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# Valuing Exchange Options Under an Ornstein-Uhlenbeck Covariance Model

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**Abstract:** In this paper we study the pricing of exchange options between two underlying assets whose dynamic show a stochastic correlation with random jumps. In particular, we consider a Ornstein-Uhlenbeck covariance model, with Levy Background Noise Processes driven by Inverse Gaussian subordinators. We use expansions in terms of Taylor polynomials and cubic splines to approximately compute the price of the derivative contract. Our findings show that the later approach provides an efficient way to compute the price when compared with a Monte Carlo method, while maintaining an equivalent degree of accuracy.

**Keywords:** exchange options; Ornstein-Uhlenbeck; Taylor expansion; splines

## 1. Introduction

In this paper we study the pricing of exchange options when the underlying assets have stochastic correlation with random jumps. More specifically, we consider an Ornstein-Uhlenbeck covariance process with Background Noise Levy Process (BNLP) driven by Inverse Gaussian subordinators. In order to calculate the price of the derivative contract we use expansions of the conditional price in terms of Taylor and cubic spline polynomials and compare the results with a computationally expensive Monte Carlo method.

Recently, the price of exchanges in models with stochastic volatilities and random jumps has been studied in several papers. Researches differ in the modeling of the correlation, volatilities and jumps, as well as in their numerical approaches. In [Garces and Cheang \(2021\)](#) for example *the method of lines* is successfully implemented. See also [Alos and Rheinlander \(2017\)](#); [Kim and Park \(2017\)](#); [Pasricha and Goel \(2022\)](#) for other techniques. Interesting applications in a context of credit risk can be found in [Kim \(2020\)](#); [Pasricha and Goel \(2019\)](#).

The exchange of two assets can be used to hedge against the changes in the price of underlying assets by betting on the difference between both. The price of these instruments has been first considered in [Margrabe \(1978\)](#) under a bivariate Black-Scholes model, where a closed-form formula for the pricing is provided. Those results have been extended in [Caldana and Fusai \(2013\)](#); [Caldana et al. \(2015\)](#); [Cheang and Chiarella \(2011\)](#) to price the exchange in the case of a jump-diffusion model, while [Bernard and Cui \(2010\)](#) considers the pricing of the derivative contract under stochastic interest rates.

On the other hand, it is well known that constant correlation, constant volatilities and continuous trajectories are features not supported by empirical evidence. Some dynamic stochastic models for the covariance have been previously proposed, see for example [Da Fonseca et al. \(2008\)](#) for the popular Wishart model, [Pigorsch and Stelzer \(2007\)](#) for an Ornstein-Uhlenbeck Levy type model, [Olivares et al. \(2010\)](#) for a simple model based on a linear combination of Cox-Ingersoll-Ross processes and finally an extension of [Barndorff-Nielsen and Shephard \(2001\)](#) to a multivariate setting proposed in [Barndorff-Nielsen et al. \(2002\)](#). Based on the later we study the integrated characteristic function, moments and pricing of the exchange.



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Unfortunately, a closed-form pricing formula is not available when stochastic covariance and random jumps are considered. Approximations based on polynomial expansions of the price, after conditioning on the integrated covariance, allow for efficient and accurate calculations.

Starting with a pioneer idea in Hull and White (1987), Taylor expansions have been used to compute the price of spread options and other multivariate contracts. For example, a second order Taylor expansion has been successfully used in Li et al. (2008, 2010) to price spread options under a multivariate Black-Scholes model. As it is possible, based on the knowledge of the characteristic function, to compute mixed moments for the integrated covariance model an approach following the same idea seems feasible to be applied for the case studied in this paper. Moreover, other polynomial expansions such as cubic splines are also considered.

Our approach is in essence a combination of conditioning, polynomial expansions and a FFT inversion. Together with the existence of a closed-form expression for the price in the Black-Scholes setting it allows to value exchange options when considering stochastic correlation.

The organization of the paper is as follows. In Section 2 we introduce the main notations and discuss the pricing of the exchange option by polynomial expansions. In Section 3 we define the Ornstein-Uhlenbeck covariance process and compute the characteristic function of the integrated process and its moments. In Section 4 we discuss the implementation of the method, while numerical results allowing a comparison between the price obtained by Monte Carlo and polynomial approximations are shown in Section 5. Proofs of theoretical results are deferred to the appendix.

## 2. Pricing Exchange Options in Models with Stochastic Covariance

First, we introduce some notations. We denote by  $C_l$  a matrix having ones in position  $(l, l)$  and zeros otherwise. For a matrix  $A$  its trace is denoted by  $tr(A)$  and its transpose by  $A'$ . For a vector  $V$  the expression  $diag(V)$  denotes a diagonal matrix whose elements in the diagonal are the components of  $V$ . For two vectors  $x$  and  $y$ ,  $xy$  represents its scalar product.

When  $l$  is an integer number,  $D^l$  represents the  $l$ -th order derivative operator. To simplify notations we make  $D^1 = D$ .

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$  be a filtered probability space. We denote by  $\mathcal{Q}$  an equivalent martingale measure (EMM), and by  $r$  the (constant) interest rate or a vector with components equal to  $r$ . The filtration  $(\mathcal{F}_t)_{t \geq 0}$  is assumed to verify the usual conditions, i.e., it is right-continuous and contains all events of probability zero.

The  $\sigma$ -algebra  $\mathcal{F}^{X_t}$  is defined for any  $t > 0$  as the  $\sigma$ -algebra generated by the random variables  $(X_s)_{0 \leq s \leq t}$ .

Also, we define the increments of the process  $(X_t)_{t \geq 0}$  as  $\Delta X_t = X_t - \lim_{s \uparrow t} X_s$ . For two squared integrable semi-martingales  $X$  and  $Y$ ,  $\langle X, Y \rangle$  defines their quadratic covariation process. The functions  $\varphi_X(u)$  and  $\varphi_X(u, a, b)$  represent respectively the characteristic function of the random variable  $X$  and the characteristic function of the random variable constrained to the interval  $[a, b]$ , both under the chosen EMM.

A two-dimensional adapted stochastic process  $(S_t)_{t \geq 0} = (S_t^{(1)}, S_t^{(2)})_{t \geq 0}$ , where their components are prices of certain assets, is defined on the filtered probability space.

We describe the prices by:

$$S_t^{(j)} = S_0^{(j)} \exp(Y_t^{(j)}) \quad j = 1, 2. \tag{1}$$

where  $Y = (Y_t^{(1)}, Y_t^{(2)})_{t \geq 0}$  is the process of log-prices.

We assume that the process of log-prices has a dynamic under  $\mathcal{Q}$  given by:

$$dY_t = (r - q - \frac{1}{2}diag(\Sigma_t))dt + \Sigma_t^{\frac{1}{2}}dB_t \tag{2}$$

while  $(\Sigma_t^{1/2})_{0 \leq t \leq T}$  is a matrix-valued stochastic process such that  $\Sigma_t^{1/2} \Sigma_t^{1/2} = \Sigma_t$ . Its components are denoted as  $(\sigma_t)_{jk}$ , for  $0 \leq j, k \leq 2$ .

Under  $\mathcal{Q}$ , the process  $(B_t)_{t \geq 0} = (B_t^{(1)}, B_t^{(2)})_{t \geq 0}$  is a standard two-dimensional Brownian motion with independent components. The vector  $q = (q_1, q_2)$  represents dividends on both assets. For any  $T > 0$  the conditional joint distribution of  $Y_T$  and its characteristic function are given in the elementary lemma below.

**Lemma 1.** Let  $(S_t)_{0 \leq t \leq T}$  be a process driven by Equations (1) and (2) under an EMM  $\mathcal{Q}$ . Then, conditionally on  $\mathcal{F}^{\Sigma_T}$ , the random variable  $Y_T$  follows a bivariate normal distribution. More precisely:

$$Y_T \sim N\left((r - q)T - \frac{1}{2} \text{diag}(\Sigma_T^+, \Sigma_T^+)\right)$$

where  $\Sigma_T^+ = (\sigma_T^+)_{jk}$  has components

$$(\sigma_T^+)_{jk} = \int_0^T (\sigma_t)_{jk} dt, \text{ for } 0 \leq j, k \leq 2$$

In particular, for a constant covariance process  $\Sigma_T^+ = \Sigma T$ .

Moreover, the characteristic function of  $Y_t$  is:

$$\varphi_{Y_T}(u) = \exp(iu(r - q)) \varphi_{\Sigma_T^+}\left(-\frac{1}{2}\theta(u)\right)$$

where:

$$\theta(u) = \begin{pmatrix} u_1(1 - iu_1) & -iu_1u_2 \\ -iu_1u_2 & u_2(1 - iu_2) \end{pmatrix}$$

**Proof.** From Equation (2) we have:

$$Y_T = (r - q)T - \frac{1}{2} \text{diag}(\Sigma_T^+, \Sigma_T^+) + \int_0^T \Sigma_t^{1/2} dB_t \tag{3}$$

The third term in the equation above follows a bivariate normal distribution, conditionally on  $\mathcal{F}^{\Sigma_T}$ , with zero mean and elements of the covariance matrix given by:

$$\begin{aligned} \text{Var}\left(\left(\int_0^T (\Sigma_t^{1/2}) dB_t\right)_j / \mathcal{F}^{\Sigma_T}\right) &= \left\langle \int_0^T (\Sigma_t^{1/2})_{j1} dB_t^{(1)} + \int_0^T (\Sigma_t^{1/2})_{j2} dB_t^{(2)} \right\rangle \\ &= \int_0^T [(\Sigma_t^{1/2})_{j1}^2 + (\Sigma_t^{1/2})_{j2}^2] dt = \sigma_T^{jj+}, \quad j = 1, 2. \end{aligned}$$

Similarly:

$$\text{cov}\left(\left(\int_0^T \Sigma_t^{1/2} dB_t\right)_1, \left(\int_0^T \Sigma_t^{1/2} dB_t\right)_2 / \mathcal{F}^{\Sigma_T}\right) = \sigma_T^{12+}$$

On the other hand, from Equation (3) and the conditional normality of the log-prices:

$$\begin{aligned} \varphi_{Y_T}(u) &= E_{\mathcal{Q}}\left[E_{\mathcal{Q}}\left(\exp(iuY_T) / \mathcal{F}^{\Sigma_T}\right)\right] \\ &= \exp(iu(r - q)) E_{\mathcal{Q}}\left[\exp\left(-\frac{1}{2}iu \text{diag}(\Sigma_T^+) - \frac{1}{2}u \Sigma_T^+ u'\right)\right] \\ &= \exp(iu(r - q)) \varphi_{\Sigma_T^+}\left(-\frac{1}{2}\theta(u)\right) \end{aligned}$$

□

The payoff of a European exchange option, with maturity at time  $T > 0$  is

$$h(Y_T) = (cS_0^{(1)} \exp(Y_T^{(1)}) - mS_0^{(2)} \exp(Y_T^{(2)}))_+$$

where  $m$  is the number of assets of type two exchanged against  $c$  assets of type one.

A closed-form formula for the price of an exchange under a bivariate Black-Scholes model, i.e., the model given by Equation (2) with a constant covariance, starting at  $t = 0$  has been found in Margrabe (1978). This price, called Margrabe price, is denoted by  $C_M := C_M(\Sigma)$ .

On the other hand, the price of the exchange option under the full model, i.e., the one driven by equation Equation (2), after conditioning on  $\mathcal{F}^{\Sigma_T}$  is denoted by  $C_{MT} := C_{MT}(\Sigma_T^+)$ . Notice that when conditioning on the covariance process the price process becomes a bivariate Gaussian model with deterministic and time-dependent volatilities and correlation.

Both prices are related by  $C_{MT}(\Sigma_T^+) = C_M(\frac{1}{T}\Sigma_T^+)$ , as the price of the later is equivalent to the Margrabe price with constant covariance matrix  $\frac{1}{T}\Sigma_T^+$ .

Additionally, we denote by  $C_{MS}$  the unconditional price of the contract. To be more precise, the price of the exchange under the model (2) is given by:

$$C_{MS} = E_Q[C_{MT}(\Sigma_T^+)] \tag{4}$$

where:

$$C_{MT}(\Sigma_T^+) = \exp^{-rT} E_Q \left[ (cS_0^{(1)} \exp(-(r - q_1)T) \exp(Y_T^{(1)}) - mS_0^{(2)} \exp(-(r - q_2)T) \exp(Y_T^{(2)}))_+ | \mathcal{F}^{\Sigma_T} \right] \tag{5}$$

From the remark above and Lemma 1 a simple extension of Margrabe formula to the case of time-dependent deterministic covariance is given by:

$$C_{MT}(\Sigma_T^+) = ce^{-(r-q_1)T} S_0^{(1)} N(d_1) - me^{-(r-q_2)T} S_0^{(2)} N(d_2)$$

$$d_1 = \frac{\log\left(\frac{cS_0^{(1)}}{mS_0^{(2)}}\right) + (q_1 - q_2)T + \frac{1}{2}v_T^+}{\sqrt{v_T^+}}$$

$$d_2 = \frac{\log\left(\frac{cS_0^{(1)}}{mS_0^{(2)}}\right) + (q_1 - q_2)T - \frac{1}{2}v_T^+}{\sqrt{v_T^+}} = d_1 - \sqrt{v_T^+}$$

with  $v_T^+ = \sigma_T^{11+} + \sigma_T^{22+} - 2\sigma_T^{12+}$ .

**Remark 1.** The conditional Margrabe price  $C_{MT}$  depends on  $\Sigma_T^+$  through the quantity  $v_T^+$ . Consequently we write  $C_{MT}(\Sigma_T^+) = C_{MT}(v_T^+)$ .

### Pricing by Polynomial Expansions

In the general case of stochastic correlation there is not analogous to Margrabe pricing formula. It is possible to approximate the price of the exchange by a suitable expansion of  $C_{MT}(v_T^+)$  in terms of Taylor polynomials around a point  $v^*$ , typically around the mean value of the integrated process given by  $v^* = E_Q(v_T^+)$ , or using a family of polynomials such cubic splines. We study in some details both approximations.

- (i) Taylor approximation.

The one-dimensional Taylor expansion of  $n$ -th order, denoted  $C_{MS}(v, v^*)$ , around the value  $v^*$  is given by:

$$C_{MS}(v, v^*) = \sum_{l=0}^n \frac{D^l C_{MT}(v^*)}{l!} (v - v^*)^l$$

A Taylor approximation of the price, taking into account Equation (4), is defined by:

$$C_{MS}(v^*) = \sum_{l=0}^n \frac{D^l C_{MT}(v^*)}{l!} E_{\mathcal{Q}}(v_T^+ - v^*)^l \tag{6}$$

**Remark 2.** Notice that, in order to implement the approximation above we need the derivatives of the  $C_{MT}(v, v^*)$  up to order  $n$  and the mixed moments of the components in the integrated covariance matrix  $\Sigma_T^+$ .

**Remark 3.** Sensitivities with respect to the parameters in the contract can be obtained in a similar way. For example, approximations of the deltas are:

$$\frac{\partial \hat{C}_{MS}^{(n)}}{\partial s^{(j)}}(v^*) = \sum_{l=0}^n \frac{D^l}{l!} \frac{\partial C_{MT}(v^*)}{\partial s^{(j)}} E_{\mathcal{Q}}(v_T^+ - v^*)^l, \quad j = 1, 2.$$

(ii) Approximation by cubic splines.

On an interval  $[a, b]$  we consider a partition  $a = v_0 \leq v_1 \leq \dots \leq v_N \leq b$ .

An approximation of  $C_{MT}(v)$  based on cubic splines is thus given by:

$$C^{spl}(v) = \sum_{j=0}^{N-1} \sum_{l=0}^3 \alpha_{l,j} 1_{[v_j, v_{j+1})} v^l \tag{7}$$

The coefficients  $\alpha_{l,j}$  depend on the partition.

To smooth the curve additional conditions on the derivatives are usually imposed. Namely,  $D_- C_{MT}(v_j) = D_+ C_{MT}(v_j), j = 1, 2, \dots, N, l = 1, 2$ , where  $D_- C_{MT}(v_j)$  and  $D_+ C_{MT}(v_j)$  are respectively the derivatives from the left and the right of the function  $C_{MT}$  at point  $v = v_j$ .

Moreover, for end points in the interval we set  $D^2 C_{MT}(a) = D^2 C_{MT}(y_N) = 0$ . See [Arcangeli et al. \(2004\)](#) for a general account on splines and its implementation.

On the other hand, this approach requires the constrained moments of  $v_T^+ = tr(M\Sigma_T^+)$  up to order  $n$  where the matrix  $M$  is:

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

To this end we first compute the corresponding characteristic function of the covariance process constrained to  $[a, b]$ . Notice that:

$$\begin{aligned} \varphi_{v_T^+}(u) &= \varphi_{\Sigma_T^+}(Mu) \\ \varphi_{v_T^+}(u, a, b) &= \varphi_{\Sigma_T^+}(Mu, a, b) = E_{\mathcal{Q}} \left[ \exp(i \operatorname{tr}(Mu\Sigma_T^+)) 1_{[a,b]}(\operatorname{tr}(M\Sigma_T^+)) \right] \end{aligned} \tag{8}$$

The constrained moments of  $v_T^+$ , assuming they exist, can be obtained by differentiating Equation (8) with respect to  $u$  and evaluating at  $u = 0$ .

We replace the function  $C_{MT}(v)$  by its approximation given in Equation (7) to obtain the following estimated of the price:

$$C^{spl} = \sum_{j=0}^{N-1} \sum_{l=0}^3 \alpha_{l,j} \tilde{m}_{v_T^+}(l, v_j, v_{j+1}) \tag{9}$$

where:

$$\tilde{m}_{v_T^+}(l, a, b) = E_{\mathcal{Q}}[(v_T^+ - a)^l 1_{[a,b)}(v_T^+)], \quad a, b \in \mathbb{R} \tag{10}$$

are the constrained moments of order  $l$  on  $[a, b)$  of  $v_T^+$  centered at  $a$ . Their calculation is discussed in Section 4.

### 3. An Ornstein-Uhlenbeck Stochastic Covariance Model

Our model is based on the general Ornstein-Uhlenbeck process with a Levy Background Noise Process (BDLP) as studied in [Barndorff-Nielsen and Shephard \(2001\)](#). It has been extended to a multidimensional setting in [Barndorff-Nielsen et al. \(2002\)](#).

We define a matrix-valued covariance process, based on independent Levy processes  $(F_t)_{t \geq 0} = (F_t^{(1)}, F_t^{(2)})_{t \geq 0}$  and  $(V_t)_{t \geq 0} = (V_t^{(1)}, V_t^{(2)})_{t \geq 0}$ , with respective characteristic exponents  $\psi_F$  and  $\psi_V$ .

The covariance process is defined for any  $t \geq 0$  by:

$$\Sigma_t = \text{diag}(F_t) + A \text{diag}(V_t) A' \tag{11}$$

where  $A = (a_{ij})$  is a  $2 \times 2$  deterministic orthonormal loading matrix.

The processes  $F$  and  $V$  correspond to idiosyncratic and common covariance factors respectively. Furthermore, we assume  $F^{(l)}$  and  $V^{(l)}$  are Ornstein-Ulenbeck Levy processes given by:

$$dF_t^{(l)} = -\lambda_{F,l} F_t^{(l)} dt + dZ_{\lambda_{F,l}t}^{(F,l)} \tag{12}$$

$$dV_t^{(l)} = -\lambda_{V,l} V_t^{(l)} dt + dZ_{\lambda_{V,l}t}^{(V,l)} \tag{13}$$

with BDLP denoted respectively by  $(Z_{\lambda_{F,l}t}^{(F,l)})$  and  $(Z_{\lambda_{V,l}t}^{(V,l)})$ ,  $\lambda_{F,l} > 0, \lambda_{V,l} > 0$ , for  $l = 1, 2$ .

After applying Ito formula we have that the integrated processes corresponding to Equations (12) and (13) are:

$$F_t^{(l,+)} = \lambda_{F,l}^{-1} (1 - \exp(-\lambda_{F,l}t)) F_0^{(l)} + \lambda_{F,l}^{-1} \int_0^t (1 - \exp(-\lambda_{F,l}(t-s))) dZ_{\lambda_{F,l}s}^{(F,l)} \tag{14}$$

$$V_t^{(l,+)} = \lambda_{V,l}^{-1} (1 - \exp(-\lambda_{V,l}t)) V_0^{(l)} + \lambda_{V,l}^{-1} \int_0^t (1 - \exp(-\lambda_{V,l}(t-s))) dZ_{\lambda_{V,l}s}^{(V,l)} \tag{15}$$

To be more specific we consider Inverse Gaussian subordinators with respective characteristic exponents:

$$\Psi_{Z_t^{F,l}}(\theta) = -a_{F,l} \left( \sqrt{-2i\theta + b_{F,l}^2} - b_{F,l} \right) \tag{16}$$

$$\Psi_{Z_t^{V,l}}(\theta) = -a_{V,l} \left( \sqrt{-2i\theta + b_{V,l}^2} - b_{V,l} \right) \tag{17}$$

The integrated covariance process is then given by:

$$\Sigma_t^+ = \int_0^t (\text{diag}(F_s) + A \text{diag}(V_s) A') ds = \text{diag}(F_t^+) + A \text{diag}(V_t^+) A' \tag{18}$$

Its characteristic function is computed in the proposition below:

**Theorem 1.** Let  $\Sigma_t^+$  be the integrated covariance processes defined by Equation (18), with  $F = (F_t)_{t \geq 0}$  and  $V = (V_t)_{t \geq 0}$  following Ornstein-Ulenbeck processes having initial deterministic values  $F_0$  and  $V_0$  and independent Inverse Gaussian subordinators as BDLPs.

Denote by  $\varphi_{\Sigma_t^+}$  the characteristic functions, let  $\theta = (\theta_{kj})_{k,j=1,2}$  be a  $2 \times 2$  matrix and  $\tilde{\theta} \in \mathbb{R}$ . Then, for  $\theta \neq 0$ :

$$\varphi_{\Sigma_t^+}(\theta) = \exp(K_1^+(\theta) + K_2^+(\theta))$$

with:

$$K_1^+(\theta) = i \sum_{l=1}^2 \theta_{ll} \lambda_{F,l}^{-1} (1 - \exp(-\lambda_{F,l}t)) F_0^{(l)} + \sum_{l=1}^2 I_F^{(l)}(\lambda_{F,l}t, \theta_{ll})$$

$$K_2^+(\theta) = i \sum_{l=1}^2 \text{tr}(\theta AC_l A') \lambda_{V,l}^{-1} ((1 - \exp(-\lambda_{V,l}t)) V_0^{(l)} + \sum_{l=1}^2 I_V^{(l)}(\lambda_{V,l}t, \text{tr}(\theta AC_l A')))$$

$$I_F^{(l)}(\lambda_{F,l}t, \tilde{\theta}) = -\frac{2a_{F,l}}{\sqrt{i\lambda_{F,l}}} \left[ -T_2^{F,l}(\tilde{\theta}) + \sqrt{i\lambda_{F,l}} b_{F,l} + \frac{i}{2} T_1^{F,l}(\tilde{\theta}) G^{F,l}(\tilde{\theta}) \right] + \lambda_{F,l} a_{F,l} b_{F,l} t \tag{19}$$

$$I_V^{(l)}(\lambda_{V,l}t, \tilde{\theta}) = -\frac{2a_{V,l}}{\sqrt{i\lambda_{V,l}}} \left[ -T_2^{V,l}(\tilde{\theta}) + \sqrt{i\lambda_{V,l}} b_{V,l} + \frac{i}{2} T_1^{V,l}(\tilde{\theta}) G^{V,l}(\tilde{\theta}) \right] + \lambda_{V,l} a_{V,l} b_{V,l} t \tag{20}$$

and

$$G^{F,l}(\tilde{\theta}) = \log \left( \exp(-\lambda_{F,l}t) \frac{(T_1^{F,l}(\tilde{\theta}) + i\sqrt{i\lambda_{F,l}} b_{F,l})^2}{(T_1^{F,l}(\tilde{\theta}) + iT_2^{F,l}(t, \tilde{\theta}))^2} \right)$$

$$T_1^{F,l}(\tilde{\theta}) = \sqrt{-2\tilde{\theta} - i\lambda_{F,l} b_{F,l}^2}$$

$$T_2^{F,l}(t, \tilde{\theta}) = \sqrt{2\tilde{\theta}(1 - e^{-\lambda_{F,l}t}) + i\lambda_{F,l} b_{F,l}^2}$$

Analogous expressions for  $T_1^{V,l}, T_2^{V,l}$  and  $G^{V,l}$  are obtained after replacing  $F$  by  $V$ .

**Proof.** See Appendix A.  $\square$

Moments of the integrated process can be obtained from the derivatives of the integrated characteristic function evaluated at zero. To this end we need to compute the derivatives of expressions (19) and (20).

For simplicity we provisionally drop the dependence on  $V$  and  $F$ . Notice that  $I(\lambda t, \tilde{\theta} \lambda^{-1} (1 - \exp(-\lambda t + s)))$  is differentiable with respect to  $\tilde{\theta}$  in a vicinity of zero. Moreover, at points  $\tilde{\theta}$  different from zero:

$$\frac{\partial \Psi_Z(\tilde{\theta} \lambda^{-1} (1 - \exp(-\lambda t + s)))}{\partial \tilde{\theta}} = -\frac{a(1 - \exp(-\lambda t + s))}{\sqrt{i\lambda} \sqrt{2\tilde{\theta}(1 - \exp(-\lambda t + s)) + i\lambda b^2}}$$

For the case  $\tilde{\theta} = 0$  we take into account that  $\Psi_Z(0) = 0$  to have:

$$\begin{aligned} \frac{\partial \Psi_Z(\tilde{\theta} \lambda^{-1} (1 - \exp(-\lambda t + s)))}{\partial \tilde{\theta}} \Big|_{\tilde{\theta}=0} &= \frac{a}{\sqrt{\lambda}} \lim_{\tilde{\theta} \rightarrow 0} \frac{i(1 - \exp(-\lambda t + s))}{\sqrt{-2i\tilde{\theta}(1 - \exp(-\lambda t + s)) + \lambda b^2}} \\ &= \frac{ia(1 - \exp(-\lambda t + s))}{\lambda b} \end{aligned}$$

The fact that the function  $\Psi_Z$  is continuously differentiable on a vicinity of zero and continuous on the variable  $s$  on the interval  $[0, \lambda t]$  allows to interchange derivative and integration by Lebesgue Dominated Convergence Theorem. Therefore, for  $\tilde{\theta} \neq 0$ :

$$\frac{\partial I(\lambda t, \tilde{\theta})}{\partial \tilde{\theta}} = -\frac{a}{\sqrt{i\lambda}} \int_0^{\lambda t} \frac{1 - \exp(-\lambda t + s)}{\sqrt{2\tilde{\theta}(1 - \exp(-\lambda t + s)) + i\lambda b^2}} ds \tag{21}$$

At  $\tilde{\theta} = 0$ :

$$\frac{\partial I(\lambda t, \tilde{\theta})}{\partial \tilde{\theta}} \Big|_{\tilde{\theta}=0} = i \frac{a}{\lambda b} (\lambda t - (1 - \exp(-\lambda t)))$$

Moreover, for  $n \geq 2$  the  $n$ -th derivative is obtained as:

$$\frac{\partial^n I(\lambda t, \tilde{\theta})}{\partial \tilde{\theta}^n} = (-1)^n \prod_{k=2}^n (2k - 3) \frac{a}{\sqrt{i\lambda}} \int_0^{\lambda t} \frac{(1 - \exp(-\lambda t + s))^n}{(2\tilde{\theta}(1 - \exp(-\lambda t + s)) + i\lambda b^2)^{\frac{2n-1}{2}}} ds$$

and evaluating at  $\tilde{\theta} = 0$ :

$$\begin{aligned} \frac{\partial^n I(\lambda t, \tilde{\theta})}{\partial \tilde{\theta}^n} \Big|_{\tilde{\theta}=0} &= (-1)^n \prod_{k=2}^n (2k - 3) \frac{a}{(i\lambda)^n b^{2n-1}} \int_0^{\lambda t} (1 - \exp(-\lambda t + s))^n ds \\ &= (-1)^n \prod_{k=2}^n (2k - 3) \frac{a}{(i\lambda)^n b^{2n-1}} \left[ \sum_{k=1}^n \binom{n}{k} (-1)^k \frac{(1 - \exp(-k\lambda t))}{k} + \lambda t \right] \end{aligned}$$

**Proposition 1.** Let  $(\Sigma_t^+)_{t \geq 0}$  be the integrated covariance processes given by Equation (18), where  $F = (F_t)_{t \geq 0}$  and  $V = (V_t)_{t \geq 0}$  follow Ornstein-Uhlenbeck processes with initial deterministic values  $F_0$  and  $V_0$  and independent Inverse Gaussian subordinators as BDLPs. Then, the first two moments of the elements in  $(\Sigma_t^+)_{t \geq 0}$  are given by:

$$\begin{aligned} E_{\mathcal{Q}}(\sigma_t^{kk+}) &= \lambda_{F,k}^{-1} (1 - \exp(-\lambda_{F,k} t)) F_0^{(k)} - i \frac{\partial I_F^{(k)}(\lambda_{F,k} t, \theta_{kk})}{\partial \theta_{kk}} \Big|_{\theta_{kk}=0} \\ &+ \sum_{l=1}^2 a_{kl}^2 \lambda_{V,l}^{-1} (1 - \exp(-\lambda_{V,l} t)) V_0^{(l)} - i \sum_{l=1}^2 a_{kl}^2 \frac{\partial I_V^{(l)}(\lambda_{V,l} t, \text{tr}(\theta A C_l A'))}{\partial \theta_{kk}} \Big|_{\theta=0} \end{aligned} \tag{22}$$

$$E_{\mathcal{Q}}(\sigma_t^{12+}) = \sum_{l=1}^2 a_{1l} a_{2l} \lambda_{V,l}^{-1} (1 - \exp(-\lambda_{V,l} t)) V_0^{(l)} - i \sum_{l=1}^2 a_{1l} a_{2l} \frac{\partial I_V^{(l)}(\lambda_{V,l} t, \text{tr}(\theta A C_l A'))}{\partial \theta_{12}} \Big|_{\theta=0} \tag{23}$$

where for  $k, l, j = 1, 2$ :

$$\begin{aligned} \frac{\partial I_F^{(k)}(\lambda_{F,k} t, \tilde{\theta})}{\partial \tilde{\theta}}(\lambda_{F,k} t, 0) &:= \frac{\partial I_F^{(k)}(\lambda_{F,k} t, \tilde{\theta})}{\partial \tilde{\theta}} \Big|_{\tilde{\theta}=0} \\ &= \frac{ia_{F,k}}{\lambda_{F,k} b_{F,k}} (\lambda_{F,k} t - (1 - \exp(-\lambda_{F,k} t))) \\ \frac{\partial I_V^{(l)}(\lambda_{V,l} t, \text{tr}(\theta A C_l A'))}{\partial \theta_{kj}}(\lambda_{V,l} t, 0) &:= \frac{\partial I_V^{(l)}(\lambda_{V,l} t, \text{tr}(\theta A C_l A'))}{\partial \theta_{kj}} \Big|_{\theta=0} \\ &= \frac{ia_{V,l}}{\lambda_{V,l} b_{V,l}} (\lambda_{V,l} t + \exp(-\lambda_{V,l} t) - 1) \end{aligned}$$

Moreover, for  $k, l = 1, 2$ :

$$\begin{aligned}
 E_{\mathcal{Q}}(\sigma_t^{kk+})^2 &= - \left( i\lambda_{F,k}^{-1}(1 - \exp(-\lambda_{F,k}t))F_0^{(k)} + \frac{\partial I_F^{(k)}(\lambda_{F,k}t, 0)}{\partial \theta_{kk}} \right. \\
 &\quad \left. + i \sum_{l=1}^2 a_{kl}^2 \lambda_{V,l}^{-1}(1 - \exp(-\lambda_{V,l}t))V_0^{(l)} + \sum_{l=1}^2 a_{kl}^2 \frac{\partial I_V^{(l)}(\lambda_{V,l}t, 0)}{\partial \theta_{kk}} \right)^2 \\
 &\quad - \left( \frac{\partial^2 I_F^{(k)}(\lambda_{F,k}t, \theta_{kk})}{\partial \theta_{kk}^2} \Big|_{\theta_{kk}=0} + \sum_{l=1}^2 a_{kl}^4 \frac{\partial^2 I_V^{(l)}(\lambda_{V,l}t, 0)}{\partial \theta_{kk}^2} \right) \tag{24}
 \end{aligned}$$

$$\begin{aligned}
 E_{\mathcal{Q}}[(\sigma_t^{12+})^2] &= - \sum_{l=1}^2 a_{1l}^2 a_{2l}^2 \frac{\partial^2 I_V^{(l)}(\lambda_{V,l}t, 0)}{\partial \theta_{12}^2} \\
 &\quad - \left( i \sum_{l=1}^2 a_{1l} a_{2l} \lambda_{V,l}^{-1}(1 - \exp(-\lambda_{V,l}t))V_0^{(l)} + \sum_{l=1}^2 a_{1l} a_{2l} \frac{\partial I_V^{(l)}(\lambda_{V,l}t, 0)}{\partial \theta_{12}} \right)^2 \tag{25}
 \end{aligned}$$

$$\begin{aligned}
 E_{\mathcal{Q}}(\sigma_t^{kk+} \sigma_t^{12+}) &= - \sum_{l=1}^2 a_{kl}^2 a_{1l} a_{2l} \frac{\partial^2 I_V^{(l)}(\lambda_{V,l}t, 0)}{\partial \theta_{kk}^2} \\
 &\quad - \left( i\lambda_{F,k}^{-1}(1 - \exp(-\lambda_{F,k}t))F_0^{(k)} + \frac{\partial I_F^{(k)}(\lambda_{F,k}t, 0)}{\partial \theta_{kk}} \right) \\
 &\quad + i \sum_{l=1}^2 a_{kl}^2 \lambda_{V,l}^{-1}(1 - \exp(-\lambda_{V,l}t))V_0^{(l)} + \sum_{l=1}^2 a_{kl}^2 \frac{\partial I_V^{(l)}(\lambda_{V,l}t, \text{tr}(\theta A C_l A'))}{\partial \theta_{kk}} \Big|_{\theta=0} \\
 &\quad \left( i \sum_{l=1}^2 a_{1l} a_{2l} \lambda_{V,l}^{-1}(1 - \exp(-\lambda_{V,l}t))V_0^{(l)} + \sum_{l=1}^2 a_{1l} a_{2l} \frac{\partial I_V^{(l)}(\lambda_{V,l}t, 0)}{\partial \theta_{12}} \right)
 \end{aligned}$$

for  $k = 1, 2$ .

$$\begin{aligned}
 E_{\mathcal{Q}}(\sigma_t^{11+} \sigma_t^{22+}) &= - \sum_{l=1}^2 a_{1l}^2 a_{2l}^2 \frac{\partial^2 I_V^{(l)}(\lambda_{V,l}t, 0)}{\partial \theta_{11} \partial \theta_{22}} \\
 &\quad - \left( i\lambda_{F,1}^{-1}(1 - \exp(-\lambda_{F,1}t))F_0^{(1)} + \frac{\partial I_F^{(1)}(\lambda_{F,1}t, 0)}{\partial \theta_{11}} \right) \\
 &\quad + i \sum_{l=1}^2 a_{1l}^2 \lambda_{V,l}^{-1}(1 - \exp(-\lambda_{V,l}t))V_0^{(l)} + \sum_{l=1}^2 a_{1l}^2 \frac{\partial I_V^{(l)}(\lambda_{V,l}t, 0)}{\partial \theta_{11}} \\
 &\quad \left( i\lambda_{F,2}^{-1}(1 - \exp(-\lambda_{F,2}t))F_0^{(2)} + \frac{\partial I_F^{(2)}(\lambda_{F,2}t, 0)}{\partial \theta_{22}} \right) \\
 &\quad + i \sum_{l=1}^2 a_{2l}^2 \lambda_{V,l}^{-1}(1 - \exp(-\lambda_{V,l}t))V_0^{(l)} + \sum_{l=1}^2 a_{2l}^2 \frac{\partial I_V^{(l)}(\lambda_{V,l}t, 0)}{\partial \theta_{22}} \tag{26}
 \end{aligned}$$

where for  $k, l, j = 1, 2$ :

$$\begin{aligned}
 \frac{\partial^2 I_F^{(k)}(\lambda_{F,k}t, \theta_{kk})}{\partial \theta_{kk}^2}(\lambda_{F,k}t, 0) &:= \frac{\partial^2 I_F^{(k)}(\lambda_{F,k}t, \theta_{kk})}{\partial \theta_{kk}^2} \Big|_{\theta_{kk}=0} \\
 &= \frac{a_{F,k}}{\lambda_{F,k}^2 b_{F,k}^3} \left[ \lambda_{F,k}t - 2(1 - \exp(-\lambda_{F,k}t)) + \frac{1}{2}(1 - \exp(-2\lambda_{F,k}t)) \right]
 \end{aligned}$$

$$\frac{\partial^2 I_V^{(1)}(\lambda t, \text{tr}(\theta AC_l A'))}{\partial \theta_{kj}^2} \Big|_{\theta=0} = \frac{a_{V,l}}{\lambda_{V,l}^2 b_{V,l}^3} \left[ \lambda_{V,l} t - 2(1 - \exp(-\lambda_{V,l} t)) + \frac{1}{2}(1 - \exp(-2\lambda_{V,l} t)) \right]$$

**Proof of Proposition 1.** See Appendix A. □

The constrained moments of  $v_T^+$  are needed in the cubic spline approaches. They are obtained via the constrained characteristic function in the proposition above.

**Proposition 2.** Let  $\Sigma_t^+$  be the integrated covariance processes defined by Equation (18), with  $F = (F_t)_{t \geq 0}$  and  $V = (V_t)_{t \geq 0}$  following Ornstein-Uhlenbeck processes having initial deterministic values  $F_0$  and  $V_0$  and independent Inverse Gaussian subordinators as BDLPs.

Denote by  $\varphi_{v_T^+}(u, a, b)$  the constrained characteristic function of  $v_T^+ = \text{tr}(M\Sigma_T^+)$ . Then:

$$\varphi_{v_T^+}(u, a, b) = -\frac{i}{2\pi} \int_{\mathbb{R}} f(y, a, b) g(u - y) dy \tag{27}$$

where  $g(x) = \exp(K_1^+(Mx) + K_2^+(Mx))$  and

$$f(y, a, b) = \begin{cases} \frac{\exp(iby) - \exp(iay)}{y} & y \neq 0 \\ i(b - a) & y = 0 \end{cases}$$

Moreover, derivatives of the constrained characteristic function with respect to  $u$  evaluated at zero can be computed as:

$$D^n \varphi_{v_T^+}(u, a, b) \Big|_{u=0} = -\frac{i}{2\pi} \sum_{j=0}^{n-1} \binom{n-1}{j} \int_{\mathbb{R}} f(y, a, b) [D^{j+1}K_1^+(-My) + D^{j+1}K_2^+(-My)] D^{n-j-1}g(-y) dy \tag{28}$$

for  $j = 2, 3, \dots, n - 1$ .

Here  $D^n$  is the  $n$ -th derivative with respect to the variable  $u$  and  $D^j K_l^+(M(u - y))$  is the  $j$ -th derivative of  $K_l^+(M(u - y))$ , also with respect to  $u$  and evaluated at  $u = 0$ .

**Proof.** See Appendix A. □

#### 4. Implementing Polynomial Expansions

In this section we precise the pricing formulas under the two approximations considered. Namely Taylor and splines approximation.

First, we implement the Taylor method based on Equation (6). To this end we first compute the Margrabe price  $C_{MT}(v)$  under a model with time-dependent and deterministic volatilities and correlation, together with its derivatives evaluated at  $v = v^*$ .

In order to simplify notations we introduce the following constants:

$$M_1 = c \exp(-(r - q_1)T) S_0^{(1)}, M_2 = m \exp(-(r - q_2)T) S_0^{(2)}$$

$$M_3 = \log\left(\frac{c S_0^{(1)}}{m S_0^{(2)}}\right) + (q_1 - q_2)T$$

Then, by elementary calculations it follows that:

$$D^k d_1(v^*) = M_3 T^{-\frac{1}{2}} (-1)^k \prod_{j=0}^{k-1} \left(\frac{1}{2} + j\right) (v^*)^{-\frac{1}{2}-k} + \frac{1}{2} \sqrt{T} \prod_{j=0}^{k-1} \left(\frac{1}{2} - j\right) (v^*)^{\frac{1}{2}-k}$$

Hence, differentiating the Margrabe formula:

$$\begin{aligned}
 D^k C_{MT}(v^*) &= M_1 \sum_{j=0}^{k-1} \binom{k-1}{j} D^j f_Z(d_1(v^*)) D^{k-j} d_1(v^*) \\
 &- M_2 \sum_{j=0}^{k-1} \binom{k-1}{j} D^j [f_Z(d_1(v^*) - \sqrt{v^*} \sqrt{T})] D^{k-j} d_1(v^*) \\
 &+ M_2 \sqrt{T} \sum_{j=0}^{k-1} \binom{k-1}{j} D^j [f_Z(d_1(v^*) - \sqrt{v^*} \sqrt{T})] \prod_{l=0}^{k-j} \left(\frac{1}{2} - l\right) (v^*)^{\frac{1}{2}-k+j}
 \end{aligned} \tag{29}$$

Therefore, the price based on the first order Taylor expansion can be computed as:

$$\begin{aligned}
 \hat{C}_{MS}^{(1)}(v^*) &= A_0^{(1)} C_M(v^*) + D^1 C_M(v^*) E_Q(\sigma_T^{11+}) + D^1 C_M(v^*) E_Q(\sigma_T^{22+}) \\
 &- 2D^1 C_M(v^*) E_Q(\sigma_T^{12+})
 \end{aligned}$$

where  $A_0^{(1)} = 1 - v^* D^1$ .

For the second order expansion we compute:

$$\begin{aligned}
 E_Q(v_T^+ - v^*)^2 &= E_Q(\sigma_T^{11+})^2 + 2E_Q(\sigma_T^{11+} \sigma_T^{22+}) - 4E_Q(\sigma_T^{11+} \sigma_T^{12+}) \\
 &+ E_Q[(\sigma_T^{22+})^2] - 4E_Q(\sigma_T^{22+} \sigma_T^{12+}) + 4E_Q[(\sigma_T^{12+})^2] \\
 &- 2v^* E_Q(\sigma_T^{11+}) - 2v^* E_Q(\sigma_T^{22+}) + 4v^* E_Q(\sigma_T^{12+}) + (v^*)^2
 \end{aligned} \tag{30}$$

Then, substituting Equation (31) into Equation (6):

$$\begin{aligned}
 \hat{C}_{MS}^{(2)}(v^*) &= \hat{C}_{MS}^{(1)}(v^*) + \frac{1}{2} D^2 C_{MT}(v^*) (v^*)^2 \\
 &- v^* D^2 C_{MT}(v^*) \left[ E_Q(\sigma_T^{11+}) + E_Q(\sigma_T^{22+}) - 2E_Q(\sigma_T^{12+}) \right] \\
 &+ \frac{1}{2} D^2 C_{MT}(v^*) \left[ E_Q(\sigma_T^{11+})^2 + 2E_Q(\sigma_T^{11+} \sigma_T^{22+}) \right. \\
 &- 4E_Q(\sigma_T^{11+} \sigma_T^{12+}) + E_Q(\sigma_T^{22+})^2 - 4E_Q(\sigma_T^{22+} \sigma_T^{12+}) \\
 &\left. + 4E_Q(\sigma_T^{12+})^2 \right]
 \end{aligned}$$

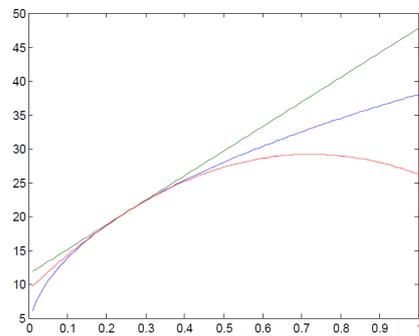
In Figure 1 we show Margrabe price values as function of the variable  $v$  (blue curve) on the interval  $(0, 1]$ . For comparison, we also show Taylor polynomials of first (green line) and second (red line) order around the average log-price  $v_0 = 0.25$  and benchmark parameters specified in section 5. Both approximations are locally accurate but, for values farther from  $v_0$  the differences are shown to be significant. It brings us the question of how often and how far departures from the average value occur.

In Figure 2 the probability density function (p.d.f.) of the random variable  $v_T^+$  obtained from  $10^5$  simulated values of  $v$  is shown. It is estimated using a non-parametric Gaussian kernel. We observe that most values concentrate around the expansion point, whereas a low but significant frequency appear far from the mean, indicating the presence of a heavy-tailed probability distribution with positive skewness.

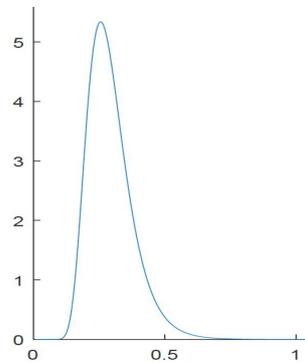
In order to overcome this potential inconvenient we consider a cubic splines approximation. The later adapts the expansion to the price behavior on different subintervals of  $[a, b)$ .

To compute the constrained moments of  $v_T^+$  we use Proposition 2, Equation (29). In order to simplify we assume initial values of the subordinators equal to zero. In Figure 3 a comparison between the Margrabe price is shown, while in Figure 4 the differences between

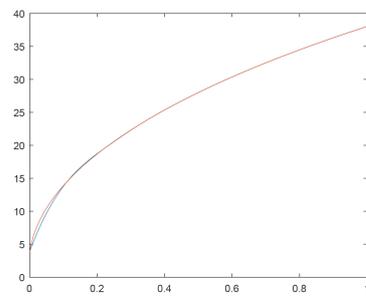
Margrabe prices and its cubic spline approximation are shown. Despite some unstable behavior at the origin the fit is reasonable accurate.



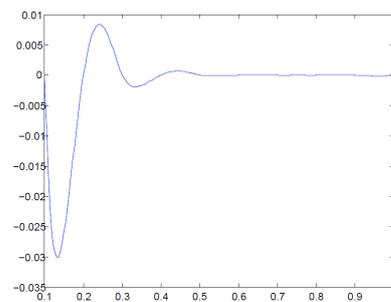
**Figure 1.** Margrabe prices as function of the parameter  $v$ . Blue line represents the price. Green and red lines are respectively the first and second order Taylor approximations.



**Figure 2.** Empirical probability density prices of  $v_T^+$  obtained after  $10^5$  simulations using Monte Carlo.



**Figure 3.** Margrabe prices (blue line) and its cubic splines approximation (red line).



**Figure 4.** Difference between Margrabe prices and its cubic spline approximation.

Therefore, we find that:

$$\begin{aligned}
 m_{v_T^+}(0, a, b) &= -\frac{i}{2\pi} \int_{\mathbb{R}} f(y, a, b)g(-y) dy \\
 m_{v_T^+}(1, a, b) &= -\frac{1}{2\pi} \int_{\mathbb{R}} f(y, a, b)(DK_1^+(-My) + DK_2^+(-My))g(-y) dy \\
 m_{v_T^+}(2, a, b) &= \frac{i}{2\pi} \left[ \int_{\mathbb{R}} f(y, a, b)(DK_1^+(-My) + DK_2^+(-My))^2g(-y) dy \right. \\
 &\quad \left. + \int_{\mathbb{R}} f(y, a, b)(D^2K_1^+(-My) + D^2K_2^+(-My))g(-y) dy \right] \\
 m_{v_T^+}(3, a, b) &= \frac{1}{2\pi} \left[ \int_{\mathbb{R}} f(y, a, b)(DK_1^+(-My) + DK_2^+(-My))^3g(-y) dy \right. \\
 &\quad + 3 \int_{\mathbb{R}} f(y, a, b)(D^2K_1^+(-My) + D^2K_2^+(-My))(DK_1^+(-My) + DK_2^+(-My))g(-y) dy \\
 &\quad \left. + \int_{\mathbb{R}} f(y, a, b)(D^3K_1^+(-My) + D^3K_2^+(-My))g(-y) dy \right]
 \end{aligned}$$

Higher moments are computed by recurrence:

$$\begin{aligned}
 m_{v_T^+}(k, a, b) &= -\frac{i^{-k+1}}{2\pi} \sum_{j=0}^{k-1} \binom{k-1}{j} \\
 &\quad \int_{\mathbb{R}} f(y, a, b)(D^{j+1}K_1^+(-My) + D^{j+1}K_2^+(-My))D^{k-j-1}g(-y) dy \\
 &\quad \text{for } k = 2, 3, \dots
 \end{aligned}$$

Finally, centered moments  $\tilde{m}_{v_T^+}(k, a, b)$  are found from:

$$\tilde{m}_{v_T^+}(k, a, b) = D^k \varphi_{v_T^+}(u)|_{u=0} \sum_{j=0}^k \binom{k}{j} (-ia)^{k-j} m_{v_T^+}(j, a, b)$$

The calculations of the functions  $K_1, K_2, \varphi_{v_T^+}$  and their derivatives are shown in the Appendix A.

The constrained moments can be directly calculated from the p.d.f. of  $v_T^+$ . In turn, the p.d.f. of  $v_T^+$  is computed via its characteristic function by inverse FFT. To this end we define the grids:

$$\begin{aligned}
 x_j &= a + \eta j, \quad j = 0, 1, \dots, n - 1 \\
 u_k &= \delta k, \quad k = 0, 1, \dots, n - 1
 \end{aligned}$$

where  $\eta = \frac{b-a}{n}$  and  $\delta = \frac{2\pi}{b-a}$  are their respective lengths.

Hence, after applying the trapezoid rule:

$$\begin{aligned}
 f_{v_T^+}(x_j) &= \frac{1}{\pi} \int_0^{+\infty} Re(\exp(-ix_j u) \varphi_{v_T^+}(u)) du \\
 &\simeq \frac{1}{\pi} \sum_{k=0}^{n-1} w_k Re(\exp(-ix_j u_k) \varphi_{v_T^+}(u_k)) \Delta u_k \\
 &= \frac{1}{\pi} Re\left(\sum_{k=0}^{n-1} w_k \delta \exp(-ia\delta k) \varphi_{v_T^+}(\delta k) \exp(-i\frac{2\pi}{n}jk)\right) \\
 &= \frac{1}{\pi} f_{ft}(h_k)
 \end{aligned}$$

with  $h_k = w_k \delta \exp(-ia\delta k) \varphi_{v_+}(\delta k)$  and  $w_0 = w_{n-1} = \frac{1}{2}$  and equal to one otherwise. The expression  $fft(h_k)$  denotes the Fast Fourier Transform of the sequence  $(h_k)$ .

See Witkovsy (2016) for FFT applications in obtaining pdf's and Hürlimann (2013) for a detailed analysis of different quadratures.

Notice that neither the joint characteristic function nor the p.d.f. of the pair of assets is known explicitly in the case of stochastic correlations. Hence, a direct application of a two dimensional FFT as in Hurd and Zhou (2010) or the approach in Caldana and Fusai (2013) to find the bounds of the price is not possible.

### 5. Numerical Results

We compare the polynomial methods and the Monte Carlo approach to pricing, for speed and accuracy. Our benchmark setting is given by a set of parameter values defining the model and the exchange contract. Contract parameters are selected within a reasonable range, according to usual practices, while the choosing of parameters in the model is made just with the purpose of illustrating the techniques. Notice that there are no benchmark values for the correlation as it depends on the specific assets considered. Nonetheless, the rationale in our choice is based on oil future prices per barrel for WTI and Brent types traded at NYSE. Initial prices are \$ 100 and \$ 96 respectively. Simulations show values within the range observed in the historic data of oil prices. On the other hand, this particular choice of parameters produces enough variability and jumps to impact the price of the derivative.

The calibration of the parameters, albeit a critical issue, is beyond the scope of the paper. See Pablo and Ciro (2023) for the use of a *Generalized Method of Moments* matching empirical and theoretical moments of both assets under a similar model.

The benchmark parameters for the model are  $a_F = (1, 1), a_V = (1, 1), b_F = (5, 5), b_V = (5, 5), \lambda_F = (1, 1), \lambda_V = (1, 1)$  and  $S_0 = (100, 96)$ . For the contract we set  $c = 1, m = 1, q = (0, 0)$  and  $T = 1$ . The interest rate is  $r = 0.04$ .

We take the loading matrix  $A$  as an orthonormal rotation matrix with an angle  $\theta, -\pi < \theta \leq \pi$ , given by:

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

A direct Monte Carlo approach is costly as trajectories for both, the covariance process and the asset process, need to be simulated a large number of times. Alternatively, the iterative Formula (4) can be used to simplify calculations as, according to Lemma 1, conditionally on the covariance process the log-prices are normally distributed. It reduces the problem to calculate the discounted average of the price of an exchange contract under a deterministic time-dependent covariance, which still has a closed-form expression given in Equation (5). Hence, only the Ornstein-Uhlenbeck covariance process needs to be simulated. We call this procedure a *partial Monte Carlo* approach.

Integrated Ornstein-Uhlenbeck process values at time  $T$ , denoted by  $\hat{F}_T^{l,+}$  and  $\hat{V}_T^{l,+}$  are computed as discrete approximations of solutions of Equations (14) and (15) with a step  $\delta$ , given respectively by:

$$\hat{F}_T^{(l,+)} = \lambda_{F,l}^{-1} \left[ (1 - \exp(-\lambda_{F,l}T)) F_0^{(l)} + \sum_{k=1}^{n_1} (1 - \exp(-\lambda_{F,l}(T - k\delta))) \Delta Z_k^{F,l} \right]$$

$$\hat{V}_T^{(l,+)} = \lambda_{V,l}^{-1} \left[ (1 - \exp(-\lambda_{V,l}T)) V_0^{(l)} + \sum_{k=1}^{n_2} (1 - \exp(-\lambda_{V,l}(T - k\delta))) \Delta Z_k^{V,l} \right]$$

where:

$$n_1 = \left\lceil \frac{\lambda_{F,l}T}{\delta} \right\rceil \text{ and } \Delta Z_k^{F,l} = Z_{k\delta}^{F,l} - Z_{(k-1)\delta}^{F,l}$$

$$n_2 = \left\lceil \frac{\lambda_{V,l}T}{\delta} \right\rceil \text{ and } \Delta Z_k^{V,l} = Z_{k\delta}^{V,l} - Z_{(k-1)\delta}^{V,l}, \quad l = 1, 2.$$

The symbol  $[x]$  is the integer part of the real value  $x$ .

Next, the integrated covariance process is computed:

$$\Sigma_T^+ = \text{diag}(F_T^+) + A \text{diag}(V_T^+) A'$$

The price of the derivative contract is estimated from Equation (4) by the simulation of the covariance process and then computing the discounted average of the Margrabe prices evaluated at these simulated volatilities.

As an illustration, in Figure 5 three trajectories of an Inverse Gaussian process  $(Z_t)_{0 \leq t \leq 1}$  with parameters  $a = 1, b = 5$  are shown. Next, we generate the corresponding Ornstein-Uhlenbeck process  $(F_t)_{0 \leq t \leq 1}$  as shown in Figure 6, starting at zero.

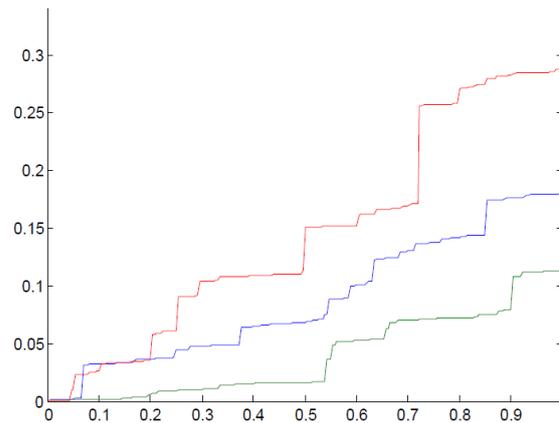


Figure 5. Three realizations of the inverse Gamma process with parameters  $a = 1, b = 5$  and  $\lambda = 1$ .

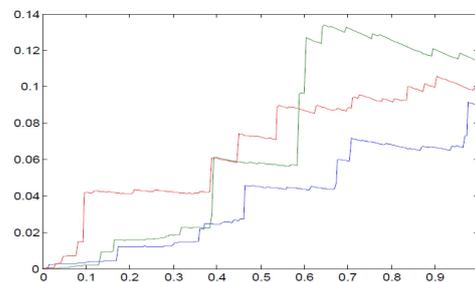
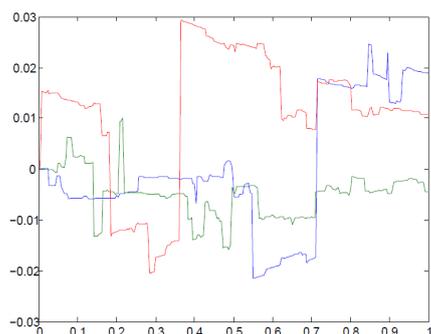


Figure 6. Three realizations of the Ornstein-Uhlenbeck process.

Finally, in Figure 7, we show the trajectories of the correlation process obtained by dividing the covariance process  $(\sigma_t^{12})_{0 \leq t \leq 1}$  by the product of volatilities from the underlying assets, with a load matrix defined by the angle  $\theta = \frac{\pi}{6}$ . Both processes  $(F_t)_{0 \leq t \leq 1}$  and  $(V_t)_{0 \leq t \leq 1}$  are generated with the benchmark model parameters. Notice the correlation process exhibits jumps at random times, accounting for the effect of unexpected events.

In Table 1 different prices of the exchange contract for some notable values of the angle in the loading matrix are shown. In the case of the Monte Carlo approach we also calculate a 95% confidence interval for the price after 1 million simulations. We see that all methods, except the second order Taylor expansion, are within a similar range. First order Taylor price presents inaccuracies for other parameters of the subordinator processes, while the ones based on cubic splines and FFT developments are quite stable, their relative average error are approximately 0.018% when compared with the Monte Carlo price.



**Figure 7.** Trajectories of the correlation process for the benchmark parameters.

On the other hand, in Table 2 we can see the execution time (in sec.) for all five methods. The code has been written on a surface pro 4 i7 using MATLAB language. Cubic splines and FFT methods work, on average, respectively 16,488.8 and 18,408.4 times faster than Monte Carlo. The fact that FFT is slightly faster than cubic splines approximation comes at no surprise. It is well known that the former has a  $O(n \log n)$  complexity compared with a  $O(n^2)$  of the later.

In implementing both approaches some quantities driving the numerical approximations are required. Namely, we need to decide on the truncation interval  $[a, b]$ , in fixing a number of points in the grid for the Fourier transform and the number of points in the spline interpolation. The three factors require a compromise between accuracy and the amount of computation time. In the choice of the truncation interval we have tried to cover most of the support of the p.d.f. of the random variable  $v_T^+$  (see Figure 4) which in turn depends on the parameters of both Inverse Gaussian subordinator processes. In our setting the interval  $[0, 5)$  was a reasonable trade-off. Of course, most of the time these parameters need to be estimated. In any case a significant probability mass is present in a neighborhood of zero, therefore  $a = 0$  seems a natural choice.

Regarding the number of points in the grid of the FFT calculation we have tested several powers of 2, ranging from  $2^8$  to  $2^{14}$ . There is not a significant change in the price across these values. We have set an intermediate value of  $2^{12}$ . After this value the computation time explodes without a significant gain in accuracy. In order to implement the approach based on spline polynomials, we explored a range from  $2^4$  to  $2^8$  of interpolation points. After  $2^6$  the price values are in close agreement with Monte Carlo and FFT. Numerical results improve if first derivatives at the end points of the expansion intervals are taken into account. They are available via Formula (30).

Notice that the FFT approach refers to the way the p.d.f. of the integrated stochastic volatility is obtained. It differs from the standard FFT pricing technique based on the Fourier transform of the payoff, see Carr and Madan (1999) for the latter. To calculate the unconditional expected value of the price an expansion based on Taylor or splines is still needed.

Alternatively, it may be possible to integrate directly Equation (4) once the p.d.f. of the integrated price is calculated. It leads to a double quadrature of a non-linear function, i.e., the Margrabe price, while avoiding the polynomial expansions. On the other hand, the expansions requires a single quadrature but at the expense of a lost in accuracy resulting from its inherent truncation.

**Table 1.** Prices obtained from different approximations and different values of  $\theta$  using the benchmark parameters.

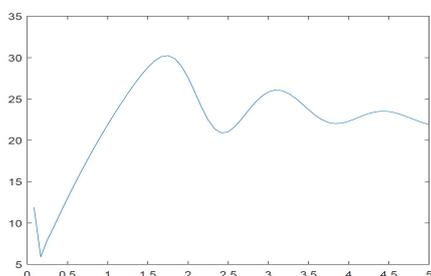
$\theta$	MC	Taylor (First Order)	Taylor (Sec. Order)	Spl.	Spl. via FFT
$\frac{\pi}{6}$	21.8643 (21.8589, 21.8697)	22.1774	19.8441	21.8969	21.8990
$\frac{\pi}{3}$	21.8610 (21.8557, 21.8664)	22.1773	19.8441	21.8969	21.8990
$\frac{\pi}{2}$	21.9191 (21.9144, 21.9238)	22.1773	20.6861	21.9506	21.9521
$\pi$	21.9163 (21.9117, 21.9210)	22.1774	20.6861	21.9506	21.9521

**Table 2.** Average computer time (in seconds) for different pricing methods using the benchmark parameters and  $\theta = \frac{\pi}{6}$ .

	MC	Taylor	Spl.	Spl. via FFT
<b>Run time</b>	179.6294	0.010847	0.010894	0.009758

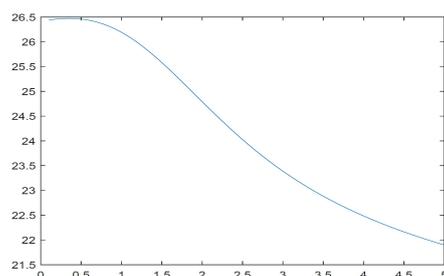
Next, we analyze the sensitivities of the price of an exchange contract as function of the maturity and the difference between the parameters driving the volatilities of both assets.

Figure 8 shows the prices under a set of maturities ranging from 1 month to 5 years while keeping constant the remaining parameters in the benchmark set. Notice the damped oscillations of the price as the maturity increases, in contrast with the standard Margrabe exchange contract, showing the stabilizing effect of the integrated volatility for large values of  $T$ .



**Figure 8.** Price of an exchange contract as function of maturity date.

In Figure 9 the curve shows the price in function of the parameter  $b$  in the inverse Gaussian distribution for the first asset, which drives down its volatility as it increases. As expected when the volatility decreases, i.e., the parameter  $b$  increases, anything else constant, the price of the exchange contract decreases.



**Figure 9.** Price of an exchange contract as function of the volatility.

### 6. Conclusions

We have discussed the pricing of exchange contracts under a dynamic model with a complex correlation structure capturing random jumps, heavy-tails, asymmetric and stochastic behavior. To this end we have proposed two approximate closed-form pricing methods based on cubic splines and Taylor approximations. To the extend of the investigation and the range of parameters considered, in the model, the contract and the numerical implementation, both approaches provide accurate and fast pricing estimates. Splines via FFT seems to have a slight edge over the direct application of cubic splines and Taylor expansion.

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### Appendix A

**Proof of Theorem 1.** From Equations (14) and (15), and Levy-Khinchine formula we have, for  $\tilde{\theta} \in \mathbb{R}$ :

$$\begin{aligned} \varphi_{F_t^{(l,+)}}(\tilde{\theta}) &= \exp(i\tilde{\theta}\lambda_{F,l}^{-1}(1 - \exp(-\lambda_{F,l}t))F_0^{(l)})E_{\mathcal{Q}}\left[\exp(i\tilde{\theta}\lambda_{F,l}^{-1}\int_0^t(1 - \exp(-\lambda_{F,l}(t-s)))dZ_{\lambda_{F,l}s}^{F,l})\right] \\ &= \exp(i\tilde{\theta}\lambda_{F,l}^{-1}(1 - \exp(-\lambda_{F,l}t))F_0^{(l)}) + \int_0^{\lambda_{F,l}t}\Psi_{Z^{F,l}}(\lambda_{F,l}^{-1}\tilde{\theta}(1 - \exp(-\lambda_{F,l}t+s)))ds \end{aligned}$$

where the last equality follows from the identity:

$$E_{\mathcal{Q}}\left(\exp(i\int_0^t f(s)dL_s)\right) = \exp\left(\int_0^t \Psi_L(f(s)) ds\right)$$

for a Levy process  $L$  with characteristic exponent  $\Psi_L$  and a continuous function  $f$ , see for example Cont and Tankov (2003). Similarly:

$$\begin{aligned} \varphi_{V_t^{(l,+)}}(\text{tr}(\theta AC_l A')) &= \exp(i\text{tr}(\theta AC_l A')\lambda_{V,l}^{-1}(1 - \exp(-\lambda_{V,l}t))V_0^{(l)}) \\ &\quad + \int_0^{\lambda_{V,l}t}\Psi_{Z^{V,l}}(\text{tr}(\theta AC_l A')\lambda_{V,l}^{-1}(1 - \exp(-\lambda_{V,l}t+s)))ds \end{aligned}$$

Next, the characteristic function for the integrated covariance process can be computed as:

$$\begin{aligned} \varphi_{\Sigma_t^+}(\theta) &= E_{\mathcal{Q}}\exp(i\text{tr}(\theta \text{diag}(F_t^+)))E_{\mathcal{Q}}\exp(i\text{tr}(\theta A \text{diag}(V_t^+)A')) \\ &= \exp\left(i\sum_{l=1}^2\lambda_{F,l}^{-1}(1 - \exp(-\lambda_{F,l}t))\theta_{ll}F_0^{(l)} + \int_0^{\lambda_{F,l}t}\Psi_{Z^{F,l}}(\lambda_{F,l}^{-1}\theta_{ll}(1 - \exp(-\lambda_{F,l}t+s)))ds\right) \\ &\quad \exp\left(i\sum_{l=1}^d\text{tr}(\theta AC_l A')\lambda_{V,l}^{-1}(1 - \exp(-\lambda_{V,l}t))\right) \\ &\quad + \sum_{l=1}^2\int_0^{\lambda_{V,l}t}\Psi_{Z^{V,l}}(\lambda_{V,l}^{-1}\text{tr}(\theta AC_l A')(1 - \exp(-\lambda_{V,l}t+s)))ds \end{aligned}$$

On the other hand, from Equation (16) we have:

$$\Psi_{ZF,l}(\tilde{\theta}\lambda_{F,l}^{-1}(1 - \exp(-\lambda_{F,l}t + s))) = -\frac{a_{F,l}}{\sqrt{i\lambda_{F,l}}}\sqrt{2\tilde{\theta}(1 - \exp(-\lambda_{F,l}t + s)) + i\lambda_{F,l}b_{F,l}^2} + a_{F,l}b_{F,l}$$

Then, for  $\tilde{\theta} \neq 0$ :

$$\begin{aligned} I_F^{(l)}(\lambda_{F,l}t, \tilde{\theta}) &= -\frac{a_{F,l}}{\sqrt{i\lambda_{F,l}}}\int_0^{\lambda_{F,l}t}\sqrt{2\tilde{\theta}(1 - \exp(-\lambda_{F,l}t + s)) + i\lambda_{F,l}b_{F,l}^2}ds + \lambda_{F,l}a_{F,l}b_{F,l}t \\ &= -\frac{2a_{F,l}}{\sqrt{i\lambda_{F,l}}}\left[-T_2^{F,l}(\tilde{\theta}) + T_1^{F,l}(\tilde{\theta})\arctan\left(\frac{T_2^{F,l}(\tilde{\theta})}{T_1^{F,l}(\tilde{\theta})}\right)\right. \\ &+ \left.\sqrt{i\lambda_{F,l}b_{F,l}} - T_1^{F,l}(\tilde{\theta})\arctan\left(\frac{\sqrt{i\lambda_{F,l}b_{F,l}}}{T_1^{F,l}(\tilde{\theta})}\right)\right] + \lambda_{F,l}a_{F,l}b_{F,l}t \end{aligned} \tag{A1}$$

In what follows we use the identity:

$$\arctanz = -\frac{i}{2}\log\left(\frac{(1 + iz)^2}{1 + z^2}\right) \quad z \in \mathbb{C}\{-i, i\}$$

where the complex-valued logarithmic function is defined according to the principal value of the argument.

Notice that:

$$\frac{\sqrt{i\lambda_{F,l}b_{F,l}}}{T_1^{F,l}(\tilde{\theta})} = i \iff \tilde{\theta} = 0$$

Similarly:

$$\frac{T_2^{F,l}(\tilde{\theta})}{T_1^{F,l}(\tilde{\theta})} = i \iff 2\tilde{\theta}(1 - e^{-\lambda_{F,l}t}) + i\lambda_{F,l}b_{F,l}^2 = -(-2\tilde{\theta} - i\lambda_{F,l}b_{F,l}^2)$$

whose solution is again  $\tilde{\theta} = 0$ .

An analysis at  $-i$  leads to the same conclusion.

Hence, for  $\tilde{\theta} \neq 0$  we have:

$$\begin{aligned} \arctan\left(\frac{\sqrt{i\lambda_{F,l}b_{F,l}}}{T_1^{F,l}(\tilde{\theta})}\right) &= -\frac{i}{2}\log\left[\frac{\left(1 + \frac{i\sqrt{i\lambda_{F,l}b_{F,l}}}{T_1^{F,l}(\tilde{\theta})}\right)^2}{1 + \left(\frac{\sqrt{i\lambda_{F,l}b_{F,l}}}{T_1^{F,l}(\tilde{\theta})}\right)^2}\right] \\ &= -\frac{i}{2}\left[2\log\left(T_1^{F,l}(\tilde{\theta}) + i\sqrt{i\lambda_{F,l}b_{F,l}}\right) - \log(-2\tilde{\theta})\right] \end{aligned}$$

and

$$\arctan\left(\frac{T_2^{F,l}(\tilde{\theta})}{T_1^{F,l}(\tilde{\theta})}\right) = -\frac{i}{2}\left[2\log\left(T_1^{F,l}(\tilde{\theta}) + iT_2^{F,l}(\tilde{\theta})\right) - \log(-2\tilde{\theta}\exp(-\lambda t))\right]$$

Then, substituting the expressions for arctan above into Equation (A1) we obtain Equation (19) after noticing that:

$$\begin{aligned}
 I_F^{(l)}(\lambda_{F,l}t, \tilde{\theta}) &= -\frac{2a_{F,l}}{\sqrt{i\lambda_{F,l}}} \left[ -T_2^{F,l}(\tilde{\theta}) + \sqrt{i\lambda_{F,l}}b_{F,l} \right. \\
 &+ \left. \frac{i}{2}T_1^{F,l}(\tilde{\theta}) \log \left( \frac{(T_1^{F,l}(\tilde{\theta}) + i\sqrt{i\lambda_{F,l}}b_{F,l})^2(-2\tilde{\theta} \exp(-\lambda_{F,l}t))}{(T_1^{F,l}(\tilde{\theta}) + iT_2^{F,l}(\tilde{\theta}))^2(-2\tilde{\theta})} \right) \right] \\
 &+ \lambda_{F,l}a_{F,l}b_{F,l}t
 \end{aligned}$$

In a similar way we obtain Equation (20). Notice that in the case of  $I_V^{(l)}(\theta)$ , the equality is valid for all values of the matrix  $\theta$  except when  $tr(\theta AC_l A') = 0$ , which is equivalent to  $\theta_{ll} = 0$ .  $\square$

**Proof of Proposition 1.** The proof is straightforward. It is based on computing the first and second derivatives of the characteristic function evaluated at zero.

Hence, for the first moments we notice that:

$$\begin{aligned}
 \frac{\partial K_1^+(\theta)}{\partial \theta_{kk}} &= i\lambda_{F,k}^{-1}(1 - \exp(-\lambda_{F,k}t))F_0^{(k)} + \frac{\partial I_F^{(k)}(\lambda_{F,k}t, \theta_{kk})}{\partial \theta_{kk}} \\
 \frac{\partial K_1^+(\theta)}{\partial \theta_{kj}} &= 0, \quad k \neq j \\
 \frac{\partial K_2^+(\theta)}{\partial \theta_{kj}} &= i \sum_{l=1}^2 a_{kl}a_{jl}\lambda_{V,l}^{-1}(1 - \exp(-\lambda_{V,l}t))V_0^{(l)} \\
 &+ \sum_{l=1}^2 a_{kl}a_{jl} \frac{\partial I_V^{(l)}(\lambda_{V,l}t, tr(\theta AC_l A'))}{\partial \theta_{kj}}
 \end{aligned}$$

Differentiating the characteristic function with respect to the components of the matrix  $\theta$  and evaluating at  $\theta = 0$  we have:

$$\begin{aligned}
 E_{\mathcal{Q}}(\sigma_t^{kk+}) &= -i \left( \frac{\partial K_1^+(\theta)}{\partial \theta_{kk}} \Big|_{\theta=0} + \frac{\partial K_2(\theta)}{\partial \theta_{kk}} \Big|_{\theta=0} \right) \\
 E(\sigma_t^{12+}) &= -i \frac{\partial K_2^+(\theta)}{\partial \theta_{12}} \Big|_{\theta=0}
 \end{aligned}$$

From which, Equations (23) and (24) follow.

On the other hand the second partial derivatives after evaluating at zero are:

$$\begin{aligned}
 \frac{\partial^2 K_1^+(\theta)}{\partial \theta_{kj} \partial \theta_{mn}} \Big|_{\theta=0} &= 0, \text{ if at least one of subscripts is different} \\
 \frac{\partial^2 K_1^+(\theta)}{\partial \theta_{kk}^2} \Big|_{\tilde{\theta}=0} &= \frac{\partial^2 I_F^{(l)}(\lambda_{F,l}t, \theta_{kk})}{\partial \theta_{kk}^2} \Big|_{\theta=0} \\
 \frac{\partial^2 K_2^+(\theta)}{\partial \theta_{kj} \partial \theta_{mn}} \Big|_{\theta=0} &= \sum_{l=1}^2 a_{kl}a_{jl}a_{ml}a_{nl} \frac{\partial^2 I_V^{(l)}(\lambda_{V,l}t, tr(\theta AC_l A'))}{\partial \theta_{kj} \partial \theta_{mn}} \Big|_{\theta=0}
 \end{aligned}$$

Therefore, we have:

$$\begin{aligned}
 E_{\mathcal{Q}}(\sigma_t^{kj+} \sigma_t^{mn+}) &= - \left( \frac{\partial^2 K_1^+(\theta)}{\partial \theta_{kj} \partial \theta_{mn}} + \frac{\partial^2 K_2^+(\theta)}{\partial \theta_{kj} \partial \theta_{mn}} \right) \Big|_{\theta=0} \\
 &- \left( \frac{\partial K_1^+(\theta)}{\partial \theta_{kj}} + \frac{\partial K_2^+(\theta)}{\partial \theta_{kj}} \right) \Big|_{\theta=0} \left( \frac{\partial K_1^+(\theta)}{\partial \theta_{mn}} + \frac{\partial K_2^+(\theta)}{\partial \theta_{mn}} \right) \Big|_{\theta=0}
 \end{aligned}$$

In particular:

$$\begin{aligned}
 E_{\mathcal{Q}}(\sigma_t^{12+})^2 &= -\frac{\partial^2 K_2^+(\theta)}{\partial \theta_{12}^2} \Big|_{\theta=0} - \left( \frac{\partial K_2^+(\theta)}{\partial \theta_{12}} \right)^2 \Big|_{\theta=0} \\
 E_{\mathcal{Q}}(\sigma_t^{kk+} \sigma_t^{12+}) &= -\left( \frac{\partial^2 K_1^+(\theta)}{\partial \theta_{kk} \partial \theta_{12}} + \frac{\partial^2 K_2^+(\theta)}{\partial \theta_{kk} \partial \theta_{12}} \right) \Big|_{\theta=0} \\
 &\quad - \left( \frac{\partial K_1^+(\theta)}{\partial \theta_{kk}} + \frac{\partial K_2^+(\theta)}{\partial \theta_{kk}} \right) \Big|_{\theta=0} \left( \frac{\partial K_1^+(\theta)}{\partial \theta_{12}} + \frac{\partial K_2^+(\theta)}{\partial \theta_{12}} \right) \Big|_{\theta=0} \\
 &= -\frac{\partial^2 K_2^+(\theta)}{\partial \theta_{kk} \partial \theta_{12}} \Big|_{\theta=0} - \left( \frac{\partial K_1^+(\theta)}{\partial \theta_{kk}} + \frac{\partial K_2^+(\theta)}{\partial \theta_{kk}} \right) \Big|_{\theta=0} \frac{\partial K_2^+(\theta)}{\partial \theta_{12}} \Big|_{\theta=0}
 \end{aligned}$$

Also:

$$\begin{aligned}
 E_{\mathcal{Q}}(\sigma_t^{11+} \sigma_t^{22+}) &= -\frac{\partial^2 K_2(\theta)}{\partial \theta_{11} \partial \theta_{22}} \Big|_{\theta=0} \\
 &\quad - \left( \frac{\partial K_1^+(\theta)}{\partial \theta_{11}} + \frac{\partial K_2^+(\theta)}{\partial \theta_{11}} \right) \Big|_{\theta=0} \left( \frac{\partial K_1^+(\theta)}{\partial \theta_{22}} + \frac{\partial K_2^+(\theta)}{\partial \theta_{22}} \right) \Big|_{\theta=0}
 \end{aligned}$$

Finally:

$$E_{\mathcal{Q}}(\sigma_t^{kk+})^2 = -\left( \frac{\partial K_1^+(\theta)}{\partial \theta_{kk}} \Big|_{\theta=0} + \frac{\partial K_2^+(\theta)}{\partial \theta_{kk}} \Big|_{\theta=0} \right)^2 - \left( \frac{\partial^2 K_1^+(\theta)}{\partial \theta_{kk}^2} + \frac{\partial^2 K_2^+(\theta)}{\partial \theta_{kk}^2} \right) \Big|_{\theta=0}$$

□

**Proof of Proposition 2.** Denote by  $\hat{1}_{[a,b]}$  the Fourier transform of  $1_{[a,b]}$ . Notice that:

$$\hat{1}_{[a,b]}(y) = -i \frac{\exp(i by) - \exp(i ay)}{y}, \quad y \neq 0$$

and equal to  $b - a$  if  $y = 0$ .

Hence:

$$\begin{aligned}
 \varphi_{v_T^+}(u, a, b) &= \int_{\mathbb{R}} \exp(iux) 1_{[a,b]}(x) \mathcal{Q}_{v_T^+}(dx) \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}} \exp(iux) \left[ \int_{\mathbb{R}} \exp(-iyx) \hat{1}_{[a,b]}(y) dy \right] \mathcal{Q}_{v_T^+}(dx) \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \exp(i(u-y)x) \mathcal{Q}_{v_T^+}(dx) \right] \hat{1}_{[a,b]}(y) dy \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_{\Sigma_T^+}(M(u-y)) \hat{1}_{[a,b]}(y) dy \\
 &= -\frac{i}{2\pi} \int_{\mathbb{R}} \varphi_{\Sigma_T^+}(M(u-y)) f(y) dy
 \end{aligned}$$

Equation (27) easily follows from Theorem 1. The second part of the proposition follows from elementary differentiation. From which Equations (25)–(27) easily follow. □

Preliminary Calculations for the Constrained Moments

Some preliminary calculations of the constrained moments are given below.

$$\begin{aligned}
 T_1^{F,l}(-y) &= \sqrt{2y - i\lambda_{F,l} b_{F,l}^2} \\
 T_2^{F,l}(T, -y) &= \sqrt{-2y(1 - e^{-\lambda_{F,l} T}) + i\lambda_{F,l} b_{F,l}^2} \\
 G^{F,l}(-y) &= \log \left( \exp(-\lambda_{F,l} T) \frac{(T_1^{F,l}(-y) + i\sqrt{i\lambda_{F,l}} b_{F,l})^2}{(T_1^{F,l}(-y) + iT_2^{F,l}(T, -y))^2} \right) \\
 \theta_l &= \text{tr}(AMC_l A') = (a_{1l} - a_{2l})^2, \quad l = 1, 2. \\
 T_1^{V,l}(-\theta_l y) &= \sqrt{2\theta_l y - i\lambda_{V,l} b_{V,l}^2} \\
 T_2^{V,l}(T, -\theta_l y) &= \sqrt{-2\theta_l y(1 - e^{-\lambda_{V,l} T}) + i\lambda_{V,l} b_{V,l}^2} \\
 G^{V,l}(-\theta_l y) &= \log \left( \exp(-\lambda_{V,l} T) \frac{(T_1^{V,l}(-\theta_l y) + i\sqrt{i\lambda_{V,l}} b_{V,l})^2}{(T_1^{V,l}(-\theta_l y) + iT_2^{V,l}(T, -\theta_l y))^2} \right)
 \end{aligned}$$

Which leads to:

$$\begin{aligned}
 I_F^{(l)}(\lambda_{F,l} T, -y) &= -\frac{2a_{F,l}}{\sqrt{i\lambda_{F,l}}} \left[ -T_2^{F,l}(T, -y) + \sqrt{i\lambda_{F,l}} b_{F,l} + \frac{i}{2} T_1^{F,l}(-y) G^{F,l}(-y) \right] + \lambda_{F,l} a_{F,l} b_{F,l} T \\
 I_V^{(l)}(\lambda_{V,l} T, -\theta_l y) &= -\frac{2a_{V,l}}{\sqrt{i\lambda_{V,l}}} \left[ -T_2^{V,l}(T, -\theta_l y) + \sqrt{i\lambda_{V,l}} b_{V,l} + \frac{i}{2} T_1^{V,l}(-\theta_l y) G^{V,l}(-\theta_l y) \right] + \lambda_{V,l} a_{V,l} b_{V,l} T \\
 DI_F^{(l)}(\lambda_{F,l} T, -y) &= -\frac{a_{F,l}}{\sqrt{i\lambda_{F,l}}} \int_0^{\lambda_{F,l} T} \frac{1 - \exp(-\lambda_{F,l} T + s)}{\sqrt{-2y(1 - \exp(-\lambda_{F,l} T + s)) + i\lambda_{F,l} b_{F,l}^2}} ds \\
 &= \frac{a_{F,l}}{\sqrt{i\lambda_{F,l}}} \int_0^{1 - \exp(-\lambda_{F,l} T)} \frac{v}{(v - 1) \sqrt{-2yv + i\lambda_{F,l} b_{F,l}^2}} dv \\
 D^n I_F^{(l)}(\lambda_{F,l} T, -y) &= (-1)^n \prod_{k=2}^n (2k - 3) \frac{a_{F,l}}{\sqrt{i\lambda_{F,l}}} \int_0^{-\lambda_{F,l} T} \frac{(1 - \exp(-\lambda_{F,l} T + s))^n}{(-2y(1 - \exp(-\lambda_{F,l} T + s)) + i\lambda_{F,l} b_{F,l}^2)^{\frac{2n-1}{2}}} ds \\
 &= (-1)^{n+1} \prod_{k=2}^n (2k - 3) \frac{a_{F,l}}{\sqrt{i\lambda_{F,l}}} \int_0^{1 - \exp(\lambda_{F,l} T)} \frac{v^n}{(v - 1) (-2yv + i\lambda_{F,l} b_{F,l}^2)^{\frac{2n-1}{2}}} dv \\
 DI_V^{(l)}(\lambda_{V,l} T, -\theta_l y) &= -\frac{a_{V,l}}{\sqrt{i\lambda_{V,l}}} \theta_l^2 \int_0^{-\lambda_{V,l} T} \frac{1 - \exp(-\lambda_{V,l} T + s)}{\sqrt{-2\theta_l y(1 - \exp(-\lambda_{V,l} T + s)) + i\lambda_{V,l} b_{V,l}^2}} ds \\
 &= -\frac{a_{V,l}}{\sqrt{i\lambda_{V,l}}} \theta_l^2 \int_0^{1 - \exp(\lambda_{V,l} T)} \frac{v}{(v - 1) \sqrt{-2\theta_l yv + i\lambda_{V,l} b_{V,l}^2}} dv \\
 D^n I_V^{(l)}(\lambda_{V,l} T, -\theta_l y) &= (-1)^n \prod_{k=2}^n (2k - 3) \frac{a_{V,l}}{\sqrt{i\lambda_{V,l}}} \theta_l^{2n} \int_0^{\lambda_{V,l} T} \frac{(1 - \exp(-\lambda_{V,l} T + s))^n}{(-2\theta_l y(1 - \exp(-\lambda_{V,l} T + s)) + i\lambda_{V,l} b_{V,l}^2)^{\frac{2n-1}{2}}} ds \\
 &= (-1)^{n+1} \prod_{k=2}^n (2k - 3) \frac{a_{V,l}}{\sqrt{i\lambda_{V,l}}} \theta_l^{2n} \int_0^{1 - \exp(-\lambda_{V,l} T)} \frac{v^n}{(v - 1) (-2\theta_l yv + i\lambda_{V,l} b_{V,l}^2)^{\frac{2n-1}{2}}} dv
 \end{aligned}$$

$$\begin{aligned}
K_1^+(-My) &= \sum_{l=1}^2 I_F^{(l)}(\lambda_{F,l}T, -y) \\
K_2^+(-My) &= \sum_{l=1}^2 I_V^{(l)}(\lambda_{V,l}T, -\theta_l y) \\
D^n K_1^+(-My) &= \sum_{l=1}^2 D^n I_F^{(l)}(\lambda_{F,l}T, -y) \\
D^n K_2^+(-My) &= \sum_{l=1}^2 D^n I_V^{(l)}(\lambda_{V,l}T, -\theta_l y)
\end{aligned}$$

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