

Article

Efficient Computation of Null Affine Parameters

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Abstract: Finding affine parameters for null geodesics is often of considerable physical importance, especially when studying null geodesics or dealing with conservation laws and/or averaged energy conditions. But explicitly finding null affine parameters is also often quite tedious and can sometimes even be somewhat tricky. Herein we shall demonstrate that the existence of a conformally related spacetime containing a conformal Killing vector, timelike in the domain of outer communication, is quite sufficient to define a preferred set of spatial three-slices—on which a well-defined “affine” three-metric can be introduced to capture the notion of affine null parameter—*before* explicitly finding the null geodesics. The construction depends on the properties of conformal transformations and on the conserved quantity associated with the conformal Killing vector. Having the affine null parameter in hand before attempting to find the actual null geodesics often quite radically simplifies other parts of the analysis. We emphasize that the successful identification of affine null parameters is a general-purpose tool of wide applicability in both general relativistic and astrophysical settings.

Keywords: null geodesics; affine parameters; Killing vectors; conformal Killing vectors; conformal transformations

1. Introduction

Null curves, and more specifically null geodesics, and the affine parameters that are always associated with null geodesics, are standard tools in both general relativity and astrophysics [1–15]. At a technical level: For any smooth null curve $x^a(\lambda)$ with a null tangent vector $k^a = dx^a/d\lambda$ and some arbitrary parameterization λ , the general geodesic equation is $\nabla_k k \propto k$. It is only for the so-called affine parameterizations that the geodesic equation specifically reduces to $\nabla_k k = 0$. See, for instance, any of the references [1–15]¹. In any fixed but arbitrary spacetime, once a specific null geodesic has been found, finding the corresponding affine parameter is a straightforward if tedious textbook exercise—amounting to solving a first-order ordinary differential equation [1–15]. But can this process be simplified and rearranged? Under what conditions is it possible to choose an affine parameter first, *before* explicitly finding the null geodesics?

Identifying an appropriate affine null parameter is essential, for instance, when one is attempting to implement the averaged null energy condition (ANEC) or any of its variants [16–42]. The crucial point is that for the arbitrary parameterizations of the null curve, the ANEC is essentially content-free since the ANEC integral could then be arbitrarily adjusted by simply changing the geodesic parameterization. Only for affine parameters, since the relation between differing affine parameters is then very tightly constrained (via an affine rescaling $\lambda \rightarrow a\lambda + b$), is the positivity of the ANEC an invariant concept. Carefully formulating the ANEC (or its variants) is, in turn, essential for understanding the relevant singularity theorems [42–52], topological censorship [53–60], and/or Hawking’s chronology protection conjecture [61–68]. Further afield, the use of null affine parameters is extremely common in ongoing research, both theoretical and observational, such as, for instance, in [69–89].

Herein we shall demonstrate that the existence of a conformally related spacetime containing a (conformal) Killing vector, timelike in the domain of outer communication, is



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quite sufficient to allow us to make some quite general and powerful observations in terms of an “affine” three-metric defined on the spatial slices.

We shall consider both the usual ADM slicings, see [90–92] and various textbook presentations such as [1–15], and also the complementary “threaded” (Kaluza–Klein-like) formulations, see [93–95]. After some preliminary comments on generalities and nonaffine parameterizations, let us look at six cases of increasing levels of generality:

- Manifestly ultrastatic spacetimes.
- Manifestly static spacetimes.
- Stationary spacetimes.
- Spacetimes with a timelike conformal Killing vector.
- Conformal deformations.
- The completely general case.

As long as there is at least some symmetry, one can be reasonably explicit regarding the affine parameter. But ultimately, we shall see that in the completely general case, one can at best nonconstructively argue for the *existence* of a purely formal Finslerian [96–103] result.

2. Generalities

In complete generality, for any differentiable vector field $K^a(x)$ and any curve with tangent vector $k^a(\lambda)$, we have the standard mathematical identity

$$\frac{d}{d\lambda} \left[g \left(K, \frac{dx^a}{d\lambda} \right) \right] = \frac{d}{d\lambda} [K_a k^a] = K_{a;b} \frac{dx^b}{d\lambda} k^a + K_a \frac{d}{d\lambda} k^a = K_{(a;b} k^a k^b + K_a k^a{}_{;b} k^b. \quad (1)$$

That is,

$$\frac{d}{d\lambda} \left[g \left(K, \frac{dx^a}{d\lambda} \right) \right] = K_{(a;b} k^a k^b + K_a (\nabla_k k)^a. \quad (2)$$

We shall systematically apply and reapply this general identity in the discussion below. In particular, for an affinely parameterized geodesic (either null or timelike), since $\nabla_k k = 0$, we have [1–15]

$$\frac{d}{d\lambda} \left[g \left(K, \frac{dx^a}{d\lambda} \right) \right] = K_{(a;b} k^a k^b. \quad (3)$$

Ultimately, it is the presence of the quantity $K_{(a;b)}$ in this purely mathematical identity that focuses attention on Killing vectors, on conformal Killing vectors, and ultimately on spacetimes conformal to spacetimes containing a conformal Killing vector.

3. Nonaffine Parameterizations

It is sometimes (albeit somewhat rarely) useful to adopt a nonaffine parameterization. Let us see what happens in this situation, and see what we are seeking to *avoid* with the approach to be developed below. For a null geodesic with the tangent vector $k^a(\lambda) = dx^a(\lambda)/d\lambda$ that is not affinely parameterized, one has $\nabla_k k = f(\lambda) k(\lambda)$, so that the general identity (2) becomes

$$\frac{d}{d\lambda} \left[g \left(K, \frac{dx^a}{d\lambda} \right) \right] = K_{(a;b} k^a k^b + f(\lambda) K_a k^a. \quad (4)$$

This can be rewritten as

$$\exp \left(+ \int^\lambda f(\bar{\lambda}) d\bar{\lambda} \right) \frac{d}{d\lambda} \left[\exp \left(- \int^\lambda f(\bar{\lambda}) d\bar{\lambda} \right) g \left(K, \frac{dx^a}{d\lambda} \right) \right] = K_{(a;b} k^a k^b. \quad (5)$$

Comparing this with the affine case of Equation (3), we see that (up to an irrelevant overall constant normalization), we can set

$$\frac{d\lambda}{d\lambda_{\text{affine}}} = \exp \left(- \int^\lambda f(\bar{\lambda}) d\bar{\lambda} \right), \quad (6)$$

whence

$$\lambda_{\text{affine}}(\lambda) = \int^{\lambda} \exp\left(+ \int^{\tilde{\lambda}} f(\tilde{\lambda}) d\tilde{\lambda}\right) d\tilde{\lambda}. \quad (7)$$

While this can always be constructed, it is precisely the need for evaluating this somewhat messy double-iterated integral that we will seek to evade developing the formalism and discussion below.

Given the above discussion, one might reasonably ask whether or not nonaffine parameterizations are ever useful? One specific situation in which nonaffine parameters are arguably useful is in working with coordinates specifically adapted to match measurements carried out in a laboratory setting. To see how this works, recall that the equation for a geodesic in arbitrary nonaffine parameterization is

$$\frac{d^2 x^a}{d\lambda^2} + \Gamma^a_{bc} \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda} = f(\lambda) \frac{dx^a}{d\lambda}. \quad (8)$$

Now suppose that we can, in some region, “naturally” assign a time-coordinate t and three spatial coordinates x^i and adopt a coordinate patch (chart) of the form (for convenience, we set $c = 1$):

$$x^a = (t, x^i). \quad (9)$$

This is an extremely mild condition: we are simply choosing a coordinate patch in which one of the coordinates is timelike—and this can *always* be conducted locally. For instance, this is exactly what one would do in a laboratory setting. But this condition is more general than merely applying to laboratory situations and might also be profitably adopted for investigating solar system physics. At least locally, it is truly a generic choice, not a physical constraint.

Let us now agree to parameterize all curves by the nonaffine parameter t , denote coordinate derivatives with respect to t by an overdot^{2,3}, and so define

$$\frac{dx^a}{dt} = \left(\frac{dt}{dt}; \frac{dx^i}{dt} \right) = (1; \dot{x}^i). \quad (10)$$

Then, if we split the geodesic Equation (10) into time+space components, we find, from the first (zeroth) component, that

$$\frac{d^2 t}{dt^2} + \Gamma^t_{bc} \frac{dx^b}{dt} \frac{dx^c}{dt} = f(t) \frac{dt}{dt}. \quad (11)$$

whence

$$0 + \Gamma^t_{tt} + [\Gamma^t_{ti} + \Gamma^t_{it}] \dot{x}^i + \Gamma^t_{ij} \dot{x}^i \dot{x}^j = f(t). \quad (12)$$

That is, in these specific coordinates and with this specific choice of parameterization, the function $f(t)$ is completely specified by the equation

$$f(t) = \Gamma^t_{bc} \frac{dx^b}{dt} \frac{dx^c}{dt}. \quad (13)$$

Equivalently,

$$f(t) = \Gamma^t_{tt} + [\Gamma^t_{ti} + \Gamma^t_{it}] \dot{x}^i + \Gamma^t_{ij} \dot{x}^i \dot{x}^j. \quad (14)$$

So $f(t)$ is, in this framework, quadratic in the coordinate three-velocity.

Now substituting this back into the spatial part of the geodesic Equation (10), we see

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{bc} \frac{dx^b}{dt} \frac{dx^c}{dt} = f(t) \frac{dx^i}{dt}. \quad (15)$$

thence, the coordinate three-acceleration is

$$\ddot{x}^i = -\Gamma^i_{tt} - [\Gamma^i_{tj} + \Gamma^i_{jt}] \dot{x}^j - \Gamma^i_{jk} \dot{x}^j \dot{x}^k + [\Gamma^t_{tt} + [\Gamma^t_{tj} + \Gamma^t_{jt}] \dot{x}^j + \Gamma^t_{jk} \dot{x}^j \dot{x}^k] \dot{x}^i. \quad (16)$$

This can be rewritten as

$$\ddot{x}^i = -\Gamma^i_{tt} - [\Gamma^i_{tj} + \Gamma^i_{jt} - \Gamma^t_{tt} \delta^i_j] \dot{x}^j - [\Gamma^i_{jk} - (\Gamma^t_{tj} + \Gamma^t_{jt}) \delta^i_k] \dot{x}^j \dot{x}^k + [\Gamma^t_{jk}] \dot{x}^j \dot{x}^k \dot{x}^i. \quad (17)$$

Thus, in this coordinate system, with this choice of parameterization, a geodesic experiences a (cubic) velocity-dependent (but nevertheless composition-independent and universal) “coordinate 3-acceleration”. This is *not* the three-acceleration; it is a coordinate three-acceleration. We are looking at a geodesic, so the four-acceleration is identically zero⁴.

If the object in question is a massive particle that is momentarily at rest ($\dot{x}^i = 0$), then

$$\ddot{x}^i \rightarrow -\Gamma^i_{tt}. \quad (18)$$

Note that this statement is *exact*; it relies on picking an appropriate coordinate system in which the particle is momentarily at rest but does not make any weak-field approximation. As a counterpoint, for a massless particle such as a photon, we always have $\dot{x}^i \neq 0$, and the three-velocity dependent terms cannot be neglected. This gives a *qualitative* and *generic* explanation for why the three-acceleration of photons differs from the three-acceleration of planets. With a little extra work, the discussion can be made fully *quantitative* and *explicit*.

Overall, while nonaffine parameterizations are sometimes useful for specific purposes, they tend to be highly coordinate-dependent and are not particularly easy to generalize. For instance, in the current framework, we have the rather messy (but fully explicit) result

$$\lambda_{\text{affine}}(t) = \int^t \exp\left(+ \int^{\tilde{t}} \left\{ \Gamma^t_{tt} + [\Gamma^t_{ti} + \Gamma^t_{it}] \dot{x}^i + \Gamma^t_{ij} \dot{x}^i \dot{x}^j \right\}_{\tilde{t}} d\tilde{t}\right) d\tilde{t}. \quad (19)$$

In subsequent sections below, we shall use symmetries and partial symmetries to (as much as possible) side step the need for explicit computations of this type.

4. Manifestly Ultrastatic Spacetimes

An ultrastatic spacetime is characterized by the global existence of a hypersurface-orthogonal timelike unit-norm Killing vector, satisfying $g(K, K) = -1$. See, for instance, references [104–111]. In any ultrastatic spacetime, one can (without a loss of generality) choose coordinates to make the metric block diagonal and the ultrastatic condition manifest, such that

$$g_{ab} = \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & g_{ij} \end{array} \right]. \quad (20)$$

Here, the three-metric $g_{ij}(\vec{x})$ is time-independent. We emphasize that while the metric can be put in this form, there is an infinity of other coordinate systems that do not share this property. If the metric is presented in the form above, then we say that the ultrastatic symmetry is manifest. Working in other coordinate systems is possible but may or may not be particularly useful depending on the questions being addressed. In the manifestly ultrastatic coordinate system for any null curve, we have

$$-dt^2 + g_{ij} dx^i dx^j = 0, \quad (21)$$

implying (up to an unimportant sign)

$$dt = \sqrt{g_{ij} dx^i dx^j}. \quad (22)$$

Thence, in the current context, for any affinely parameterized null geodesic, one has the conservation law [1–15]

$$g\left(K, \frac{dx^a}{d\lambda}\right) = g_{ta} \frac{dx^a}{d\lambda} = -\frac{dt}{d\lambda} = \text{constant} = \epsilon. \quad (23)$$

Thence, along any affinely parameterized null geodesic in an ultrastatic spacetime, up to an undetermined constant, one has

$$d\lambda \propto dt = \sqrt{g_{ij} dx^i dx^j}. \quad (24)$$

That is, in any ultrastatic spacetime, the preferred time coordinate, or equivalently the spatial distance traveled, can be used as an affine parameter. While this particular ultrastatic situation is almost trivial, the discussion now readily generalizes to both static and stationary spacetimes—and beyond.

A specific class of examples of ultrastatic spacetimes that arises from outside the general relativity community is the class of Lorentzian metrics describing index-gradient methods in optics. Let $n(\vec{x})$ be a position-dependent refractive index (achieved, for instance, by doping the background optical medium with various admixtures of other materials). Then, define the ultrastatic metric

$$g_{ij}(\vec{x}) = n^2(\vec{x}) \delta_{ij}; \quad g_{ab}(\vec{x}) = -1 \oplus g_{ij}(\vec{x}). \quad (25)$$

This ultrastatic metric adequately describes the propagation of light in the ray optics limit, and from the above discussion, we see

$$d\lambda \propto dt = n(\vec{x}) ||d\vec{x}||. \quad (26)$$

Thence, in index-gradient optics, the affine parameter is simply the optical distance.

5. Manifestly Static Spacetimes

A static spacetime is fully characterized by the global existence of a hypersurface orthogonal Killing vector K , which is timelike (that is, $g(K, K) < 0$) in the domain of outer communication. See, for instance, references [1–15,112–117]. In a static spacetime, one can (without a loss of generality) choose coordinates to make the metric block diagonal and the static condition manifest, such that

$$g_{ab} = \left[\begin{array}{c|c} -N^2 & 0 \\ \hline 0 & g_{ij} \end{array} \right], \quad (27)$$

where both $N(\vec{x})$ and $g_{ij}(\vec{x})$ are time-independent (again, while this can be conducted, it is not strictly speaking necessary to do so. But making the symmetry manifest by the judicious choice of coordinates simplifies the discussion).

Then, for any null curve

$$-N^2 dt^2 + g_{ij} dx^i dx^j = 0, \quad (28)$$

implying (up to an unimportant sign)

$$N dt = \sqrt{g_{ij} dx^i dx^j}. \quad (29)$$

For any affinely parameterized null geodesic, one has the conservation law

$$g\left(K, \frac{dx^a}{d\lambda}\right) = g_{ta} \frac{dx^a}{d\lambda} = -N^2 \frac{dt}{d\lambda} = \text{constant} = \epsilon. \quad (30)$$

Thence, along any affinely parameterized null geodesic, up to some undetermined arbitrary nonzero constant, one has

$$d\lambda \propto N^2 dt = N \sqrt{g_{ij} dx^i dx^j} = \sqrt{N^2 g_{ij} dx^i dx^j}. \quad (31)$$

This can also be written as

$$d\lambda \propto -g_{tt} dt = \sqrt{-g_{tt} g_{ij} dx^i dx^j} = \sqrt{h_{ij} dx^i dx^j}. \quad (32)$$

Note carefully that the “affine” three-metric $h_{ij} = -g_{tt} g_{ij}$ appearing above is *not* what is sometimes called the “optical metric”. The optical metric would instead be the conformally related quantity

$$g_{ij}^{optical} = \frac{g_{ij}}{N^2} = -\frac{g_{ij}}{g_{tt}} = \frac{h_{ij}}{g_{tt}^2}. \quad (33)$$

While the optical metric is useful for determining paths of light rays, it is the affine three-metric that is useful for determining affine parameters. More on these issues later.

Examples:

- Schwarzschild spacetime in curvature coordinates:

One has

$$ds^2 = -(1 - 2m/r)dt^2 + \frac{dr^2}{1 - 2m/r} + r^2 d\Omega^2; \quad (34)$$

then

$$d\lambda \propto \sqrt{dr^2 + (r^2 - 2mr)d\Omega^2}. \quad (35)$$

So in particular, for radial null geodesics, without any further calculation, $\lambda \propto r$ (this is tolerably well-known, though it is usually checked by a brute-force analysis of the radial null geodesics).

- Reissner–Nordström spacetime in curvature coordinates:

One has

$$ds^2 = -(1 - 2m/r + q^2/r^2)dt^2 + \frac{dr^2}{1 - 2m/r + q^2/r^2} + r^2 d\Omega^2; \quad (36)$$

then

$$d\lambda \propto \sqrt{dr^2 + (r^2 - 2mr + q^2)d\Omega^2}. \quad (37)$$

So in particular, for radial null geodesics, without any further calculation, $\lambda \propto r$ (this is tolerably well-known, though it is usually checked by a brute-force analysis of the radial null geodesics).

- Static spherical symmetry (in curvature coordinates):

One has

$$ds^2 = -e^{-2\Phi(r)}[1 - 2m(r)/r]dt^2 + \frac{dr^2}{1 - 2m(r)/r} + r^2 d\Omega^2; \quad (38)$$

then

$$d\lambda \propto \sqrt{e^{2\Phi(r)}\{dr^2 + [1 - 2m(r)/r]r^2 d\Omega^2\}}. \quad (39)$$

So in particular, for radial null geodesics, the affine parameter is

$$d\lambda \propto e^{\Phi(r)} dr; \quad \lambda \propto \int e^{\Phi(r)} dr. \quad (40)$$

This result does not appear to be particularly well-known (but see, for instance [15]).

- Static spherical symmetry (in Buchdahl coordinates):
One has [118–120]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \Sigma(r)^2 d\Omega^2; \quad (41)$$

then

$$d\lambda \propto \sqrt{dr^2 + f(r)\Sigma(r)^2 d\Omega^2}. \quad (42)$$

So in particular, for radial null geodesics in Buchdahl coordinates, the affine parameter is simply the radial coordinate

$$d\lambda \propto dr; \quad \lambda \propto r. \quad (43)$$

This result does not appear to be at all well-known.

- Exponential metric:
One has [121]

$$ds^2 = -e^{-2m/r}dt^2 + e^{2m/r}\{dr^2 + r^2 d\Omega^2\}; \quad (44)$$

then

$$d\lambda \propto \sqrt{dr^2 + r^2 d\Omega^2}. \quad (45)$$

So for the exponential metric, the null affine parameter is simply the naive distance $ds_0 = \sqrt{dr^2 + r^2 d\Omega^2}$. This result does not appear to be particularly well-appreciated.

- Black bounce spacetimes:
One has [122,123]

$$ds^2 = -\left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{\sqrt{r^2 + a^2}}} + (r^2 + a^2) d\Omega^2; \quad (46)$$

then

$$d\lambda \propto \sqrt{dr^2 + \left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right)(r^2 + a^2)d\Omega^2}. \quad (47)$$

So for the black bounce metric, the radial null affine parameter is simply $\lambda \propto r$. This result does not appear to be particularly well-appreciated.

- More generally, consider Synge's "conformastat" metrics [124] (signifying: static, spatially conformally flat). The line element can then without a loss of generality be cast in the form

$$ds^2 = -e^{2\Phi_0(\vec{x})} dt^2 + e^{2\Phi_1(\vec{x})} d\vec{x} \cdot d\vec{x}. \quad (48)$$

That is, the spacetime metric can be put into the form

$$g_{ab} = \left[\begin{array}{c|c} -e^{2\Phi_0(\vec{x})} & 0 \\ \hline 0 & e^{2\Phi_1(\vec{x})} \delta_{ij} \end{array} \right]. \quad (49)$$

Then, the affine three-metric for these conformastat spacetimes is itself conformally flat:

$$h_{ij} = e^{2[\Phi_0(\vec{x}) + \Phi_1(\vec{x})]} \delta_{ij}, \quad (50)$$

corresponding to

$$d\lambda = e^{[\Phi_0(\vec{x}) + \Phi_1(\vec{x})]} ||d\vec{x}||. \quad (51)$$

So for the conformastat metrics, the null affine parameter is simply proportional (in a position-dependent manner) to the naive distance $ds_0 = ||d\vec{x}||$.

- A special case of the conformastat metrics is $\Phi_0(\vec{x}) = -\Phi_1(\vec{x})$ (this is a generalization of the exponential metric mentioned above). The line element can then without a loss of generality be cast in the form

$$ds^2 = -e^{2\Phi(\vec{x})} dt^2 + e^{-2\Phi(\vec{x})} d\vec{x} \cdot d\vec{x}. \quad (52)$$

That is, the spacetime metric can be put into the form

$$g_{ab} = \left[\begin{array}{c|c} -e^{2\Phi(\vec{x})} & 0 \\ \hline 0 & e^{-2\Phi(\vec{x})} \delta_{ij} \end{array} \right]. \quad (53)$$

Then, the affine three-metric for this specific subclass of conformastat spacetimes is flat:

$$h_{ij} = \delta_{ij}, \quad d\lambda = ||d\vec{x}||. \quad (54)$$

So for this specific subclass of conformastat metrics, the null affine parameter is simply the naive distance $ds_0 = ||d\vec{x}||$.

Many other manifestly static examples could be considered, but instead let us now consider the mathematically more general, and astrophysically more important, case of stationary (nonstatic) spacetimes.

6. Stationary Spacetimes

A stationary spacetime is characterized by the global existence of a Killing vector K (not necessarily hypersurface orthogonal), which is timelike (that is, $g(K, K) < 0$) in the domain of outer communication. See references [125–138].

6.1. ADM Form

In any stationary spacetime, one can without a loss of generality choose coordinates such that the metric is of ADM form, see references [1–15,90–92], and all the metric components are time-independent

$$g_{ab} = \left[\begin{array}{c|c} -N^2 + g^{kl}v_kv_l & v_i \\ \hline v_j & g_{ij} \end{array} \right]. \quad (55)$$

For completeness, note

$$g^{ab} = \left[\begin{array}{c|c} -N^{-2} & v_i/N^2 \\ \hline v_j/N^2 & g^{ij} - v^iv^j/N^2 \end{array} \right]. \quad (56)$$

Here, $g^{ij} = [g_{ij}]^{-1}$ and $v^i = g^{ij}v_j$. Then, for any null curve

$$-(N^2 - g^{kl}v_kv_l)dt^2 + 2v_idx^i dt + g_{ij}dx^i dx^j = 0. \quad (57)$$

For any affinely parameterized null geodesic, one has the conservation law

$$g\left(K, \frac{dx^a}{d\lambda}\right) = g_{ta} \frac{dx^a}{d\lambda} = -(N^2 - g^{kl}v_kv_l) \frac{dt}{d\lambda} + v_i \frac{dx^i}{d\lambda} = \text{constant} = \epsilon. \quad (58)$$

Let us rewrite this as

$$-\sqrt{N^2 - g^{kl}v_kv_l} \frac{dt}{d\lambda} + \frac{v_i}{\sqrt{N^2 - g^{kl}v_kv_l}} \frac{dx^i}{d\lambda} = \frac{\epsilon}{\sqrt{N^2 - g^{kl}v_kv_l}}. \quad (59)$$

Squaring both sides:

$$(N^2 - g^{kl}v_kv_l) \left(\frac{dt}{d\lambda}\right)^2 - 2v_i \frac{dx^i}{d\lambda} \frac{dt}{d\lambda} + \frac{\left(v_i \frac{dx^i}{d\lambda}\right)^2}{(N^2 - g^{kl}v_kv_l)} = \frac{\epsilon^2}{(N^2 - g^{kl}v_kv_l)}. \quad (60)$$

That is

$$(N^2 - g^{kl}v_kv_l)dt^2 - 2v_id x^i dt + \frac{(v_id x^i)^2}{(N^2 - g^{kl}v_kv_l)} = \frac{\epsilon^2}{(N^2 - g^{kl}v_kv_l)}d\lambda^2. \quad (61)$$

Now add the two Equations (57) and (61) derived above to obtain

$$g_{ij}dx^i dx^j + \frac{(v_id x^i)^2}{(N^2 - g^{kl}v_kv_l)} = \frac{\epsilon^2}{(N^2 - g^{kl}v_kv_l)}d\lambda^2. \quad (62)$$

Rearranging

$$\epsilon^2 d\lambda^2 = (N^2 - g^{kl}v_kv_l)g_{ij}dx^i dx^j + (v_id x^i)^2. \quad (63)$$

That is

$$\epsilon^2 d\lambda^2 = \{(N^2 - g^{kl}v_kv_l)g_{ij} + v_iv_j\} dx^i dx^j. \quad (64)$$

Thence, along any affinely parameterized null geodesic in any stationary spacetime, one has

$$d\lambda \propto \sqrt{\{(N^2 - g^{kl}v_kv_l)g_{ij} + v_iv_j\} dx^i dx^j}. \quad (65)$$

Defining the affine three-metric

$$h_{ij} = (N^2 - g^{kl}v_kv_l)g_{ij} + v_iv_j = -g_{tt}g_{ij} + g_{ti}g_{tj}, \quad (66)$$

we see that for any affinely parameterized null geodesic in any stationary spacetime, one has

$$d\lambda \propto \sqrt{h_{ij}dx^i dx^j}. \quad (67)$$

We emphasize: the affine three-metric is very simply determined by the lapse, shift, and physical three-metric appearing in the ADM decomposition.

6.2. Threaded Form

Alternatively, one can choose to put the metric in so-called “threaded” (Kaluza–Klein-like) form [93–95]:

$$ds^2 = -V^2(dt + \omega_i dx^i)^2 + \gamma_{ij}dx^i dx^j. \quad (68)$$

That is,

$$g_{ab} = \left[\begin{array}{c|c} -V^2 & -V^2\omega_j \\ \hline -V^2\omega_i & \gamma_{ij} - V^2\omega_i\omega_j \end{array} \right]. \quad (69)$$

Qualitatively similar metrics, though typically in higher space–time dimensions, commonly arise in Kaluza–Klein theory [139–143]. For completeness, note

$$g^{ab} = \left[\begin{array}{c|c} -V^{-2} + g^{ij}\omega_i\omega_j & -\omega^j \\ \hline -\omega^i & \gamma^{ij} \end{array} \right]. \quad (70)$$

Here, $\gamma^{ij} = [\gamma_{ij}]^{-1}$ and $\omega^i = \gamma^{ij}\omega_j$. That is, the threaded form of the metric is in some sense dual to the ADM form of the metric.

Then, in threaded form, the affine three-metric is particularly simple:

$$h_{ij} = -g_{tt}g_{ij} + g_{ti}g_{tj} = V^2(\gamma_{ij} - V^2\omega_i\omega_j) + (V^2\omega_i)(V^2\omega_j) = V^2\gamma_{ij}. \quad (71)$$

Thence,

$$d\lambda \propto |V|\sqrt{\gamma_{ij}dx^i dx^j}. \quad (72)$$

We emphasize: the affine three-metric is very simply determined by the quantities appearing in the threading decomposition.

These two analyses completely settle the stationary case.

6.3. Examples

- Strong Painlevé–Gullstrand metrics (static but not manifestly so):
In this situation, one has [144–147]

$$g_{ab} = \left[\begin{array}{c|c} -1 + \delta^{kl} v_k v_l & v_i \\ \hline v_j & \delta_{ij} \end{array} \right]. \quad (73)$$

Thence,

$$g_{ij} \rightarrow \delta_{ij}; \quad N \rightarrow 1; \quad d\lambda \propto \sqrt{\{(1 - \delta^{kl} v_k v_l) \delta_{ij} + v_i v_j\}} dx^i dx^j. \quad (74)$$

That is,

$$d\lambda \propto \sqrt{(\delta_{ij} - |v|^2 \delta_{ij} - \hat{v}_i \hat{v}_j)} dx^i dx^j. \quad (75)$$

Then, for null geodesics parallel to the flow $\hat{v}$, we have $d\lambda \propto \sqrt{\delta_{ij} dx^i dx^j}$, while for null geodesics perpendicular to the flow $\hat{v}$, we have $d\lambda \propto \sqrt{-g_{tt} \delta_{ij} dx^i dx^j}$. That is, $d\lambda \propto \sqrt{1 - |v|^2} \sqrt{\delta_{ij} dx^i dx^j}$.

- Weak Painlevé–Gullstrand metrics (static but not manifestly so):
In this situation, one has [144–147]

$$g_{ab} = \left[\begin{array}{c|c} -N^2 + \delta^{kl} v_k v_l & v_i \\ \hline v_j & \delta_{ij} \end{array} \right]. \quad (76)$$

Thence,

$$g_{ij} \rightarrow \delta_{ij}; \quad d\lambda \propto \sqrt{\{(N^2 - \delta^{kl} v_k v_l) \delta_{ij} + v_i v_j\}} dx^i dx^j. \quad (77)$$

That is

$$d\lambda \propto \sqrt{(N^2 \delta_{ij} - |v|^2 \delta_{ij} - \hat{v}_i \hat{v}_j)} dx^i dx^j. \quad (78)$$

Then, for null geodesics parallel to the flow $\hat{v}$, we have $d\lambda \propto N \sqrt{\delta_{ij} dx^i dx^j}$, while for null geodesics perpendicular to the flow $\hat{v}$, we have $d\lambda \propto \sqrt{-g_{tt} \delta_{ij} dx^i dx^j}$. That is, $d\lambda \propto \sqrt{N^2 - |v|^2} \sqrt{\delta_{ij} dx^i dx^j}$.

- Boyer–Lindquist metrics (not necessarily limited to Kerr or Kerr–Newman spacetime):
One has, see references [125–138, 148–152],

$$g_{ij} = \left[\begin{array}{ccc} g_{rr} & 0 & 0 \\ 0 & g_{\theta\theta} & 0 \\ 0 & 0 & g_{\phi\phi} \end{array} \right]_{ij}; \quad v_i = (0, 0, v_\phi) = (0, 0, g_{t\phi}) \quad (79)$$

Then,

$$d\lambda \propto \sqrt{\{-g_{tt} g_{ij} + v_i v_j\}} dx^i dx^j. \quad (80)$$

That is,

$$d\lambda \propto \sqrt{\left[\begin{array}{ccc} -g_{tt} g_{rr} & 0 & 0 \\ 0 & -g_{tt} g_{\theta\theta} & 0 \\ 0 & 0 & -g_{tt} g_{\phi\phi} + g_{t\phi}^2 \end{array} \right]_{ij}} dx^i dx^j. \quad (81)$$

The affine three-metric is singular at both the ergosurfaces, where $g_{tt} = 0$, and at the horizons, where $g_{tt} g_{\phi\phi} - g_{t\phi}^2 = 0$.

- For the Kerr spacetime itself, the affine three-metric is particularly simple. A brief calculation (in Boyer–Lindquist coordinates) yields

$$d\lambda \propto \sqrt{\begin{bmatrix} \frac{r^2-2mr+a^2\cos^2\vartheta}{r^2-2mr+a^2} & 0 & 0 \\ 0 & r^2-2mr+a^2\cos^2\vartheta & 0 \\ 0 & 0 & (r^2-2mr+a^2)\sin^2\vartheta \end{bmatrix}}_{ij} dx^i dx^j. \quad (82)$$

This can also be written as

$$d\lambda \propto \sqrt{\begin{bmatrix} 1 - \frac{a^2\sin^2\vartheta}{r^2-2mr+a^2} & 0 & 0 \\ 0 & r^2-2mr+a^2\cos^2\vartheta & 0 \\ 0 & 0 & (r^2-2mr+a^2)\sin^2\vartheta \end{bmatrix}}_{ij} dx^i dx^j. \quad (83)$$

It is somewhat unusual to have any Kerr-related result looking relatively simple. In particular, for photons emitted along the axis of rotation, we have $\lambda \propto r$, as we saw happens for Schwarzschild. More subtly, for null geodesics confined to the equatorial plane $\vartheta = \pi/2$, we can restrict the spatial coordinates to $x^i = (t, \phi)$ and introduce an affine two-metric

$$d\lambda \propto \sqrt{\begin{bmatrix} 1 - \frac{a^2}{r^2-2mr+a^2} & 0 \\ 0 & (r^2-2mr+a^2) \end{bmatrix}}_{ij} dx^i dx^j. \quad (84)$$

Many other examples of stationary spacetimes could be considered, but let us now turn to some time-dependent situations.

7. Spacetimes with Timelike Conformal Killing Vector

Suppose now that we have a preferred timelike vector field K^a , which we do not necessarily insist is a Killing vector.

7.1. ADM Form

Consider the timelike integral curves of K and take a spatial slice transverse to them; note that we only require transversality, not orthogonality. Then, without any further loss of generality, let us set up an ADM coordinate system in which both

$$g_{ab}(t, \vec{x}) = \left[\begin{array}{c|c} -N^2 + g^{kl}v_kv_l & v_i \\ \hline v_j & g_{ij} \end{array} \right], \quad (85)$$

and

$$K^a(t, \vec{x}) = (e^\Psi; 0, 0, 0). \quad (86)$$

(Here, all of these quantities can now be both space- and time-dependent).

Let us now assume K^a is a conformal Killing vector, $K_{(a;b)} = \mathcal{L}_K g_{ab} \propto g_{ab}$. We will not need to know what the proportionality factor is. That is,

$$K^c \partial_c g_{ab} + (\partial_a K^c) g_{cb} + (\partial_b K^c) g_{ac} \propto g_{ab}. \quad (87)$$

By constructing $\partial_a K^c = (\partial_a \Psi) K^c$, the conformal Killing equation reduces to

$$K^c \partial_c g_{ab} + (\partial_a \Psi) K_b + K_a (\partial_b \Psi) \propto g_{ab}. \quad (88)$$

Now for convenience, define $T^a = (1; 0, 0, 0)$ so that $K^a = e^\Psi T^a$; we have

$$\partial_t g_{ab} + (\partial_a \Psi) T_b + T_a (\partial_b \Psi) \propto g_{ab}. \quad (89)$$

As long as K^a is a conformal Killing vector, $K_{(a;b)} = \mathcal{L}_K g_{ab} \propto g_{ab}$, then we still have a conservation law for any affinely parameterized null geodesic:

$$K^a g_{ab} \frac{dx^b}{d\lambda} = e^\Psi g_{tb} \frac{dx^b}{d\lambda} = e^\Psi \left\{ -(N^2 - g^{kl} v_k v_l) \frac{dt}{d\lambda} + v_i \frac{dx^i}{d\lambda} \right\} = \text{constant} = \epsilon. \quad (90)$$

This is a slight modification to Equation (58) above, now allowing for a nontrivial conformal distortion e^Ψ . Thence,

$$\left\{ -(N^2 - g^{kl} v_k v_l) \frac{dt}{d\lambda} + v_i \frac{dx^i}{d\lambda} \right\} = e^{-\Psi} \epsilon. \quad (91)$$

Now, carry through essentially the same algebra as for the stationary case. We still have

$$-(N^2 - g^{kl} v_k v_l) dt^2 + 2v_i dx^i dt + g_{ij} dx^i dx^j = 0, \quad (92)$$

but now

$$(N^2 - g^{kl} v_k v_l) (dt)^2 - 2v_i dx^i dt + \frac{(v_i dx^i)^2}{(N^2 - g^{kl} v_k v_l)} = \frac{\epsilon^2 e^{-2\Psi}}{(N^2 - g^{kl} v_k v_l)} d\lambda^2. \quad (93)$$

This implies

$$g_{ij} dx^i dx^j + \frac{(v_i dx^i)^2}{(N^2 - g^{kl} v_k v_l)} = \frac{\epsilon^2 e^{-2\Psi}}{(N^2 - g^{kl} v_k v_l)} d\lambda^2, \quad (94)$$

which we can rewrite as

$$\epsilon^2 d\lambda^2 = e^{2\Psi} \left\{ (N^2 - g^{kl} v_k v_l) g_{ij} dx^i dx^j + (v_i dx^i)^2 \right\}. \quad (95)$$

That is,

$$\epsilon^2 d\lambda^2 = e^{2\Psi} \{ (N^2 - g^{kl} v_k v_l) g_{ij} + v_i v_j \} dx^i dx^j. \quad (96)$$

Therefore, along any affinely parameterized null geodesic, one still has

$$d\lambda \propto \sqrt{e^{2\Psi} \{ (N^2 - g^{kl} v_k v_l) g_{ij} + v_i v_j \} dx^i dx^j} \quad (97)$$

This can also be written as

$$d\lambda \propto \sqrt{e^{2\Psi} \{ -g_{tt} g_{ij} + g_{ti} g_{tj} \} dx^i dx^j} \quad (98)$$

The affine three-metric is now

$$h_{ij} = e^{2\Psi} \{ -g_{tt} g_{ij} + g_{ti} g_{tj} \}. \quad (99)$$

Note the extra conformal factor $e^{2\Psi}$ now appearing in the affine three-metric. Note that this conformal factor $e^{2\Psi}$ is not coming directly from the spacetime metric; it is coming indirectly from the assumed conformal Killing vector.

7.2. Threaded Form

Alternatively, one can choose to put the metric in “threaded” [93–95] (Kaluza–Klein-like) form:

$$g_{ab} = \left[\begin{array}{c|c} -V^2 & -V^2 \omega_j \\ \hline -V^2 \omega_i & \gamma_{ij} - V^2 \omega_i \omega_j \end{array} \right]. \quad (100)$$

Then, in threaded form, the affine three-metric is particularly simple:

$$h_{ij} = e^{2\Psi} \{ -g_{tt} g_{ij} + g_{ti} g_{tj} \} = e^{2\Psi} \{ V^2 (\gamma_{ij} - V^2 \omega_i \omega_j) + (V^2 \omega_i) (V^2 \omega_j) \} = e^{2\Psi} V^2 \gamma_{ij}. \quad (101)$$

That is,

$$h_{ij} = e^{2\Psi} V^2 \gamma_{ij}. \quad (102)$$

Again, note the extra conformal factor $e^{2\Psi}$ now appearing in the affine three-metric. Again, note that this conformal factor $e^{2\Psi}$ is not coming directly from the spacetime metric; it is coming indirectly from the assumed conformal Killing vector.

7.3. Examples

- FLRW cosmology:

$$ds^2 = -dt^2 + a(t)^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right\}; \quad K^a = (a(t); 0, 0, 0). \quad (103)$$

Then

$$d\lambda \propto \sqrt{a(t)^4 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]} = a(t)^2 ds_0 = a(t) dt, \quad (104)$$

where s_0 is the distance in the normalized hypersphere/flat/hyperbolic plane with a spatial three-metric $[g_0]_{ij}$. The affine three-metric can be written as

$$h_{ij}(t, \vec{x}) = a(t)^4 [g_0]_{ij}(x). \quad (105)$$

To check this, work in conformal time $dt = a(\eta)d\eta$ wherein

$$ds^2 = a(\eta)^2 \left\{ -d\eta^2 + \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right\}; \quad K^a = (1; 0, 0, 0). \quad (106)$$

(working in these coordinates, it is particularly easy to check that K is indeed a conformal Killing vector). Then,

$$d\lambda \propto \sqrt{a(\eta)^4 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]} = a(\eta)^2 d\eta = a(t) dt. \quad (107)$$

- Any constant-spatial-shape cosmology:

$$ds^2 = -dt^2 + a(t)^2 \{ [g_0]_{ij}(\vec{x}) dx^i dx^j \}; \quad K^a = (a(t); 0, 0, 0). \quad (108)$$

(the key constraint here is that the shape of the spatial slices is assumed to be governed by $[g_0]_{ij}(\vec{x})$, which is assumed to be an arbitrary time-independent three-metric). Then,

$$d\lambda \propto \sqrt{a(t)^4 \{ [g_0]_{ij}(\vec{x}) dx^i dx^j \}} = a(t)^2 ds_0 = a(t) dt. \quad (109)$$

where s_0 is the distance in the normalized constant-spatial-shape slices with the spatial three-metric $[g_0]_{ij}$. The affine three-metric can be written as

$$h_{ij}(t, \vec{x}) = a(t)^4 [g_0]_{ij}(x). \quad (110)$$

To check this work in conformal time, $dt = a(\eta)d\eta$, wherein

$$ds^2 = a(\eta)^2 \left\{ -d\eta^2 + [g_0]_{ij}(\vec{x}) dx^i dx^j \right\}; \quad K^a = (1; 0, 0, 0). \quad (111)$$

(working in these coordinates, it is particularly easy to check that K is indeed a conformal Killing vector). Then,

$$d\lambda \propto \sqrt{a(\eta)^4 \{ [g_0]_{ij}(\vec{x}) dx^i dx^j \}} = a(\eta)^2 d\eta = a(t) dt. \quad (112)$$

Many other examples along these lines could be developed.

8. Geometrical (3 + 1) Interpretation

Can we give the affine three-metric h_{ij} a geometrical (3 + 1) spacetime interpretation? Consider the (3 + 1) spacetime metric

$$g_{ab}(x, t) = \left[\begin{array}{c|c} -N^2 + g^{kl}v_kv_l & v_i \\ \hline v_j & g_{ij} \end{array} \right], \quad (113)$$

with

$$K^a(x, t) = (e^\Psi; 0, 0, 0), \quad (114)$$

and define the four-index tensor

$$Z_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}. \quad (115)$$

This tensor shows up quite naturally in curved spacetime electromagnetism.

Now define the two-index tensor h_{ab} obtained by projecting the four-index tensor Z_{abcd} onto the spatial hyperplanes orthogonal to the conformal timelike Killing vector:

$$h_{ac} = Z_{abcd}K^bK^d = g_{ac}(g_{bd}K^bK^d) - (g_{ad}K^d)(g_{bc}K^c). \quad (116)$$

Here,

$$(g_{bd}K^bK^d) = e^{2\Psi}g_{tt}; \quad \text{and} \quad (g_{ad}K^d) = e^\Psi(g_{tt}, v_i). \quad (117)$$

Thence,

$$\begin{aligned} h_{cd} &= Z_{abcd}K^bK^d \\ &= e^{2\Psi}\{g_{tt}g_{ac} - (g_{tt}, v_i)_a(g_{tt}, v_j)_c\} \\ &= e^{2\Psi}\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & g_{tt}g_{ij} - v_iv_j \end{array} \right]_{ac} \\ &= -\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & h_{ij} \end{array} \right]_{ac}. \end{aligned} \quad (118)$$

This gives a (3 + 1) spacetime perspective on just where the affine three-metric

$$h_{ij} \propto e^{2\Psi}\{-g_{tt}g_{ij} + v_iv_j\} = e^{2\Psi}\{-g_{tt}g_{ij} + g_{ti}g_{tj}\} = e^{2\Psi}V^2\gamma_{ij} \quad (119)$$

was coming from. Note that

$$\mathcal{L}_K h_{ac} = (\mathcal{L}_K Z_{abcd})K^bK^c \propto Z_{abcd}K^bK^c = h_{ac}, \quad (120)$$

so K^b is also a conformal Killing vector of the singular metric h_{ac} .

9. Conformal Transformations

Now consider an arbitrary spacetime with metric $\hat{g}_{ab}$ and no particular symmetries. Under the conformal transformation $g_{ab} = e^{2\Theta}\hat{g}_{ab}$, we have

$$\Gamma^a_{bc} = \hat{\Gamma}^a_{bc} + (\delta^a_b\Theta_{,c} + \delta^a_c\Theta_{,b} - \Theta^a\hat{g}_{bc}). \quad (121)$$

Thence, if $\hat{k}$ is an affinely parameterized null geodesic, $\hat{\nabla}_{\hat{k}}\hat{k} = 0$, we have

$$[\nabla_{\hat{k}}\hat{k}]^a = (\delta^a_b\Theta_{,c} + \delta^a_c\Theta_{,b} - \Theta^a\hat{g}_{bc})\hat{k}^b\hat{k}^c = 2(\hat{k}^c\Theta_{,c})\hat{k}^a. \quad (122)$$

Now define k by $\hat{k} = fk$, then

$$[\nabla_{fk}(fk)]^a = f^2[\nabla_k k]^a + f(k \cdot \nabla f)k^a = 2f^2(k^c\Theta_{,c})k^a \quad (123)$$

That is

$$[\nabla_k k]^a = 2(k^c \Theta_{,c})k^a - (f^{-1}k \cdot \nabla f)k^a \quad (124)$$

Thence, k will be affinely parameterized with respect to g provided we pick $f = e^{2\Theta}$. This is equivalent to setting $d\lambda/d\hat{\lambda} = e^{2\Theta}$, or more explicitly

$$d\lambda = e^{2\Theta} d\hat{\lambda}. \quad (125)$$

Even more explicitly: if we can construct an “affine 3-metric” $\hat{h}_{ij}$ for the original spacetime metric $\hat{g}_{ab}$ so that

$$d\hat{\lambda} = \sqrt{\hat{h}_{ij} dx^i dx^j} \quad (126)$$

then for the conformally transformed spacetime metric $g_{ab} = e^{2\Theta} \hat{g}_{ab}$, we have⁵

$$h_{ij} = e^{4\Theta} \hat{h}_{ij}. \quad (127)$$

In particular, given our previous results, this construction will work for any spacetime that is conformal to a spacetime containing a timelike conformal Killing vector.

Examples:

- As a consistency check, note that the observations above are compatible with what we already saw happening for FLRW spacetimes and for constant-spatial-shape cosmologies.
- CFLRW spacetimes: As a further generalization, consider the conformally FLRW spacetimes of reference [153]:

$$ds^2 = e^{2\Theta(x)} \left[-dt^2 + a(t)^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right\} \right]. \quad (128)$$

Here, $\Theta(x)$ is allowed to depend on both space and time. The CFLRW spacetimes are of interest because they have the same null geodesics (though not the same null affine parameter) as FLRW. In view of the preceding discussion, for any such spacetime, we can identify a suitable affine parameter as

$$d\lambda = \sqrt{\hat{h}_{ij} dx^i dx^j} \propto \sqrt{e^{4\Theta(x)} a(t)^4 \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right\}}. \quad (129)$$

We can also unwrap this as

$$d\lambda \propto e^{2\Theta(x)} a(t) dt. \quad (130)$$

More examples along these lines could be developed.

10. General Case

The completely general case is considerably more subtle, and we can at best develop an implicit nonconstructive formalism. As long as there was at least some symmetry—at least having the spacetime conformally related to a spacetime containing a conformal timelike Killing vector—then we have already demonstrated the existence of an affine three-metric h_{ij} completely characterizing the possible null affine parameters. If we try to dispense with symmetries altogether, then the best that can be conducted is an implicit Finsler-like construction [96–103]. Indeed, if we write

$$g_{ab}(x, t) = \left[\frac{-N^2 + g^{kl} v_k v_l}{v_j} \middle| \frac{v_i}{g_{ij}} \right], \quad (131)$$

then for any null curve, we still have

$$-(N^2 - g^{kl} v_k v_l) dt^2 + 2v_i dx^i dt + g_{ij} dx^i dx^j = 0. \quad (132)$$

Can we now argue for some generalization of our previous construction? Perhaps something along the lines of

$$d\lambda \propto \sqrt{(\text{something}) \{-g_{tt}g_{ij} + v_i v_j\} dx^i dx^j} = \sqrt{(\text{something}) \{V^2 \gamma_{ij}\} dx^i dx^j} ? \quad (133)$$

Indeed, let $T^a = (1, 0, 0, 0)$, and for any affinely parameterized null geodesic, define

$$\mathcal{Z}(x, dx/d\lambda) = T^a g_{ab} \frac{dx^b}{d\lambda} = g_{tb} \frac{dx^b}{d\lambda} = \left\{ -(N^2 - g^{kl} v_k v_l) \frac{dt}{d\lambda} + v_i \frac{dx^i}{d\lambda} \right\}. \quad (134)$$

We do not know how to calculate $\mathcal{Z}(x, dx/d\lambda)$ explicitly, but we know it must exist. Unfortunately, in the general case (without appealing to any Killing symmetries), this quantity $\mathcal{Z}(x, dx/d\lambda)$ can depend on the specific null geodesic in question; it is no longer a function on the manifold, but is instead a function on the tangent bundle.

Then, similarly to what we saw for the (conformal) timelike Killing vectors, we see

$$\frac{\mathcal{Z}(x, dx/d\lambda) d\lambda}{\sqrt{(N^2 - g^{kl} v_k v_l)}} = \left\{ -\sqrt{N^2 - g^{kl} v_k v_l} dt + \frac{v_i}{\sqrt{N^2 - g^{kl} v_k v_l}} dx^i \right\}. \quad (135)$$

Squaring both sides, this then implies

$$\left(\frac{\mathcal{Z}(x, dx/d\lambda) d\lambda}{\sqrt{(N^2 - g^{kl} v_k v_l)}} \right)^2 = (N^2 - g^{kl} v_k v_l) dt^2 - 2v_i dx^i dt + \frac{v_i v_j dx^i dx^j}{N^2 - g^{kl} v_k v_l} \quad (136)$$

Add this to the null cone condition (132), and we find

$$\left(\frac{\mathcal{Z}(x, dx/d\lambda) d\lambda}{\sqrt{(N^2 - g^{kl} v_k v_l)}} \right)^2 = +g_{ij} dx^i dx^j + \frac{v_i v_j dx^i dx^j}{N^2 - g^{kl} v_k v_l} \quad (137)$$

Rearranging

$$\mathcal{Z}^2(x, dx/d\lambda) d\lambda^2 = \left\{ (N^2 - g^{kl} v_k v_l) g_{ij} + v_i v_j \right\} dx^i dx^j \quad (138)$$

Then

$$d\lambda^2 = \mathcal{Z}(x, dx/d\lambda)^{-2} \{-g_{tt}g_{ij} + g_{ti}g_{tj}\} dx^i dx^j \quad (139)$$

So we (nonconstructively) deduce the *existence* of a Finslerian affine three-metric h_{ij} , defined up to a (in general unknown) conformal factor defined over the tangent bundle:

$$h_{ij} = \frac{\{-g_{tt}g_{ij} + g_{ti}g_{tj}\}}{\mathcal{Z}(x, dx/d\lambda)^2} = \frac{V^2 \gamma_{ij}}{\mathcal{Z}(x, dx/d\lambda)^2}. \quad (140)$$

Note that the Finslerian behavior has at least been isolated in the conformal factor—this is the type of structure explored by Bekenstein in reference [96]: a Riemannian metric (more precisely, a Lorentzian metric) distorted by a Finslerian conformal factor.

Unfortunately, proving the *existence* of this Finslerian affine three-metric seems to be as far as one can go—without imposing at least some symmetry on the spacetime, it seems to be impossible to explicitly determine (or at the very least, significantly constrain) the Finslerian conformal factor

$$\mathcal{Z}(x, dx/d\lambda) = T^a g_{ab} \frac{dx^b}{d\lambda}. \quad (141)$$

11. Discussion

In this article, we set up a quite general formalism for the a priori identification of affine null parameters in spacetimes conformally related to spacetimes with (at least) a

timelike conformal Killing vector. The formalism is sufficiently general so as to immediately yield useful information in many physically interesting situations. This complements the more traditional a posteriori approach, where one first on a case-by-case basis identifies null geodesics and then on a case-by-case basis attempts to find an affine null parameter. A number of explicit examples were provided to illustrate the utility of the procedure.

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Abbreviations

The following abbreviations are used in this manuscript:

ANEC	Averaged null energy condition
ADM	Arnowitt–Deser–Misner
FLRW	Friedmann–Lemaître–Robertson–Walker
CFLRW	Conformally Friedmann–Lemaître–Robertson–Walker

Notes

- ¹ The situation for nongeodesic null curves is, if anything, worse. Changing the parameterization of a nongeodesic null curve will again modify the “acceleration” $\nabla_k k$ by terms proportional to the tangent vector k , but there is now no longer a simple prescription for choosing a preferred affine parameterization.
- ² We emphasize: The overdots are not proper time derivatives or affine derivatives. The overdots are intimately entangled with the specific choice of coordinate system.
- ³ We emphasize: this is a choice, not a physical constraint.
- ⁴ The only even mildly surprising part of the result is the presence in the three-acceleration of a term cubic in the three-velocity.
- ⁵ Note the perhaps naively unexpected occurrence of the square $e^{4\Theta}$ of the conformal factor $e^{2\Theta}$.

References

1. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*; Freeman: San Francisco, CA, USA, 1972; Reprinted: Princeton, NJ, USA, 2018; ISBN 13 978-0691177793.
2. Wald, R.M. *General Relativity*; University of Chicago Press: Chicago, IL, USA, 1984. [\[CrossRef\]](#)
3. Carroll, S.M. *Spacetime and Geometry: An Introduction to General Relativity*; Addison-Wesley: Boston, MA, USA, 2003; ISBN 13 978-0805387322.
4. Hartle, J.B. *Gravity: An Introduction to Einstein’s General Relativity*; Pearson: London, UK, 2002; ISBN 13 978-0805386622.
5. Schutz, B. *A First Course in General Relativity*; Cambridge University Press: Cambridge, UK, 1985; ISBN 13 978-0521887052.
6. Weinberg, S. *Gravitation and Cosmology: Principles and Applications*; John Wiley: Hoboken, NJ, USA, 1972; ISBN 10 0-471-92567-5.
7. Weinberg, S. *Cosmology*; Oxford University Press: Oxford, UK, 2008; ISBN 13 978-0-19-852682-7.
8. Peebles, P.J.E. *Principles of Physical Cosmology*; Princeton University: Princeton, NJ, USA, 1993; ISBN 10 0-961-07428-3.
9. Padmanabhan, T. *Gravitation: Foundations and Frontiers*; Cambridge University Press: Cambridge, UK, 2010; ISBN 13 978-0-521-88223-1.
10. Hobson, M.P.; Efstathiou, G.P.; Lasenby, A.N. *General Relativity: An Introduction for Physicists*; Cambridge University Press: Cambridge, UK, 2006; ISBN 13 978-0521829519.
11. Sachs, R.K.; Wu, H.-H. *General Relativity for Mathematicians*; Springer: Berlin/Heidelberg, Germany, 1977; ISBN 13 978-1-4612-9905-9 (softcover)/13 978-1-4612-9903-5 (e-book). [\[CrossRef\]](#)
12. Stephani, H. *Relativity: An Introduction to Special and General Relativity*; Cambridge University Press: Cambridge, UK, 1982; ISBN 13 978-0521010696.
13. D’Inverno, R. *Introducing Einstein’s Relativity*; Oxford University Press: Oxford, UK, 1992; ISBN 13 978-0198596868.
14. Poisson, E. *A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics*; Cambridge University Press: Cambridge, UK, 2004; ISBN 13 978-0511606601. [\[CrossRef\]](#)
15. Visser, M. *Lorentzian Wormholes: From Einstein to Hawking*; AIP Press [Now Springer]: New York, NY, USA, 1996; ISBN 10 1-56396-653-0.

16. Flanagan, E.E.; Wald, R.M. Does back reaction enforce the averaged null energy condition in semiclassical gravity? *Phys. Rev. D* **1996**, *54*, 6233–6283. [[CrossRef](#)] [[PubMed](#)]
17. Yurtsever, U. Does quantum field theory enforce the averaged weak energy condition? *Class. Quant. Grav.* **1990**, *7*, L251–L258. [[CrossRef](#)]
18. Wald, R.M.; Yurtsever, U. General proof of the averaged null energy condition for a massless scalar field in two-dimensional curved space-time. *Phys. Rev. D* **1991**, *44*, 403–416. [[CrossRef](#)] [[PubMed](#)]
19. Borde, A. Geodesic focusing, energy conditions and singularities. *Class. Quant. Grav.* **1987**, *4*, 343–356. [[CrossRef](#)]
20. Ford, L.H.; Roman, T.A. Averaged energy conditions and quantum inequalities. *Phys. Rev. D* **1995**, *51*, 4277–4286. [[CrossRef](#)] [[PubMed](#)]
21. Ford, L.H.; Roman, T.A. Averaged energy conditions and evaporating black holes. *Phys. Rev. D* **1996**, *53*, 1988–2000. [[CrossRef](#)] [[PubMed](#)]
22. Graham, N.; Olum, K.D. Achronal averaged null energy condition. *Phys. Rev. D* **2007**, *76*, 064001. [[CrossRef](#)]
23. Wall, A.C. Proving the Achronal Averaged Null Energy Condition from the Generalized Second Law. *Phys. Rev. D* **2010**, *81*, 024038. [[CrossRef](#)]
24. Kontou, E.A.; Olum, K.D. Proof of the averaged null energy condition in a classical curved spacetime using a null-projected quantum inequality. *Phys. Rev. D* **2015**, *92*, 124009. [[CrossRef](#)]
25. Capozziello, S.; Lobo, F.S.N.; Mimoso, J.P. Energy conditions in modified gravity. *Phys. Lett. B* **2014**, *730*, 280–283. [[CrossRef](#)]
26. Capozziello, S.; Lobo, F.S.N.; Mimoso, J.P. Generalized energy conditions in Extended Theories of Gravity. *Phys. Rev. D* **2015**, *91*, 124019. [[CrossRef](#)]
27. Curiel, E. A Primer on Energy Conditions. *Einstein Stud.* **2017**, *13*, 43–104. [[CrossRef](#)]
28. Hochberg, D.; Visser, M. Dynamic wormholes, anti-trapped surfaces, and energy conditions. *Phys. Rev. D* **1998**, *58*, 044021. [[CrossRef](#)]
29. Barceló, C.; Visser, M. Twilight for the energy conditions? *Int. J. Mod. Phys. D* **2002**, *11*, 1553–1560. [[CrossRef](#)]
30. Fewster, C.J.; Roman, T.A. Null energy conditions in quantum field theory. *Phys. Rev. D* **2003**, *67*, 044003; Erratum in *Phys. Rev. D* **2009**, *80*, 069903. [[CrossRef](#)]
31. Kar, S.; Dadhich, N.; Visser, M. Quantifying energy condition violations in traversable wormholes. *Pramana* **2004**, *63*, 859–864. [[CrossRef](#)]
32. Martín-Moruno, P.; Visser, M. Classical and semi-classical energy conditions. *Fundam. Theor. Phys.* **2017**, *189*, 193–213. [[CrossRef](#)]
33. Visser, M. Scale anomalies imply violation of the averaged null energy condition. *Phys. Lett. B* **1995**, *349*, 443–447. [[CrossRef](#)]
34. Visser, M.; Barceló, C. Energy conditions and their cosmological implications. In *COSMO99*; World Scientific: Singapore, 2000. [[CrossRef](#)]
35. Yurtsever, U. A note on the averaged null energy condition in quantum field theory. *Phys. Rev. D* **1995**, *52*, R564–R568. [[CrossRef](#)]
36. Yurtsever, U. The Averaged null energy condition and difference inequalities in quantum field theory. *Phys. Rev. D* **1995**, *51*, 5797–5805. [[CrossRef](#)]
37. Fewster, C.J.; Olum, K.D.; Pfenning, M.J. Averaged null energy condition in spacetimes with boundaries. *Phys. Rev. D* **2007**, *75*, 025007. [[CrossRef](#)]
38. Graham, N.; Olum, K.D. Plate with a hole obeys the averaged null energy condition. *Phys. Rev. D* **2005**, *72*, 025013. [[CrossRef](#)]
39. Kontou, E.A. Averaged null energy condition and quantum inequalities in curved spacetime. *arXiv* **2015**, arXiv:1507.06299.
40. Penrose, R.; Sorkin, R.D.; Woolgar, E. A Positive mass theorem based on the focusing and retardation of null geodesics. *arXiv* **1993**, arXiv:gr-qc/9301015.
41. Morris, M.S.; Thorne, K.S.; Yurtsever, U. Wormholes, Time Machines, and the Weak Energy Condition. *Phys. Rev. Lett.* **1988**, *61*, 1446–1449. [[CrossRef](#)] [[PubMed](#)]
42. Witten, E. Light Rays, Singularities, and All That. *Rev. Mod. Phys.* **2020**, *92*, 045004. [[CrossRef](#)]
43. Senovilla, J.M.M. Singularity Theorems and Their Consequences. *Gen. Relativ. Grav.* **1998**, *30*, 701. [[CrossRef](#)]
44. Hawking, S.; Ellis, G. *The Large Scale Structure of Space-Time*; Cambridge University Press: Cambridge, UK, 1973; ISBN 13 978-0521099066.
45. Roman, T.A.; Bergmann, P.G. Stellar collapse without singularities? *Phys. Rev. D* **1983**, *28*, 1265–1277. [[CrossRef](#)]
46. Roman, T.A. Quantum Stress Energy Tensors and the Weak Energy Condition. *Phys. Rev. D* **1986**, *33*, 3526–3533. [[CrossRef](#)]
47. Roman, T.A. On the ‘Averaged Weak Energy Condition’ and Penrose’s Singularity Theorem. *Phys. Rev. D* **1988**, *37*, 546–548. [[CrossRef](#)]
48. Bojowald, M. Singularities and Quantum Gravity. *AIP Conf. Proc.* **2007**, *910*, 294–333. [[CrossRef](#)]
49. Parker, L.; Fulling, S.A. Quantized matter fields and the avoidance of singularities in general relativity. *Phys. Rev. D* **1973**, *7*, 2357–2374. [[CrossRef](#)]
50. Fewster, C.J.; Galloway, G.J. Singularity theorems from weakened energy conditions. *Class. Quant. Grav.* **2011**, *28*, 125009. [[CrossRef](#)]
51. Abreu, G.; Visser, M. Some generalizations of the Raychaudhuri equation. *Phys. Rev. D* **2011**, *83*, 104016. [[CrossRef](#)]
52. Ford, L.H. The Classical singularity theorems and their quantum loop holes. *Int. J. Theor. Phys.* **2003**, *42*, 1219–1227. [[CrossRef](#)]
53. Friedman, J.L.; Schleich, K.; Witt, D.M. Topological censorship. *Phys. Rev. Lett.* **1993**, *71*, 1486–1489; Erratum in *Phys. Rev. Lett.* **1995**, *75*, 1872. [[CrossRef](#)] [[PubMed](#)]

54. Galloway, G.J.; Schleich, K.; Witt, D.M.; Woolgar, E. Topological censorship and higher genus black holes. *Phys. Rev. D* **1999**, *60*, 104039. [[CrossRef](#)]
55. Jacobson, T.; Venkataramani, S. Topology of event horizons and topological censorship. *Class. Quant. Grav.* **1995**, *12*, 1055–1062. [[CrossRef](#)]
56. Shapiro, S.L.; Teukolsky, S.A.; Winicour, J. Toroidal black holes and topological censorship. *Phys. Rev. D* **1995**, *52*, 6982–6987. [[CrossRef](#)]
57. Chruściel, P.T.; Galloway, G.J.; Solis, D. Topological censorship for Kaluza-Klein space-times. *Ann. Henri Poincaré* **2009**, *10*, 893–912. [[CrossRef](#)]
58. Browdy, S.F.; Galloway, G.J. Topological censorship and the topology of black holes. *J. Math. Phys.* **1995**, *36*, 4952–4961. [[CrossRef](#)]
59. Eichmair, M.; Galloway, G.J.; Pollack, D. Topological censorship from the initial data point of view. *J. Diff. Geom.* **2013**, *95*, 389–405. [[CrossRef](#)]
60. Chruściel, P.T.; Galloway, G.J. Roads to topological censorship. *arXiv* **2019**, arXiv:1906.02151.
61. Hawking, S.W. The Chronology protection conjecture. *Phys. Rev. D* **1992**, *46*, 603–611. [[CrossRef](#)] [[PubMed](#)]
62. Visser, M. From wormhole to time machine: Comments on Hawking’s chronology protection conjecture. *Phys. Rev. D* **1993**, *47*, 554–565. [[CrossRef](#)] [[PubMed](#)]
63. Kay, B.S.; Radzikowski, M.J.; Wald, R.M. Quantum field theory on space-times with a compactly generated Cauchy horizon. *Commun. Math. Phys.* **1997**, *183*, 533–556. [[CrossRef](#)]
64. Visser, M. Traversable wormholes: The Roman ring. *Phys. Rev. D* **1997**, *55*, 5212–5214. [[CrossRef](#)]
65. Visser, M. The Reliability horizon for semiclassical quantum gravity: Metric fluctuations are often more important than back reaction. *Phys. Lett. B* **1997**, *415*, 8–14. [[CrossRef](#)]
66. Visser, M. The Quantum physics of chronology protection. *arXiv* **2002**, arXiv:gr-qc/0204022.
67. Friedman, J.L.; Higuchi, A. Topological censorship and chronology protection. *Ann. Phys.* **2006**, *15*, 109–128. [[CrossRef](#)]
68. Liberati, S. Do not mess with time: Probing faster than light travel and chronology protection with superluminal warp drives. *arXiv* **2016**, arXiv:1601.00785.
69. Eling, C.; Jacobson, T. Spherical solutions in Einstein-aether theory: Static aether and stars. *Class. Quant. Grav.* **2006**, *23*, 5625–5642; Erratum in *Class. Quant. Grav.* **2010**, *27*, 049801. [[CrossRef](#)]
70. Olmo, G.J.; Rubiera-Garcia, D. Nonsingular Black Holes in $f(R)$ Theories. *Universe* **2015**, *1*, 173–185. [[CrossRef](#)]
71. Hollands, S.; Wald, R.M.; Zahn, J. Quantum instability of the Cauchy horizon in Reissner–Nordström–deSitter spacetime. *Class. Quant. Grav.* **2020**, *37*, 115009. [[CrossRef](#)]
72. Bejarano, C.; Olmo, G.J.; Rubiera-Garcia, D. What is a singular black hole beyond General Relativity? *Phys. Rev. D* **2017**, *95*, 064043. [[CrossRef](#)]
73. Ashtekar, A.; Pretorius, F.; Ramazanoglu, F.M. Evaporation of 2-Dimensional Black Holes. *Phys. Rev. D* **2011**, *83*, 044040. [[CrossRef](#)]
74. Menchon, C.; Olmo, G.J.; Rubiera-Garcia, D. Nonsingular black holes, wormholes, and de Sitter cores from anisotropic fluids. *Phys. Rev. D* **2017**, *96*, 104028. [[CrossRef](#)]
75. Flanagan, É.É.; Grant, A.M.; Harte, A.I.; Nichols, D.A. Persistent gravitational wave observables: General framework. *Phys. Rev. D* **2019**, *99*, 084044. [[CrossRef](#)]
76. Maeda, K.i.; Nozawa, M. Black Hole in the Expanding Universe with Arbitrary Power-Law Expansion. *Phys. Rev. D* **2010**, *81*, 124038. [[CrossRef](#)]
77. Yang, X.; Wang, J. YNOGK: A new public code for calculating null geodesics in the Kerr spacetime. *Astrophys. J. Suppl.* **2013**, *207*, 6. [[CrossRef](#)]
78. Sanghai, V.A.A.; Fleury, P.; Clifton, T. Ray tracing and Hubble diagrams in post-Newtonian cosmology. *J. Cosmol. Astropart. Phys.* **2017**, *7*, 028. [[CrossRef](#)]
79. Arrechea, J.; Barceló, C.; Carballo-Rubio, R.; Garay, L.J. Semiclassical constant-density spheres in a regularized Polyakov approximation. *Phys. Rev. D* **2021**, *104*, 084071. [[CrossRef](#)]
80. Fuentes, J.L.; Hidalgo, J.C.; Malik, K.A. Galaxy number counts at second order: An independent approach. *Class. Quant. Grav.* **2021**, *38*, 065014. [[CrossRef](#)]
81. Anastopoulos, C.; Savvidou, N. Classification theorem and properties of singular solutions to the Tolman–Oppenheimer–Volkoff equation. *Class. Quant. Grav.* **2021**, *38*, 075024. [[CrossRef](#)]
82. White, C.J. Blacklight: A General-relativistic Ray-tracing and Analysis Tool. *Astrophys. J. Suppl.* **2022**, *262*, 28. [[CrossRef](#)]
83. Boyanov, V. Vacuum polarisation and regular gravitational collapse. *arXiv* **2023**, arXiv:2306.07169.
84. Arrechea, J.; Barceló, C.; Boyanov, V. After collapse: On how a physical vacuum can change the black hole paradigm. *arXiv* **2023**, arXiv:2307.13416.
85. Dexter, J.; Agol, E. A Fast New Public Code for Computing Photon Orbits in a Kerr Spacetime. *Astrophys. J.* **2009**, *696*, 1616–1629. [[CrossRef](#)]
86. Jacobson, T. When is $g_{(tt)}g_{(rr)} = -1$? *Class. Quant. Grav.* **2007**, *24*, 5717–5719. [[CrossRef](#)]
87. Chandrasekaran, V.; Prabhu, K. Symmetries, charges and conservation laws at causal diamonds in general relativity. *J. High Energ. Phys.* **2019**, *10*, 229. [[CrossRef](#)]

88. Chan, C.k.; Medeiros, L.; Ozel, F.; Psaltis, D. GRay2: A General Purpose Geodesic Integrator for Kerr Spacetimes. *Astrophys. J.* **2018**, *867*, 59. [\[CrossRef\]](#)
89. Preston, B.; Poisson, E. Light-cone coordinates based at a geodesic world line. *Phys. Rev. D* **2006**, *74*, 064009. [\[CrossRef\]](#)
90. Arnowitt, R.L.; Deser, S.; Misner, C.W. The Dynamics of general relativity. *Gen. Relativ. Grav.* **2008**, *40*, 1997–2027. [\[CrossRef\]](#)
- 91.ourgoulhon, E. 3+1 formalism and bases of numerical relativity. *arXiv* **2007**, arXiv:gr-qc/0703035.
92. Hawking, S.W.; Horowitz, G.T. The Gravitational Hamiltonian, action, entropy and surface terms. *Class. Quant. Grav.* **1996**, *13*, 1487–1498. [\[CrossRef\]](#)
93. Boersma, S.; Dray, T. Slicing, threading & parametric manifolds. *Gen. Relativ. Grav.* **1995**, *27*, 319–339. [\[CrossRef\]](#)
94. van Elst, H.; Uggla, C. General relativistic (1+3) orthonormal frame approach revisited. *Class. Quant. Grav.* **1997**, *14*, 2673–2695. [\[CrossRef\]](#)
95. Ellis, G.F.R.; van Elst, H. Cosmological models: Cargese lectures 1998. *NATO Sci. Ser. C* **1999**, *541*, 1–116. [\[CrossRef\]](#)
96. Bekenstein, J.D. The Relation between physical and gravitational geometry. *Phys. Rev. D* **1993**, *48*, 3641–3647. [\[CrossRef\]](#)
97. Perlick, V. Fermat principle in Finsler spacetimes. *Gen. Relativ. Grav.* **2006**, *38*, 365–380. [\[CrossRef\]](#)
98. Gibbons, G.W.; Gomis, J.; Pope, C.N. General very special relativity is Finsler geometry. *Phys. Rev. D* **2007**, *76*, 081701. [\[CrossRef\]](#)
99. Gibbons, G.W.; Herdeiro, C.A.R.; Warnick, C.M.; Werner, M.C. Stationary Metrics and Optical Zermelo-Randers-Finsler Geometry. *Phys. Rev. D* **2009**, *79*, 044022. [\[CrossRef\]](#)
100. Skakala, J.; Visser, M. The causal structure of spacetime is a parameterized Randers geometry. *Class. Quant. Grav.* **2011**, *28*, 065007. [\[CrossRef\]](#)
101. Lammerzahl, C.; Perlick, V.; Hasse, W. Observable effects in a class of spherically symmetric static Finsler spacetimes. *Phys. Rev. D* **2012**, *86*, 104042. [\[CrossRef\]](#)
102. Lämmerzahl, C.; Perlick, V. Finsler geometry as a model for relativistic gravity. *Int. J. Geom. Methods Mod. Phys.* **2018**, *15*, 1850166. [\[CrossRef\]](#)
103. Pfeifer, C. Finsler spacetime geometry in Physics. *Int. J. Geom. Methods Mod. Phys.* **2019**, *16*, 1941004. [\[CrossRef\]](#)
104. Page, D.N. Thermal Stress Tensors in Static Einstein Spaces. *Phys. Rev. D* **1982**, *25*, 1499. [\[CrossRef\]](#)
105. Blau, S.; Visser, M.; Wipf, A. Zeta Functions and the Casimir Energy. *Nucl. Phys. B* **1988**, *310*, 163. [\[CrossRef\]](#)
106. Gusev, Y.; Zelnikov, A. Nonlocal effective action at finite temperature in ultrastatic space-times. *Class. Quant. Grav.* **1998**, *15*, L13–L19. [\[CrossRef\]](#)
107. Furlani, E.P. Quantization of massive vector fields on ultrastatic space-times. *Class. Quant. Grav.* **1997**, *14*, 1665–1677. [\[CrossRef\]](#)
108. Popov, A.A. Analytical approximation for $\langle \phi^2 \rangle$ of a quantized scalar field in ultrastatic asymptotically flat spacetimes. *Phys. Rev. D* **2004**, *70*, 084047. [\[CrossRef\]](#)
109. Fewster, C.J.; Pfenning, M.J. A Quantum weak energy inequality for spin one fields in curved space-time. *J. Math. Phys.* **2003**, *44*, 4480–4513. [\[CrossRef\]](#)
110. Sonego, S. Ultrastatic spacetimes. *J. Math. Phys.* **2010**, *51*, 092502. [\[CrossRef\]](#)
111. Fewster, C.J.; Verch, R. The Necessity of the Hadamard Condition. *Class. Quant. Grav.* **2013**, *30*, 235027. [\[CrossRef\]](#)
112. Padmanabhan, T. Gravitational entropy of static space-times and microscopic density of states. *Class. Quant. Grav.* **2004**, *21*, 4485–4494. [\[CrossRef\]](#)
113. Perlick, V. On the Exact gravitational lens equation in spherically symmetric and static space-times. *Phys. Rev. D* **2004**, *69*, 064017 [\[CrossRef\]](#)
114. Visser, M. Dirty black holes: Thermodynamics and horizon structure. *Phys. Rev. D* **1992**, *46*, 2445–2451. [\[CrossRef\]](#)
115. Hochberg, D.; Visser, M. Geometric structure of the generic static traversable wormhole throat. *Phys. Rev. D* **1997**, *56*, 4745–4755. [\[CrossRef\]](#)
116. Rahman, S.; Visser, M. Space-time geometry of static fluid spheres. *Class. Quant. Grav.* **2002**, *19*, 935–952. [\[CrossRef\]](#)
117. Martin, D.; Visser, M. Bounds on the interior geometry and pressure profile of static fluid spheres. *Class. Quant. Grav.* **2003**, *20*, 3699–3716. [\[CrossRef\]](#)
118. Boonserm, P.; Visser, M. Buchdahl-like transformations for perfect fluid spheres. *Int. J. Mod. Phys. D* **2008**, *17*, 135–163. [\[CrossRef\]](#)
119. Boonserm, P.; Visser, M. Buchdahl-like transformations in general relativity. *Thai J. Math.* **2007**, *5*, 209–223.
120. Lobo, F.S.N.; Rodrigues, M.E.; Silva, M.V.d.; Simpson, A.; Visser, M. Novel black-bounce spacetimes: Wormholes, regularity, energy conditions, and causal structure. *Phys. Rev. D* **2021**, *103*, 084052. [\[CrossRef\]](#)
121. Boonserm, P.; Ngampitipan, T.; Simpson, A.; Visser, M. Exponential metric represents a traversable wormhole. *Phys. Rev. D* **2018**, *98*, 084048. [\[CrossRef\]](#)
122. Simpson, A.; Visser, M. Black-bounce to traversable wormhole. *J. Cosmol. Astropart. Phys.* **2019**, *2*, 042. [\[CrossRef\]](#)
123. Bronnikov, K.A.; Walia, R.K. Field sources for Simpson-Visser spacetimes. *Phys. Rev. D* **2022**, *105*, 044039. [\[CrossRef\]](#)
124. Synge, J.L. *Relativity: The General Theory*; North Holland: Amsterdam, The Netherlands, 1960.
125. Kerr, R.P. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.* **1963**, *11*, 237–238; Reprinted in Cambridge, UK, 2009; pp. 347–348. [\[CrossRef\]](#)
126. Kerr, R.P. Gravitational collapse and rotation. In *Quasi-Stellar Sources and Gravitational Collapse: Proceedings of the First Texas Symposium*, Robinson, I., Schild, A., Schücking, E. L., Eds.; University of Chicago Press: Chicago, IL, USA, 1965; Reprinted in Cambridge, UK, 2009; pp. 349–353.

127. Kerr, R.P. *Discovering the Kerr and Kerr-Schild Metrics*; Cambridge University Press: Cambridge, UK, 2009; pp. 38–72.
128. Wiltshire, D.L.; Visser, M.; Scott, S.M. *The Kerr Spacetime: Rotating Black Holes in General Relativity*; Cambridge University Press: Cambridge, UK, 2009; ISBN 13 978-0-521-88512-6.
129. Visser, M. *The Kerr Spacetime: A Brief Introduction*; Cambridge University Press: Cambridge, UK, 2009; pp. 3–37.
130. O’Niel, B. *The Geometry of Kerr Black Holes*; Dover: New York, NY, USA, 1995; ISBN 13 978-0-486-49342-8.
131. Newman, E.T.; Couch, R.; Chinnapared, K.; Exton, A.; Prakash, A.; Torrence, R. Metric of a Rotating, Charged Mass. *J. Math. Phys.* **1965**, *6*, 918–919. [[CrossRef](#)]
132. Newman, E.T.; Janis, A.I. Note on the Kerr spinning particle metric. *J. Math. Phys.* **1965**, *6*, 915–917. [[CrossRef](#)]
133. Doran, C. A New form of the Kerr solution. *Phys. Rev. D* **2000**, *61*, 067503. [[CrossRef](#)]
134. Medved, A.J.M.; Martin, D.; Visser, M. Dirty black holes: Symmetries at stationary nonstatic horizons. *Phys. Rev. D* **2004**, *70*, 024009. [[CrossRef](#)]
135. Liberati, S.; Tricella, G.; Visser, M. Towards a Gordon form of the Kerr spacetime. *Class. Quant. Grav.* **2018**, *35*, 155004. [[CrossRef](#)]
136. Baines, J.; Visser, M. Physically motivated ansatz for the Kerr spacetime. *Class. Quant. Grav.* **2022**, *39*, 235004. [[CrossRef](#)]
137. Teukolsky, S.A. The Kerr Metric. *Class. Quant. Grav.* **2015**, *32*, 124006. [[CrossRef](#)]
138. Adamo, T.; Newman, E.T. The Kerr-Newman metric: A Review. *Scholarpedia* **2014**, *9*, 31791. [[CrossRef](#)]
139. Kaluza, T. Zum Unitätsproblem der Physik. *Sitzungsber. Preuss. Akad. Wiss. Berl. (Math. Phys.)* **1921**, 1921, 966–972. [[CrossRef](#)]
140. Klein, O. Quantum Theory and Five-Dimensional Theory of Relativity. *Z. Phys.* **1926**, *37*, 895–906. [[CrossRef](#)]
141. Overduin, J.M.; Wesson, P.S. Kaluza-Klein gravity. *Phys. Rept.* **1997**, *283*, 303–380. [[CrossRef](#)]
142. Witten, E. Search for a Realistic Kaluza-Klein Theory. *Nucl. Phys. B* **1981**, *186*, 412. [[CrossRef](#)]
143. Salam, A.; Strathdee, J.A. On Kaluza-Klein Theory. *Ann. Phys.* **1982**, *141*, 316–352. [[CrossRef](#)]
144. Martel, K.; Poisson, E. Regular coordinate systems for Schwarzschild and other spherical space-times. *Am. J. Phys.* **2001**, *69*, 476–480. [[CrossRef](#)]
145. Volovik, G.E. Simulation of Painleve-Gullstrand black hole in thin He-3—A film. *JETP Lett.* **1999**, *69*, 705–713. [[CrossRef](#)]
146. Gaur, R.; Visser, M. Cosmology in Painlevé-Gullstrand coordinates. *J. Cosmol. Astropart. Phys.* **2022**, *9*, 030. [[CrossRef](#)]
147. Visser, M.; Liberati, S. Painleve-Gullstrand coordinates versus Kerr spacetime geometry. *Gen. Relativ. Grav.* **2022**, *54*, 145. [[CrossRef](#)]
148. Boyer, R.H.; Lindquist, R.W. Maximal analytic extension of the Kerr metric. *J. Math. Phys.* **1967**, *8*, 265. [[CrossRef](#)]
149. Rajan, D.; Visser, M. Global properties of physically interesting Lorentzian spacetimes. *Int. J. Mod. Phys. D* **2016**, *25*, 1650106. [[CrossRef](#)]
150. Baines, J.; Berry, T.; Simpson, A.; Visser, M. Darboux diagonalization of the spatial 3-metric in Kerr spacetime. *Gen. Relativ. Grav.* **2021**, *53*, 3. [[CrossRef](#)]
151. Schuster, S.; Visser, M. Boyer-Lindquist space-times and beyond: Meta-material analogues. *arXiv* **2018**, arXiv:1802.09807.
152. Bambi, C.; Modesto, L. Rotating regular black holes. *Phys. Lett. B* **2013**, *721*, 329–334. [[CrossRef](#)]
153. Visser, M. Conformally Friedmann–Lemaître–Robertson–Walker cosmologies. *Class. Quant. Grav.* **2015**, *32*, 135007. [[CrossRef](#)]

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