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Black Holes Hint towards De Sitter Matrix Theory

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Abstract: De Sitter black holes and other non-perturbative configurations can be used to probe the holographic degrees of freedom of de Sitter space. For small black holes, evidence was first provided in the seminal work of Banks, Fiol, and Morrise and follow-ups by Banks and Fischler, showing that dS is described by a form of matrix theory. For large black holes, the evidence provided here is new: Gravitational calculations and matrix theory calculations of the rates of exponentially rare fluctuations match one another in surprising detail. The occurrences of Nariai geometry and the "inside-out" transition are particularly interesting examples, which I explain in this paper.

Keywords: black holes; de Sitter; matrix

Citation: Susskind, L. Black Holes Hint towards De Sitter-Matrix Theory. *Universe* **2023**, *9*, 368. https://doi.org/10.3390/universe9080368

Academic Editor: David Mattingly

Received: 17 September 2021
Revised: 22 March 2022
Accepted: 29 March 2022
Published: 11 August 2023

1. Entanglement in De Sitter Space

In this paper, I will assume that there is a holographic description of the static patches of four-dimensional de Sitter space[1]; but unlike AdS, de Sitter space has no asymptotic boundary where the degrees of freedom are located. Instead, the holographic degrees of freedom are nominally located on the boundary of the static patch (SP) (see, for example, [1–6]), which involves the stretched horizon.

Static patches come in opposing pairs. To account for the pair, two sets of degrees of freedom are required. The Penrose diagram of de Sitter space in Figure 1 shows such a pair of SPs along with their stretched horizons. The center of the SPs (sometimes thought of as the points where observers are located) will be called the pode and the antipode.

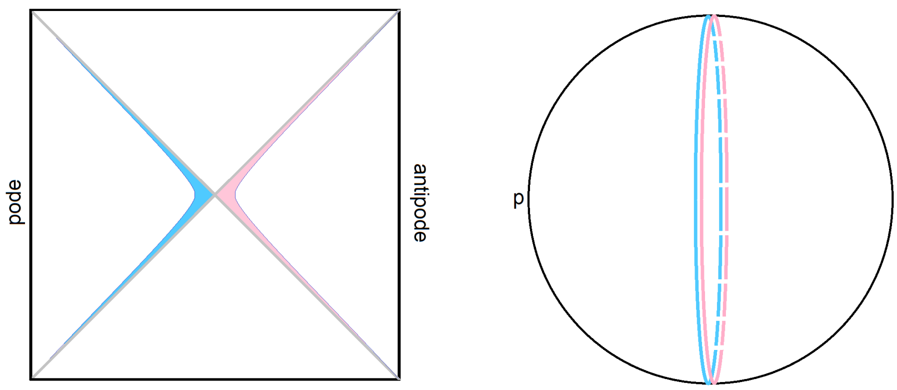

Figure 1. The left panel shows the Penrose diagram for de Sitter space with a particular choice of opposing static patches. The blue and pink regions are the stretched horizons of the two SPs. The right panel shows the spatial geometry of a time-symmetric slice. The blue and pink surfaces represent the stretched horizons.

Although it is clear from the Penrose diagram that the two SPs are entangled in the thermofield-double state, no clear framework similar to the Ryu–Takyanagi formula has been formulated for de Sitter space. This paper is not primarily about such a de Sitter generalization of the RT framework but I will briefly sketch what such a generalization looks like.

We assume that the entanglement entropy of the two sides—pode and antipode—is proportional to the minimum area of a surface homologous to the boundary of one of the two components—let us say the pode side. However, what do we mean by the boundary? The full spatial slice at $t = 0$ has no boundary, but the static patch is bounded by the blue stretched horizon. Thus, we try the following formulation.

The entanglement entropy of the pode-antipode systems is $1/4G$ *times the minimal area of a surface homologous to the stretched horizon (of either side).*

This, however, will not work. Figure 2 shows the spatial slice and the adjacent pair of stretched horizons. The dark blue curve represents a surface homologous to the pode's stretched horizon. It is obvious that that curve can be shrunk to zero, which if the above formulation were correct, would imply vanishing entanglement between the pode and antipode static patches.

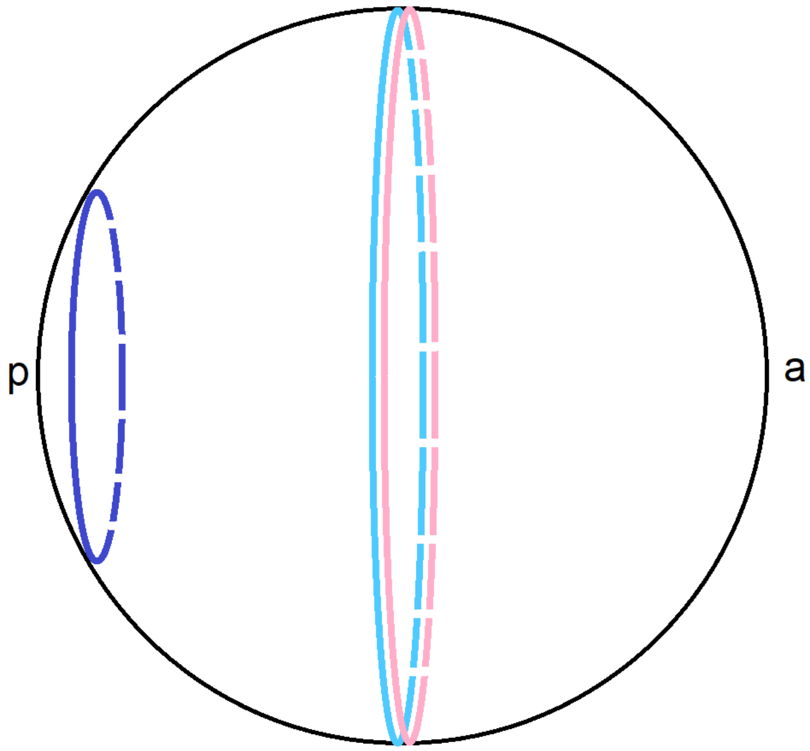

Figure 2. A $t = 0$ slice of dS and the stretched horizons shown as light blue and pink great circles. The dark blue surface is homologous to the light blue horizon. It can be shrunk to a point.

To perform better, we first separate the two stretched horizons a bit. This is a natural thing to perform since they will separate after a short period of time, as is obvious from Figure 1. Let us now reformulate a dS-improved version of the RT principle.

The entanglement entropy of the pode-antipode systems is $1/4G$ *times the minimal area of a surface homologous to the stretched horizon of the pode and lying between the two sets of degrees of freedom, i.e., between the two stretched horizons.*

This version of the RT principle is illustrated in Figure 3.

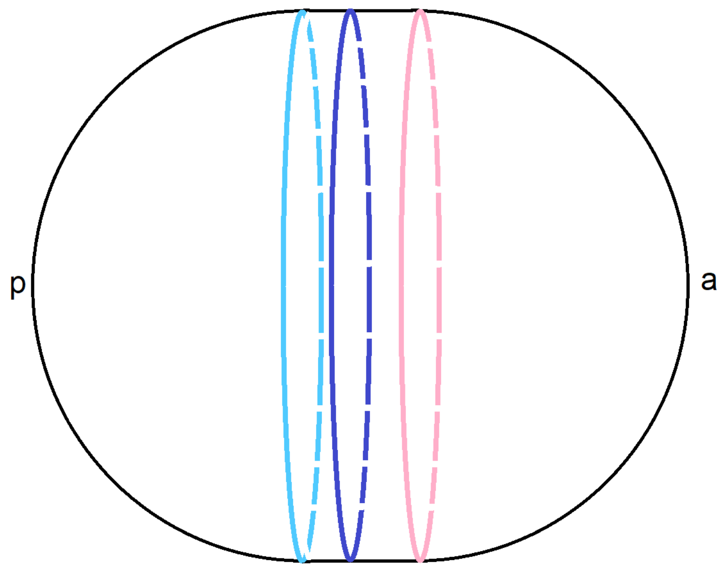

Figure 3. The dark blue curve represents the minimal surface lying between the two stretched horizons shown in light blue and pink.

It is evident from the figure that the area of the dSRT surface is the area of the horizon. This provides the entanglement entropy that we expect [7], which is the following.

$$\frac{\text{Horizon Area}}{4G}.$$

One thing to note is that in anti-de Sitter spaces, the phrase "lying between the two sets of degrees of freedom" is redundant. The degrees of freedom lie at the asymptotic boundary and any minimal surface will necessarily lie between them.

This version of the de Sitter RT formula is sufficient for time-independent geometries. A more general "maxmin" formulation is described as follows: Pick a time on the stretched horizons and anchor a three-dimensional surface Σ connecting the two. This is shown in Figure 4.

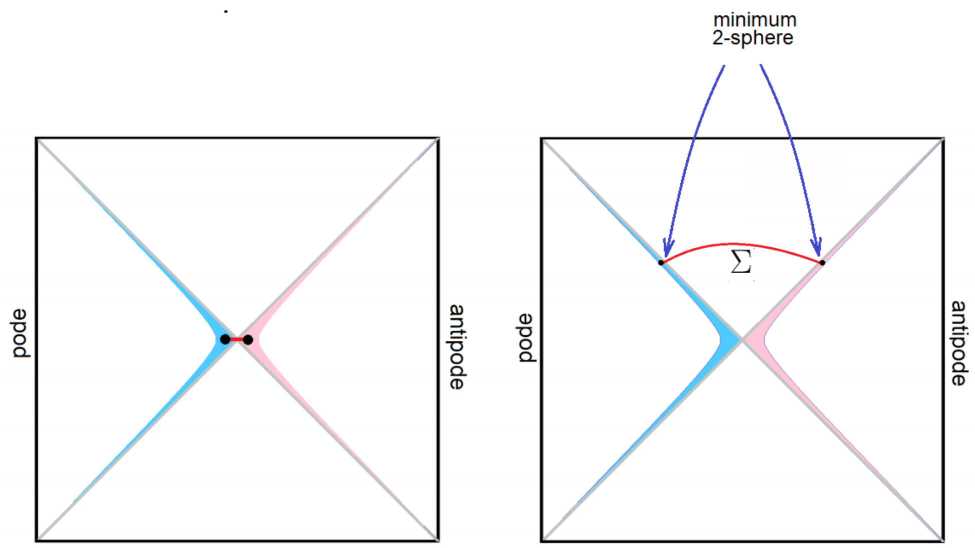

Figure 4. In both panels, the black dots represent the anchoring points of a space-like surface Σ connecting the two horizons at a particular time. The minimal two-sphere cutting Σ lies at the anchoring points.

Find the minimum-area two-dimensional sphere that cuts the three-dimensional surface Σ and call its area $A_{min}(\Sigma)$. It is not hard to show that the minimum area sphere hugs one of the two horizons, as shown in Figure 4. The reason is that in de Sitter space, the local two-sphere grows (exponentially) as one moves behind the horizon.

Now maximize $A_{min}(\Sigma)$ over all space-like Σ. Call the resulting area the following.

$$A_{maxmin}.$$

The entanglement entropy between the pode and antipode static patches is as follows.

$$S_{ent} = \frac{A_{maxmin}}{4G}. \tag{1}$$

Because $A_{min}(\Sigma)$ occurs at the anchoring points, the maximization of $A_{min}(\Sigma)$ is redundant in the case depicted in Figure 1.

It should be possible to generalize the dSRT formula to include bulk entanglement term, but I will save this for another time.

Now, we turn to the main subject of this paper—dS black holes and their implications for dS holography.

2. From Small Black Holes to Nariai

The properties of black holes in four-dimensional de Sitter space provide hints about the holographic degrees of freedom and their dynamics. These hints will lead us to a remarkably general conclusion: the underlying holographic description of de Sitter space must be a form a matrix quantum mechanics.

The Schwarzschild de Sitter metric is given by the following:

$$\begin{aligned} ds^2 &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \\ f(r) &= 1 - \frac{r^2}{R^2} - \frac{2MG}{r} \end{aligned} \tag{2}$$

where R is the de Sitter radius, M is the black hole mass, and G is Newton's constant.

There are two horizons: the larger cosmic horizon and the smaller black hole horizon. The horizons are defined by $f(r) = 0$. Defining $g(r) = rf(r)$, the horizon condition becomes the following.

$$g(r) = r - r^3/R^2 - 2MG = 0. \tag{3}$$

The function $g(r)$ is shown in Figure 5.

We have values of M satisfying the following.

$$0 < \frac{MG}{R} < \frac{1}{3\sqrt{3}}, \tag{4}$$

Equation (3) has three solutions, two with positive values of r, and one with negative r. The two positive solutions, r_- and r_+, define the black hole horizon and the cosmic horizon, respectively. The negative solution, $r = -r_0$, is unphysical. Outside the range (4), the metric has a naked singularity. Given the values of R (or equivalently the cosmological constant) and G, there is only one parameter in the metric, namely M. Alternatively, we may choose the independent parameter to be either r_+, r_-, or the dimensionless parameter x defined by the following.

$$x \equiv (r_+ - r_-)/R. \tag{5}$$

The variable x runs from $x = -1$ to $x = +1$. Over this range, the mass M runs over its allowable values (4) twice: once for $x < 0$ and once for $x > 0$. As x increases from -1 to 0, the black hole horizon r_- grows, and the cosmic horizon r_+ shrinks so that the two become equal at $x = 0$. When x becomes positive, the two horizons are exchanged so that $r_- > r_+$. Beyond that r_+ becomes the black hole horizon, and r_- the cosmic horizon.

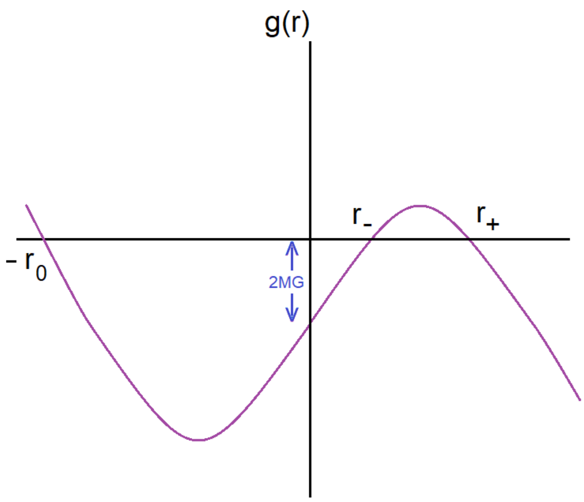

Figure 5. The function $g(r)$ and its zeros, r_0 and $r_\pm$.

There are two possible ways to think about this. In the first, we assume that the range $x > 0$ is redundant and simply describes the same states that were covered for $x < 0$; roughly speaking we think of the choice of the sign of x as a gauge choice. The second possibility is that the two ranges are physically different configurations. We will adopt the latter viewpoint in this paper.

The cubic function $g(r)$ in (3) may be written as a product.

$$g(r) = -\frac{(r - r_+)(r - r_-)(r + r_0)}{R^2} \tag{6}$$

Matching (3) with (6), we find the following relations.

$$\begin{aligned} r_0 &= r_+ + r_- && \text{(a)} \\ R^2 &= r_+^2 + r_-^2 + r_+ r_- && \text{(b)} \\ 2MGR^2 &= r_+ r_- (r_+ + r_-) && \text{(c)} \\ x^2 &= \frac{r_+^2 + r_-^2 - 2r_+ r_-}{R^2} && \text{(d)} \end{aligned} \tag{7}$$

The last of these equations—(7d)—is just the square of the defining relation (5). By combining (7d) and (7b), we find the following relation.

$$R^2 - r_+^2 - r_-^2 = \frac{R^2}{3}\left(1 - x^2\right) \tag{8}$$

The significance of this equation will become clear in Section 3.2.

3. Entropy Deficit

In thermal equilibrium, the entropy is maximized subject to the constraint of a given average energy. In the context of the static patch, the equilibrium entropy is the usual de Sitter entropy, which I will call S_0.

$$S_0 = \frac{\text{area}}{4G} = \frac{\pi R^2}{G}. \tag{9}$$

Fluctuations may occur in which entropy is decreased to a smaller value of S_1. The probability for such a fluctuation is provided in terms of the entropy deficit:

$$\Delta S \equiv S_0 - S_1 \tag{10}$$

by the following.

$$\text{Probability} = e^{-\Delta S} \tag{11}$$

3.1. Small Black Holes

By a small black hole, I mean one for which its mass in Planck units is fixed as R becomes large. It may also be defined by its entropy being parametrically of an order of 1 as the de Sitter entropy is taken to infinity.

Let us consider the probability for a fluctuation in which a black hole of mass M appears at the pode[2]. The thermal equilibrium entropy of de Sitter space is provided in (9). To compute entropy S_1 of a state with a black hole at the pode, we use Equation (3) to compute r_+. The lowest order in M we found is as follows.

$$r_+ = R - MG. \tag{12}$$

The area and entropy of the cosmic horizon to lowest order are as follows:

$$\begin{aligned} \text{area} &= 4\pi(R^2 - 2RMG) \\ S &= \frac{\pi(R^2 - 2RMG)}{G}, \end{aligned} \tag{13}$$

and the entropy deficit is the following.

$$\Delta S = 2\pi RM. \tag{14}$$

The probability for the fluctuation is the following:

$$\text{Prob} = e^{-\Delta S} = e^{-2\pi RM}, \tag{15}$$

which is also the Boltzmann weight $e^{-\beta M}$.

Another interesting form for the entropy deficit for small black holes can easily be derived and is given by the following:

$$\Delta S = \sqrt{Ss}, \tag{16}$$

where s is the black hole's entropy $4\pi M^2 G$.

Note that for fluctuations involving small black holes of fixed mass, the entropy deficit proceeds as $S^{1/2}$. For example, the probability of a Planck mass black hole ($s = 1$) is as follows[3],

$$\text{Prob} \sim \exp\left(-\sqrt{S}\right). \tag{17}$$

Equation (16) is a strong hint about the holographic degrees of freedom of de Sitter space. It is quite unusual in the manner it depends on the entropies of not only the black hole but also on the entropy of the cosmic horizon. The question it raises is the following: without direct reference to gravity, what form of holographic degrees of freedom can result in such a relation?

Following [8], we will see in Section 5 that the answer involves matrix degrees of freedom similar to those of the BFSS M(atrix) theory. Moreover, (17) is suggestive of instantons in large-N matrix quantum mechanics and large-N gauge theories. In Section 3.2, we will see another more detailed relation for the entropy deficit of large black holes that even more strongly supports the claim for the matrix degrees of freedom.

3.2. Large Black Holes

By a large black hole, I mean one with Schwarzschild radius parametrically of the order the de Sitter radius R. Equivalently, the entropy of a large black hole is a fixed fraction of the de Sitter entropy. Such black holes are characterized by fixed values of the dimensionless variable x defined by (5). At $x \approx -1$, the black hole's radius is very small compared to the radius of the cosmic horizon. At $x = 0$, the two horizon areas become equal and of order R^2. At that point, the geometry is called the Nariai geometry. Its properties are reviewed in the Appendices A–C.

For $x > 0$, the black hole and cosmic horizon switch roles. As mentioned earlier, we will consider $x = -1$ and $x = 1$ to be different states.

The entropy of a horizon of radius r is given by the following.

$$S = \frac{\text{area}}{4G} = \pi r^2. \tag{18}$$

Applying this to the original de Sitter horizon with radius R and to the horizons $r_\pm$ of the Schwarzschild-dS geometry, we have the following.

$$\begin{aligned} S_0 &= \frac{\pi R^2}{G} \\ S_+ &= \frac{\pi r_+^2}{G} \\ S_- &= \frac{\pi r_-^2}{G}. \end{aligned} \tag{19}$$

The total entropy is the sum of the black hole and cosmic entropies.

$$S(x) = S_+(x) + S_-(x). \tag{20}$$

Armed with these relations, we may write Equation (8) in the following surprisingly simple form.

$$\begin{aligned} \Delta S(x) &= \pi R^2 \left(\frac{1 - x^2}{3G} \right) \\ &= \frac{S_0}{3}(1 - x^2) \end{aligned} \tag{21}$$

Equation (21) is illustrated in Figure 6.

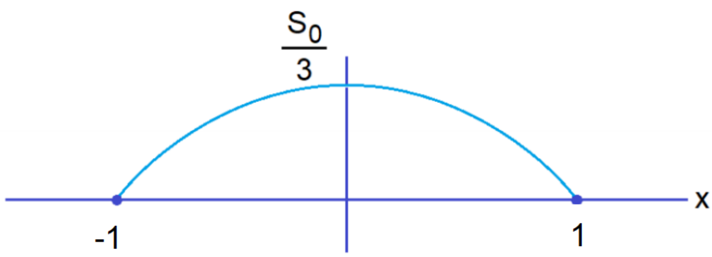

Figure 6. The entropy deficit as a function of x.

Equation (21) may also be written as follows:

$$\Delta S(x) = \Delta S_N \, (1 - x^2), \tag{22}$$

where $\Delta S_N = \frac{1}{3} S_0$ is the entropy deficit of the Nariai geometry. Note that ΔS is symmetric under $x \to -x$ and perfectly smooth at $x = 0$, which is the point where the black hole and cosmic horizons cross.

Equation (22) provides a detailed relation for how the entropy deficit varies with parameter x. In principle, the ratio $\frac{\Delta S(x)}{\Delta S_N}$ could have been a good deal more complicated, depending in an arbitrary way on x and entropy S. We will observe in Section 5 that the relation in (22) is characteristic of theories with certain matrix degrees of freedom.

4. Probabilities and the Entropy Deficit

The importance of the entropy deficit is that it determines the probabilities for Boltzmann fluctuations through formula (11)

For example, (21) implies that the probability for the occurrence of a freak fluctuation in which a black hole of mass M appears at the pode is as follows.

$$\text{Probability}(x) = e^{-\Delta S(x)} = e^{\pi R^2 (x^2 - 1)/3G}. \tag{23}$$

The location of the black hole need not be exactly at the pode. Let us introduce Cartesian coordinates X_i centered at the pode. The entropy will then depend not only on x but also X_i. By a suitable normalization of the coordinates, the entropy deficit in (23) can be generalized to [6] the following:

$$\text{Probability}(X) = e^{-\Delta S(X)} = e^{\pi R^2 (X^2 - 1)/3G}. \tag{24}$$

where X represents the four component object (x, X_i).

Now, consider the total probability for a black hole to nucleate anywhere in the static patch. It is given by an integral of the following form.

$$\text{Probability} = \int \frac{d^4 X}{R^4} e^{\pi R^2 (X^2 - 1)/3G}. \tag{25}$$

The range of the integration is from $X = 0$ to $X \sim 1$. The details of the boundary at $X \sim 1$ are not important as long as the components of X are of order 1. At the boundary of the integration, the black hole is very small ($x \sim 1$) or its location is close to the horizon.

Defining $u = R^2 X^2$, this may be written as follows.

$$\text{Probability} = e^{-\pi R^2/3} \int_0^{R^2} du \; u e^{\pi u/3G}. \tag{26}$$

The integral is straightforward and provides the following.

$$\text{Probability} = \left(\frac{3}{S_0} - \frac{9}{S_0^2} \right) + \left(\frac{9}{S_0^2} \right) e^{-S_0/3}. \tag{27}$$

Let us rewrite (27) using $S_0 = \pi R^2 / G$.

$$\text{Probability} = \left(\frac{3G}{\pi R^2} - \frac{9G^2}{\pi^2 R^4} \right) + \left(\frac{9G^2}{\pi^2 R^4} \right) e^{-\frac{\pi R^2}{3G}}. \tag{28}$$

The first term in (28) appears to be perturbative in the Newton constant. It represents contributions from very small black holes that appear close to the horizon and then fall back in. However, one might argue that this is misleading and that we should cut off the integral when the mass of the black hole becomes microscopic. In that case the first term in (28) would be replaced by $\exp(-\sqrt{S})$. This contribution numerically dominates the second term but is non-universal—it depends sensitively on microphysics.

The second term, although very sub-leading, is what really interests us. It is non-perturbative in G and this is due to a saddle point in the integrand at $X = 0$. This saddle point represents the contribution of the Nariai geometry to the path integral. It is universal and independent of any micro-physics.

One may wonder whether there is any process for which the non-universal small black hole contribution vanishes and the Nariai geometry dominates. The answer is yes; the Nariai geometry provides the leading contribution to the "inside-out" process (see Section 7).

5. dS-Matrix Theory

One can argue on the basis of entropy bounds that the holographic degrees of freedom live at the horizon of the static patch, but that argument does not tell us anything about the nature of those degrees of freedom. I will not try to provide a detailed model here, but the properties of de Sitter black holes can tell us more. What we will learn is that the degrees of freedom must be matrices [2–5], and the Hamiltonian should include a term for which its role is to enforce certain constraints.

5.1. Small Blocks

Let us return to Equation (16) for small black holes.

$$\Delta S = \sqrt{Ss},$$

This relation contains a hint about the nature of the holographic degrees of freedom of a de Sitter space. Following Banks and collaborators [2], it motivates us to conjecture that the degrees of freedom are matrices (see also [3–5]) in the same sense as in a M(atrix) theory [8]. To see why, let us assume that the horizon degrees of freedom are a collection of $N \times N$ Hermitian matrices.

$$A_{m,n} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,N} \end{pmatrix} \tag{29}$$

An implicit index runs over some finite range and may include both bosonic and fermionic matrices. Taking a cue from BFSS M(atrix) theory [8], we may think of the index m as running over a set of N $D0$-branes. Very roughly, the diagonal elements represent positions of the branes, while the off-diagonal elements represent operators that create and annihilate strings connecting the $D0$-branes.

For now, we will not specify any particular form for the Hamiltonian but we will assume that one exists, as well as a thermal ensemble at the appropriate temperature. We also assume that the total entropy in thermal equilibrium is proportional to the number of degrees of freedom:

$$S_0 = \sigma N^2 \tag{30}$$

where σ is the entropy per degree of freedom.

Consider a state with a black hole of entropy as follows:

$$s = \sigma m^2 \tag{31}$$

which is located at the pode. Motivated by the M(atrix) Theory, we assume that the degrees of freedom split into block-diagonal form with the cosmic horizon degrees of freedom forming an $(N-m)\times(N-m)$ block, and the black hole degrees of freedom forming an $m\times m$ block. The indicies labeling the large block and small blocks will be called I and i, respectively. The entries in the large and small blocks are a_{IJ} and a_{ij}. The off-diagonal elements connecting the two blocks are a_{iJ} and a_{Ij}.

Again, motivated by M(atrix) theory, we will assume that in a state composed of two well-separated components—in this case the small black hole and the large cosmic horizon—the off-diagonal degrees of freedom a_{iJ} and a_{Ij} are constrained[4] to be in their ground states and, therefore, carry no entropy [2–5]. In other words, the state is constrained by $2m(N-m)$ constraints, which express the condition that there are no strings connecting the $D0$-branes in the two blocks. Classically, these constraints take the following form.

$$a_{iJ} = a_{Ij} = 0. \tag{32}$$

Subject to these constraints, the entropy of this state is the following.

$$S_1 = \sigma(N-m)^2 + \sigma m^2 \tag{33}$$

Assuming $m << N$ and working toward the lowest order in m, the entropy deficit is as follows.

$$\Delta S = 2\sigma N m. \tag{34}$$

Using (30) and (31) provides the following:

$$\Delta S = 2\sqrt{Ss} \tag{35}$$

which reproduces (16) within a factor of 2. I do not know of any other mechanisms that can accomplish this.

The factor of two discrepancy between (16) and (35) may be another significant hint for which we will discuss in Section 6.

5.2. Remark on Higher Dimensions

In higher dimensions, the situation becomes more complicated. I will quote the d dimensional generalization of (16), which is valid for $s << S_0$.

$$\Delta S = \left(\frac{d-2}{2}\right) S_0^{\frac{1}{d-2}}\, s^{\frac{d-3}{d-2}}. \tag{36}$$

This formula, derived from the d-dimensional Schwarzschild solution, can be reproduced with matrix degrees of freedom but at a cost. It is necessary to allow the entropy per degree of freedom to depend on N according to[5] the following.

$$\sigma(N) \sim \frac{1}{N^{\left(\frac{d-4}{d-3}\right)}}. \tag{37}$$

I will leave any further dicussions of the higher dimensional generalization to future work.

5.3. Large Blocks

Returning to the case $d = 4$, let us now consider fixed values of s/S_0. The entropy deficit is proportional to the number of constraints.

$$\Delta S = 2\sigma m(N-m). \tag{38}$$

Following (5) and using the fact that $r_\pm$ is proportional to the square roots of the entropies of the horizons, we define the following.

$$x = \frac{\sqrt{s} - \sqrt{S}}{\sqrt{S_0}}. \tag{39}$$

We use the following:

$$\begin{aligned} S &= \sigma(N-m)^2 \\ s &= \sigma m^2 \end{aligned} \tag{40}$$

and we find the following:

$$x = \frac{2m}{N} - 1, \tag{41}$$

which leads to a relation identical to (22).

$$\begin{aligned} \Delta S(x) &= \Delta S_N(1 - x^2), \\ \Delta S_N &= \frac{1}{2} S_0. \end{aligned} \tag{42}$$

As in the earlier small-block case (35), the only difference between the gravitationl result and the dS-matrix result is the numerical factor. At the Nariai point $x = 0$, the entropy deficit in the matrix theory is $S_0/2$ instead of $S_0/3$. In Section 6, we will see that there is room in the matrix theory to decrease the numerical constant in (42) and bring it closer to its gravitational value of $1/3$.

There is nothing inevitable about the relation (42). It does not follow from any general statistical or thermodynamic principles. It is a consequence of the matrix degrees of freedom and the particular assumptions concerning the manner the system is decomposed into subsystems. The close correspondence between the matrix theory and general relativity calculations of $\Delta S(x)$ seems remarkable to me. I do not know any other holographic mechanism that can lead to it. However it is important to explain the discrepancy between the numerical factors. In the next section, we will observe that there is plenty of room in the matrix theory to modify the constants and bring them into alignment with their gravitational counterparts.

6. Dynamics of the Constraints

The degrees of freedom and Hamiltonian of the static patch are highly constrained by the symmetries of de Sitter space [6]. Implementing those symmetries is a very hard problem, which I will not try to solve in this paper. My purpose is more modest: namely, to illustrate a dynamical mechanism for how the constraints (32) can be enforced by energy considerations.

Let us add (29) one more $N \times N$ matrix to the matrix degrees of freedom, denoted by $\mathcal{R}$.

$$\mathcal{R}_{m,n} = \begin{pmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,N} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m,1} & r_{m,2} & \cdots & r_{m,N}. \end{pmatrix} \tag{43}$$

The notation is chosen to indicate that the eigenvalues of $\mathcal{R}$ represent a radial position in the static patch.

To enforce the constraints, we will assume that the matrix-theory Lagrangian contains the following term:

$$L = \sum \mathrm{Tr}\left(c^2 R^4 \dot{A}^2 - [\mathcal{R}, A][A, \mathcal{R}]\right) \tag{44}$$

where c is a numerical constant, and the sum in (44) is over all the other matrices—bosonic and fermionic[6] —that comprise the degrees of freedom of the matrix theory.

Now, consider a configuration representing an object well separated from the cosmic horizon. For simplicity, the object could be at the pode at $r = 0$. The cosmic horizon is at $r = R$. To represent this, we assume matrix $\mathcal{R}$ has an approximately block-diagonal form:

$$\begin{aligned} r_{IJ} &= R\delta_{IJ} + \epsilon_{IJ} \\ r_{ij} &= \epsilon_{ij} \\ r_{iJ} &= \epsilon_{iJ} \\ r_{Ij} &= \epsilon_{Ij}. \end{aligned} \tag{45}$$

where ϵ is a numerically small matrix representing quantum fluctuations.

Ignoring ϵ, the commutator term in (44) provides the following.

$$\mathrm{Tr}\sum[\mathcal{R}, A][A, \mathcal{R}] = R^2 \sum |a_{iJ}|^2. \tag{46}$$

Combining this with the kinetic term in (44) provides the following.

$$L = \sum\left(c^2 R^4 \dot{a}_{iJ}\dot{a}_{Ji} - R^2 a_{iJ} a_{Ji}\right) \tag{47}$$

The effective Hamiltonian for the off-diagonal elements is a sum of harmonic oscillator Hamiltonians with the following frequency.

$$\omega = \frac{1}{cR}. \tag{48}$$

If ω is much larger than the other energy scales, the oscillators will be forced to their ground states and the off-diagonal degrees of freedom will carry no entropy. In that case, the analysis leading up to equations (35) and (42) applies unmodified.

The energy scale with which ω is to be compared is temperature $T = 1/(2\pi R)$. If numerical constant c is much smaller than 2π, then the constraints will be tightly enforced, but the more interesting situation is when $c \sim 2\pi$. In that case, the constraints will not be tightly enforced; the off diagonal elements will carry some entropy, but only with a fraction of σ (the thermal entropy per degree of freedom in (30)). If we carry out the analysis leading up to equations (35) and (42), we will find that the only effect of relaxing the constraints is to change the numerical coefficients in these equations. For example, it should be possible to choose c so as to change the constant in (35) from 2 to the gravitational value 1. At the same time, that will decrease the value of factor $1/2$ in (42), but to bring it to exactly $1/3$ would require subtle and possibly finetuned properties of H. One might speculate that if the Hamiltonian satisfies the symmetry requirements of de Sitter space, this would be automatic.

7. The Inside-Out Process

Quantum mechanically, Nariai geometry is unstable[7] due to evaporation [10]. Initially, the two horizons are at the same temperature (see Appendix C) Now suppose a statistical fluctuation occurs and the left horizon emits a bit of energy that is absorbed by the right horizon. The effect is to increase T_L and decrease T_R. This creates a tendency (heat flows from hot to cold) for more energy to flow from left to right. The statistical tendency is for the left horizon to shrink down to a small black hole, while the right horizon grows to the full size of the de Sitter horizon. Eventually, the small black hole will disappear, transferring all its energy to the cosmic horizon on the right side. Of course it could have happened the other way—the right horizon shrinking and the left growing.

How long does the entire process of evaporation take? The answer is roughly Page time $t_p \sim S_0 R$. Note that this process does not violate the second law— entropy increases from $\frac{2}{3}S_0$ to S_0.

However, now instead of running the system forward in time with e^{-iHt}, we run it backwards with e^{iHt}. What will happen is the time reverse in which the system back-evolves to some microstate of de Sitter space with either the left or right horizon growing. This implies that there are fluctuations in the thermal state, which begin with dS, pass through Nariai space and eventually decay back to dS. The entire history from dS to N to dS is a massive Boltzmann fluctuation in which the de Sitter horizon emits a small black hole which then grows to the Nariai size, and then one of the two Nariai horizons shrinks back to nothing, while the other grows back to a dS size.

In particular, the process can proceed so that the two horizons are exchanged. One may think of it, in terms of the diagram in Figure 6, as a process in which the system migrates from $x = -1$ to $x = +1$, passing through the Nariai state at $x = 0$. This process of exchange of the horizons is the "inside-out" process. An observer (Figure 7) watching this take place would literally observe the dS turn itself inside out—the tiny black hole growing and becoming the surrounding cosmic horizon while the cosmic horizon shrinks to a tiny black hole (or no black hole at all).

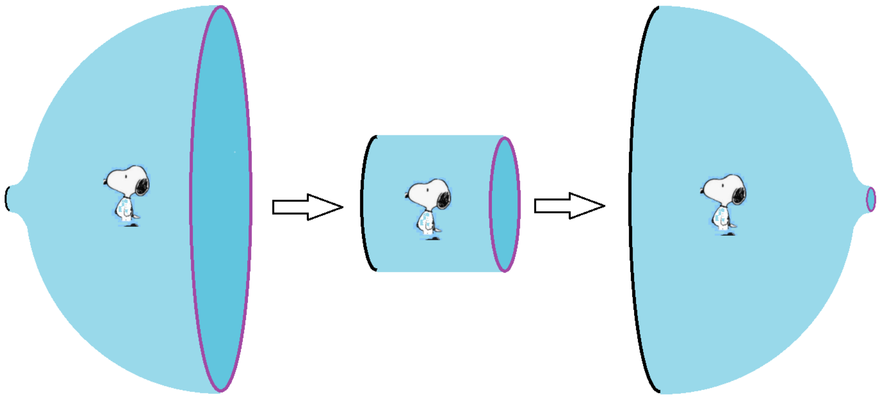

Figure 7. The inside-out transition as seen by an observer in the static patch.

For the inside-out process to take place, the system must pass through the Nariai state at $x = 0$. Since the probability for this is $e^{-S_0/3}$, it is obviously not allowed perturbatively. Passing through the Nariai state provides the leading contribution to the transition $(x = -1) \rightarrow (x = 1)$.

It is tempting to think of the inside-out process as a quantum tunneling event mediated by some kind of conventional instanton, i.e., a solution of the classical Euclidean equations of motion interpolating from $x = -1$ to $x = +1$. This is not correct—there is no such solution. What does exist is the classical Nariai solution eternally sitting at the point $x = 0$. This is similar to a process in which a system gradually thermally up-tunnels over a broad potential barrier, mediated by an so-called Hawking Moss instanton [11]. In the Hawking-Moss framework, the exponential of the Euclidean action (in this case, the action of the Euclidean Nariai geometry) provides the probability to find the system at the top of the potential [12] (in other words, at the Nariai point). The probability is given by the following $e^{-S_0/3}$.

The HM instanton does control the rate at which such inside-out processes occur. There are two time scales of interest. The first, which i will call δt, is how long does the process take from beginning $(x = -1)$ to end $(x = 1)$? The answer is that it takes a time of the order of Page time, $\delta t \sim t_{page} \sim S_0 R$. The other time scale, Δt, is the average time between inside-out events. That time period is very much longer: δt is essentially instantaneous on the longer time scale Δt. This is shown in Figure 8.

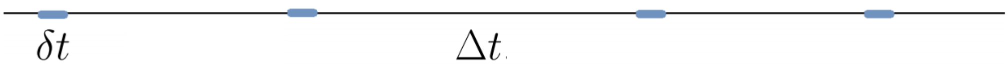

Figure 8. Time scales for the inside-out process. The short blue intervals δt represent the duration of process which takes place over a time of order the Page time. The long intervals Δt between them represent the times between inside-out transitions.

Under this circumstance, the probability of finding the system close to the Nariai state would be the following ratio.

$$\text{Probability} = \frac{\delta t}{\Delta t} \tag{49}$$

The probability to find the Nariai state is of order $e^{-S_0/3}$, from which we conclude the following.

$$\Delta t \sim R S_0 e^{S_0/3}. \tag{50}$$

The prefactor in (50) is not very reliable, but it does show that the rate of inside-out events is determined by the exponential $e^{-S_0/3}$. This can be compared with the longest possible decay time for Coleman DeLuccia tunneling to a terminal vacuum, if in fact such decays are allowed. That time scale can in principle be as long as e^{S_0} although it can be much shorter. If we suppose the decay rate relative to terminal vacua is as long as possible, then there is plenty of time for the inside-out process to occur many times before de Sitter vacuum decays.

The inside-out process is particularly interesting because its rate is controlled by the Nariai saddle at $x = 0$, with no contribution from small black holes. In Section 4, the Nariai saddle was a tiny subleading effect in the probability for a black hole fluctuation, but the inside-out transition can only occur if the system passes through the Nariai point. Therefore, the rate is determined by the universal saddle at $x = 0$.

It is obvious what the inside-out transition means in the dS-matrix theory. The matrix representation of the unconstrained thermal equilibrium state has all N^2 degrees of freedom fluctuating in thermal equilibrium. The state with a small black hole is a constrained state [2–5] represented by block-diagonal matrices; one small block for the black hole, and one large block for the cosmic horizon. In the inside-out process, the small block grows while the large block shrinks until they become equal and then continues until the blocks are exchanged. In the process, the system must pass through the configuration with two equal blocks, which is the matrix version of Nariai geometry.

8. Instantons and Giant Instantons

The processes of small black hole formation and the inside-out transition exhibit some interesting parallels with instanton-mediated processes in large-N gauge and matrix theories.

8.1. Some Probabilities

This subsection summarizes the results of some probability calculations in gravity and dS-matrix theory so that we can compare them with instanton amplitudes.

The thermal formation of the smallest black hole—one with entropy of order unity—has a matrix theory description in which the small block is a single matrix element and the number of constrained is $\sim 2N$. The entropy deficit is the following:

$$\Delta S \sim N \tag{51}$$

and the corresponding probability is as follows.

$$P = \exp(-N). \tag{52}$$

Now consider the probability in de Sitter gravity for a minimal size black hole with $s = 1$.

$$P = \exp(-\sqrt{S}). \tag{53}$$

Using $S = \sigma N^2$, we see that (52) and (53) are essentially the same.

Next, we have a bigger fluctuation in the matrix theory, namely a fluctuation that occrs all the way to the matrix version of the Nariai state $m = N/2$, in which the two blocks are equal. The entropy deficit is the following.

$$\Delta S = \frac{\sigma N^2}{2} \tag{54}$$

We compare that with the gravity result for the same process.

$$\Delta S = S_0/3. \tag{55}$$

Using $S = \sigma N^2$ shows that (54) and (55) scale in the same manenr.

8.2. Instantons

Now, let us turn to instantons: first in matrix quantum mechanics and then in gauge theories. The simplest example is single-matrix quantum mechanics with the following Lagrangian:

$$L = \frac{1}{2g^2}\left(\mathrm{Tr}\dot{A}^\dagger\dot{A} - \mathrm{Tr}\,V(A)\right), \tag{56}$$

and V is a double-well potential such as the one in Figure 9.

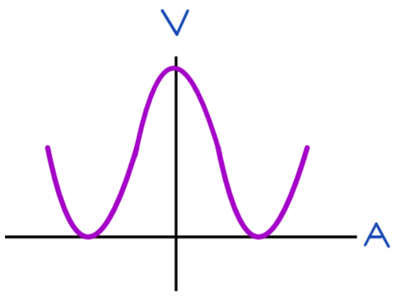

Figure 9. A double-well potential for a matrix model.

By standard arguments, this can be reduced to the quantum mechanics of a one-dimensional system of N fermions, which represent the eigenvalues of A.

An individual eigenvalue can tunnel from the left well to the right well with the probability given by an instanton. The probability for a single eigenvalue tunneling is the following.

$$P_1 \sim \exp(-1/g^2). \tag{57}$$

In the 't Hooft large-N limit, we have 't Hooft.

$$g^2 N = \lambda \qquad (\lambda \sim 1). \tag{58}$$

We find 't Hooft .

$$P_1 \sim \exp(-N). \tag{59}$$

This simple instanton process scales with N the same as in (52), suggesting that the formation of a Planck-mass black hole is an instanton-mediated process in the dS-matrix theory.

8.3. Giant Instantons

We may also consider a process in which all eigenvalues tunnel from one side to the other. I will call it a "giant instanton". The action for a giant instanton is N-times larger than the simple instanton, and the probability for the "giant transition" is as follows.

$$P_N \sim \exp(-N^2). \tag{60}$$

The probability for the giant transition scales the same manner as the inside-out transition, namely $\exp -S_0$. We note that this transition, much similar to the inside-out transition, takes the system between states related by a form of symmetry.

Instantons and giant instanton transitions also exist in the Yang–Mills theory. Recall that an instanton in an $SU(N)$ theory lives in an $SU(2)$ subgroup and describes a tunneling transition of the $SU(2)$ Chern-Simons invariant by one unit. The rate also scales as $\exp(-N)$.

One can also consider a transition in which all $N/2$-commuting $SU(2)$ subgroups tunnel. The rate for such giant instantons is $\exp(-N^2)$. Thus, we observe a common pattern governing non-perturbative transition rates in large-N gauge theories, matrix theories, and also Boltzmann fluctuations in de Sitter space.

9. Remarks about the Holographic Principle in dS

Semiclassically, the static patch of de Sitter space is a holographic quantum system with the degrees of freedom localized at the stretched horizon. This is reasonable in semiclassical-gravity, but things are more complicated in the full non-perturbative theory. Large Boltzmann fluctuations can lead to higher topologies such as the Nariai geometry, and the horizon can break up into multiple horizons. Where, under those circumstances, do the holographic degrees of freedom reside? On the outermost or largest horizon? On the union of all of the horizons? Or is the hologram more abstract and not localized at all?

1. The outermost horizon? Consider a state very near the Nariai limit but with one horizon being slightly larger than the other. In this case, the largest horizon has an entropy that is slightly greater than $S_0/3$. This is clearly not enough degrees of freedom to describe both horizons that, in sum, have entropy $2S/3$. Moreover, the sizes of the horizons can change with time; the largest can become the smallest and vice versa.
2. The union of horizons? This also does not seem right. The corresponding matrix description would be that the hologram is the union of blocks, but the Hilbert space does not factor into Hilbert spaces for the blocks. This is obvious from the fact that there are off-diagonal components a_{Ij} and a_{Ji}. In an approximation, these elements may be unexcited if the constraints are tight, but in order to match the numerical coefficients. the constraints cannot be infinitely tight.
3. The dS-matrix theory shows that the hologram is a single system, with a number of degrees of freedom as large as the largest area of the cosmic horizon when it forms a single connected whole. It is large enough to describe any state of the system but is not much larger. In that sense, it may be identified with the horizon of the dominant saddle point in the path integral. In individual branches of the wave function, no single component of the horizon may be large enough to describe the whole, but the hologram itself is described.

Funding: This research received no external funding.

Acknowledgments: I thank Adam Brown for the many discussions on the material in this paper. L.S. was supported in part by NSF grant PHY-1720397.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Nariai Geometry

As the mass M of the black hole increases, the two solutions r_- and r_+ come together. The limiting geometry is the Nariai solution. It occurs at the point where $dg/dr = 0$. From (3).

$$r_N = \frac{R}{\sqrt{3}}. \tag{A1}$$

The entropy of each horizon is given by $\frac{\text{area}}{4G} = \pi r_N^2 = S_0/3$. The combined entropy of the two horizons is the total Nariai entropy S_N.

$$\begin{aligned} S_N &= \frac{2}{3}S_0 \\ &= \frac{2\pi}{3}R^2. \end{aligned} \tag{A2}$$

The entropy deficit of a state ρ is defined as the difference of the de Sitter entropy and the entropy of ρ. For the Nariai state, the entropy deficit is as follows.

$$\begin{aligned} \Delta S_N &= S_0 - S_N \\ &= \frac{S_0}{3}. \end{aligned} \tag{A3}$$

To understand Nariai geometry, we begin with the near-Nariai geometry in which the two roots $r_\pm$ in Figure 5 are very close, as can be seen from Figure 5. The function $f(r)$ in the small interval between the roots may be expanded to quadratic order. We define the following.

$$r - \frac{R}{\sqrt{3}} = u. \tag{A4}$$

Then, f is given by the following:

$$f = \frac{3}{R^2}(\epsilon^2 - u^2) \tag{A5}$$

and the following as well.

$$ds^2 = -\frac{3}{R^2}(\epsilon^2 - u^2) + \frac{R^2}{3(\epsilon^2 - u^2)}du^2 + \frac{R^2}{3}d\Omega^2. \tag{A6}$$

Now, we define the following:

$$\begin{aligned} v &= \frac{u}{\epsilon} \\ \tau &= \frac{3\epsilon}{R^2}t \end{aligned} \tag{A7}$$

and the Nariai metric is obtained.

$$\frac{R^2}{3}\left(-(1-v^2)d\tau^2 + \frac{1}{1-v^2}dv^2 + d\Omega^2\right). \quad \text{(A8)}$$

The Nariai geometry is $dS_2 \times S_2$. The Euclidean continuation is simply $S_2 \times S_2$ with both S_2-factors having radius $R/\sqrt{3}$.

This casts a new light on the second term of (28). It is evident that the saddle point's contribution to the path integral comes from classical Nariai geometry—a geometry with a different topology. The contribution to the path integral can be worked out by calculating action I of the classical $S_2 \times S_2$ geometry. One finds the following unsurprising result.

$$I_{Nariai} = S_N. \quad \text{(A9)}$$

Consider the contributions to the Euclidean path integral from the original de Sitter space S_4, and Nariai space $S_2 \times S_2$. Schematically (ignoring prefactors), the path integral is given by the following.

$$e^{S_0} + e^{2S_0/3} = e^{S_0}\left(1 + e^{-S_0/3}\right). \quad \text{(A10)}$$

The geometry of de Sitter space is the non-compact goup $O(4,1)$, which is a continuation of the compact group $O(5)$. The compact group describes the symmetry of the Euclidean continuation of dS, namely the 4-sphere. $O(4,1)$ and $O(5)$. are the symmetries of the semiclassical theory. The geometry of Nariai space[8] is $dS(2) \times S(2)$, or in the Euclidean signature, $dS(2) \times S(2)$. The symmetry of Nariai space is $O(2,1) \times O(3)$ or $O(3) \times O(3)$. The dS symmetry is larger than Nariai symmetry, but it does not contain Nariai symmetry as a subgroup.

In the semiclassical limit in which entropy is infinite, the probability of transitions between the two geometries is zero. There is no obstacle to the symmetries being realized. However, as we have observed, in the full quantum theory, there are transitions, and it does not seem possible for either $O(4,1)$ or $O(2,1) \times O(3)$ to be exact. This clash was discussed in [6].

Appendix B. The Equilibrium Shell

A non-relativistic particle at rest in static coordinates will be in equilibrium at a point where $\frac{df(r)}{dr} = 0$. For an ordinary Schwarzschildblack hole in flat space, no such point exists; $f(r)$ is monotonic in that case. However, the Schwarzschild-de Sitter black hole does have an equilibrium point. From (2), we see that the (unstable) equilibrium shell is given by the following.

$$r_{eq}^3 = MGR^2 \quad \text{(A11)}$$

Consider the Penrose diagram for a Schwarzschild–de Sitter black hole in Figure A1.

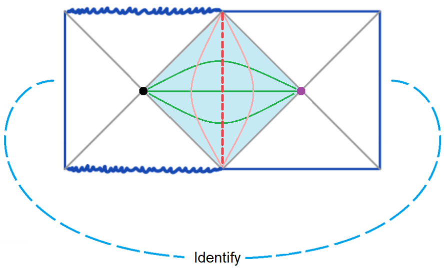

Figure A1. Penrose diagram for Schwarzschild-de Sitter black hole and a static patch along with the equilibrium surface shown as a red dashed line.

The static patch shown in light blue surrounds the black hole so that the black hole remains static at the center of the static patch. The dotted red line is the equilibrium shell that also surrounds the black hole at the equilibrium position.

The equilibrium shell is a natural place to introduce observers. We can think of it as a substitute for the pode. If the observer looks in one direction, he sees the black hole's horizon, and in the other direction, he sees the cosmic horizon surrounding him.

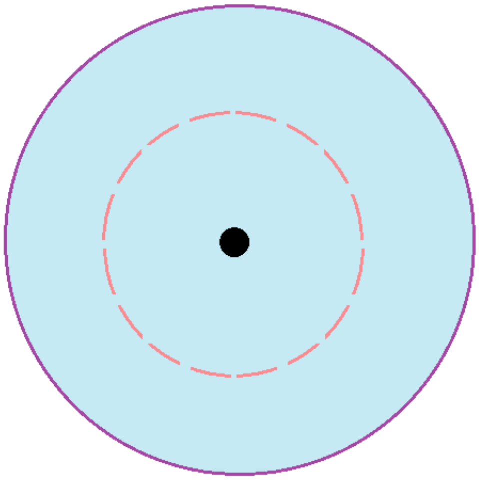

Figure A2. A small black hole in the static patch. The red dashed curve is the equilibrium surface.

Figure A2 shows the static patch surrounded by the cosmic horizon and the black hole at the center of the patch. The dotted red circle (really a sphere) represents the equilibrium shell.

In the Nariai limit, the metric takes form (A8) and one sees that the equilibrium shell is at the symmetry point $v = 0$ midway between the horizons. One should note that although the value of r is the same at the two horizons, the distance between them is not zero. It is given by the following.

$$\begin{aligned} \text{distance} &= \frac{R}{\sqrt{3}} \int_{-1}^{1} \frac{1}{\sqrt{1-v^2}} dv \\ &= \frac{\pi R}{\sqrt{3}}. \end{aligned} \tag{A12}$$

Appendix C. The Temperature of the Nariai Geometry

Now, let us consider the Minkowski-signature Nariai geometry (A8). In this limit, the two horizons become equal and the geometry is symmetric with respect to a reflection about $v = 0$. The equilibrium shell is the two-sphere at $v = 0$.

Let us consider the temperature of the Nariai geometry. The temperature of a black hole in flat or AdS space is usually defined as the temperature registered by a thermometer located at spatial infinity. In de Sitter space, there is no asymptotic spatial infinity, so we must choose another rule for defining temperature. One possibility is to locate the thermometer at the equilibrium shell. To compute the temperature, we may consider the Euclidean continuation and compute the circumference of the time-circle and identify it with inverse temperature. Alternatively, we may use the fact that the Minkowski geometry is $dS_2 \times S_2$ with a radius $R/\sqrt{3}$. It follows from both arguments that the temperature at the equilibrium shell is as follows:

$$T_N = \frac{\sqrt{3}}{2\pi R} \tag{A13}$$

(which is larger by a factor $\sqrt{3}$ than the temperature of the original de Sitter space.) The observer at the equilibrium position is bathed in radiation at temperature T_N.

When at the same temperature, the two horizons are in thermal equilibrium with each other, but the equilibrium is unstable.

We should note that the pode of the two dimensional de Sitter space is exactly at the point $v = 0$, so in that sense, the temperature at the equilibrium shell is the temperature at the pode of dS_2. The temperature of the original dS_4 is the proper temperature at the 4-D pode, and it is given by the following.

$$T = \frac{1}{2\pi R.} \tag{A14}$$

Notes

[1] The various mechanisms and calculations described in this paper apply to four dimensions. Generalization to other number of dimensions is non-trivial, and I will not undertake the task here.

[2] The center of the causal patch at $r = 0$. The center of the opposing static patch is called the antipode [6].

[3] In this paper, notation exp is synonymous with "exponential in". Thus, e^S, $e^{S/3}$, and e^{2S} are all exp S.

[4] Banks and Fischler have proposed that the connection between localized objects and constrained states of holographic variables is the basis for understanding locality on scales smaller than that set by the cosmological constant [4,5].

[5] A different view of the holography of higher-dimensional dS based on multidimensional matrices was given in [4,5].

[6] For fermionic matrices, the quadratic kinetic term in (44) should be replaced by the usual Dirac term linear in time derivatives.

[7] In an earlier version of this paper, I had claimed that Nariai geometry is classically stable. I am grateful to the referee for pointing out the existence of a possible classical instability. I quote from the referee's report: "In fact, it is unstable with respect to homogeneous anisotropic perturbations that results either in its Kasner-like collapse with the power-law exponents $(-1/3, 2/3, 2/3)\ldots$" [9].

[8] A generalization to a covering of Nariai space is possible in which one of the compact spatial coordinates is continued to a non-periodic coordinate. See, for example, [13]. The generalization plays no role but I mention it to avoid ambiguity. I thank the referee for reminding me of this fact.

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