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Supplements Related to Normal π-Projective Hypermodules

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Abstract: In this study, the role of supplements in Krasner hypermodules is examined and related to normal π-projectivity. We prove that the class of supplemented Krasner hypermodules is closed under finite sums and under quotients. Moreover, we give characterizations of finitely generated supplemented and amply supplemented Krasner hypermodules. In the second part of the paper we relate the normal projectivity to direct summands and supplements in Krasner hypermodules.

Keywords: direct summand; normal π-projective hypermodule; supplement subhypermodule; small subhypermodule

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1. Introduction

Hypercompositional algebra, a new branch of abstract algebra, started its development in 1934, when F. Marty introduced the concept of hypergroup as a natural generalization of the concept of group. The law of synthesis of two elements was extended, in the sense that the operation (defined on a group) was substituted with a multivalued operation (called hyperoperation), i.e., the result of the hyperoperation being a subset of the underlying set. As a consequence, new algebraic hypercompositional structures are defined and the properties of the classical structures are conserved, or not, for similar hyperstructures. This is also the case of the modules, extended to hypermodules, introduced firstly by Krasner [1], and known today as Krasner hypermodules. Their additive part is a canonical hypergroup. The fundamental aspects of the theory of hypermodules are very well covered, for example, by the studies of Massouros [2], Nakassis [3], Anvariyeh [4,5], Ameri and Shojaei [6], and Bordbar and Cristea [7–9].

Recently, the concept of smallness in module theory has been transported and investigated by Moniri et al. [10] in the class of hypermodules. Similarities and differences of this concept in both theories have been clearly highlighted and supported by several examples. As it was defined in [11] and then recalled in [12] already in the 1960s, a left R-submodule N of an R-module M, where R is an arbitrary unitary associative ring, is *small* if $N + K = M$, for any R-submodule K of M, implies $K = M$, and it is denoted by $N \ll M$ [13]. An R-module M is called a *hollow* if every proper R-submodule of M is small in M. In a similar way, we may define these two concepts in hypermodule theory, but we must pay attention, as it is explained in [10], to their meaning in a Krasner hypermodule (where the additive part is a canonical hypergroup) and in a general hypermodule having the additive part an arbitrary hypergroup (that can be also non-commutative). In addition, an R-hypermodule M, with the property that the intersection of its two R-subhypermodules is again an R-subhypermodule, is called *supplemented* if for each proper R-subhypermodule N of M there exists a proper R-subhypermodule K of M such that $K + N = M = N + K$ and $N \cap K \ll K$. In a Krasner hypermodule, the intersection of two subhypermodules is always a subhypermodule, while in a general hypermodule this property may not hold for arbitrary subhypermodules, only for closed subhypermodules [10].

In this paper, we aim to obtain more properties of supplements in Krasner R-hypermodules and understand their role related to projective hypermodules, in particular with normal π-projective hypermodules. After a brief introduction on hypermodules, homomorphisms, and supplements in hypermodules, in Section 3, we provide new properties of supplemented Krasner R-hypermodules. We prove that any quotient hypermodule of a supplemented hypermodule is again supplemented (see Theorem 1), and, similarly, the sum of two supplemented hypermodules is supplemented, too (see Theorem 2). In addition, we will provide also a new characterization of the finitely generated supplemented hypermodules (see Theorem 4) and finitely generated amply supplemented hypermodules (Theorems 5 and 6). In Section 4 we define the normal π-projective R-hypermodule and present several properties related to direct summands and supplements. The main results are represented by Proposition 2 and Corollary 2. The concept of normal projectivity has been recently introduced by Ameri and Shojaei [6], using different kinds of epimorphisms defined in the Krasner hypermodule category. Then, Bordbar and Cristea [14] provided their characterization by mean of chains of hypermodules. This study is a step forward in the theory of projective Krasner hypermodules.

2. Preliminaries and Notation

In this section, we briefly recall the main concepts and results related to Krasner hypermodules that we will use throughout this paper. For a better understanding of the topic, we start with some fundamental definitions in hypercompositional algebra presented in several books [15,16] and overview articles [3,17,18]. We refer the reader also to the first chapters of the book [13], containing an up-to-date account on lifting modules that generalize the projective supplemented modules, and to the book [19] for an introduction to module theory.

Hypermodules. Let H be a nonempty set and $\mathcal{P}^*(H)$ be the set of all nonempty subsets of H. The couple $(H,\circ)$ is a *hypergroupoid*, where the hyperoperation on H is a function $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$. For any nonempty subsets X and Y of H, one defines $X \circ Y = \cup_{x \in X, y \in Y} x \circ y$. We simply write $a \circ X$ and $X \circ a$ instead of $\{a\} \circ X$ and $X \circ \{a\}$, respectively, for any $a \in H$ and any nonempty subset X of H. A hypergroupoid $(H,\circ)$ is called a *semihypergroup* if the hyperoperation $\circ$ is associative, i.e., for every $a,b,c \in H$, we have $a \circ (b \circ c) = (a \circ b) \circ c$. A hypergroupoid $(H,\circ)$ is called a *quasihypergroup* if the reproduction law holds, i.e., for every $x \in H$, $x \circ H = H = H \circ x$. If the hypergroupoid $(H,\circ)$ is a semihypergroup and quasihypergroup, then it is called a *hypergroup*. A nonempty subset S of a hypergroup $(H,\circ)$ is called a *subhypergroup* of H if, for every $a \in S$, $a \circ S = S = S \circ a$. A *canonical hypergroup* is a hypergroup $(H,\circ)$ satisfying the following conditions: (i) it is commutative, i.e., for every $a,b \in H$, $a \circ b = b \circ a$; (ii) there exists $e \in H$ such that $\{a\} = (a \circ e) \cap (e \circ a)$ for every $a \in H$ (such an element e is called an *identity* of the hypergroup); (iii) for every $a \in H$ there exists a unique $a^{-1} \in H$ such that $e \in a \circ a^{-1}$ (the element a' is called the *inverse* of a); (iv) for every $a,b,c \in H$, if $c \in a \circ b$, then $a \in c \circ b^{-1}$ and $b \in a^{-1} \circ c$.

An algebraic system $(R,+,\cdot)$ is called a *Krasner hyperring* if

1. $(R,+)$ is a canonical hypergroup;
2. $(R,\cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $a \cdot 0 = 0 = 0 \cdot a$ for any $a \in R$;
3. The multiplication distributes over the addition on both sides, i.e., for any $a,b,c \in R$, $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a$;

while $(R,+,\cdot)$ is called a *general hyperring* (or simply, a hyperring) if

1. $(R,+)$ is a canonical hypergroup with the scalar identity 0_R;
2. $(R,\cdot)$ is a semihypergroup;

3. The multiplication distributes over the addition on both sides.

A hyperring R is called *commutative* if it is commutative with respect to the multiplication. If $a \in a \cdot 1_R \cap 1_R \cdot a$ for every $a \in R$, then the element 1_R is called a *unit element* of the hyperring R.

Now, let R be a hyperring with the identity element 1_R. A *left R-hypermodule* is defined as an algebraic system $(M, +, \circ)$, where the hypergroup $(M, +)$ is endowed with an external multivalued operation $\circ$, i.e., $\circ : R \times M \longrightarrow \mathcal{P}^*(M)$ such that, for every $x, y \in R$ and $a, b \in M$, the following statements hold:

1. $x \circ (a + b) = x \circ a + x \circ b$;
2. $(x + y) \circ a = x \circ a + y \circ a$;
3. $(x \cdot y) \circ a = x \circ (y \circ a)$;
4. $a \in 1_R \circ a$.

Similarly, the concept of *right R-hypermodule* is defined and we say that $(M, +, \circ)$ is an R-hypermodule if it is a left and right one. Some authors call this hypercompositional structure a general hypermodule. A nonempty subset N of an R-hypermodule M is called a *subhypermodule* of M if N is an R-hypermodule under the same hyperoperations of M, and we denote this as $N \leq M$. In other words, N is a subhypermodule of M if and only if $x \circ a \subseteq N$ and $a - b \in N$ for every $x \in R$ and $a, b \in N$ [20]. A hypermodule M having the additive part of a canonical hypergroup is called a *canonical* R-hypermodule if it is a hypermodule over a Krasner hyperring $(R, +, \cdot)$.

If we consider a Krasner hyperring R, then we may endow a canonical hypergroup $(M, +)$ with an external operation $\cdot : R \times M \longrightarrow M$ defined as $(r, m) \longmapsto r \cdot m \in M$. If, for every $x, y \in R$ and $a, b \in M$, the following statements hold:

1. $x \cdot (a + b) = x \cdot a + x \cdot b$;
2. $(x + y) \cdot a = x \cdot a + y \cdot a$;
3. $(x \cdot y) \cdot a = x \cdot (y \cdot a)$;
4. $a = 1_R \cdot a$;
5. $x \cdot 0_M = 0_R$;

then M is called a *Krasner left R-hypermodule*. Similarly, a right Krasner R-hypermodule is defined and it is called a Krasner R-hypermodule (or simply a Krasner hypermodule) if it is both left and right.

Let $\{N_i\}_{i \in I}$ be a family of subhypermodules of an R-hypermodule M. The set $\sum_{i \in I} N_i = \cup \{ \sum_{i \in I} a_i \,|\, a_i \in N_i$ for every $i \in I$ such that $\exists n \in \mathbb{N} : a_i = 0,$ for all but finitely many $i \geq n \}$ is a subhypermodule of M. A nonempty subset J of a commutative hyperring R is called a *hyperideal*, if $x - y \subseteq J$ and $a \cdot x \subseteq J$ for every $a \in R$ and $x, y \in J$. Recall that every hyperideal J of a hyperring R is a subhypermodule of the R-hypermodule R.

Let M be a left Krasner hypermodule over a Krasner hyperring R and K be a subhypermodule of M. Consider the set $\frac{M}{K} = \{ a + K \,|\, a \in M \}$. Then, $\frac{M}{K}$ is a left Krasner hypermodule over R under the hyperoperation defined as $+ : \frac{M}{K} \times \frac{M}{K} \longrightarrow \mathcal{P}^*(\frac{M}{K})$ and the external operation $\odot : R \times \frac{M}{K} \longrightarrow \frac{M}{K}$ defined as $(a + K) + (a' + K) = \{ b + K \mid b \in a + a' \}$ and $x \odot (a + K) = \{ b + K \mid b \in x \cdot a \}$ for every $a, a', b \in M$ and $x \in R$. The Krasner hypermodule $\frac{M}{K}$ is called the *quotient hypermodule* of the hypermodule M. Note that $a + K = K$ if and only if $a \in K$.

A nonzero Krasner R-hypermodule M is called *simple* [20] if the only subhypermodules of M are $\{0_M\}$ and M itself. We denote by $S(M)$ the set of all simple subhypermodules of the Krasner R-hypermodule M.

The following technical result will be often used in the next sections.

Lemma 1 ((Modularity law) [21])**.** *Suppose that M is a Krasner R-hypermodule and A, B, and C are subhypermodules of M such that $B \leq A$. Then, $A \cap (B + C) = B + (A \cap C)$.*

Small subhypermodules. A subhypermodule N of a left Krasner R-hypermodule M is called a *small subhypermodule* of M and denoted by $N \ll M$, if $N + L \neq M$ for every proper subhypermodule L of M. We refer the reader to [21] for basic properties related to small subhypermodules. We recall here some basic properties of small subhypermodules that will be used throughout the paper.

Lemma 2 ([21]). *Let M be a hypermodule and $X \leq Y$ be subhypermodules of M. Then*

(1) $Y \ll M$ if and only if $X \ll M$ and $\frac{Y}{X} \ll \frac{M}{X}$.
(2) Any finite sum of small subhypermodules of M is again small in M.
(3) If Y is a direct summand of M and $X \ll M$, then $X \ll Y$.

A left Krasner R-hypermodule M is called a *hollow* [10] if every proper subhypermodule of M is small in M. Similarly to module theory, a left Krasner R-hypermodule is a hollow if and only if the sum of any of its proper subhypermodules is a proper subhypermodule. Moreover, M is called *local* if it has a proper subhypermodule that contains all proper subhypermodules of M.

Let M be a left Krasner R-hypermodule. We will denote by $Rad(M)$ the sum of all small subhypermodules of M, that is, $Rad(M) = \sum_{L \ll M} L$. If M has no small subhypermodules of M, then we set $Rad(M) = M$. Notice that $Rad(M)$ is always a subhypermodule of the left Krasner R-hypermodule M and M is local if and only if M is hollow and $Rad(M) \neq M$ [10].

Homomorphisms. Let M and $M^{'}$ be two left Krasner R-hypermodules. A function $f : M \longrightarrow M^{'}$ is called a *homomorphism* if for every $a, b \in M$ and $r \in R$, it holds $f(a + b) \subseteq f(a) + f(b)$ and $f(r \circ a) = r \circ f(a)$, while it is called a *strong homomorphism* if $f(a + b) = f(a) + f(b)$ and $f(r \circ a) = r \circ f(a)$. For any subhypermodule N of a left Krasner R-hypermodule M, the image $f(N)$ is a subhypermodule of $M^{'}$ and the kernel $ker(f) = \{ a \in M \mid f(a) = 0_{M'} \}$ is a subhypermodule of M. If $f : M \longrightarrow M'$ is a strong epimorphism, i.e., a strong surjective homomorphism, and $ker(f) \ll M$, then f is called a *small strong epimorphism*. A subhypermodule U of a Krasner R-hypermodule M is called *fully invariant* in M if $\alpha(U)$ is a subhypermodule of U, for every strong endomorphism $\alpha : M \longrightarrow M$.

Supplements. Two subhypermodules N and $N^{'}$ of a left Krasner R-hypermodule M are called *independent* if $N \cap N^{'} = \{0_M\}$, and in this case, their sum $N + N^{'}$ is denoted by $N \oplus N^{'}$ and called *direct sum*. Moreover, a subhypermodule N of M is called a *direct summand* of M if $M = N \oplus K$ for some subhypermodule K of M [21]. A left Krasner R-hypermodule M is called *semisimple*, if its subhypermodules are direct summands in M [20]. As a generalization of semisimple hypermodules, in [10], the class of supplemented hypermodules was introduced. Let M be a left Krasner R-hypermodule and U, V be subhypermodules of M. V is called a *supplement* of U in M if it is a minimal element in the set $\{ L \leq M \mid L + U = M \}$. Then M is called *supplemented* if every subhypermodule of M has a supplement in M [10]. Thus, it is clear that V is a supplement of U in M if and only if $V + U = M$ and $U \cap V \ll V$, i.e., the canonical map $V \longrightarrow \frac{M}{U}$ is a small strong epimorphism. Moreover, U has *amply supplements* in M if, whenever $U + V = M$, V contains a supplement V' of U in M. The left Krasner R-hypermodule M is called *amply supplemented* if every subhypermodule has amply supplements in M. These definitions have been initially introduced in [10] for general hypermodules, and several examples have been illustrated there.

It is clear that semisimple modules and hollow modules are examples of amply supplemented Krasner R- hypermodules. Moreover, a supplemented Krasner R-hypermodule M with zero $Rad(M)$ is semisimple. Thus, we can write the following implications between the three classes of Krasner R-hypermodules:

semisimple hypermodules$\Longrightarrow$ amply supplemented hypermodules$\Longrightarrow$ supplemented hypermodules.

3. Some Results of (Amply) Supplemented Hypermodules

Throughout this paper, we work with left Krasner R-hypermodules, that we briefly call hypermodules.

In this section, some basic properties of (amply) supplemented hypermodules are presented. For a better understanding of the concept, we will start with one example of amply supplemented hypermodule.

Example 1 ([10])**.** *Take the set $R = \{0,1,2,3\}$ equipped with the hyperoperation $+$ and operation $\cdot$ defined as follows:*

$+$	0	1	2	3
ine 0	0	1	2	3
ine 1	1	0, 1	3	2, 3
ine 2	2	3	0	1
ine 3	3	2, 3	1	0, 1
ine				

$$\text{and} \quad r \cdot s = \begin{cases} 2, & \text{if } r, s \in \{2,3\} \\ 0, & \text{otherwise.} \end{cases}$$

Then R is a Krasner hyperring and $M = R$ is a left Krasner R-hypermodule with the proper subhypermodules $\{0\}$, $K = \{0,1\}$, and $L = \{0,2\}$. Since $L + K = M$, it follows that $\{0\}$ is the only small subhypermodule of M. In addition, all subhypermodules are direct summands of M and thus M is amply supplemented.

Recall here a result on the smallness property in quotient hypermodules.

Proposition 1 ([10])**.** *Let M be a hypermodule and $U \subset L$ be subhypermodules of M. Then $\frac{L}{U}$ is a small subhypermodule of $\frac{M}{U}$ if and only if for all subhypermodules K of M the equality $L + K = M$ implies $U + K = M$.*

In the following auxiliary result, we will present some properties of the supplements of a hypermodule and of the set $Rad(M) = \sum_{L \ll M} L$. Notice that very often we make use of the second isomorphism theorem [22].

Lemma 3. *Let M be a hypermodule and K, L two subhypermodules such that L is a supplement of K in M.*

1. *If U is a subhypermodule of L, then $\frac{L}{U}$ is not small in $\frac{M}{U}$.*
2. *If U is a subhypermodule of L and U is a small subhypermodule in M, then U is a small subhypermodule in L.*
3. $Rad(L) = L \cap Rad(M)$,
4. $Rad(\frac{M}{K}) = \frac{Rad(M)+K}{K}$.
5. $Rad(M) = (L + Rad(M)) \cap (K + Rad(M)) = (L \cap Rad(M)) + (K \cap Rad(M))$.

Proof. (1) Let U be a subhypermodule of L. If $\frac{L}{U}$ is small in $\frac{M}{U}$, i.e., $\frac{L}{U} \ll \frac{M}{U}$, then, by Proposition 1, it follows that $K + U = M$, which contradicts with the minimality of L as a supplement of K. Thus, $\frac{L}{U}$ is not small in $\frac{M}{U}$.

(2) Suppose that $U + T = L$, for some subhypermodule T of L. Then $(U + T) + K = L + K = M$. Therefore $U + (T + K) = M$, and since $U \ll M$, it follows that $T + K = M$, with L a supplement of K in M. Thus, by the minimality of L, we have $T = L$. Hence, $U \ll L$.

(3) It is clear that $Rad(L) \subseteq L \cap Rad(M)$. Conversely, let $a \in L \cap Rad(M)$. Since $Rad(M)$ is the sum of all small subhypermodules of M, it follows that $Ra \ll M$. Then, by (2), we obtain $Ra \ll L$, i.e., $a \in Rad(L)$. Thus, $L \cap Rad(M) \subseteq Rad(L)$, and therefore $Rad(L) = L \cap Rad(M)$.

(4) Since L is a supplement of K in M, the canonical map $L \longrightarrow \frac{M}{K}$ is a small strong epimorphism. From $\frac{L}{L \cap K} \cong \frac{M}{K}$, we have that $Rad(\frac{L}{L \cap K}) \cong Rad(\frac{M}{K})$. Since $K \cap L \subseteq Rad(L)$, it follows that $Rad(\frac{L}{L \cap K}) = \frac{Rad(L)}{L \cap K}$. Therefore, every maximal subhypermodule of L contains

$K \cap L$. In addition, we have $\frac{M}{K} = \frac{K+L}{K} \cong \frac{L}{K \cap L}$ which implies $Rad(\frac{M}{K}) = \frac{Rad(L)+K}{K}$. By using the canonical strong epimorphism $M \longrightarrow \frac{M}{K}$, on one side we have $\frac{Rad(M)+K}{K} \leq \frac{Rad(L)+K}{K}$. On the other side, $Rad(L) + K \leq Rad(M) + K$, and therefore $Rad(\frac{M}{K}) = \frac{Rad(M)+K}{K}$.

(5) Let $N = Rad(M)$ and $\psi : M \longrightarrow \frac{M}{K}$ be the strong canonical epimorphism. Since $\psi(Rad(L)) = \dfrac{Rad(L)+K}{K} \cong \dfrac{Rad(L)}{K \cap Rad(L)} = \dfrac{Rad(L)}{K \cap L \cap Rad(M)} = \dfrac{Rad(L)}{K \cap L} = Rad(\dfrac{L}{K \cap L}) \cong Rad(\dfrac{M}{K})$ and knowing (4), it follows that $\dfrac{Rad(M)+K}{K} = \dfrac{Rad(L)+K}{K}$. Thus, we can write

$$N + K = Rad(M) + K = Rad(L) + K = (L \cap N) + K.$$

Then, $N + (N \cap K) = N \cap (N + K) = N \cap [(L \cap N) + K]$, which implies that $N = (L \cap N) + (K \cap N)$, meaning that $Rad(M) = (L \cap Rad(M)) + (K \cap Rad(M))$. It remains to prove the first part of the formula. Again we use the modularity law and we obtain $L \cap (N + K) = L \cap (L \cap N + K) = (L \cap N) + (L \cap K) = L \cap N$. Therefore, $(N + K) \cap (N + L) = N + ((N + K) \cap L) = N + (L \cap N) = N$, meaning that

$$Rad(M) = (L + Rad(M)) \cap (K + Rad(M)).$$

Now the proof is completed. □

Recall from [6] that an R-hypermodule P is *normal A-projective* if, for every strong epimorphism $g \in Hom_R(A, B)$ and every strong homomorphism $f \in Hom_R(P, B)$, there exists $\overline{f} \in Hom_R(P, A)$ such that $g \circ \overline{f} = f$. If P is normal A-projective in the category *Hmod* for every hypermodule A, then P is called a *normal projective* hypermodule. In addition, from [21], we know that, for any hypermodules M and N, a strong epimorphism $g : M \longrightarrow N$ is a small strong epimorphism if and only if for every strong homomorphism f; if $g \circ f$ is a strong epimorphism, then f is a strong epimorphism, too. Moreover, if $f : P \longrightarrow M$ is a small strong epimorphism and P is a projective R-hypermodule, then P is called a *projective cover* of M.

Some fundamental results of supplements related to homomorphisms and projective covers are gathered in the next result.

Lemma 4. *1. In the following commutative diagram, suppose that γ and θ are strong epimorphisms, while α is a small strong epimorphism related to the hypermodules U, V, W, S.*

$$\begin{tikzcd} U \arrow[r, "\beta"] \arrow[d, "\gamma"'] & V \arrow[d, "\theta"] \\ W \arrow[r, "\alpha"'] & S \end{tikzcd}$$

If X is a supplement of $ker(\gamma)$ in U, then $\beta(X)$ is a supplement of $ker(\theta)$ in V.

2. *If X is a subhypermodule of the hypermodule M and $\frac{M}{X}$ has a projective cover, then X has a supplement in the hypermodule M.*
3. *If L is a supplement of K in a hypermodule M and $\eta : M \longrightarrow M$ is a strong endomorphism with $Im(1 - \eta)$ a subhypermodule of K, then $\eta(L)$ is a supplement of K in M.*
4. *If L is a supplement of K in a hypermodule M and X is a subhypermodule of K, then $\frac{L+X}{X}$ is a supplement of $\frac{K}{X}$ in $\frac{M}{X}$.*

Proof. (1) By the hypothesis, we have $X + ker(\gamma) = U$ and $X \cap ker(\gamma) \ll X$. Let us consider the following short exact sequence constructed with the hypermodules and their strong homomorphisms: $X \longrightarrow \frac{U}{ker(\gamma)} \longrightarrow W \longrightarrow S \equiv U \longrightarrow \beta(X) \longrightarrow \frac{V}{ker(\theta)} \longrightarrow S$. Since

$\beta(X) + ker(\theta) = V$ and $\beta(X) \cap ker(\theta) \ll \beta(X)$, it follows that $\beta(X)$ is a supplement of $ker(\theta)$ in V.

(2) Let P be a projective cover of $\frac{M}{X}$, i.e., the function $f : P \longrightarrow \frac{M}{X}$ is a strong epimorphism and P is a projective hypermodule. Then, for the strong epimorphism $h \in Hom_R(M, \frac{M}{X})$, there exists $\beta \in Hom_R(P, M)$ such that $g \circ \beta = f$, i.e., the following diagram commutes.

$$\begin{tikzcd} P \arrow[r, "\beta"] \arrow[d, "id"'] & M \arrow[d, "g"] \\ P \arrow[r, "f"] & \frac{M}{X} \end{tikzcd}$$

Since $ker(id) = \{0_P\}$, it follows that P is a supplement of $\{0_P\} = ker(id)$ in P and by item (1) it results that $\beta(P)$ is a supplement of $X = ker(g)$.

(3) Let L be a supplement of K in the R-hypermodule M. Since $Im(1-\eta)$ is a subhypermodule of K, it follows by item (1) that the following diagram

$$\begin{tikzcd} M \arrow[r, "\eta"] \arrow[d] & M \arrow[d] \\ \frac{M}{K} \arrow[r] & \frac{M}{K} \end{tikzcd}$$

is commutative and $\eta(L)$ is a supplement of K in M.

(4) The statement follows by applying item (1) to the following diagram

$$\begin{tikzcd} L \subset M \arrow[r, "\psi"] \arrow[d] & \frac{M}{X} \arrow[d, "\alpha"] \\ \frac{M}{K} \arrow[r] & \frac{M}{K} \end{tikzcd}$$

where $\psi(L) = \frac{L+X}{X}$ and $Ker\alpha = \frac{K}{X}$. $\square$

The last item of Lemma 4 can be written as follows.

Theorem 1. *Every quotient hypermodule of a supplemented hypermodule is supplemented, too.*

In order to characterize the sum of supplemented hypermodules, we first prove the following auxiliary result.

Lemma 5. *Let M be an R-hypermodule.*

1. *Let N, K, L be three subhypermodules of M such that $N + K + L = M$. If N is a supplement of $K + L$ in M and K is a supplement of $N + L$ in M, then $N + K$ is a supplement of L in M.*
2. *Let N and K be two subhypermodules of M such that N is supplemented. If $N + K$ has a supplement in M, then K has a supplement in M, too.*

Proof. (1) Since N is a supplement of $K + L$ in M, it follows that $(K + L) \cap N \ll N$ and, similarly, $(N + L) \cap K \ll K$. We will prove that $L \cap (N + K) \ll N + K$.

By the modularity law we have $L \cap (N + K) = N + (L \cap K)$ and $K \cap (N + L) = N + (K \cap L)$. Therefore, $L \cap (N + K) \subseteq [N \cap (L + K)] + [K \cap (L + N)]$ and since the sum of small subhypermodules is a small subhypermodule, we have that $L \cap (N + K) \ll N + K$.

(2) Let X be a supplement of $N+K$ in M. Thus, $N+K+X=M$ and $(N+K)\cap X \ll X$. We know that N is supplemented, therefore $N\cap(K+X)$ has a supplement Y in N, i.e., $N\cap(K+X)+Y=N$ and $N\cap(K+X)\cap Y=(K+X)\cap Y \ll Y$. Then we have

$$M=N+K+X=N\cap(K+X)+Y+(K+X)=K+X+Y$$

and

$$K\cap(X+Y)\subseteq[X\cap(K+Y)]+[Y\cap(K+X)]\subseteq[X\cap(K+N)]+[Y\cap(K+X)].$$

Since $X\cap(K+N)\ll X$ and $Y\cap(K+X)\ll Y$, it follows that $K\cap(X+Y)\ll X+Y$ (the sum of small subhypermodules is a small subhypermodule). Thus, $X+Y$ is a supplement of K in M. □

Theorem 2. *The sum of two supplemented hypermodules is supplemented, too.*

Proof. Let M_1 and M_2 be two supplemented hypermodules. We will prove that $M=M_1+M_2$ is supplemented, too. Let U be a subhypermodule of M. Since M_2 is supplemented, it follows that its subhypermodule $(M_1+U)\cap M_2$ has a supplement V in M_2. Then $M=M_1+M_2=M_1+(M_1+U)\cap M_2+V=M_1+U+V$. In addition, since V is a supplement of $(M_1+U)\cap M_2$ in M_2, we have $(M_1+U)\cap V=((M_1+U)\cap M_2)\cap V \ll V$. This means that V is a supplement of M_1+U in M. Since M_1 is supplemented, by Lemma 5 (2), it follows that U has a supplement in M. Therefore, M is supplemented. □

Corollary 1. *If in the exact sequence $0\longrightarrow U\longrightarrow M\longrightarrow \frac{M}{U}\longrightarrow 0$ of hypermodules U and $\frac{M}{U}$ are supplemented and U has a supplement in every subhypermodule X, with $U<X<M$, then the hypermodule M is supplemented.*

Proof. Let V be a subhypermodule of M, $\frac{X}{U}$ be a supplement of $\frac{V+U}{U}$ in $\frac{M}{U}$, and Y be a supplement of U in X. We have $U+Y=X$, $U\cap Y\ll Y$, $\frac{V+U}{U}+\frac{X}{U}=\frac{M}{U}$ and $\frac{V+U}{U}\cap\frac{X}{U}\ll\frac{X}{U}$. Hence $V+U+X=M$, $(V+U)\cap Y\le(V+U)\cap X$, so $\frac{(V+U)\cap Y}{U}\le\frac{(V+U)\cap X}{U}\ll\frac{X}{U}$. Thus, we suppose that there exists a subhypermodule T of Y such that $(V+U)\cap Y+T=Y$. Then $\frac{(V+U)\cap Y}{U}+\frac{U+T}{U}=\frac{X}{U}$. By using $\frac{(V+U)\cap Y}{U}\ll\frac{X}{U}$, it follows that $\frac{U+T}{U}=\frac{X}{U}$. Thus, $U+T=X$. Since Y is a supplement of U in X, we have $T=Y$. Therefore, Y is a supplement of $V+U$ in M, i.e., $Y\cap(V+U)\ll Y$. Then, by Lemma 5 (2), we conclude that V has a supplement in M. □

Recall that an R-hypermodule M is local if and only if M is hollow and $Rad(M)\neq M$ [10]. It can be easily seen here that if a hypermodule M is a hollow and $Rad(M)\neq M$, then M is cyclic and $Rad(M)$ is the largest subhypermodule of M.

Theorem 3. *If M is a direct sum of hollow hypermodules and $Rad(M)$ is small in M, then M is supplemented.*

Proof. Let $M=\oplus_{\gamma\in\Omega}M_\gamma$, with M_γ a hollow hypermodule and $Rad(M)\ll M$. Defining $\overline{M_\gamma}=\frac{M_\gamma+Rad(M)}{Rad(M)}$, we obtain $\overline{M_\gamma}\cong\frac{M_\gamma}{M_\gamma\cap Rad(M)}=\frac{M_\gamma}{Rad(M_\gamma)}$ for every $\gamma\in\Omega$. Since M_γ is a hollow hypermodule for every $\gamma\in\Omega$, it follows that $\frac{M_\gamma}{Rad(M_\gamma)}$ is hollow, too, accordingly with [21], Proposition 2.4. Then $Rad(M)=\oplus_{\gamma\in\Omega}Rad(M_\gamma)$ and $Rad(M)\ll M$. Moreover, since M_γ is a hollow hypermodule and $Rad(M_\gamma)\le M_\gamma$ for every $\gamma\in\Omega$, it follows that $Rad(M_\gamma)\ll M_\gamma$. Thus, $Rad(M_\gamma)\neq M_\gamma$ for every $\gamma\in\Omega$ and $Rad(M_\gamma)$ is the largest subhypermodule of M_γ for every $\gamma\in\Omega$. Therefore, $\frac{M_\gamma}{Rad(M_\gamma)}$ is simple for every $\gamma\in\Omega$. This implies that $\overline{M}=\frac{M}{Rad(M)}=\oplus_{\gamma\in\Omega}\frac{M_\gamma}{Rad(M_\gamma)}\cong\oplus_{\gamma\in\Omega}\overline{M_\gamma}$. Therefore, for an arbitrary subhypermodule U of M, there exists $\Phi\subset\Omega$ with $\overline{M}=(\oplus_{\gamma\in\Phi}\frac{M_\gamma}{Rad(M_\gamma)})\oplus\overline{U}$. Let

$V = \oplus_{\gamma\in\Phi} M_\gamma$. Then, $\overline{M} = \overline{U} \oplus \overline{V}$ and $U \cap V \leq Rad(M)$. So $U \cap V \ll M$. Because $Rad(M)$ is small in M, we have $U \cap V \ll M$ and since V is a direct summand of M, we conclude that $U \cap V \ll V$. Therefore, V is a supplement of U in M. Hence M is supplemented. □

Theorem 4. *For a finitely generated hypermodule M, the following statements are equivalent:*
(a) *M is supplemented.*
(b) *Every maximal subhypermodule of M has a supplement in M.*
(c) *M is a sum of hollow subhypermodules.*

Proof. $(a) \Rightarrow (b)$ This implication is clear.

$(b) \Rightarrow (c)$ Let $S = \sum\{U \leq M \mid U$ is a hollow subhypermodule of $M\}$. Then $S \subseteq M$. Now suppose $S \subset M$. Since M is finitely generated, by Zorn's lemma we know that S is contained in a maximal subhypermodule N of M. By hypothesis (b), N has a supplement K in M, i.e., $M = N + K$ and $N \cap K \ll K$. Since $\frac{M}{N} = \frac{N+K}{N} \cong \frac{K}{N\cap K}$, it follows that $N \cap K$ is a maximal subhypermodule of K and $N \cap K \ll K$. Thus, K is local with the largest subhypermodule $N \cap K$. From $K < S < N$ it follows that $M = N + K = N$, which is a contradiction. Then $S = M$.

$(c) \Rightarrow (a)$ We know that M is finitely generated and $M = \sum_{\gamma\in\Omega} M_\gamma$, with Ω a finite set, where each M_γ is a hollow subhypermodule and $Rad(M) \ll M$. Let K be any proper subhypermodule of M. We can write $\frac{M}{Rad(M)} = \sum_{\gamma\in\Omega} \frac{M_\gamma + Rad(M)}{Rad(M)}$. Since $Rad(M_\gamma) \subseteq M_\gamma \cap Rad(M)$ and $\frac{M_\gamma+Rad(M)}{Rad(M)} \cong \frac{M_\gamma}{M_\gamma\cap Rad(M)}$, these factors are simple or zero. We gain the equation $\frac{M}{Rad(M)} = \oplus_{\theta\in\Omega'} \frac{M_\theta+Rad(M)}{Rad(M)}$, and since $Rad(M) \ll M$, we conclude that $M = \sum_{\theta\in\Omega'} M_\theta$ with local subhypermodules M_θ for any $\theta \in \Omega' \subset \Omega$. Thus, K is contained in a maximal one, and K has a supplement in M, as we saw in Theorem 3, so M is supplemented. □

Example 2 (See [10], Example 2.4)**.** *Let $(\mathbb{Z}_2 \times \mathbb{Z}_4, *, \diamond)$ be a hypermodule over the hyperring $(\mathbb{Z}, \oplus, \odot)$, where $(a,b) * (c,d) = \{(a,b), (c,d))\}$, $n \diamond (a,b) = \{n(a,b)\}$, $n \oplus m = \{n,m\}$ and $n \odot m = \{nm\}$ for all $(a,b), (c,d) \in \mathbb{Z}_2 \times \mathbb{Z}_4$ and $n,m \in \mathbb{Z}$. Since every proper subhypermodule of $(\mathbb{Z}_2 \times \mathbb{Z}_4, *, \diamond)$ is small, it follows that $(\mathbb{Z}_2 \times \mathbb{Z}_4, *, \diamond)$ is a hollow. By using Theorem 4, we conclude that $(\mathbb{Z}_2 \times \mathbb{Z}_4, *, \diamond)$ is also supplemented.*

We conclude this section with some characterizations of amply supplemented hypermodules.

Theorem 5. *For a hypermodule M, the following statements are equivalent.*
(a) *M is amply supplemented.*
(b) *Every subhypermodule N of M is of the form $N = N_1 + N_2$ with N_1 supplemented and $N_2 \ll M$.*
(c) *For every proper subhypermodule N of M, there exists a supplemented proper subhypermodule N_1 of N with $\frac{N}{N_1} \ll \frac{M}{N_1}$.*

Proof. $(a) \Rightarrow (b)$ Let M be an amply supplemented hypermodule and N be a proper subhypermodule of M. Then N has an ample supplement K in M, i.e., $M = K + N$ and there exists a supplement N_1 of K in M which lies in N. It follows that $N \cap (K + N_1) = N \cap M = N$, while by the modular law we have $N \cap (K + N_1) = N_1 + (K \cap N)$. Thus, $N_1 + (K \cap N) = N$. Denote $N_2 = K \cap N$, which is small in M. It remains to be shown that N_1 is supplemented. By the hypothesis for a subhypermodule A of N_1, let L be a supplement of $A + K$ in M that is contained in N_1. Then L is also a supplement of A in N_1 because $L \cap A \ll L$, and from the minimality of N_1, it follows that $L + A = N_1$.

$(b) \Rightarrow (c)$ If $N = N_1 + N_2$ with $N_2 \ll M$, it follows immediately that $\frac{N}{N_1} \ll \frac{M}{N_1}$.

$(c) \Rightarrow (a)$ Let N be a subhypermodule of N. By hypothesis, there exists a supplemented subhypermodule N_1 of N with $\frac{N}{N_1} \ll \frac{M}{N_1}$. It follows that $U + N_1 = M$, and if $N^{'}$ is a supplement of $U \cap N_1$ in N_1, then using the small strong epimorphism $N^{'} \longrightarrow \frac{N_1}{U \cap N_1} \cong \frac{M}{U}$, we conclude that $N^{'}$ is a supplement of U in M and it is contained in N_1. Thus, N has an ample supplement in M. □

Theorem 6. *A finitely generated hypermodule M is amply supplemented if and only if every maximal subhypermodule has ample supplements in M.*

Proof. For an arbitrary ample supplemented hypermodule M, first we prove the following property. If $A + B = M$ and both subhypermodules A and B have ample supplements in M, then so has $A \cap B$. Indeed, from $(A \cap B) + C = M$, it follows that $A + (B \cap C) = M = B + (A \cap C)$, and since M is ample supplemented, we have a supplement $B^{'} < B \cap C$ of A and a supplement $A^{'} < A \cap C$ of B in M. It follows that $A^{'} + B^{'} < C$ is a supplement of $A \cap B$ in M.

Suppose now that M is finitely generated and every maximal subhypermodule has ample supplements in M. By Theorem 4, we know that M is supplemented, hence $\frac{M}{Rad(M)}$ is semisimple. Thereby, for every subhypermodule U of M, the factor hypermodule $\frac{M}{Rad(M)+U}$ is semisimple and finitely generated. Since $\frac{M}{Rad(M)+U}$ is semisimple, it follows that $Rad(\frac{M}{Rad(M)+U}) = 0$. In addition, $Rad(\frac{M}{Rad(M)+U}) = \cap_{i \in I} \frac{X_i}{Rad(M)+U} = \frac{\cap_{i \in I} X_i}{Rad(M)+U}$ for every maximal subhypermodule $\frac{X_i}{Rad(M)+U}$ of $\frac{M}{Rad(M)+U}$. Thus, $Rad(M) + U = \cap_{i \in I} X_i$ for all maximal subhypermodules X_i of M. Since $\frac{M}{Rad(M)+U}$ is finitely generated, there exists a finite subset $I^{'}$ of I such that $Rad(M) + U = \cap_{i \in I^{'}} X_i$. Thus, by the first part of the proof, we conclude that $Rad(M) + U$ has ample supplements in M, so also U. □

4. Normal π-Projective Hypermodules

The aim of this section is to introduce the notion of normal π-projective hypermodule, to find its properties related to direct summands and supplements, and to provide a relationship between direct summands and supplements for this particular case of hypermodules.

Definition 1. *An R-hypermodule M is called* normal π-projective *if for every pair (U, V) of subhypermodules of M satisfying $U + V = M$, there exists a strong homomorphism $\eta : M \longrightarrow M$ with $Im(\eta) \leq U$ and $Im(1 - \eta) \leq V$, where 1 denotes the identity strong homomorphism of M.*

Subhypermodules U and V are called normal mutually-projective *if U is normal V-projective and V is normal U-projective [6].*

Lemma 6. *For a normal π-projective hypermodule M, the following statements hold:*

1. *If $U + V = M$ and U is a direct summand in M, there exists a subhypermodule $V^{'}$ of V with $U \oplus V^{'} = M$.*
2. *If $U + V = M$ and U and V are direct summands in M, then so is $U \cap V$.*
3. *If $U \oplus V = M$ and $\alpha : U \longrightarrow V$ is a strong homomorphism with a direct summand $Im(\alpha)$ in V, then $ker(\alpha)$ is a direct summand in U.*
4. *If $U \oplus V = M$ and a subhypermodule $U^{'}$ of U exists such that $\frac{U}{U^{'}}$ is isomorphic to a direct summand in V, then $U^{'}$ is a direct summand in U.*

Proof. (1) Let $M = U + V$ and $M = U \oplus X$ for some subhypermodule X of M. Since M is a normal π-projective hypermodule, there exists a strong homomorphism $\eta : M = V + U \longrightarrow M$ with $Im(1 - \eta) \leq U$ and $Im(\eta) \leq V$. Therefore $\eta(X) \leq \eta(M) \leq V$, meaning that $\eta(X)$ is a subhypermodule of V. Then, it follows that $M = U \oplus \eta(X)$. Thus, there exists a subhypermodule $V^{'} = \eta(X)$ of V with $U \oplus V^{'} = M$.

(2) Suppose that $U + V = M$ and U and V are direct summands in M. Then, by (1), there exist a subhypermodule $U^{'}$ of U and a subhypermodule $V^{'}$ of V such that $M = U \oplus V^{'} = U^{'} \oplus V$. It follows that $M = (U \cap V) \oplus (U^{'} + V^{'})$.

(3) Suppose that $U \oplus V = M$ and $Im(\alpha)$ is a direct summand in V. Then there exists a subhypermodule V' of V such that $Im(\alpha) \oplus V^{'} = V$. Let $A = U + V^{'}$ and $B = \{ u + \alpha(u) \mid u \in U \}$. Thus $M = A + B = A \oplus Im(\alpha) = B \oplus V$ and $A \cap B = ker(\alpha)$, so applying (2), we obtain that $ker(\alpha)$ is a direct summand in M and also in U.

(4) Suppose that $U \oplus V = M$, and $\frac{U}{U'}$ is isomorphic to a direct summand in V, where $U^{'}$ is a subhypermodule of U. Therefore there exists a strong injective homomorphism $\beta : \frac{U}{U'} \longrightarrow V$ such that $Im(\beta)$ is a direct summand in V. Now consider the strong canonical epimorphism $\pi : U \longrightarrow \frac{U}{U'}$ and let $\alpha = \beta \circ \pi : U \longrightarrow V$. By (3), we obtain that $ker(\alpha) = U^{'}$ is a direct summand in U. □

Example 3. *Let I and J be right hyperideals in a hyperring R, with $I \subset J \subseteq I^{'}$, where $I^{'}$ is the intersection of all maximal hyperideals containing I. Consider the hypermodule $M := \frac{R}{I} \times \frac{R}{J}$ and the subhypermodules $A = R \cdot (1,0)$, $C = R \cdot (1,1)$ and $C = R \cdot (0,1)$. Since $I \subset J$, it follows immediately that $M = A + B = A \oplus C = B \oplus C$ and $A \cap B = \{0_R\} \cdot \frac{J}{I}$. Because $J \subseteq I^{'}$, it follows that $\frac{J}{I} \subseteq Rad(\frac{R}{I})$, thus $A \cap B \ll M$ and therefore $A \cap B \neq \{0_M\}$ is not a direct summand in M. Moreover, the subhypermodules A and B are mutual supplements in M, that is $M = A + B$, $A \cap B \ll B$ and $A \cap B \ll A$.*

Lemma 7. *If $M = U \oplus V$ is a normal π-projective hypermodule, then the subhypermodules U and V are normal π-projective, too. In addition, they are normal mutually-projective.*

Proof. To show the normal π-projectivity of U, suppose that $X + Y = U$, where X and Y are subhypermodules of U. Since $X + (Y + V) = M$ and M is a normal π-projective hypermodule, it follows that there exists a strong endomorphism α of M such that $Im(\alpha)$ is a subhypermodule of X and $Im(1 - \alpha)$ is a subhypermodule of $Y + V$. This induces a map $\eta : U \longrightarrow U$ defined by $\eta(u) = \alpha(u)$, for each $u \in U$. Then we have $Im(\eta) = \alpha(U) \leq X$ and $Im(1 - \eta) = Im(1 - \alpha) \leq Y$, since $U \cap V = \{0\}$. Therefore, U is a normal π-projective subhypermodule, and similarly we can prove the assertion for V.

It remains to be proved that V is U-projective. For this, for an arbitrary hypermodule Q, consider an arbitrary strong epimorphism $\beta : U \longrightarrow Q$ and an arbitrary strong homomorphism $\Phi : V \longrightarrow Q$. Therefore $Y = \{ u - v \mid u \in U, v \in V \text{ and } \beta(u) = \Phi(v) \}$ is a subhypermodule of M such that $U + Y = M$. Hence, since M is a π-projective hypermodule, a strong endomorphism α of M exists such that $Im(\alpha) \leq U$ and $Im(1 - \alpha) \leq Y$. Therefore, the map $\gamma : V \longrightarrow U$ induced by α, i.e., $\gamma(v) = \alpha(v)$, for any $v \in V$, the equality $\beta\gamma = \Phi$ holds. □

Lemma 8. *For a normal π-projective hypermodule M, the following statements hold:*

(1) *If $U + V = M$ and U has a supplement in M, then U has a supplement contained in V.*
(2) *If $U + V = M$ and U and V have supplements in M, then $U \cap V$ also has a supplement in M.*
(3) *If $U + V = M$ and V is a fully invariant subhypermodule in M, then every supplement of U lies in V.*
(4) *If U and V are mutual supplements in M, then $M = U \oplus V$.*

Proof. (1) By hypothesis, the hypermodule $M = U + V$ is π-projective, so there exists a strong endomorphism α of M such that $Im\alpha \leq V$ and $Im(1 - \alpha) \leq U$. Since U has a supplement W in M, by Lemma 4 (3) it follows that $\alpha(W)$ is a supplement of U and $\alpha(W) \leq V$.

(2) Accordingly with point (1), the subhypermodule U has a supplement $V^{'}$ in M, with $V^{'} \leq V$, and the subhypermodule V has a supplement $U^{'}$ in M, with $U^{'} \leq U$.

Therefore, $M = U + V' = V + U'$, with $U \cap V' \ll V'$ and $V \cap U' \ll U'$. By the modularity law, we can write $U = U \cap M = U \cap (V + U') = (U \cap V) + U'$ and similarly, $V = (U \cap V) + V'$. Therefore,

$$M = U + V = [(U \cap V) + U'] + [(U \cap V) + V'] = U \cap V + (U' + V')$$

and

$$\begin{aligned}(U' + V') \cap (U \cap V) &= [(U' + V') \cap U] \cap V \\ &= (U' + V' \cap U) \cap V \\ &= (U' \cap V) + (V' \cap U) \\ &\ll U' + V'.\end{aligned}$$

It follows that $U' + V'$ is a supplement of $U \cap V$ in M.

(3) Let W be a supplement of U in M. Since M is a normal π-projective hypermodule, there exists a strong endomorphism $\alpha : M = W + U \longrightarrow M$, with $Im(1 - \alpha) \leq W$ and $Im(\alpha) \leq U$. Therefore, taking $\eta = 1 - \alpha$, we obtain that $\eta(V) = W \leq V$, since V is a fully invariant subhypermodule of M.

(4) Let U and V be mutual supplements in M. It is enough to show that $U \cap V = 0$, since clearly, $M = U + V$. Consider the strong epimorphisms $\alpha : U \times V \longrightarrow \frac{M}{U} \times \frac{M}{V}$, defined by $\alpha(u, v) = (v + U, u + V)$, for all $(u, v) \in U \times V$, $\beta : U \times V \longrightarrow M$, defined as $\beta(u, v) = u + v$, for all $(u, v) \in U \times V$ and $\pi : M \longrightarrow \frac{M}{U} \times \frac{M}{V}$, with the definition law $\pi(m) = (m + U, m + V)$, for all $m \in M$. Since

$$\begin{aligned}(\pi\beta)(u, v) &= \pi(u + v) = ((u + v) + U, (u + v) + V) = (v + U, u + V) \\ &= \alpha(u, v),\end{aligned}$$

it follows that the following diagram

$$\begin{tikzcd} U \times V \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \frac{M}{U} \times \frac{M}{V} \\ M = U + V \arrow[ur, dashed, "\pi"'] & \end{tikzcd}$$

is commutative. It can be seen that $ker(\alpha) = (U \cap V) \times (U \cap V)$. Since $U \cap V \ll U$ and $U \cap V \ll V$, we obtain that $ker(\alpha)$ is small in $U \times V$. It means that α is a small strong epimorphism. Since M is normal π-projective, there exists a strong homomorphism $\eta : M \longrightarrow M$ with $Im(\eta) \leq U$ and $Im(1 - \eta) \leq V$. Let $f : M \longrightarrow U \times V$ be defined by $f(m) = (\eta(m), (1 - \eta)(m))$, for all $m \in M$. Then $(\beta f)(m) = \beta(f(m)) = \beta(\eta(m), (1 - \eta)(m)) = \eta(m) + (1 - \eta)(m) = m = I_M(m)$ and so β splits. It means that $ker(\beta)$ is a direct summand of $U \times V$. Therefore $ker(\beta) \leq ker(\alpha)$. It follows that $ker(\beta) \ll U \times V$ and so $ker(\beta) = 0$. Hence $U \cap V = 0$. □

Proposition 2. *For a normal π-projective hypermodule M, the following assertions are equivalent.*

1. *If $U + V = M$ and $U \cap V$ has a supplement in M, then U and V have a supplement in M, too.*
2. *If $U + V = M$ and $U \cap V$ has a supplement in M, there exist $U' \leq U$ and $V' \leq V$ with $U' \oplus V' = M$.*
   Moreover, if $Rad(M) \ll M$, then these two assertions are further equivalent to the next three, where $\overline{M} = \frac{M}{Rad(M)}$.
3. *If $U < M$ and $\overline{U}$ is a direct summand in $\overline{M}$, then U has a supplement in M.*
4. *Every direct summand of $\overline{M}$ is the image of a direct summand in M.*
5. *Every decomposition of $\overline{M}$ is induced by a decomposition of M.*

Proof. (1) $\Rightarrow$ (2) By hypothesis, U has a supplement in M. Thus, accordingly to Lemma 6 (1), there exists $V' \leq V$ such that $U + V' = M$ and $U \cap V' \ll V'$, so U has a supplement V' in M. Similarly, V' has a supplement $U' \leq U$, and therefore $U' \oplus V' = M$ by Lemma 8 (4).

(2) $\Rightarrow$ (1) Let $U + V = M$ and W be a supplement of $U \cap V$ in M. Therefore, $M = (U \cap V) + W$. Based on the modularity law, we can write $V = M \cap V = [(U \cap V) + W] \cap V = (U \cap V) + (V \cap W)$ and similarly, $U = (U \cap V) + (U \cap W)$. Therefore,

$$
\begin{aligned}
M &= U + V = [(U \cap V) + (V \cap W)] + [(U \cap V) + (U \cap W)] \\
&= U \cap V + V \cap W + U \cap W \\
&= U + (V \cap W).
\end{aligned}
$$

Since W is a supplement of $U \cap V$ in M, it follows that $(U \cap V) \cap W = U \cap (V \cap W)$ is a small subhypermodule of M. By hypothesis, there exist subhypermodules $U' \leq U$ and $V' \leq V \cap W$ such that $U' \oplus V' = M$. Now it is clear that $M = U + V' = V + U'$. Since $U \cap V' \leq U \cap (V \cap W)$ is a small subhypermodule of M and V' is a direct summand of M, we obtain that $U \cap V'$ is a small subhypermodule of V'. Hence V' is a supplement of U in M. In the same way, it can be shown that U' is a supplement of V in M.

Suppose now that $Rad(M) \ll M$.

(2) $\Rightarrow$ (5) If $\overline{U} \oplus \overline{V} = \overline{M}$ with $Rad(M) \leq U$ and $V \leq M$, then there exist $U' \leq U$ and $V' \leq V$ with $U' \oplus V' = M$ by the hypothesis, and hence it follows that $\overline{U'} = \overline{U}$ and $\overline{V'} = \overline{V}$.

(5) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (3) For $\overline{W} \oplus \overline{U} = \overline{M}$, there exists a direct summand V of M with $\overline{V} = \overline{W}$ by the hypothesis. It follows that V is a supplement of U in M.

(3) $\Rightarrow$ (1) Let $U + V = M$ and W be a supplement of $U \cap V$ in M. It follows for $V_1 := V \cap W$ that $U + V_1 = M$ and $U \cap V_1 \subseteq Rad(M)$. Because M is normal π-projective and $Rad(M)$ is a fully invariant subhypermodule in M, it can be shown that $(U + Rad(M)) \cap (V_1 + Rad(M)) = Rad(M)$, that is, $\overline{U} \oplus \overline{V_1} = \overline{M}$, so U has a supplement in M by the hypothesis. Similarly, the property holds for V. $\square$

As a direct consequence of Proposition 2, we state the following necessary and sufficient condition for a normal π-projective hypermodule with small radical to be supplemented.

Corollary 2. *A normal π-projective hypermodule M with small radical is supplemented if and only if the quotient hypermodule $\overline{M} = \frac{M}{Rad(M)}$ is semisimple and every direct summand of $\overline{M}$ is the image of a direct summand in M.*

5. Conclusions

In classical algebra there is a unique concept of module over a ring, while in hypercompositional algebra we must distinguish between the general hypermodule and the Krasner hypermodule, depending on their additive structure: if the additive part is a canonical hypergroup, then we talk about a Krasner hypermodule. Thus, some properties hold only in Krasner hypermodules and not in general ones, such as, for example, the following one. The sum of two arbitrary Krasner subhypermodules is always a Krasner subhypermodule, whereas, the sum of subhypermodules of a general hypermodule may not be a subhypermodule. As a consequence, $Rad(M)$, which is the sum of all small subhypermodules of a hypermodule M (a general one or a Krasner hypermodule), plays a fundamental role in the characterization of hollow hypermodules. These are hypermodules with the property that every subhypermodule is small.

In this article, we have focused on Krasner hypermodules and in particular we have related the notions of supplement and direct summand to normal projectivity. Especially, we have proved that the class of supplemented Krasner hypermodules is closed under finite sums and under quotients. In addition, we have showed that a finitely generated Krasner hypermodule is supplemented if and only if it is a sum of hollow subhypermodules. Some characterizations of amply supplemented Krasner subhypermodules have been provided. One of them says that a finitely generated hypermodule M is amply supplemented if and only if every maximal subhypermodule has ample supplements in M. After presenting some fundamental properties of normal π-projective hypermodules related to the behavior of direct summands and supplements, we have concluded our study with a necessary and sufficient condition for a normal π-projective hypermodule M with small radical $Rad(M)$ to be supplemented.

We believe that this study could open new lines of research, one being related with embeddings. It would be useful to know that any Krasner R-hypermodule is embedding in a normal π-projective hypermodule, because then we can easily work with the characterizations provided in this article for normal π-projective hypermodules. Another future research idea could be related with the category of Krasner R-hypermodules. If we consider a normal π-projective R-hypermodule M with strong endomorphism hyperring $S = End(M)$, then we may ask about the relationship between the class $hom_R^S(M, N)$ of all strong R-homomorphisms from M to an arbitrary subhypermodule N and S as an S-subhypermodule.

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