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Review

Kerner Equation for Motion in a Non-Abelian Gauge Field

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Abstract: The equations of motion of an isospin-carrying particle in a Yang–Mills and gravitational field were first proposed in 1968 by Kerner, who considered geodesics in a Kaluza–Klein-type framework. Two years later, the flat space Kerner equations were completed by also considering the motion of the isospin by Wong, who used a field-theoretical approach. Their groundbreaking work was then followed by a long series of rediscoveries whose history is reviewed. The concept of isospin charge and the physical meaning of its motion are discussed. Conserved quantities are studied for Wu–Yang monopoles and diatomic molecules by using van Holten's algorithm.

Keywords: Kerner-Wong equation; isospin motion; non-Abelian gauge field

1. Introduction: A Short History of the Isospin

Certain ideas are put forward, then forgotten, and then reproposed again by various authors who ignore previous, and indeed each other's work. A typical example is that of an *isospin-carrying particle moving in a Yang–Mills field*, first studied by Kerner [1] Figure 1:

RYSZARD KERNER
Generalization of the Kaluza-Klein theory for an arbitrary non-abelian gauge group
Annales de l'I. H. P., section A, tome 9, n° 2 (1968), p. 143-152
<http://www.numdam.org/item?id=AIHPA_1968__9_2_143_0>

Figure 1. Kerner's paper in which the equation for particle in a Non-Abelian gauge field was first proposed.

Two years after Kerner's pioneering paper, Wong [2], who ignored all of Kerner's work and used a different field-theoretical framework, completed the Kerner Equation (17), below, which describes motion in ordinary space–time, with one for the dynamics of the isospin, Equation (18).

Their work was subsequently continued by many other researchers [3–15]. Jackiw and Manton [16], searching for a physical interpretation of some of the quantities found in the study of symmetries of gauge fields (re)covered the Kerner Equation (17) from a partial variational principle while assuming the isospin Equation (18).

These studies were parallelled by physical applications that include motion in the field of a non-Abelian monopole [17–19], which requires an extension to the Yang–Mills–Higgs systems [20–23].

Yet another application is to the *non-Abelian Aharonov experiment* proposed by Wu and Yang [24,25], elaborated in [26][1], which will be further studied in [28]. The effect is related to topological defects [29–31] and, more recently, to artificial gauge fields that can be produced in a laboratory [32–39].

As a physical illustration, we derived conserved quantities for Wu–Yang monopoles [40] and diatomic molecules [41,42].

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This review celebrates the 80th birthday of Richard Kerner by recounting the fascinating story of the isospin-carrying particles initiated by him when his given name was still "Ryszard".

2. Gauge Theory and the Kaluza–Klein Framework

2.1. Yang–Mills Theory

The concept of isotopic spin (in short: isospin) was introduced by Heisenberg in 1932 [43], who argued that a proton and a neutron should be viewed as two different states of the same particle, related by an "internal" SU(2) rotation[2].

Let us recall that electrodynamics is an Abelian gauge theory: it is described by a real 1-form $A = A_\mu dx^\mu$ called the vector potential, which is, however, determined only up to a gauge transformation, such as

$$A_\mu \to A_\mu - ig^{-1}\partial_\mu g\,, \tag{1}$$

where $g(x^\mu)$ is a U(1)-valued function in space–time.

Twenty years later, Yang and Mills (YM) generalized Maxwell's theory to non-Abelian fields, which take their values in the Lie algebra $\mathfrak{G} = \mathrm{su}(2)$ and can thus be acted upon by $G = \mathrm{SU}(2)$-valued gauge transformations [27,45]. In detail, YM fields are described by the Yang–Mills potential, represented either by a 3-vector $\boldsymbol{A} = (A^a_\mu)$, $a = 1,2,3$ or, alternatively, by anti-hermitian su(2) matrices, $A_\mu = A^a_\mu \frac{1}{2i}\sigma_a$, where the sigmas are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

which satisfy $[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c$. In what follows, we shall use mainly the matrix-formalism. Space–time indices will be denoted by Greek letters, typically $\mu, \nu, \ldots$ etc. Latin characters $a, b \ldots$ are used for the internal isospin indices. The Lie bracket in su(2) is $\big([A,B]\big)^a = \epsilon^a_{\ bc}A^b B^c$. The field strength of a Yang–Mills field is

$$F = \tfrac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu \quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\mu\right]. \tag{2}$$

The Lie algebra carries a metric given by the trace form $g_{ab}A^a B^b = -2\mathrm{tr}(AB)$ that we also denote by $A \cdot B$. For su(2), $g_{ab} = \delta_{ab}$.

The fundamental property of the Yang–Mills theory is its behavior under an SU(2)-valued gauge transformation [27], such as

$$A_\mu \to g^{-1}A_\mu\, g + g^{-1}\partial_\mu g\,, \qquad F_{\mu\nu} \to g^{-1}F_{\mu\nu}\, g\,, \tag{3}$$

where $g(x^\mu) \in \mathrm{SU}(2)$. A particle is coupled to the electromagnetic field A_μ by *minimal coupling*, which amounts to replacing ordinary derivatives with gauge-covariant derivatives [45],

$$\partial_\mu \to \partial_\mu - i\,A_\mu\,, \tag{4}$$

where the electric charge is scaled to one. In [1], Kerner argued that in a YM gauge field, this prescription should be replaced by an expression that (i) describes the properties of proton/neutron-type "particles with internal YM structure", and (ii) couples such a particle to the non-Abelian gauge potential: Rule (4) should be generalized to

$$\partial_\mu \to \partial_\mu + A_\mu \tag{5}$$

acting on fields in the fundamental representation. The non-Abelian coupling constant is scaled to unity.

What are the dynamics of such an isospin-carrying particle (also called a particle with an internal YM structure)? Kerner answers the question by considering a non-Abelian generalization of the Kaluza–Klein (KK) theory [46,47].

2.2. Abelian Kaluza–Klein Theory

Electromagnetism and gravitation were unified into a geometrical framework (now called fiber bundle theory) by Kaluza [46] and Klein [47] about 100 years ago[3].

It is assumed that the world has *four spatial* dimensions. However, one of them, which we denote by x^5, curled up to form a circle so small as to be unobservable. The basic assumption is that the correct vacuum is $M^4 \times \mathbb{S}^1_R$, the product of four-dimensional Minkowski space with the coordinates x^μ, $\mu = 0,1,2,3$, and with an internal circle of radius R.

Then, general relativity in five dimensions contains a local U(1) gauge symmetry arising from the isometry of the hidden fifth dimension. The extra components of the metric tensor constitute the gauge fields and could be identified with the electromagnetic vector potential.

The theory is invariant under general coordinate transformations that are independent of x^5. In addition to the ordinary four-dimensional coordinate transformations, we have a U(1) local gauge transformation

$$x^5 \to x^5 + \Lambda(x^\mu) \tag{6}$$

under which the $g_{\mu 5}$ component transforms to a $\mathrm{U}(1)$ gauge field,

$$g_{\mu 5}(x) \to g_{\mu 5}(x) + \partial_\mu \Lambda. \tag{7}$$

We write the metric with indices $A = \mu, 5$ as

$$ds^2 = g_{AB} dx^A dx^B \quad \text{where} \quad g_{AB} = \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}, \tag{8}$$

i.e.,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + (dx^5 + A_\mu dx^\mu)^2 . \tag{9}$$

Expressing the five-dimensional scalar curvature R_5 in four-dimensional terms, $R_5 = R_4 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ where R_4 is the *four*-dimensional curvature, the effective low-energy theory is described by the four-dimensional action

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-\det\left(g_{\mu\nu}\right)} \left(R_4 + \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \tag{10}$$

where $G = G_K / 2\pi R$ is Newton's constant. The internal radius R is determined by the electric charge. The motion is given by a five-dimensional geodesic,

$$\frac{d^2 x^A}{d\tau^2} + \Gamma^A_{BC} \frac{dx^B}{d\tau} \frac{dx^C}{d\tau} = 0 . \tag{11}$$

The KK space–time possesses a Killing vector, namely

$$K^A \frac{\partial}{\partial x^A} = \frac{\partial}{\partial x^5} , \tag{12}$$

which implies that

$$q = K_A \frac{dx^A}{d\tau} = \frac{dx^5}{d\tau} + A_\mu \frac{dx^\mu}{d\tau} \tag{13}$$

is a *constant of the motion* identified with the conserved electric charge. The remaining equations of motion then take the form of

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = q \left(g^{\mu\alpha} F_{\alpha\nu} \right) \frac{dx^\nu}{d\tau} , \tag{14}$$

where $\Gamma^{\mu}_{\alpha\beta}$ is the Levi–Civita connection constructed from the four-dimensional metric $g_{\mu\nu}$. On the right, we recognize the Lorentz force of electromagnetism.

2.3. Non-Abelian Generalization

Kerner, in his groundbreaking paper [1], proposed to derive the dynamics of an isospin-carrying particle in a Yang–Mills (YM) field by generalizing the Abelian KK framework to non-Abelian gauges. His framework was further generalized [4] and applied later to particle motion in a Yang–Mills field by projecting the geodesic motion to 4D space [15]. His idea (eqn. #(12) of [1]) was to replace the internal circle U(1) in the fifth dimension with the non-Abelian gauge group, SU(2), and the gauge potential in (8) by its non-Abelian counterpart.

The new key ingredient w.r. t. electromagnetism is the *isospin*, represented by a su(2) matrix,

$$Q = Q^a \,\frac{1}{2i}\sigma_a \in \mathrm{su}(2)\,, \tag{15}$$

which couples the particle to the YM field introduced in Section 2.1, A^a_μ and $F^a_{\alpha\beta}$, respectively. The covariant derivative is

$$D_\mu Q = \partial_\mu Q + [A_\mu, Q]\,. \tag{16}$$

The su(2)-valued YM potential is implemented in the isospin $Q \in \mathrm{su}(2)$ by commutation.

In a judicious coordinate system chosen by Kerner [1], the equations of motion for a test particle in the combined gravitational and gauge fields simplify to his eqn. # (34),

$$\boxed{\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = \left(g_{ab}Q^b\right)\left(g^{\mu\alpha}F^a_{\ \alpha\beta}\right)\frac{dx^\beta}{ds}\,.} \tag{17}$$

Generalizing the gauge group from the U(1) of electromagnetism to the Yang–Mills gauge group SU(2) is not without consequence though: unlike the electric charge in the electromagnetic theory, which is a conserved scalar, the isospin has its own dynamics indeed—it is *not* a constant but a vector that (as Kerner puts it) "*rotates, depending on the external field*".

The equations for the motion of the isospin,

$$\dot{Q} = [Q, A_\nu \dot{x}^\nu]\,, \tag{18}$$

where the "dot" is d/ds, were spelled out two years later by Wong [2]. In geometric language, the isospin is parallel-transported along the space–time trajectory, $x(t) = (x^\mu)$. Written in terms of the covariant derivative (16),

$$\boxed{D_s Q \;\equiv\; \dot{Q} + [A_\nu \dot{x}^\nu, Q] = 0} \tag{19}$$

this equation states that the isospin is *covariantly* (but not ordinarily) conserved. Equation (18) is consistent with Kerner's words, though, and also with what Yang and Mills state in [27], where they mention "isospin rotation".

One can wonder why Kerner did *not* spell out the equations of motion for the isospin explicitly. A real answer can be given only by him; however, one can try to guess what he might have had in his mind. One good reason might well have been that considering the isospin as a non-constant, non-Abelian analog of the constant electric charge could have appeared too radical and even shocking, and be therefore discarded[4].

Other subtle reasons related to the *gauge invariance* and the consequent problems of physical interpretation might also exist [49–53]. Another one could come from the experimental side.

Wong's approach [2] is radically different from that of Kerner: Instead of generalizing the classical dynamics of a charged particle moving in a curved space, he "dequantizes" the Dirac equation. Balachandran et al. [5,6] studied particles with an internal structure that were then recast in a symplectic framework by Sternberg [7–9], Weinstein [10], and Montgomery [14]. Duval [11–13] extended Souriau's approach [54] to particles with spin [12]—hitting yet another shocking idea: physicists, referring to Landau–Lifshitz, were firmly convinced that *classical spin just does not exist* and rejected Souriau's ideas [54] that were rooted in the representation theory.

Gauge fields with spontaneous symmetry breaking admit finite-energy static solutions with magnetic charges referred to as *non-Abelian monopoles* [19]. For an isospin-carrying particle in the field of a self-dual monopole [20–22], Fehér found, moreover, that outside the monopole core, where the $\mathrm{SU}(2)$ symmetry is spontaneously broken to $\mathrm{U}(1)$, the dynamics of a particle with isospin reduces to that of an electrically charged particle in the field of a Dirac monopole, combined with specific scalar potentials, familiar from the Abelian theory [55,56].

2.4. Fiber Bundles and a Symplectic Framework

Trautman [3] and Cho [4] reformulated the non-Abelian KK theory in terms of fiber bundles [57,58]: For the gauge group G, the field is described by a Lie algebra-valued connection form an α on a principal bundle $\mathfrak{P}$ with the structure group G over space–time, M. The YM potential A in Section 2 is the pull-back to M of the connection-1 formed by the section $M \to \mathfrak{P}$ of the bundle. A gauge transformation amounts to changing the section and results in (3). Choosing a section yields a local trivialization $\mathfrak{P} = M \times G$, and the YM connection form is written as

$$\alpha = A_\mu dx^\mu + g^{-1}dg\,. \tag{20}$$

Recall that the Maurer–Cartan form $g^{-1}dg$ takes its values in the Lie algebra $\mathfrak{G}$ of G. Using fiber bundles for gauge theory was advocated by T. T. Wu and C.N. Yang [24,25,59] in the monopole context[5]; see also [60,61].

A comprehensive KK unification of non-Abelian gauge fields with gravity in principal fiber bundle terms was put forward by Cho in [4], who derived a unified Einstein–Hilbert action in (4+n)-dimensions, both in the basis used by Kerner and also in a horizontal-lift basis, which diagonalizes the KK metric and generalizes (10).

Duval et al. [12,13] proposed an alternative, symplectic version, "à la Souriau" [54], reminiscent of but different from the Kaluza–Klein approach. Both theories use a higher-dimensional fiber bundle extension of the conventional space–time structure. Below, we summarize the main features of the Souriau framework:

1. The system is described by a fiber bundle $\mathcal{V}$ over space–time M called an *evolution space*—Souriau's "espace d'évolution".
2. The dynamics are discussed in terms of differential forms. The main tool is a 1-form $\tilde{\omega}$ on $\mathcal{V}$, whose exterior derivative $\tilde{\Omega} = d\tilde{\omega}$ is, in Souriau's language, "presymplectic", i.e., a closed 2-form that has a constant rank, $\dim \mathrm{Ker}\,\tilde{\Omega} = \mathrm{const}$. Then, the motions are the projections onto M of the integral submanifolds of the characteristic foliation of $\mathrm{Ker}\,\tilde{\Omega}$. Factoring out $\mathrm{Ker}\,\tilde{\Omega}$ yields $\mathcal{U}$, the *space of motions* (an abstract substitute for phase space—Souriau's "espace des mouvements" [54]). The presymplectic form $\tilde{\Omega}$ projects onto $\mathcal{U}$ as a symplectic form, i.e., one which is closed and has no kernel, as illustrated in Figure 2.
3. Group S is a symmetry for a system if it acts on the space of motions $\mathcal{U}$ by preserving the symplectic structure.
4. A system is *elementary* with respect to a symmetry group S if the action of the latter on $\mathcal{U}$ is transitive. Souriau's orbit construction [54] applies to an arbitrary symmetry group: The space of motions of an elementary system is, conversely, a (co)adjoint

orbit $\mathcal{O} = \{g^{-1}Q_0 g \,|\, g \in S\}$ of a basepoint Q_0 chosen in the Lie algebra $\in \mathfrak{S}$ of the symmetry group. $\mathcal{O}$ is endowed with its canonical symplectic form:

$$\tilde{\Omega} = d\tilde{\omega}, \qquad \tilde{\omega} = Q_0 \cdot (g^{-1}dg)\,. \tag{21}$$

In particular, applying the general construction to gauge group G endows the orbit in lieu of the Lie algebra $\mathfrak{G}$ with its canonical symplectic form.

5. The symmetry group S w.r.t., where the system is elementary, can be viewed as evolution space, $\mathcal{V} = S$ [62]; S is a principal fiber bundle over its (co)adjoint orbit $\mathcal{O}$.

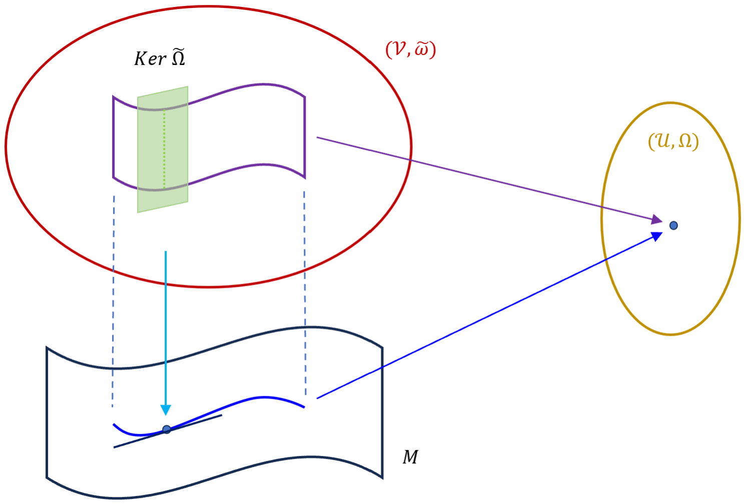

Figure 2. Souriau's framework: the **worldline** in **M** is the projection of a **characteristic sheet** of the 2-form $\tilde{\boldsymbol{\Omega}} = \mathbf{d}\tilde{\omega}$ on the **evolution space**, $\mathcal{V}$. Factoring out the characteristic foliation tangent to ker $\tilde{\boldsymbol{\omega}}$, $\mathcal{V}$ projects to the **space of motions**, $\mathcal{U}$, to which the 2-form $d\tilde{\omega}$ projects as a symplectic form $d\tilde{\omega}$ and corresponds to the worldlines in M.

Now, we spell out a simplified form of the Souriau–Duval framework in flat space. For further details, the reader is advised to consult [12,13].

• A massive free relativistic particle. The Poincaré group (P) is a fiber bundle over Minkowski spacetime M with the Lorentz group as structure group [12,62]. We represent the Poincaré group by 5×5 matrices $\begin{pmatrix} L & x \\ 0 & 1 \end{pmatrix}$, where the 4×4 matrix L belongs to the Lorentz subgroup and $x = (x^\mu) \in M$. Then,

$$\text{Poincaré}/\text{Lorentz} = M = \text{Minkowski}\,. \tag{22}$$

Moreover, we chose the basepoint Q_m in the Poincaré Lie algebra,

$$Q_m = \begin{pmatrix} 0 & x_m \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad x_m = m\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \in M \tag{23}$$

where $m = $ const, interpreted as rest mass. Writing the Maurer–Cartan form as

$$g^{-1}dg = \begin{pmatrix} L^{-1}dL & L^{-1}dx \\ 0 & 0 \end{pmatrix} \tag{24}$$

we obtain, on the Poincaré orbit $\mathcal{O}_m$ of Q_m,

$$\tilde{\omega}_m = mI_\mu dx^\mu \quad \Rightarrow \quad \tilde{\Omega}_m = mdI_\mu \wedge dx^\mu \tag{25}$$

where I_μ is a component of the Lorenz matrix, which is future pointing and belongs to the unit tangent bundle of M [12,13,62]. Then, the characteristic foliation projects in a suitable parametrisation to M onto a curve, which is a solution of

$$\dot{x}^\mu = I^\mu, \qquad \dot{I}^\mu = 0\,. \tag{26}$$

Equation (26) describes the geodesic motion in Minkowski space, i.e., the motion of a free relativistic particle with no spin[6].

• The free theory based on the Poincaré group P is readily extended to a (still free) relativistic particle with an internal structure, thus enlarging the evolution space and 1-form, P and $\tilde{\omega}_m$, respectively, to

$$\mathfrak{P} = P \times G \quad \text{and} \quad \omega = \tilde{\omega}_m + Q_0 \cdot g^{-1}dg\,, \tag{27}$$

where g takes its values in the gauge group G and the basepoint is $Q_0 \in \mathfrak{G}$ (the Poincaré part being understood).

The kernel of $\tilde{\Omega}$ in (21) implies the free Equation (26), supplemented by that for the isospin (19), whose properties will be further studied in Section 3. In geometric language, the isospin belongs to the associated bundle $\mathfrak{P} \times_G \mathcal{O}_0$, where $\mathcal{O}_0$ is the (co)adjoint orbit of Q_0 in $\mathfrak{G}$. In local coordinates, $\mathfrak{P} \times_G \mathcal{O}_0 \simeq M \times \mathcal{O}_0$ [13]. The space of motions is $\mathcal{O}_m \times \mathcal{O}_0$ endowed with the projection of $\tilde{\Omega}$ in (21).

For $G = \mathrm{U}(1)$, the free-charged particle is recovered, with Q identified as the constant electric charge.

• Minimal coupling to a Yang–Mills field amounts, in bundle language, to generalize (27) on $\mathfrak{P}$ by

$$\omega = \tilde{\omega}_m + Q_0 \cdot \alpha\,, \tag{28}$$

with $Q_0 \in \mathfrak{G}$, which, in view of (20), is indeed the geometric form of (5). In local coordinates,

$$\omega = \big(\partial_\mu + Q_0 \cdot A_\mu\big)dx^\mu + Q_0 \cdot g^{-1}dg\,. \tag{29}$$

The 2-form $\Omega = d\omega$ is, by the Cartan structure equations [57,58], p. 78,

$$\Omega = \Omega_0 + Q_0 \cdot d\alpha = \Omega_0 + Q_0 \cdot \big(D\alpha - [\alpha, \alpha]\big)\,, \tag{30}$$

where the last term also involves, in addition to the Lie bracket, the wedge product of the differential forms. Its kernel projects to the Kerner–Wong Equations (17) and (18) [13].

3. Physical Meaning of Isospin Dynamics

Limiting our investigations to flat Minkowski space, the Kerner Equation (17) simplify to

$$\ddot{x}_\mu = Q^a\, F^a_{\mu\nu}\, \dot{x}^\nu\,, \tag{31}$$

supplemented by the isospin Equation (18)[7].

To what extent is the isospin vector, Q, an analog of the *constant* electric charge? We argue that $Q = \mathrm{const}$ would be inconsistent with gauge invariance: if we had $\dot{Q} = 0$, the rhs of (18) would change under a gauge transformation as

$$0 = [Q, A_\mu \dot{x}^\mu] \to [Q, (g^{-1}A_\mu g)\dot{x}^\mu] + [Q, (g^{-1}\partial_\mu g)\dot{x}^\mu]\,,$$

and there is no reason for the rhs to vanish. The situation improves, though, if the gauge transformation is non-trivially implemented on the isospin[8]

$$Q \to g^{-1}Q\,g\,. \tag{32}$$

Then, the rhs of (18) would transform as

$$[Q, A_\mu \dot{x}^\mu] \to g^{-1}\Big\{[Q, A_\mu \dot{x}^\mu] + \Big(Q\partial_\mu g\, g^{-1} - \partial_\mu g\, g^{-1} Q\Big)\dot{x}^\mu\Big\}\, g\,.$$

The first term in the curly bracket would be perfect, but the second one would vanish only for $g = \text{const}$. However, the terms coming from $d(g^{-1}Q\,g)/ds$ cancel the unwanted terms, leaving us with the desired covariant transformation law cf. (32),

$$D_s Q \to g^{-1} D_s Q\, g\,. \tag{33}$$

Further insight into isospin dynamics can be gained by assuming, for simplicity, that the curvature of the connection form (in physical terms, the Yang–Mills field) is zero, $F = D\alpha = 0$, which is a gauge-independent statement by (3), and the space–time motion is free. Do we also have $\dot{Q} = 0$? The answer is *yes and no*. Let us explain. In topologically trivial situations[9], $F = D\alpha = 0$ implies that one can find a gauge where $A_\mu = 0$. Then, $\dot{Q} = 0$ follows obviously from the isospin Equation (18). This is a *gauge-dependent statement*, though: We are allowed to apply a gauge transformation by an arbitrary $G = \mathrm{SU}(2)$-valued function $g(x^\mu)$, which changes $A_\mu = 0$ to a pure gauge $A_\mu = g^{-1}\partial_\mu g$. *However, it also rotates the isospin* (32). Further, $d\big(g^{-1}Q\,g\big)/ds \neq 0$ in general; the gauge-covariant statement is that the isospin is covariantly conserved (19).

What is then the physical meaning of the isospin vector? First, we note that

$$|Q|^2 = Q^a Q^a \tag{34}$$

is gauge invariant, and deriving it implies, using (33), that the length $|Q|$ is conserved,

$$\frac{d|Q|}{ds} = 0 \quad \Rightarrow \quad |Q| = \text{const}\,. \tag{35}$$

The isospin is thus constrained to lie on an adjoint orbit of the gauge group G in its Lie algebra $\mathfrak{G}$, and in our case, to a sphere, $Q \in \mathcal{O} = \Big\{g^{-1}Q_0 g\,\Big|\, g \in \mathrm{SU}(2)\Big\} \approx \mathbb{S}^2$. This fact is behind the Souriau-type construction of isospin-extended models [12,13,63].

Which components of Q have a gauge invariant physical meaning? This question leads to the so-called "color problem" [65–67]. The point is the subtle difference between gauge transformations and internal symmetries [52,53].

In physical terms, can we implement an element of the gauge group on the physical fields? And if we can, will it be symmetry in the usual sense [68]? In bundle language, a gauge transformation acts on the fibers *from the right* [57,58], while symmetry should act *from the left* [52,53]. Can we transfer the right action to a left action? In geometric terms, "implementable" means that the $G = \mathrm{SU}(2)$ bundle $\mathfrak{P}$ should be reducible, and "symmetry" requires that the connection form α in (20), which represents the YM potential, should also be reducible to the reduced bundle.

When the underlying topology is non-trivial (as non-Abelian monopoles [19]), there can be an obstruction, such as *"global color can not be defined"*, as is mentioned in refs. [65–67]. Another physical instance is provided by the non-Abelian Aharonov–Bohm effect [24], for which there is no obstruction but an ambiguity of how it should be implemented [28].

4. Conservation Laws with Isospin

4.1. Van Holten's Covariant Framework

The Hamiltonian of a point particle of unit mass carrying isospin $\vec{Q} = (Q^a)$ that moves in a static YM field is

$$H = \frac{1}{2}\Big(\vec{p} - \vec{A}^a Q^a\Big)^2, \tag{36}$$

where we again scaled the coupling constant equal to one. Defining the covariant Poisson bracket as [69],

$$\{f,g\} = D_j f \frac{\partial g}{\partial \pi_j} - \frac{\partial f}{\partial \pi_j} D_j g + Q^a F^a_{jk} \frac{\partial f}{\partial \pi_j} \frac{\partial g}{\partial \pi_k} - f^{abc} \frac{\partial f}{\partial Q^a} \frac{\partial g}{\partial Q^b} Q^c, \tag{37}$$

where the f^{abc} are the structure constants of the Lie algebra, and the covariant phase-space derivative is

$$D_i f = \partial_i f - f^{abc} Q^a A^b_i \frac{\partial f}{\partial Q^c}. \tag{38}$$

The nonzero Poisson brackets are

$$\{x_i, p_j\} = \delta_{ij}, \quad \left\{Q^a, Q^b\right\} = -f^{abc} Q^c. \tag{39}$$

Then, the Hamilton equations with $\vec{\pi} = \dot{\boldsymbol{r}}$,

$$\dot{x}_i = \{x_i, H\}, \qquad \dot{\pi}_i = \{\pi_i, H\}, \qquad \dot{Q}^a = \{Q^a, H\} \tag{40}$$

allow us to recover the flat-space Kerner–Wong equations,

$$\ddot{x}^i = Q^a F^a_{ij} \dot{x}^j = Q^a \varepsilon^{ijk} \dot{x}^j B^k_a, \tag{41a}$$

$$\dot{Q}^a = f^{abc} \dot{x}^i A^i_b Q_c, \tag{41b}$$

equivalent to (31) and (18).

Following van Holten [69–72], constants of the motion can be sought by expanding into the powers of the covariant momentum,

$$q = C(\boldsymbol{r}) + C_i(\boldsymbol{r})\, \pi_i + \frac{1}{2!} C_{ij}(\boldsymbol{r})\, \pi_i \pi_j + \ldots \tag{42}$$

Skipping the Abelian case, we move directly to the non-Abelian one. Requiring q to Poisson-commute with the Hamiltonian thus yields a series of constraints, eqn. # (70) in [69].

$$\begin{aligned}
&D_i C = Q^a F^a_{ij} C_j, \\
&D_i C_k + D_j C_i = Q^a \left(F^a_{ik} C_{kj} + F^a_{jk} C_{ki}\right), \\
&D_i C_{jk} + D_j C_{ki} + D_k C_{ij} = Q^a \left(F^a_{il} C_{ljk} + F^a_{jl} C_{lki} + F^a_{kl} C_{lij}\right), \\
&\qquad\vdots \qquad\qquad\qquad\qquad\qquad \vdots
\end{aligned} \tag{43}$$

The expansion (43) can be truncated at a finite order when the covariant Killing equation is satisfied at some order n. When we have a Killing tensor, $D_{(i_1} C_{i_2 \ldots i_n)} = 0$, then we can set

$$C_{i_1 \ldots i_p} = 0 \tag{44}$$

for all $p \geq n$ and find a constant of the motion of the polynomial form [69]. Referring to the literature for details [69–72], we mention that in the Abelian theory, $\vec{Q}$ is just a constant identified with an electric charge.

$$q = \sum_{k=0}^{p-1} \frac{1}{k!} C_{i_1 \ldots i_k} \pi_{i_1} \ldots \pi_{i_k} \tag{45}$$

The van Holten algorithm can be generalized by adding a static scalar potential, which may also depend on the isospin. The Hamiltonian (36) then becomes

$$H = \frac{1}{2} \pi_i^2 + V\left(x^i, Q^a\right) \tag{46}$$

with the equations of motion

$$\ddot{x}^i = Q^a F^a_{ij} \dot{x}^j - D_i V \,, \tag{47a}$$

$$\dot{Q}^a = f^{abc} \dot{x}^i A^i_b Q_c + f^{abc} Q^b \frac{\partial V}{\partial Q^c} \,. \tag{47b}$$

A comparison with (41) then shows that (47a) picks up a covariant force term. Note also that when V does depend on $\vec{Q}$, the isospin is no longer parallel transported.

Generalizing (42) to isospin-dependent coefficients,

$$q(\boldsymbol{r}, \vec{Q}) = C(\boldsymbol{r}, \vec{Q}) + C_i(\boldsymbol{r}, \vec{Q})\, \pi_i + \frac{1}{2!} C_{ij}(\boldsymbol{r}, \vec{Q})\, \pi_i \pi_j + \ldots \tag{48}$$

the constraints (43) are also generalised [70]:

$$\begin{aligned}
&C_i D_i V + f^{abc} Q^a \frac{\partial C}{\partial Q^b} \frac{\partial V}{\partial Q^c} = 0 \,, \\
&D_i C = Q^a F^a_{ij} C_j + C_{ij} D_j V + f^{abc} Q^a \frac{\partial C_i}{\partial Q^b} \frac{\partial V}{\partial Q^c} \,, \\
&D_i C_j + D_j C_i = Q^a (F^a_{ik} C_{kj} + F^a_{jk} C_{ki}) + \; C_{ijk} D_k V + f^{abc} Q^a \frac{\partial C_{ij}}{\partial Q^b} \frac{\partial V}{\partial Q^c} \,, \\
&\qquad \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned} \tag{49}$$

New gradient-in-V terms thus arise, even when the potential does not depend on the isospin $V = V(\boldsymbol{r})$. These terms play a role for self-dual Wu–Yang monopoles [40] and diatoms [41], as seen in Sections 4.2 and 4.3, respectively.

1. When $C_i(\boldsymbol{r})$ is a Killing vector, then we have $p = 2$, and the expansion can be reduced to a linear expression,

$$q = C(\boldsymbol{r}) + C_i(\boldsymbol{r})\, \pi_i \,, \tag{50}$$

allowing us to recover the conserved and angular momentums [69]. Focusing our attention on the latter, we chose a unit vector $\vec{n}$. Then,

$$\vec{C} = \vec{n} \times \boldsymbol{r} \tag{51}$$

is a Killing vector for the rotations around $\vec{n}$ and thus generates conserved angular momentum $\vec{J}$. Furthermore, van Holten's recipe can also be applied to a Dirac monopole of charge $q = eg$, recovering the angular momentum vector

$$\vec{J} = \boldsymbol{r} \times \vec{\pi} - q\, \hat{\boldsymbol{r}} \,, \tag{52}$$

which includes the celebrated radial "spin from isospin" term [70,73].

2. Similarly, choosing unit vector $\vec{n}$ again,

$$C_{ij} = 2\delta_{ij}\, \vec{n} \cdot \boldsymbol{r} - (n_i x_j + n_j x_i) \tag{53}$$

is a Killing tensor of order 2, which generates the well-known Runge–Lenz vector of planetary motion, [69–72]

$$\vec{K} = \vec{\pi} \times \vec{J} + \alpha\, \hat{\boldsymbol{r}} \,. \tag{54}$$

More generally, the framework also applies to the so-called "MIC-Zwanziger" system [55,56], which combines a Dirac monopole of charge q with an arbitrary r^{-1} Newtonian and a fine-tuned inverse-square potential,

$$V(r) = \frac{q^2}{2r^2} + \frac{\alpha}{r} \,. \tag{55}$$

The combined system generalizes the well-known dynamical O(4)/O(3,1) symmetry of planetary motion spanned by the angular momentum and the Runge–Lenz vector, $\vec{J}$ in (52) and $\vec{K}$, respectively [55,56]. The relations

$$\vec{J} \cdot \hat{\boldsymbol{r}} = -q \quad \text{and} \quad \big[\vec{K} + \tfrac{\alpha}{q}\vec{J}\big] \cdot \boldsymbol{r} = \vec{J}^2 - q^2 \tag{56}$$

then imply that the motion is a conic section, as depicted in Figure 3.

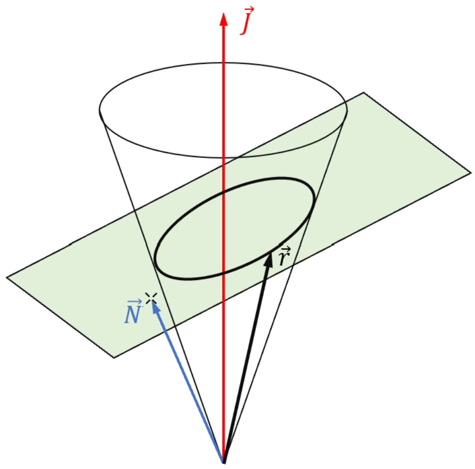

Figure 3. The conservation of the monopole angular momentum $\vec{J}$ implies that a particle moves on a cone, whose axis is $\vec{J}$. The O(4)/O(3,1) dynamical symmetry generated by the angular momentum and the Runge–Lenz vector $\vec{K}$ implies, in turn, that the trajectory lies in the plane perpendicular to $\vec{N} = \vec{K} + (\alpha/q)\vec{J}$ and is therefore a conic section.

Spin can also be considered [23].
We mentioned that the MIC–Zwanziger system is essentially equivalent to the one that describes the long-range monopole scattering [74] alias Kaluza–Klein monopole [48,75]; see also [22,76–81]. Dynamical symmetry for a self-dual Wu–Yang monopole [82] will be further analyzed in the next subsection.

3. The van Holten algorithm also applies to quantum dots, Hénon-Heiles, and Holt systems, with Killing tensors whose rank ranges from one to six are studied in [71,72].

4.2. Motion in the Wu–Yang Monopole Field

The Wu–Yang monopole [40] is given by the non-Abelian gauge potential with a "hedgehog" magnetic field:

$$A_i^a = \epsilon_{iak}\frac{x_k}{r^2}\,, \qquad F_{ij}^a = \epsilon_{ijk}\,\frac{x_k\,x_a}{r^4}\,. \tag{57}$$

The terminology is justified by presenting the field strength as

$$B_k^a = \tfrac{1}{2}\epsilon_{ijk}F_{ij}^a = \frac{x^k x^a}{r^4}\,. \tag{58}$$

The projection of the Wu–Yang magnetic field onto the "hedgehog" direction is thus

$$B_k^a \cdot \hat{x}^a = \frac{x^k}{r^3}\,, \tag{59}$$

which shows that the Wu–Yang magnetic field is that of a Dirac monopole of unit charge embedded into isospace. The remarkable feature of this expression is that the external and internal coordinates are correlated.

Let us consider an isospin-carrying particle moving in a Wu–Yang monopole field augmented with a rotationally invariant scalar potential $V(r)$ and inquire about the conserved quantities.

• A most important observation states that for an *arbitrary* radial potential $V(r)$, we can choose $C = \vec{Q} \cdot \hat{\boldsymbol{r}}$, which is covariantly constant, such as

$$D_i C = 0. \tag{60}$$

Then, (49) is satisfied with $C_i = C_{ij} = \ldots = 0$. The van Holten algorithm then applies, proving that the projection of the isospin onto the radial direction

$$q = C = \vec{Q} \cdot \hat{\boldsymbol{r}} \tag{61}$$

is a constant of the motion [69].

The Wu–Yang Ansatz (57) played an important role in later developments, as it prefigured the finite-energy non-Abelien monopoles [17–19]. The "hedgehog" is the large-r behavior of the Higgs field, and (61) is identified by the electric charge outside the monopole core. See, for e.g., [19] or [81] for comprehensive reviews.

• Applied to the Killing vector (51), we obtain the *conserved angular momentum* [69]

$$\vec{J} = \boldsymbol{r} \times \vec{\pi} - q\,\hat{\boldsymbol{r}}\,, \tag{62}$$

which looks formally identical to the Abelian expression (52). Remember, however, that q here is *not* a universal constant but the (conserved) *projection of the isospin* onto the "hedgehog" direction $\hat{\boldsymbol{r}}$, which mixes internal and external coordinates. Thus, we have the familiar radial term but now in the non-Abelian context.

• We now inquire about the quantities that are quadratic in the momentum. Inserting (53) into (49) from the second-order equation, we find

$$\vec{C} = \vec{\pi} \times (q\,\hat{\boldsymbol{r}})\,. \tag{63}$$

For which potentials do we obtain a quadratic conserved quantity? Referring to [69,70] for details, we record the answer:

$$C = \alpha\,\vec{\pi} \cdot \hat{\boldsymbol{r}} \quad \text{and} \quad V(r) = \frac{q^2}{2r^2} + \frac{\alpha}{r} + \beta\,, \tag{64}$$

where α and β are arbitrary constants. The coefficient of the r^{-2} term is correlated with the conserved charge q (61) as (55) for the MIC–Zwanziger system [55,56,79]. Collecting our results,

$$\vec{K} = \vec{\pi} \times \vec{J} + \alpha\,\hat{\boldsymbol{r}} \tag{65}$$

is a conserved Runge–Lenz vector for an isospin-carrying particle in the Wu–Yang monopole field combined with the fine-tuned potential $V(r)$ in (64).

The conserved quantities $\vec{J}$ and $\vec{K}$ span an $\mathrm{O}(4)/\mathrm{O}(3,1)$ dynamical symmetry that allows us to describe the large-r motion, both classically and quantum mechanically [21,22]. The trajectories are again conic sections of the MIC–Zwanziger system in Figure 3.

This generalizes the Abelian result to an isospin-carrying particle outside the core of a self-dual non-Abelian monopole [82]. This "coincidence" is explained as follows: For large-r, the gauge field of a self-dual non-Abelian monopole of charge m [19] is of the radially symmetric Wu–Yang form, Equation (57), completed with a "hedgehog" Higgs field,

$$\Phi^a = \left(1 - \frac{m}{r}\right)\frac{x^a}{r}\,, \tag{66}$$

whose direction is precisely

$$\widehat{\Phi} = \frac{\Phi}{|\Phi|} = \hat{\boldsymbol{r}}\,. \tag{67}$$

The projection of the isospin onto $\widehat{\Phi}$, q in (61) is thus conserved, and outside the core, the motion is that of an electric charge in the MIC–Zwanziger field [21,22,55,56,79]. The isospin-dependent dynamical symmetry is analyzed in [82].

4.3. Diatomic Molecules

In Ref. [41], Moody, Shapere, and Wilczek have shown that nuclear motion in a diatomic molecule can be described by the effective non-Abelian gauge field,

$$A_i^{\,a} = (1-\kappa)\epsilon_{iaj}\,\frac{x_j}{r^2} \quad \text{and} \quad F_{ij}^{\,a} = (1-\kappa^2)\epsilon_{ijk}\frac{x_k x_a}{r^4}\,, \tag{68}$$

respectively, where κ is a real parameter. For $\kappa = 0$, (68) is the field of the Wu–Yang monopole [40], (57). For other values of κ, it is a truly non-Abelian configuration (except for $\kappa = \pm 1$, when the field strength vanishes and (68) is a gauge transform of the vacuum).

Dropping scalar potential $V(r)$, we return to the Hamiltonian of a spinless particle with a non-Abelian structure, (36),

$$H = \tfrac{1}{2}\vec{\pi}^2, \quad \vec{\pi} = \vec{p} - \vec{A}\,. \tag{69}$$

Inquiring about conserved quantities, we note first that when $\kappa \neq 0$, then q is not covariantly conserved in general,

$$D_j q = \frac{\kappa}{r}\left(Q^j - q\frac{x_j}{r}\right) \neq 0\,, \tag{70}$$

implying that q in (61) is *not conserved* for $\kappa \neq 0$,

$$\{H, q\} = -\vec{\pi}\cdot\vec{D}q \neq 0\,, \tag{71}$$

unless the isospin is also radial. The bracketed quantity in (70) is indeed the non-aligned-with-the-field piece of the isospin. When the isospin spin and magnetic field happen to be aligned, then q in (61) *is* conserved.

Nor is the length of the to-become-charge q conserved in general:

$$\{H, q^2\} = -2\kappa q\,(\vec{\pi}\cdot\vec{D}q) \neq 0\,. \tag{72}$$

Whereas the length of the isospin, $\vec{Q}^2$, *is* conserved: $\{H, \vec{Q}^2\} = 0$. Thus, electric charge non-conservation comes from *isospin precession*, as in the non-Abelian Aharonov–Bohm effect [24,28,41]. For $\kappa = 0$, we recover the Wu–Yang case when q is conserved, as seen in Section 4.2.

The gauge field (68) is rotationally symmetric, and an isospin-carrying particle submitted to it has, nevertheless, conserved angular momentum [41,42]. Its form is, however, somewhat unconventional.

Our starting point is the first-order condition in (49). We consider first $V = 0$; then, with F_{jk}^a in (68), the equation to be solved is

$$D_i C = (1-\kappa^2)\,\frac{q}{r}\,\Big((\vec{n}\cdot\hat{\boldsymbol{r}})\frac{x_i}{r} - n_i\Big). \tag{73}$$

In the Wu–Yang case, $\kappa = 0$, we have $C = -\vec{n}\cdot q\,\hat{\boldsymbol{r}}$; however, for $\kappa \neq 0$, the to-be electric charge q is not conserved. Using (71) allows us to infer [70] that

$$C = -\vec{n}\cdot\Big(q\,\hat{\boldsymbol{r}} + \kappa(\vec{Q} - q\hat{\boldsymbol{r}})\Big)\,. \tag{74}$$

The conserved angular momentum is, therefore:

$$\vec{J} = \boldsymbol{r} \times \vec{\pi} - \vec{\Psi}, \tag{75}$$

$$\vec{\Psi} = q\,\hat{\boldsymbol{r}} + \kappa\,(\vec{Q} - q\hat{\boldsymbol{r}}) = q\,\hat{\boldsymbol{r}} + \kappa\,(\hat{\boldsymbol{r}} \times \vec{Q}) \times \hat{\boldsymbol{r}}, \tag{76}$$

which are consistent with the results in [41,42]. Note, however, that the spin-from-isospin contribution changes, w.r.t. (62):

$$q\,\hat{\boldsymbol{r}} \to \vec{\Psi}\,. \tag{77}$$

For $\kappa = 0$, we recover the Wu–Yang expression, (62). Eliminating $\vec{\pi}$ in favor of $\vec{p} = \vec{\pi} + \vec{A}$ allows us to rewrite the total angular momentum as

$$\vec{J} = \boldsymbol{r} \times \vec{p} - \vec{Q}\,, \tag{78}$$

making manifest the "spin from isospin term", which is, however, *not aligned with the "hedgehog" magnetic field. Consistent with* (70)*, the non-conservation of q in* (61) *is achieved precisely for this non-alignment*.

Restoring the potential, we see that again, due to the non-conservation of q, $D_j V \neq 0$ in general. The zeroth-order condition $\vec{C} \cdot \vec{D}V = 0$ in (49) is nevertheless satisfied if V is a radial function independent of $\vec{Q}$, $V = V(r)$, since then $\vec{D}V = \vec{\nabla}V$, which is perpendicular to infinitesimal rotations $\vec{C}$. Alternatively, a direct calculation using the same formulae allows us to confirm that $\vec{J}$ commutes with the Hamiltonian, $\{J_i, H\} = 0$.

Multiplying (78) by $\hat{\boldsymbol{r}}$ yields, once again,

$$\vec{J} \cdot \hat{\boldsymbol{r}} = -q \tag{79}$$

as in the Wu–Yang case. This is, however, less useful, as before, since q is *not a constant of the motion anymore*; thus, the angle between $\vec{J}$ and the radius vector $\boldsymbol{r}(t)$ is not constant either: the motion is not confined to a cone anymore.

Our attempts to find a conserved Runge–Lenz vector for the diatomic system have failed.

5. Conclusions and Outlook

The groundbreaking work of Kerner [1] and Wong [2], and continued by many others [3–16], allows us to gain an insight into the structure of non-Abelian gauge theory [27]. Our paper reviews the KK framework, retraces the chronological order of these discoveries, and analyzes the subtle physical meaning of isospin dynamics.

In addition to the conceptual works above, we underlined that the Kerner–Wong model [1,2] admits important physical applications.

The analysis of a self-dual monopole field [22,23,78,79] could be paralleled by studying motion in pure YM configurations with no scalar field [83,84] and also the monopole scattering [74,76–78,80] alias Kaluza–Klein monopole [48,75,77].

In Section 4, we applied van Holten's algorithm [69–72] conservation laws for a particle with isospin in the non-Abelian fields exemplified by a Wu–Yang monopole [40,69], and of diatomic molecules [41,42,70]. The O(4)/O(3,1) dynamical symmetry of a Wu–Yang monopole augmented with a self-dual Higgs field and implying elliptic trajectories is broken for diatomic molecules due to the non-conservation of the to-become electric charge, q in (61).

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Notes

1 To study the Non-Abelian Aharonov–Bohm effect was suggested to one of us (PAH) in the early eighties by Tai Tsun Wu, who also insisted that we should study the original paper of Yang and Mills [27]. We are grateful for his advices and would like to congratulate him on his 90th birthday.

2 The fascinating story of gauge theory is recounted by O'Raifeartaigh [44].

3 Our outline follows [48].

4 Duval's note [11] was rejected from Comptes Rendues de l'Académie des Sciences *without refereeing*.

5 Souriau has discussed the fiber bundle description of a monopole in "Prequantization" chapter of his never completed and thus unpublished revision of his book [54].

6 Spinning particles are obtained by modify the basepoint Q_0 in (23), cf. eqn. #(3.9) in [13].

7 The Equations (18)–(31) were also studied by refining the field-theoretical arguments of Wong [49]. The classical isospin is the expectation value of the non-Abelian field, $Q^a = \frac{1}{2}\int \psi^\dagger \sigma_a \psi\,.$

8 The covariant transformation rule (32) is consistent with the geometric status of the isospin viewed as a section of the associated bundle $\mathfrak{P} \times_G \mathcal{O}$ [13,63].

9 The topologically non-trivial case is studied in [26,63,64].

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