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New Monotonic Properties of Positive Solutions of Higher-Order Delay Differential Equations and Their Applications

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Abstract: In this work, new criteria were established for testing the oscillatory behavior of solutions of a class of even-order delay differential equations. We follow an approach that depends on obtaining new monotonic properties for the decreasing positive solutions of the studied equation. Moreover, we use these properties to provide new oscillation criteria of an iterative nature. We provide an example to support the significance of the results and compare them with the related previous work.

Keywords: oscillation criteria; higher-order; delay differential equations

MSC: 34C10; 34K11

1. Introduction

The aim of this study is to obtain new monotonic properties of positive solutions to even-order delay differential equations (DDE)

$$\left(b(l)\left(y^{(n-1)}(l)\right)^{\kappa}\right)' + q(l)y^{\kappa}(g(l)) = 0,\ l \geq l_0, \tag{1}$$

where $n \geq 4$ is an even natural number, κ is a ratio of odd positive integers ($\in \mathbb{Q}^{+}_{\text{odd}}$), b, q, $g \in C[l_0, +\infty)$, $b(l) > 0$, $b'(l) \geq 0$, $q(l) \geq 0$, $g'(t) \geq 0$, $g(l) \leq l$, and $\lim_{l\to+\infty} g(l) = +\infty$. We, moreover, establish a new oscillation criterion for the solutions of (1) in the non-canonical case, that is,

$$\int_{l_0}^{+\infty} b^{-1/\kappa}(\eta)\mathrm{d}\eta < +\infty. \tag{2}$$

By a proper solution of (1), we mean a function $y \in C^{n-1}[l_0, +\infty)$ which has the properties

$$b\left(y^{(n-1)}\right)^{\kappa} \in C^1([l_0, +\infty)),\ \sup\{|y(l)| : l \geq l_*\} > 0,\ \text{for } l_* \in [l_0, +\infty),$$

and y satisfies (1) on $[l_0, +\infty)$. A solution y of (1) is called *non-oscillatory* if it is eventually positive or eventually negative; otherwise, it is called *oscillatory*.

DDEs are a type of functional differential equations and are also a better method to model natural phenomena. This is because the DDEs take into account the temporal memory of the phenomena. There is an active research movement to verify the qualitative properties (oscillatory, periodicity, stability, boundedness, etc.) for solutions of these equations, see for example [1–18] as well as the references listed in them. Oscillation theory is one of the branches of the qualitative theory of differential equations, which deals with the issue of oscillatory and non-oscillatory behavior of solutions to differential equations, as well as discusses the issue of the zeros of the solutions and the distances between them.

In canonical case, namely,

$$\int_{l_0}^{+\infty} b^{-1/\kappa}(\eta)\mathrm{d}\eta = +\infty,$$

many works dealt with the issue of the oscillatory and non-oscillatory behavior of solutions of DDE

$$\left(\left(y^{(n-1)}(l)\right)^{\kappa}\right)' + q(l)y^{\gamma}(g(l)) = 0, \tag{3}$$

and special cases of it, where $\gamma \in \mathbb{Q}^{+}_{\mathrm{odd}}$, see for example [19–21]. In [22], Grace et al. studied the oscillation of the more general form of canonical DDE

$$\left(b_3\left(b_2\left(b_1 y'\right)'\right)'\right)'(l) + q(l)y(g(l)) = 0,$$

under the condition

$$\int_{l_0}^{+\infty} b_i^{-1}(\eta)\mathrm{d}\eta = +\infty,\ i = 1, 2, 3.$$

For the non-canonical case (2), Baculikova et al. [23] presented comparative results for (1) with three DDEs of first order. For the reader's convenience, we review Corollary 4 in [23].

Theorem 1. *Let (2) hold, and assume that, for some $\varrho \in (0,1)$, and every $l_1 \geq l_0$, both*

$$\liminf_{l\to+\infty} \int_{g(l)}^{l} q(\eta)\frac{\varrho}{(n-1)!}\frac{g^{n-1}(\eta)}{b^{1/\kappa}(g(\eta))}\mathrm{d}\eta > \frac{1}{\mathrm{e}} \tag{4}$$

and

$$\liminf_{l\to+\infty} \int_{g(l)}^{l} \frac{1}{b^{1/\kappa}(u)}\left(\int_{l_0}^{u} q(\eta)\left(\frac{\varrho}{(n-2)!}g^{n-2}(\eta)\right)^{\kappa}\mathrm{d}\eta\right)^{1/\kappa}\mathrm{d}u > \frac{1}{\mathrm{e}}, \tag{5}$$

are satisfied. Then, every nonoscillatory solution of (1) tends to zero as $l \to +\infty$.

Assume, in addition, that there exists $\xi(l) \in C([l_0, +\infty))$ where $\xi_1(l) = \xi(l)$ and $\xi_{i+1}(l) = \xi_i(\xi(l))$, such that

$$\xi(l) \text{ is nondecreasing, } \xi(l) > l \text{ and } \xi_{n-2}(\tau(l)) < l$$

and

$$\liminf_{l\to+\infty} \int_{\xi_{n-2}(\tau(l))}^{l} b^{-1/\kappa}(u)\left(\int_{l_1}^{u} q(\eta)\mathrm{d}\eta\right)^{1/\kappa}(J_{n-2}(g(u)))^{1/\kappa}\mathrm{d}u > \frac{1}{\mathrm{e}},$$

where $J_1(l) = \xi(l) - l$ and $J_{i+1}(l) = \int_l^{\xi} J_i(s)\mathrm{d}s$.

Then, (1) is oscillatory.

Recently, Moaaz and Muhib [16] studied the oscillation of the fourth-order DDE

$$\left(b(l)\left(y'''(l)\right)^{\kappa}\right)' + f(l, y(g(l))) = 0, \tag{6}$$

where $f(l,y) \geq q(l)y^{\gamma}(l)$, and $\gamma \in \mathbb{Q}^{+}_{\text{odd}}$. They used a generalized Riccati substitution and presented nontraditional oscillation conditions, as in the following theorem:

Theorem 2. *Assume that (2) hold, and the following DDE is oscillatory for some $\lambda_0 \in (0,1)$:*

$$y'(l) + q(l)\left(\frac{\lambda_0 g^3(l)}{3! b^{1/\kappa}(g(l))}\right)^{\gamma} y^{\gamma/\kappa}(g(l)) = 0. \tag{7}$$

If there are $\rho, \theta \in C^1([l_0, +\infty), \mathbb{R}^+)$ with

$$\lim_{l\to+\infty} \sup \frac{\varphi_0^{\kappa}(l)}{\rho(l)} \int_{l_0}^{l} \left(\rho(\eta)q(\eta)h(\eta)\left(\frac{\lambda}{2!}g^2(\eta)\right)^{\gamma} - \frac{b(\eta)(\rho'(\eta))^{\kappa+1}}{(\kappa+1)^{(\kappa+1)}\rho^{\kappa}(\eta)}\right)\mathrm{d}\eta > 1 \tag{8}$$

and

$$\lim_{l\to+\infty} \sup \frac{\varphi_2^{\kappa}(l)}{\theta(l)} \int_{l_0}^{l} \left(\theta(\eta)q(\eta)\mu(\eta) - \frac{(\theta'(\eta))^{\kappa+1}}{(\kappa+1)^{(\kappa+1)}\theta^{\kappa}(\eta)\varphi_1^{\kappa}(\eta)}\right)\mathrm{d}\eta > 1 \tag{9}$$

for some $\lambda_1 \in (0,1)$, and any positive constants c_i and k_i, then (6) is oscillatory, where

$$\varphi_0(l) := \int_l^{+\infty} \frac{1}{b^{1/\kappa}(\eta)}\mathrm{d}\eta, \quad \varphi_m(l) := \int_l^{+\infty} \varphi_{m-1}(\eta)\mathrm{d}\eta \ \text{ for } m = 1,2, \tag{10}$$

$$h(l) := \begin{cases} c_1^{\kappa-\gamma} & \text{if } \kappa > \gamma \\ c_2\, \varphi^{\gamma-\kappa}(l) & \text{if } \kappa < \gamma \end{cases} \quad \text{and} \quad \mu(l) := \begin{cases} k_1^{\kappa-\gamma} & \text{if } \kappa > \gamma \\ k_2\, \varphi_2^{\gamma-\kappa}(l) & \text{if } \kappa < \gamma \end{cases}.$$

On the other hand, recently, many interesting works have appeared which contribute significantly to the development of the study of second-order DDEs. From these works, Baculikova [5] presented a new approach based on the improvement of the monotonic properties of a class of positive solutions of linear DDE

$$\left(b(l)\left(y'(l)\right)\right)' + q(l)y(g(l)) = 0.$$

In this paper, as an extension of Baculikova's results in [5] to the quasi-linear case and the higher-order, we present new monotonic properties of the decreasing positive solutions of (1) in the non-canonical case. We obtain a comparison result in which oscillation (1) is deduced from oscillation of a first-order DDE in addition to some previous conditions. Our new results improve Theorems 1 and 2. Finally, an example is provided to support the significance of the new results.

2. Main Results

For the convenience of presenting the results, we define

$$\varphi_0(l) := \int_l^{+\infty} b^{-1/\kappa}(\eta)\mathrm{d}\eta,$$

and

$$\varphi_j(l) := \int_l^{+\infty} \varphi_{j-1}(\eta)\mathrm{d}\eta, \ \text{for } j = 1,2,\ldots,n-2.$$

Lemma 1. *Assume that $y \in S^+$ and y satisfies*

$$y^{(s)}(l)y^{(s+1)}(l) < 0 \ \textit{for } s = 0,1,\ldots,n-2, \tag{11}$$

for $l \geq l_1 \in [l_0, +\infty)$. *If*

$$\int_{l_0}^{+\infty} \left(\frac{1}{b(u)} \int_{l_0}^{u} q(\eta) \mathrm{d}\eta \right)^{1/\kappa} \mathrm{d}u = +\infty, \tag{12}$$

then

$$(-1)^{j+1} y^{(n-j-2)}(l) \leq b^{1/\kappa}(l) y^{(n-1)}(l) \varphi_j(l) \text{ for } j = 0, 1, \ldots, n-2, \tag{13}$$

$$\lim_{l \to +\infty} y(l) = 0 \tag{14}$$

and

$$\left(\frac{y(l)}{\varphi_{n-2}(l)} \right)' \geq 0. \tag{15}$$

Proof. Assume that $y \in S^+$ and satisfies (11) for $l \geq l_1$. for some $l_1 \in [l_0, +\infty)$. Then, there is a $l_2 \geq l_1$ with $y(g(l)) > 0$ for all l_2, and hence, from (1), we have

$$\left(b(l) \left(y^{(n-1)}(l) \right)^{\kappa} \right)' = -q(l) y^{\kappa}(g(l)) \leq 0.$$

From (11), we get

$$-y^{(n-2)}(l) \leq y^{(n-2)}(+\infty) - y^{(n-2)}(l) = \int_{l}^{+\infty} \frac{b^{1/\kappa}(\eta) y^{(n-1)}(\eta)}{b^{1/\kappa}(\eta)} \mathrm{d}\eta \leq b^{1/\kappa}(l) y^{(n-1)}(l) \varphi_0(l),$$

and so

$$y^{(n-2)}(l) \geq -b^{1/\kappa}(l) y^{(n-1)}(l) \varphi_0(l). \tag{16}$$

Integrating (16) $n-2$ times over $[l, +\infty)$, and using (11), we arrive at (13).

Since $y' < 0$, we get that $\lim_{l \to +\infty} y(l) = k \geq 0$. Let $k > 0$, and so, there is a $l_2 \geq l_1$ with $y(l) \geq k$ for $l \geq l_2$. Then, (1) becomes

$$\left(b(l) \left(y^{(n-1)}(l) \right)^{\kappa} \right)' \leq -k^{\kappa} q(l).$$

Integrating the above inequality over $[l_2, l)$, we obtain

$$b(l) \left(y^{(n-1)}(l) \right)^{\kappa} - b(l_2) \left(y^{(n-1)}(l_2) \right)^{\kappa} \leq -\int_{l_2}^{l} k^{\kappa} q(\eta) \mathrm{d}\eta.$$

From (11), we have $y^{(n-1)}(l) < 0$ for $l \geq l_1$. Then, $b(l_2) \left(y^{(n-1)}(l_2) \right)^{\kappa} < 0$, and so

$$y^{(n-1)}(l) \leq -k \left(\frac{1}{b(l)} \int_{l_2}^{l} q(\eta) \mathrm{d}\eta \right)^{1/\kappa}. \tag{17}$$

Integrating (17) over $[l_2, l)$, we have

$$y^{(n-2)}(l) \leq y^{(n-2)}(l_2) - k \int_{l_2}^{l} \left(\frac{1}{b(u)} \int_{l_2}^{u} q(\eta) \mathrm{d}\eta \right)^{1/\kappa} \mathrm{d}u,$$

which with (12) gives $\lim_{l \to +\infty} y^{(n-2)}(l) = -\infty$, a contradiction. Therefore, $\lim_{l \to +\infty} y(l) = 0$.

Now, using (13) at $j = 0$, we get that

$$\left(\frac{y^{(n-2)}(l)}{\varphi_0(l)} \right)' = \frac{1}{\varphi_0^2(l)} \left(\varphi_0(l) y^{(n-1)}(l) + b^{-1/\kappa}(l) y^{(n-2)}(l) \right) \geq 0,$$

which leads to

$$-y^{(n-3)}(l) \geq \int_l^{+\infty} \varphi_0(\eta) \frac{y^{(n-2)}(\eta)}{\varphi_0(\eta)} \mathrm{d}\eta \geq \frac{y^{(n-2)}(l)}{\varphi_0(l)} \varphi_1(l).$$

This implies

$$\left(\frac{y^{(n-3)}(l)}{\varphi_1(l)} \right)' = \frac{1}{\varphi_1^2(l)} \left(\varphi_1(l) y^{(n-2)}(l) + \varphi_0(l) y^{(n-3)}(l) \right) \leq 0.$$

By repeating a similar approach, we obtain (15). This proves the lemma. □

Theorem 3. *Assume that $y \in S^+$ and satisfies (11). If (12) holds and*

$$\varphi_{n-2}^{1+\kappa}(l) \varphi_{n-3}^{-1}(l) q(l) \geq \kappa \varrho_0, \tag{18}$$

for some $\varrho_0 \in (0,1)$, then

$$\begin{array}{ll} (\mathbf{A}_{1,0}) & \left(y(l)/\varphi_{n-2}^{\varrho_0}(l) \right)' \leq 0, \\ (\mathbf{A}_{2,0}) & \lim_{l \to +\infty} y(l)/\varphi_{n-2}^{\varrho_0}(l) = 0. \end{array}$$

Proof. Assume that $y \in S^+$ and satisfies (11) for $l \geq l_1$ for some $l_1 \in [l_0, +\infty)$. Then, from Lemma 1, we have that (13)–(15) hold.

$(\mathbf{A}_{1,0})$: The quantity $(b(l)\left(y^{(n-1)}(l)\right)^{\kappa})'$ can be written in the following form

$$(b(l)\left(y^{(n-1)}(l)\right)^{\kappa})' = \left(\left(b^{1/\kappa}(l) y^{(n-1)}(l) \right)^{\kappa} \right)' = \kappa \left(b^{1/\kappa}(l) y^{(n-1)}(l) \right)^{\kappa-1} \left(b^{1/\kappa}(l) y^{(n-1)}(l) \right)', \tag{19}$$

using (1) and (19), we get

$$\kappa \left(b^{1/\kappa}(l) y^{(n-1)}(l) \right)^{\kappa-1} \left(b^{1/\kappa}(l) y^{(n-1)}(l) \right)' \leq -q(l) y^{\kappa}(g(l))$$

and so

$$\left(b^{1/\kappa}(l) y^{(n-1)}(l) \right)' \leq -\frac{1}{\kappa} \left(b^{1/\kappa}(l) y^{(n-1)}(l) \right)^{1-\kappa} q(l) y^{\kappa}(g(l)), \tag{20}$$

using (13) at $j = n-2$, we have

$$-\frac{y(l)}{\varphi_{n-2}(l)} \leq b^{1/\kappa}(l) y^{(n-1)}(l)$$

and so

$$\left(\frac{y(l)}{\varphi_{n-2}(l)} \right)^{1-\kappa} = \left(-\frac{y(l)}{\varphi_{n-2}(l)} \right)^{1-\kappa} \leq \left(b^{1/\kappa}(l) y^{(n-1)}(l) \right)^{1-\kappa}. \tag{21}$$

Combining (20) and (21), we find

$$\begin{aligned} \left(b^{1/\kappa}(l) y^{(n-1)}(l) \right)' &\leq -\frac{1}{\kappa} \left(\frac{y(l)}{\varphi_{n-2}(l)} \right)^{1-\kappa} q(l) y^{\kappa}(g(l)) && (22) \\ &\leq -\frac{1}{\kappa} \left(\frac{y(l)}{\varphi_{n-2}(l)} \right)^{1-\kappa} q(l) y^{\kappa}(l) \\ &\leq -\frac{1}{\kappa} \frac{y(l)}{\varphi_{n-2}^{1-\kappa}(l)} q(l). && (23) \end{aligned}$$

Integrating (23) over $[l_2, l)$, and using (18), we get

$$\begin{aligned} b^{1/\kappa}(l)y^{(n-1)}(l) &\leq b^{1/\kappa}(l_2)y^{(n-1)}(l_2) - \frac{1}{\kappa}\int_{l_2}^{l} \frac{y(s)}{\varphi_{n-2}^{1-\kappa}(s)} q(s)\mathrm{d}s \\ &\leq b^{1/\kappa}(l_2)y^{(n-1)}(l_2) - \frac{1}{\kappa}y(l)\int_{l_2}^{l} \frac{q(s)}{\varphi_{n-2}^{1-\kappa}(s)}\mathrm{d}s \\ &\leq b^{1/\kappa}(l_2)y^{(n-1)}(l_2) - \varrho_0 y(l)\int_{l_2}^{l} \frac{\varphi_{n-3}(s)}{\varphi_{n-2}^{2}(s)}\mathrm{d}s \\ &\leq b^{1/\kappa}(l_2)y^{(n-1)}(l_2) + \varrho_0\frac{y(l)}{\varphi_{n-2}(l_2)} - \varrho_0\frac{y(l)}{\varphi_{n-2}(l)}, \end{aligned}$$

which, with (14), gives

$$b^{1/\kappa}(l)y^{(n-1)}(l) \leq -\varrho_0\frac{y(l)}{\varphi_{n-2}(l)}. \tag{24}$$

Thus, using (13) at $j = n-3$, we obtain

$$\frac{y'(l)}{\varphi_{n-3}(l)} \leq -\varrho_0\frac{y(l)}{\varphi_{n-2}(l)}.$$

Consequently,

$$\left(\frac{y(l)}{\varphi_{n-2}^{\varrho_0}(l)}\right)' = \frac{(\varphi_{n-2}(l)y'(l) + \varrho_0 y(l)\varphi_{n-3}(l))}{\varphi_{n-2}^{\varrho_0+1}(l)} \leq 0,$$

then $(\mathbf{A}_{1,0})$ holds.

$(\mathbf{A}_{2,0})$: Now, since $\left(y/\varphi_{n-2}^{\varrho_0}\right)' \leq 0$, we see that $\lim_{l\to+\infty} y(l)/\varphi_{n-2}^{\varrho_0}(l) = k_1 \geq 0$. Let $k_1 > 0$, and so, there is a $l_2 \geq l_1$ with $y(l)/\varphi_{n-2}^{\varrho_0}(l) \geq k_1$ for $l \geq l_2$. Next, we define

$$F(l) := \frac{y(l) + b^{1/\kappa}(l)y^{(n-1)}(l)\varphi_{n-2}(l)}{\varphi_{n-2}^{\varrho_0}(l)}.$$

Then, from (13), $F(l) > 0$ for $l \geq l_2$. Differentiating $F(l)$ and using (18), (23) and (13), we get

$$\begin{aligned} F'(l) &= \frac{1}{\varphi_{n-2}^{2\varrho_0}(l)}\Big[\varphi_{n-2}^{\varrho_0}(l)\Big(y'(l) - b^{1/\kappa}(l)y^{(n-1)}(l)\varphi_{n-3}(l) + \Big(b^{1/\kappa}(l)y^{(n-1)}(l)\Big)'\varphi_{n-2}(l)\Big) \\ &\quad + \varrho_0\varphi_{n-2}^{\varrho_0-1}(l)\varphi_{n-3}(l)\Big(y(l) + b^{1/\kappa}(l)y^{(n-1)}(l)\varphi_{n-2}(l)\Big)\Big] \\ &\leq \frac{1}{\varphi_{n-2}^{\varrho_0+1}(l)}\Big[-\varrho_0\varphi_{n-3}(l)y(l) + \varrho_0\varphi_{n-3}(l)\Big(y(l) + b^{1/\kappa}(l)y^{(n-1)}(l)\varphi_{n-2}(l)\Big)\Big] \\ &\leq \frac{1}{\varphi_{n-2}^{\varrho_0+1}(l)}\Big[-\varrho_0\varphi_{n-3}(l)y(l) + \varrho_0\varphi_{n-3}(l)y(l) + \varrho_0\varphi_{n-3}(l)b^{1/\kappa}(l)y^{(n-1)}(l)\varphi_{n-2}(l)\Big] \\ &\leq \frac{1}{\varphi_{n-2}^{\varrho_0}(l)}\varrho_0\varphi_{n-3}(l)b^{1/\kappa}(l)y^{(n-1)}(l). \end{aligned} \tag{25}$$

Using the fact that $y(l)/\varphi_{n-2}^{\varrho_0}(l) \geq k_1$ with (24), we obtain.

$$b^{1/\kappa}(l)y^{(n-1)}(l) \leq -\varrho_0\frac{y(l)}{\varphi_{n-2}(l)} \leq -\varrho_0 k_1\varphi_{n-2}^{\varrho_0-1}(l). \tag{26}$$

Combining (25) and (26), we get

$$F'(l) \leq -\frac{k_1}{\varphi_{n-2}(l)}\varrho_0^2 \varphi_{n-3}(l) < 0.$$

Integrating this inequality over $[l_2, l)$, we find

$$\begin{aligned} F(l) - F(l_2) &\leq -k_1 \varrho_0^2 \int_{l_2}^{l} \frac{\varphi_{n-3}(s)}{\varphi_{n-2}(s)} \mathrm{d}s \\ &\leq -k_1 \varrho_0^2 \ln \frac{\varphi_{n-2}(l_2)}{\varphi_{n-2}(l)} \end{aligned}$$

and so

$$-F(l_2) \leq -k_1 \varrho_0^2 \ln \frac{\varphi_{n-2}(l_2)}{\varphi_{n-2}(l)} \to -\infty \text{ as } l \to +\infty,$$

we arrive at a contradiction, and so $k_1 = 0$. Then, $(\mathbf{A}_{2,0})$ holds. This proves the theorem. □

The asymptotic and monotonous properties of positive solutions are of great benefit in improving oscillation criteria. So, in the following theorem, we improve the properties by assuming that

$$\varrho_1 := \varrho_0 \frac{\lambda^{\kappa \varrho_0}}{1 - \varrho_0},$$

and proving these properties for ϱ_1.

Theorem 4. *Assume that $y \in S^+$, y satisfies (11), (12) and (18) hold for some $\varrho_0 \in (0,1)$, and*

$$\frac{\varphi_{n-2}(g(l))}{\varphi_{n-2}(l)} \geq \lambda, \tag{27}$$

for some $\lambda \geq 1$. If $\varrho_0 \leq \varrho_1$, then

$$\begin{aligned} &(\mathbf{A}_{1,1}) \quad \left(y(l)/\varphi_{n-2}^{\varrho_1}(l)\right)' \leq 0, \\ &(\mathbf{A}_{2,1}) \quad \lim_{l \to +\infty} y(l)/\varphi_{n-2}^{\varrho_1}(l) = 0. \end{aligned}$$

Proof. Assume that $y \in S^+$ and satisfies (11) for $l \geq l_1$ for some $l_1 \in [l_0, +\infty)$. From Lemma 3, we have that $(\mathbf{A}_{1,0})$ and $(\mathbf{A}_{2,0})$ hold. Proceeding as in the proof of Theorem 3, we arrive at (22) holds. Integrating (22) over $[l_2, l)$, we get

$$b^{1/\kappa}(l) y^{(n-1)}(l) \leq b^{1/\kappa}(l_2) y^{(n-1)}(l_2) - \frac{1}{\kappa} \int_{l_2}^{l} \left(\frac{y(\eta)}{\varphi_{n-2}(\eta)}\right)^{1-\kappa} q(\eta) y^{\kappa}(g(\eta)) \mathrm{d}\eta. \tag{28}$$

Using $(\mathbf{A}_{1,0})$, we have that

$$y(g(l)) \geq \varphi_{n-2}^{\varrho_0}(g(l)) \frac{y(l)}{\varphi_{n-2}^{\varrho_0}(l)}.$$

Then, (28) becomes

$$\begin{aligned} b^{1/\kappa}(l) y^{(n-1)}(l) &\leq b^{1/\kappa}(l_2) y^{(n-1)}(l_2) - \frac{1}{\kappa} \int_{l_2}^{l} \frac{y^{1-\kappa}(\eta)}{\varphi_{n-2}^{1-\kappa}(\eta)} q(\eta) \left(\varphi_{n-2}^{\varrho_0}(g(l))\right)^{\kappa} \frac{y^{\kappa}(l)}{\left(\varphi_{n-2}^{\varrho_0}(l)\right)^{\kappa}} \mathrm{d}\eta \\ &\leq b^{1/\kappa}(l_2) y^{(n-1)}(l_2) - \frac{1}{\kappa} \int_{l_2}^{l} \frac{q(\eta)}{\varphi_{n-2}^{1-\kappa}(\eta)} \frac{\left(\varphi_{n-2}^{\varrho_0}(g(\eta))\right)^{\kappa}}{\left(\varphi_{n-2}^{\varrho_0}(\eta)\right)^{\kappa}} \varphi_{n-2}^{\varrho_0}(\eta) \frac{y(\eta)}{\varphi_{n-2}^{\varrho_0}(\eta)} \mathrm{d}\eta, \end{aligned}$$

which, with the fact that $y/\varphi_{n-2}^{\varrho_0}$ is a decreasing function, gives

$$b^{1/\kappa}(l)y^{(n-1)}(l) \leq b^{1/\kappa}(l_2)y^{(n-1)}(l_2) - \frac{1}{\kappa}\frac{y(l)}{\varphi_{n-2}^{\varrho_0}(l)}\int_{l_2}^{l}\frac{q(\eta)}{\varphi_{n-2}^{1-\kappa}(\eta)}\frac{\left(\varphi_{n-2}^{\varrho_0}(g(\eta))\right)^{\kappa}}{\left(\varphi_{n-2}^{\varrho_0}(\eta)\right)^{\kappa}}\varphi_{n-2}^{\varrho_0}(\eta)\mathrm{d}\eta.$$

Hence, from (18) and (27), we obtain

$$\begin{aligned} b^{1/\kappa}(l)y^{(n-1)}(l) &\leq b^{1/\kappa}(l_2)y^{(n-1)}(l_2) - \varrho_0\lambda^{\kappa\varrho_0}\frac{y(l)}{\varphi_{n-2}^{\varrho_0}(l)}\int_{l_2}^{l}\frac{\varphi_{n-3}(\eta)}{\varphi_{n-2}^{2-\varrho_0}(\eta)}\mathrm{d}\eta \\ &\leq b^{1/\kappa}(l_2)y^{(n-1)}(l_2) - \varrho_0\frac{\lambda^{\kappa\varrho_0}}{1-\varrho_0}\frac{y(l)}{\varphi_{n-2}^{\varrho_0}(l)}\left(\frac{1}{\varphi_{n-2}^{1-\varrho_0}(l)} - \frac{1}{\varphi_{n-2}^{1-\varrho_0}(l_2)}\right) \\ &\leq b^{1/\kappa}(l_2)y^{(n-1)}(l_2) + \varrho_1\frac{y(l)}{\varphi_{n-2}^{\varrho_0}(l)}\frac{1}{\varphi_{n-2}^{1-\varrho_0}(l_2)} - \varrho_1\frac{y(l)}{\varphi_{n-2}(l)}, \end{aligned}$$

which, with the fact that $\lim_{l\to+\infty} y(l)/\varphi_{n-2}^{\varrho_0}(l) = 0$, gives

$$b^{1/\kappa}(l)y^{(n-1)}(l) \leq -\varrho_1\frac{y(l)}{\varphi_{n-2}(l)}. \tag{29}$$

Thus, from (13) at $j = n-3$, we obtain

$$\frac{y'(l)}{\varphi_{n-3}(l)} \leq -\varrho_1\frac{y(l)}{\varphi_{n-2}(l)}.$$

Consequently,

$$\left(\frac{y(l)}{\varphi_{n-2}^{\varrho_1}(l)}\right)' = \frac{1}{\varphi_{n-2}^{\varrho_1+1}(l)}\left(\varphi_{n-2}(l)y'(l) + \varrho_1\varphi_{n-3}(l)y(l)\right) \leq 0,$$

then $(\mathbf{A}_{1,1})$ holds.
$(\mathbf{A}_{2,1})$: Now, since $\left(y/\varphi_{n-2}^{\varrho_1}\right)' \leq 0$, we see that $\lim_{l\to+\infty} y(l)/\varphi_{n-2}^{\varrho_1}(l) = k_2 \geq 0$. Let $k_2 > 0$, and so, there is a $l_2 \geq l_1$ with $y(l)/\varphi_{n-2}^{\varrho_1}(l) \geq k_2$ for $l \geq l_2$. Next, we define

$$F(l) := \frac{y(l) + b^{1/\kappa}(l)y^{(n-1)}(l)\varphi_{n-2}(l)}{\varphi_{n-2}^{\varrho_0}(l)}.$$

Then, from (13), $F(l) > 0$ for $l \geq l_2$. Differentiating $F(l)$ and using (18), (23) and (13), we get

$$\begin{aligned} F'(l) &= \frac{1}{\varphi_{n-2}^{2\varrho_0}(l)}\Big[\varphi_{n-2}^{\varrho_0}(l)\Big(y'(l) - b^{1/\kappa}(l)y^{(n-1)}(l)\varphi_{n-3}(l) + \left(b^{1/\kappa}(l)y^{(n-1)}(l)\right)'\varphi_{n-2}(l)\Big) \\ &\quad +\varrho_0\varphi_{n-2}^{\varrho_0-1}(l)\varphi_{n-3}(l)\left(y(l) + b^{1/\kappa}(l)y^{(n-1)}(l)\varphi_{n-2}(l)\right)\Big] \\ &\leq \frac{1}{\varphi_{n-2}^{\varrho_0+1}(l)}\left[-\varrho_0\varphi_{n-3}(l)y(l) + \varrho_0\varphi_{n-3}(l)\left(y(l) + b^{1/\kappa}(l)y^{(n-1)}(l)\varphi_{n-2}(l)\right)\right] \\ &\leq \frac{1}{\varphi_{n-2}^{\varrho_0+1}(l)}\left[-\varrho_0\varphi_{n-3}(l)y(l) + \varrho_0\varphi_{n-3}(l)y(l) + \varrho_0\varphi_{n-3}(l)b^{1/\kappa}(l)y^{(n-1)}(l)\varphi_{n-2}(l)\right] \\ &\leq \frac{1}{\varphi_{n-2}^{\varrho_0}(l)}\varrho_0\varphi_{n-3}(l)b^{1/\kappa}(l)y^{(n-1)}(l). \end{aligned} \tag{30}$$

Using the fact that $y(l)/\varphi_{n-2}^{\varrho_0}(l) \geq k_2$ with (29), we obtain.

$$b^{1/\kappa}(l)y^{(n-1)}(l) \leq -\varrho_1 \frac{y(l)}{\varphi_{n-2}(l)} \leq -\varrho_1 k_2 \varphi_{n-2}^{\varrho_0 - 1}(l). \tag{31}$$

Combining (30) and (31), we get

$$F'(l) \leq -\frac{k_2}{\varphi_{n-2}(l)} \varrho_0 \varrho_1 \varphi_{n-3}(l) < 0.$$

Integrating this inequality over $[l_2, l)$, we find

$$\begin{aligned} F(l) - F(l_2) &\leq -k_2 \varrho_0 \varrho_1 \int_{l_2}^{l} \frac{\varphi_{n-3}(s)}{\varphi_{n-2}(s)} \mathrm{d}s \\ &\leq -k_2 \varrho_0 \varrho_1 \ln \frac{\varphi_{n-2}(l_2)}{\varphi_{n-2}(l)} \end{aligned}$$

and so

$$-F(l_2) \leq -k_2 \varrho_0 \varrho_1 \ln \frac{\varphi_{n-2}(l_2)}{\varphi_{n-2}(l)} \to -\infty \text{ as } l \to +\infty,$$

we arrive at a contradiction, and so $k_2 = 0$. Then, $(\mathbf{A}_{2,1})$ holds. This proves the theorem. □

Next, by defining the nondecreasing sequence $\{\varrho_j\}_{j=0}^{m}$ by

$$\varrho_j := \varrho_0 \frac{\lambda^{\kappa \varrho_{j-1}}}{1 - \varrho_{j-1}}, \tag{32}$$

we can prove the properties

$$\begin{array}{ll} (\mathbf{A}_{1,m}) & \left(y(l)/\varphi_{n-2}^{\varrho_m}(l) \right)' \leq 0, \\ (\mathbf{A}_{2,m}) & \lim_{l \to +\infty} y(l)/\varphi_{n-2}^{\varrho_m}(l) = 0, \end{array}$$

using the same approach as in Theorem 4.

Theorem 5. *Assume that $y \in S^+$, $\kappa \geq 1$ and y satisfies (11). Let, for $\varrho_0 \in (0,1)$, (12) and (18) be satisfied. If $\varrho_{i-1} \leq \varrho_i < 1$ for all $i = 1, 2, \ldots, m-1$, then the DDE*

$$W'(l) + \frac{q(l)}{\kappa(1 - \varrho_m)} \varphi_{n-2}^{\kappa}(l) W(g(l)) = 0, \tag{33}$$

has a positive solution, where λ and ϱ_j are defined as (27) and (32), respectively.

Proof. Assume that $y \in S^+$ and satisfies (11) for $l \geq l_1$ for some $l_1 \in [l_0, +\infty)$. Then, it follows from Theorem 3 that $(\mathbf{A}_{1,m})$ and $(\mathbf{A}_{2,m})$ hold.
Now, we define

$$W(l) := b^{1/\kappa}(l) y^{(n-1)}(l) \varphi_{n-2}(l) + y(l). \tag{34}$$

Then, from (13) at $j = n-2$, $W(l) > 0$ for $l \geq l_2$, and

$$W'(l) = \left(b^{1/\kappa}(l) y^{(n-1)}(l) \right)' \varphi_{n-2}(l) - b^{1/\kappa}(l) y^{(n-1)}(l) \varphi_{n-3}(l) + y'(l), \tag{35}$$

using (13) at $j = n-3$, we have

$$\begin{aligned} W'(l) &\leq \left(b^{1/\kappa}(l)y^{(n-1)}(l)\right)'\varphi_{n-2}(l) \leq -\frac{1}{\kappa}\frac{y^{1-\kappa}(l)}{\varphi_{n-2}^{1-\kappa}(l)}q(l)y^{\kappa}(g(l))\varphi_{n-2}(l) \\ &\leq -\frac{1}{\kappa}\frac{y^{1-\kappa}(g(l))}{\varphi_{n-2}^{1-\kappa}(l)}q(l)y^{\kappa}(g(l))\varphi_{n-2}(l) \leq -\frac{1}{\kappa}y(g(l))\varphi_{n-2}^{\kappa}(l)q(l). \end{aligned} \tag{36}$$

As in the proof of Theorem 3, we arrive at (29). From (34) and (29), we have

$$W(l) \leq (1-\varrho_m)y(l).$$

Thus, (36) becomes

$$W'(l) + \frac{q(l)}{\kappa(1-\varrho_m)}\varphi_{n-2}^{\kappa}(l)W(g(l)) \leq 0. \tag{37}$$

Hence, W is a positive solution of (37). From Theorem 1 in [24], (33) has also a positive solution. Therefore, the proof is complete. □

Now, in the next part, we obtain new oscillation conditions for (1), using the previous results.

Theorem 6. *Assume that (12) and (18) hold for some $\varrho_0 \in (0,1)$, and that ϱ_j, λ are defined as in Theorem 3. If, $\varrho_{i-1} \leq \varrho_i < 1$ for all $i = 1,2,\ldots,m-1$, and all solutions of DDEs (33),*

$$w'(l) + q(l)\left(\frac{\epsilon_1 g^{n-1}(l)}{(n-1)!\left(b^{1/\kappa}(g(l))\right)}\right)^{\kappa} w(g(l)) = 0 \tag{38}$$

and

$$\Omega'(l) + \frac{\epsilon_2}{(n-2)!b^{1/\kappa}(l)}\left(\int_{l_0}^{l} q(\eta)\left(g^{n-2}(\eta)\right)^{\kappa}\mathrm{d}\eta\right)^{1/\kappa}\Omega(g(l)) = 0, \tag{39}$$

are oscillatory, for some $\epsilon_1, \epsilon_2, \varrho_m \in (0,1)$, then every solution of (1) is oscillatory.

Proof. Assume the contrary that $y \in S^+$. Then, from [25], we have the following three cases, eventually:
(**i**) $y^{(s)}(l) > 0$ for $s = 0,1,n-1$ and $y^{(n)}(l) < 0$;
(**ii**) $y^{(s)}(l) > 0$ for $s = 0,1,n-2$ and $y^{(n-1)}(l) < 0$;
(**iii**) $(-1)^s y^{(s)}(l) > 0$ for $s = 0,1,\ldots,n-1$.

In view of Theorem 3 in [23], the fact that the solutions of Equations (38) and (39) oscillate, rules out the cases (**i**) and (**ii**), respectively. Then, we have (**iii**) hold. Using Theorem 5, we get that Equation (33) has a positive solution, a contradiction. This proves the theorem. □

Corollary 1. *Assume that (12), (18), (4) and (5) hold for some $\varrho_0 \in (0,1)$, and ϱ_j and λ are defined as (32) and (27), respectively. If $\varrho_{i-1} \leq \varrho_i < 1$ for all $i = 1,2,\ldots,m-1$,*

$$\liminf_{l\to+\infty}\int_{g(l)}^{l} q(\eta)\varphi_{n-2}^{\kappa}(\eta)\mathrm{d}\eta > \frac{\kappa(1-\varrho_m)}{\mathrm{e}}, \tag{40}$$

for some $\epsilon, \varrho_m \in (0,1)$, then every solution of (1) is oscillatory.

Proof. In view of Corollary 2.1 in [26], conditions (40), (4) and (5) imply oscillation of the solutions of (33), (38) and (39), respectively. Therefore, from Theorem 6, every solution of (1) is oscillatory. □

Example 1. *For $l \geq 1$, consider the fourth-order delay differential equation*

$$\left(e^{l} y'''(l)\right)' + q_0 e^{l} y(l - g_0) = 0, \tag{41}$$

where $\kappa = 1$, $n = 4$, $q_0 \in (0,1)$, $g_0 > 0$, $b(l) = e^{l}, q(l) = q_0 e^{l}$ and $g(l) = (l - g_0)$. It is clear that $\varphi_i(u) = e^{-l}$, $i = 0,1,2$. Moreover, we find that (12) holds.

If we now set $\varrho_0 = q_0$, then we conclude that (18) is satisfied. As a result of the calculations, we see that (4) and (5) hold. Now, the condition (40) reduces to

$$q_0 g_0 > \frac{(1 - \varrho_m)}{e}. \tag{42}$$

Hence, by using Corollary 1, every solution of (41) is oscillatory if (42) satisfied.

Remark 1. *Consider the differential equation*

$$\left(e^{l} y'''(l)\right)' + q_0 e^{l} y(l - 2) = 0.$$

This table compares between our criteria and the previous related one:

Theorem 1:	$q_0 > 11.772$.
Theorem 2:	$q_0 > 0.25$.
Corollary 1:	$q_0 > 0.155\,36$.

We notice that Corollary 1 supports the most efficient condition. Thus, our results improve the results in [16,23].

3. Conclusions

We established new oscillation criteria of (1) by finding new properties of positive solutions. Our results improve and extend some of the results in the literature. It is interesting to study differential equations

$$\left(b(l)\left((y(l) + p(l)y(\sigma(l)))^{(n-1)}\right)^{\kappa}\right)' + q(l)y^{\kappa}(g(l)) = 0,$$

where $\sigma(l) \leq l$ and $0 \leq p(l) \leq p_0 < +\infty$.

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