space has been developed in recent years by Jaekel, Mund and Barata. It is particularly interesting because of its connection to thermal aspects of quantum fields, as early realized by Narnhofer, Peter and Thirring [66]. We refer to [49] for some recent references.

In addition, an important perturbative algebraic quantum field theory has been developed in the last twenty years. Its recent application to the sine-Gordon model [67] actually yielded a convergent series, which marks the success of this approach: we refer to [67] to references to the previous extensive literature. An earlier treatment of integrable two-dimensional models, with several novel structural features, both conceptual and technical, is due to Lechner and collaborators, see the review by Alazzawi and Lechner [68] and references given there.

In spite of the fact that the convergence of the perturbation series for models such as the massive Thirring-Schwinger model (qed_{1+1}) is well-known since the early days of constructive quantum field theory (1976) [69], it is important to emphasize that no singular point fields appear in the functional analytic approach of [67], so that it may be expected that this new method, invented by Fredenhagen, may be able to cope with physically relevant models in the future, such as gauge theories in four space-time dimensions.

Concerning the latter, since the gluons are, as the photon, massless, we are proposing a picture of interacting gauge theories as composed of "dressed" quarks and gluons, or dressed electrons and photons, for qed. The latter has been suggested by Lieb and Loss [34] and Jaekel and myself [35], and we suggested further that this picture is characterized by the condition $Z = 0$ [12]. Unfortunately, even in qed_{1+3} this involves a construction of fermion observable fields which has not even been achieved in perturbation theory [53]. This project is therefore very difficult for qed, and the self-interactions of the gluons in qcd make it even much more difficult there. We feel, however, that it contains a "grain of truth": in particular, Lieb and Loss provide convincing arguments to the effect that description by "dressed" photons is the more natural one in their relativistic model of qed, while we relate this phenomenon to the non-Fock structure of the representations (when the ultraviolet cutoff is removed). The latter appear in connection with a generalization of the transformations analyzed by Wightman and Schweber [70] which occur in qed_{3+1} in the Coulomb gauge [35]. Therefore, from our point of view, a first step would be a study of a simpler model displaying vacuum polarization. The simplest such is Model 2.

Although the Lee-Friedrichs model is very old (see [6] and references given there), it is one of the few approximations to relativistic quantum field models which preserve some basic physical features. This may be due to the fact that its space dimension is three, and that it retains the trilinear coupling familiar from relativistic theories. We proposed in Section 5 that the main feature distinguishing quantum field theory from many-body systems is *vacuum polarization*. When the theories are euclidean invariant, Haag's theorem, together with theorem 3 provides a way to distinguish Model 3Y from Model 4Y (the letter Y refers to the important work of Yndurain [58], who showed that the "ghosts" in the Lee model were solely due to neglecting recoil). We believe that Model 3Y is suitable to verify

the prediction $Z = 0$ for unstable particles. Model 4Y seems at present beyond any control, but understanding Model 2, which is relevant to atomic resonances, would be an enormous advance towards grasping the main features of vacuum polarization. It should be noted that Herbert Fröhlich's electron-phonon theory [71] is a true quantum field theory with vacuum polarization, and the failure of being able to handle it may well be the “penalty” for the problems found in BCS theory [64]. The study of Model 1 in [6] shows clearly that the field-theory interaction through emission and absorption of (virtual) particles is of *qualitatively* different nature from potential theory: in particular, the regeneration of the unstable state from the decay products is a virtual quantum phenomenon analogous to tunneling in potential theory which is, however, not present in potential theory. In the case of atomic resonances, Model 1 even accounts even for subtle physical aspects, such as the fact that the natural line width of Lamb states is much smaller than the corresponding (Lamb) shift, essential for the observability of the latter [6]: this is an example of the physical features mentioned in the beginning of this paragraph.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We should like to thank a referee, as well as Oscar Eboli, for useful suggestions.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Zyla, P.A.; Barnett, R.M.; Beringer, J.; Dahl, O.; Dwyer, D.A.; Groom, D.E.; Lin, C.-J.; Lugovsky, K.S.; Pianori, E.; Robinson, D.J.; et al. Review of Particle Physics *Prog. Theor. Exp. Phys.* **2020**, *8*, 083C01.
2. Weinberg, S. *The Quantum Theory of Fields Volume I—Foundations*; Cambridge University Press: Cambridge, UK, 1995.
3. Thirring, W. On the divergence of perturbation theory for quantized fields. *Helv. Phys. Acta* **1953**, *26*, 33.
4. Hooft, G. Reflections on the renormalization procedure for gauge theories. *Nucl. Phys. B* **2016**, *912*, 4. [CrossRef]
5. Wightman, A.S. Should we believe in quantum field theory? In *The Whys of Subnuclear Physics*; Zichichi, A., Ed.; Plenum Press: New York, NY, USA, 1979.
6. Wreszinski, W.F. Unstable states in a model of nonrelativistic quantum electrodynamics. *J. Stat. Phys.* **2021**, *182*, 26. [CrossRef]
7. Gavrilov, S.P.; Gitman, D.M.; Dmitriev, V.D.; Panferov, A.D.; Smolyansky, S.A. Radiation problems accompanying carrier production by an electric field in graphene. *Universe* **2020**, *6*, 205. [CrossRef]
8. Thirring, W.E. A soluble relativistic field theory. *Ann. Phys.* **1958**, *3*, 91. [CrossRef]
9. Mattis, D.C.; Lieb, E.H. Exact solution of a many-Fermion system and its associate Bose field. *J. Math. Phys.* **1965**, *6*, 304–312. [CrossRef]
10. Verbeure, A.; Zagrebnov, V.A. About the Luttinger model. *J. Math. Phys.* **1993**, *34*, 785. [CrossRef]
11. Mastropietro, V.; Mattis, D.C. *Luttinger Model—The First 50 Years and Some New Directions*; World Scientific Publ. Co.: Singapore, 2013.
12. Jaekel, C.D.; Wreszinski, W.F. A criterion to characterize interacting theories in the Wightman framework. *Quantum Stud. Math. Found.* **2021**, *8*, 51. [CrossRef]
13. Buchholz, D. Gauss' law and the infraparticle problem. *Phys. Lett. B* **1986**, *174*, 331. [CrossRef]
14. Streater, R.F.; Wightman, A.S. *PCT, Spin and Statistics and All That*; W. A. Benjamin, Inc.: Princeton, NJ, USA, 1964.
15. Sakurai, J.J. *Advanced Quantum Mechanics*; Addison Wesley: Boston, MA, USA, 1967.
16. Hardy, G.H. *Divergent Series*; Oxford at the Clarendon Press: Oxford, UK, 1949.
17. Tao, T. The Euler-Maclaurin Formula, Bernoulli Numbers, the Zeta Function and Real Variable Analytic Continuation. Available online: <https://terrytao.wordpress.com/2010/04/10/> (accessed on 4 July 2021).
18. Casimir, H.B.G. On the Attraction Between Two Perfectly Conducting Plates. *Indag. Math.* **1948**, *10*, 261.
19. Buck, R.C. *Advanced Calculus*; McGraw Hill: New York, NY, USA, 1965.
20. Abramowitz, M.; Stegun, I. *Handbook of Mathematical Functions*; Dover: 1965. Available online: <https://www.math.hkbu.edu.hk/support/aands/intro.htm> (accessed on 4 July 2021).
21. Wightman, A.S. Progress in the foundations of quantum field theory. In *Proceedings of the 1967 International Conference on Particles and Fields*; Hagen, C.R., Guralnik, G., Mathur, V.A., Eds.; Wiley-Interscience: Hoboken, NJ, USA, 1967.
22. Candelpergher, B. Ramanujan summation of divergent Series. In *Lecture Notes in Mathematics 2185*; Springer: Berlin/Heidelberg, Germany, 2017.

23. Hochstadt, H. *The Functions of Mathematical Physics*; John Wiley: Hoboken, NJ, USA, 1971.
24. Itzykson, C.; Zuber, J.B. *Quantum Field Theory*; McGraw-Hill Book Co.: New York, NY, USA, 1980.
25. Milonni, P.W. *The Quantum Vacuum: An Introduction to Quantum Electrodynamics*; Academic Press: Cambridge, MA, USA, 1994.
26. Milton, K.A. *The Casimir Effect: Physical Manifestations of Zero Point Energy*; World Scientific: Singapore, 2001.
27. Plunien, G.; Muller, B.; Greiner, W. The Casimir effect. *Phys. Rep.* **1986**, *134*, 87. [[CrossRef](#)]
28. Bordag, M.; Klimchitskaya, G.L.; Mohideen, U.; Mostepanenko, V.M. *Advances in the Casimir Effect*; Oxford Science Publications: Oxford, UK, 2009.
29. Farina, C. The Casimir effect: Some aspects. *Braz. J. Phys.* **2006**, *36*, 1137. [[CrossRef](#)]
30. Martin, P.A.; Buenzli, P.R. The Casimir effect. *Acta Phys. Pol.* **2006**, *37*, 2503.
31. Spohn, H. *Dynamics of Charged Particles and Their Radiation Field*; Cambridge University Press: Cambridge, UK, 2004.
32. Jaffe, A.M. Nine Lessons of My Teacher Arthur Strong Wightman. A Talk at Princeton University 10-3-2013. IAMP News Bulletin. April 2013. Available online: <https://www.arthurjaffe.com/Assets/pdf/Arthur%20Wightman%2010%20March%202013.pdf> (accessed on 4 July 2021).
33. Lieb, E.H. Some problems in statistical mechanics that I would like to see solved. *Physica A* **1999**, *263*, 491. [[CrossRef](#)]
34. Lieb, E.H.; Loss, M. Stability of a model of relativistic quantum electrodynamics. *Comm. Math. Phys.* **2002**, *228*, 561. [[CrossRef](#)]
35. Jaekel, C.D.; Wreszinski, W.F. Stability of relativistic quantum electrodynamics in the Coulomb gauge. *J. Math. Phys.* **2018**, *59*, 032303. [[CrossRef](#)]
36. Bohr, N.; Rosenfeld, L. Zur Frage der Messbarkeit der elektromagnetischen Feldgrößen. *Kgl. Danske Vidensk. Selsk. Mat. Fys. Med.* **1933**, *12*, 8.
37. Wightman, A.S. Quantum field theory in terms of vacuum expectation values. *Phys. Rev.* **1956**, *101*, 860. [[CrossRef](#)]
38. Scharf, G.; Wreszinski, W.F. On the Casimir effect without cutoff. *Found. Phys. Lett.* **1992**, *5*, 479. [[CrossRef](#)]
39. Brown, L.S.; Maclay, G.J. Vacuum stress between conducting plates: An image solution. *Phys. Rev.* **1969**, *184*, 1272. [[CrossRef](#)]
40. Niekerken, O. *Quantentheoretische und Klassische Vakuum—Kräfte bei Temperatur Null und bei endlicher Temperatur*; Diplomarbeit Hamburg Februar: 2009. Available online: <https://www.osti.gov/etdweb/biblio/21196771> (accessed on 4 July 2021).
41. Schwartz, L. *Mathematics for the Physical Sciences*; Dover Publ. Inc.: Mineola, NY, USA, 2008.
42. Blanchard, P.; Bruening, E. *Mathematical Methods in Physics*; Birkhaeuser: Boston, MA, USA, 2003.
43. Barton, G. *An Introduction to Advanced Field Theory*; Interscience: Hoboken, NJ, USA, 1963.
44. Weinberg, S. *The Quantum Theory of Fields Volume 2*; Cambridge University Press: Cambridge, UK, 1996.
45. Steinmann, O. Perturbation Expansions in Axiomatic Field Theory. *Lect. Notes. Phys.* **1971**, *11*. [[CrossRef](#)]
46. Wightman, A.S. Introduction to some aspects of quantized fields. In *High Energy Electromagnetic Interactions and Field Theory*; Lévy, M., Ed.; Gordon and Breach: Philadelphia, PA, USA, 1967.
47. Haag, R. *Local Quantum Physics*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 1996.
48. Wreszinski, W.F. Equal time limits of anticommutators in relativistic theories. *Nuovo. Cim. A* **1971**, *1*, 691–705. [[CrossRef](#)]
49. Jaekel, C.D.; Mund, J. Canonical interacting quantum fields in two-dimensional de Sitter space. *Phys. Lett. B* **2017**, *772*, 786–790. [[CrossRef](#)]
50. Strocchi, F.; Wightman, A.S. Proof of the charge superselection rule in local relativistic quantum field theory. *J. Math. Phys.* **1974**, *15*, 2198. [[CrossRef](#)]
51. Lehmann, H. Über Eigenschaften von Ausbreitungsfunktionen und Renormierungskonstanten quantisierter Felder. *Nuovo Cimento* **1954**, *11*, 342. [[CrossRef](#)]
52. Lowenstein, J.H.; Swieca, J.A. Quantum electrodynamics in two dimensions. *Ann. Phys.* **1971**, *68*, 172–195. [[CrossRef](#)]
53. Steinmann, O. Perturbative QED in terms of gauge invariant fields. *Ann. Phys.* **1984**, *157*, 232–254. [[CrossRef](#)]
54. Houard, J.; Jouvet, B. Étude d'un modèle de champs quantifiés à constante de renormalisation nulle. *Nuovo Cim.* **1960**, *18*, 466. [[CrossRef](#)]
55. Araki, H.; Munakata, Y.; Kawaguchi, M.; Goto, T. Quantum field theory of unstable particles. *Progr. Theor. Phys.* **1957**, *17*, 419. [[CrossRef](#)]
56. Landsman, N.P. Non-shell unstable particles in thermal field theory. *Ann. Phys.* **1988**, *186*, 141–205. [[CrossRef](#)]
57. Martin, P.; Rothen, F. *Many Body Problems and Quantum Field Theory*, 2nd ed.; An Introduction; Springer: Berlin/Heidelberg, Germany, 2004.
58. Ynduráin, F.J. Definition of the Hamiltonian in a simple field-theoretical model. *J. Math. Phys.* **1966**, *7*, 1133. [[CrossRef](#)]
59. Sewell, G.L. *Quantum Theory of Collective Phenomena*; Oxford University Press: Oxford, UK, 1986.
60. Casher, A.; Kogut, J.; Susskind, L. Vacuum polarization and the quark-parton puzzle. *Phys. Rev. Lett.* **1973**, *31*, 792. [[CrossRef](#)]
61. Swieca, J.A. Solitons and confinement. *Fortschr. Phys.* **1977**, *25*, 303. [[CrossRef](#)]
62. Buchholz, D.; Fredenhagen, K. From path integrals to dynamical algebras: A macroscopic view of quantum physics. *Found. Phys.* **2020**, *50*, 727. [[CrossRef](#)]
63. Alicki, R.; Fannes, M.; Verbeure, A. Unstable particles and the Poincaré semigroup in quantum field theory. *J. Phys. A Math. Gen.* **1986**, *19*, 919. [[CrossRef](#)]
64. Sewell, G.L. *Quantum Mechanics and Its Emergent Macrophysics*; Princeton University Press: Princeton, NJ, USA, 2002.
65. Mund, J.; Rehren, K.H.; Schroer, B. Gauss' law and string-localised quantum field theory. *J. High Energy Phys.* **2020**, *1*, 1. [[CrossRef](#)]

-
66. Narnhofer, H.; Peter, I.; Hidding, W.T. How hot is de Sitter space? *Int. J. Mod. Phys. B* **1996**, *10*, 1507. [[CrossRef](#)]
 67. Bahns, D.; Fredenhagen, K.; Rejzner, K. Local nets of von Neumann algebras in the Sine-gordon model. *Commun. Math. Phys.* **2021**, *383*, 1975. [[CrossRef](#)]
 68. Alazzawi, S.; Lechner, G. Inverse scattering and locality in integrable field theories. *arXiv* **2016**, arXiv:1608.02359.
 69. Fröhlich, J.; Seiler, E. The massive Thirring-Schwinger model: Convergence of perturbation theory and particle structure. *Helv. Phys. Acta* **1976**, *49*, 889.
 70. Wightman, A.S.; Schweber, S. Configuration-space methods in relativistic quantum field theory. *Phys. Rev.* **1955**, *98*, 812. [[CrossRef](#)]
 71. Fröhlich, H. Interactions of electrons with lattice vibrations. *Proc. R. Soc. A* **1952**, *215*, 291.