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A Note on the Laguerre-Type Appell and Hypergeometric Polynomials

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Abstract: The Laguerre derivative and its iterations have been used to define new sets of special functions, showing the possibility of generating a kind of parallel universe for mathematical entities of this kind. In this paper, we introduce the Laguerre-type Appell polynomials, in particular, the Bernoulli and Euler case, and we examine a set of hypergeometric Laguerre–Bernoulli polynomials. We show their main properties and indicate their possible extensions.

Keywords: Laguerre-type exponentials; Laguerre-type derivative; Appell polynomials; hypergeometric polynomials and numbers

MSC: 11B83; 33B10; 11B68; 33C15



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1. Introduction

Special functions and special number sequences are not only interesting from a theoretical point of view, but are also widely used in physical applications (electrodynamics, classical mechanics and quantum mechanics), engineering and applied mathematics, life sciences and many other fields of science. The vast majority of special functions, including elliptic integrals, beta functions, the incomplete gamma function, Bessel functions, Legendre functions, classic orthogonal polynomials, Kummer confluent functions, etc., are represented in terms of hypergeometric functions, through the introduction of suitable parameters. Many multivariate generalizations of hypergeometric functions have been studied in the literature, including via an extension of the Pochhammer symbol [1–9].

In the field of polynomial functions, the literature on Bernoulli polynomials and numbers is vast [10], since they frequently appear in mathematics. Along with the Stirling numbers, they play a central role in number theory. The Bernoulli numbers arose from the process of summing powers of integers; however, this sequence of numbers occurs surprisingly often in many areas of mathematics, such as the Taylor expansion in a neighborhood of the origin of circular and hyperbolic tangent and cotangent functions, and the residual term of the Euler–MacLaurin quadrature rule. Furthermore, there are intimate connections with the Riemann zeta function and even with Fermat's last theorem.

As D.E. Smith noted [11], among all the sequences of numbers “*there is hardly a species so important and so generally applicable as Bernoulli numbers.*”

Generalizations of the classical Bernoulli polynomials were previously considered in [12–15]. The hypergeometric Bernoulli polynomials have been introduced and recently studied in several articles [16–20], in connection with generalizations of the Riemann Zeta function [21–24].

In this paper, recalling the Laguerre-type derivative and the possibility of constructing special functions generated by the corresponding differential isomorphism (see [25] and the references therein), we examine the case of the Laguerre-type Appell polynomials and

consider, in particular, their extensions to the case of hypergeometric Bernoulli polynomials, restricting ourselves to the case of the first-order Laguerre derivative.

It is worth noting that several applications have been implemented using the Laguerre derivative, in connection with population dynamics [25].

In Section 2 we recall the Laguerre-type exponentials and the differential isomorphism producing the relevant special functions. Then we consider, in particular, the case of Laguerre-type Appell polynomials (Section 3), including the Bernoulli and Euler cases.

Lastly, in Section 4, we introduce the Laguerre-type hypergeometric polynomials, considering only the simplest cases, that is, for $k = 1$ and $k = 2$.

Of course, several further extensions could be made, considering higher values of k , or iterating the isomorphism described in [26] to higher-order Laguerre derivatives, as is recalled in the Conclusions, but this provides no further information, since the methodology remains essentially the same.

2. Laguerre-Type Exponentials and Special Functions

The *Laguerre derivative*, is defined by

$$D_L := Dx D = D + x D^2, \quad (1)$$

where $D = D_x = d/dx$. In [25] it is shown that, for all complex numbers a , it results in:

$$D_L e_1(ax) = a e_1(ax), \quad (2)$$

where $e_1(x) = \sum_{k=0}^{\infty} x^k / (k!)^2$ is the first-order Laguerre-type exponential [27,28].

The above property can be iterated when, defining Laguerre-type exponentials of higher order, called *L-exponentials*. In general, (see [28], Theorem 2.2).

Theorem 1. *The function*

$$e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}} \quad (3)$$

is an eigenfunction of the operator

$$\begin{aligned} D_{nL} &:= Dx \cdots Dx Dx D = D(xD + x^2 D^2 + \cdots + x^n D^n) \\ &= S(n+1, 1)D + S(n+1, 2)x D^2 + \cdots + S(n+1, n+1)x^n D^{n+1}, \end{aligned} \quad (4)$$

where $S(n+1, k)$, $k = 1, 2, \dots, n+1$ denotes the Stirling number of the second kind. That is, for every complex number a , it results in

$$D_{nL} e_n(ax) = a e_n(ax). \quad (5)$$

Remark 1. Note that the function $e_n(x)$ gives back the classical exponential when $n = 0$. Therefore, we put $e_0(x) := e^x$, $D_{0L} := D$, and for $n = 1$ we find $D_{1L} := D_L$.

Remark 2. The operator $D_{nL} := Dx \cdots Dx Dx D = D(xD + x^2 D^2 + \cdots + x^n D^n) = S(n+1, 1)D + S(n+1, 2)x D^2 + \cdots + S(n+1, n+1)x^n D^{n+1}$, where the $S(n, k)$ coefficients are the second kind of Stirling numbers, is a particular case of the hyper-Bessel differential operators introduced in [29] (putting $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 1$). The Bessel-type differential operators of arbitrary order n were considered by I. Dimovski, in 1966 [30] and later studied in 1994 by V. Kiryakova (see [31] and the references therein). See also [32].

In [26] it is proven that, in the space \mathcal{A}_x of analytic functions having the same radius of convergence, the correspondence

$$D \rightarrow D_L, \quad x \cdot \rightarrow D_x^{-1}, \quad (6)$$

where

$$D_x^{-1}f(x) = \int_0^x f(\xi) d\xi, \quad D_x^{-n}f(x) = \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1} f(\xi) d\xi, \quad (7)$$

introduces isomorphism of topological vector spaces, denoted by \mathcal{T}_x , (preserving differentiation), defined as

$$\mathcal{T}_x(x^n) = D_x^{-n}(1) = \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1} d\xi = \frac{x^n}{n!}. \quad (8)$$

This kind of isomorphism is widely used in operational calculus and differential equations also under the name of the transmutation or similarity operator, since it transforms operators and eigenfunctions into each other.

In this isomorphism we have the correspondences

$$\mathcal{T}_x(e^x) = \sum_{k=0}^{\infty} \frac{\mathcal{T}_x(x^k)}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} = e_1(x)$$

and in general

$$\mathcal{T}_x^m(e^x) = \sum_{k=0}^{\infty} \frac{\mathcal{T}_x(x^k)}{(k!)^m} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{m+1}} = e_m(x).$$

Correspondingly, the derivative operator is transformed into

$$\begin{aligned} D_L &= Dx D, & D_{2L} &= D_L D_x^{-1} D_L = Dx Dx D, \\ D_{3L} &= D_L D_x^{-1} D_L D_x^{-1} D_L = Dx Dx Dx D, \end{aligned} \quad (9)$$

and so on.

The L -exponentials of higher order are obtained by an iterative application of the considered differential isomorphism.

Furthermore, the Hermite polynomial $H_n^{(1)}(x, y) := (x-y)^n$ becomes, under the \mathcal{T}_x transformation, the Laguerre polynomial $\mathcal{L}_n(x, y)$, since

$$\mathcal{T}_x H_n^{(1)}(x, y) = \mathcal{L}_n(x, y) := n! \sum_{r=0}^n \frac{(-1)^r y^{n-r} x^r}{(n-r)!(r!)^2},$$

and many other applications can be derived.

The above method has been exploited to define the Laguerre-type special functions, and introduced and studied in several articles, including for extensions to the multivariate case [33]. Applications were shown in [25,26].

Remark 3. Note that the operator D_{nL} is different from $(DxD)^n$. In fact, according to Viskov [34], it results in

$$(DxD)^n = D^n x^n D^n,$$

and using Leibniz's rule, we find the expression:

$$\begin{aligned} (DxD)^n &= D^n (x^n D^n) = \sum_{k=0}^n \binom{n}{k} D^{n-k} x^n D^{n+k} \\ &= \sum_{k=0}^n \left[\binom{n}{k} \right]^2 (n-k)! x^k D^{n+k} = \sum_{k=0}^n \frac{n!}{k!} \binom{n}{k} x^k D^{n+k}. \end{aligned}$$

3. The Laguerre-Type Appell Polynomials

Definition 1. The Laguerre-type Appell polynomials (shortly L -Appell polynomials) ${}_L a_n(x)$ are defined by means of the exponential generating function [1]:

$$A(t) e_1(xt) = \sum_{n=0}^{\infty} {}_L a_n(x) \frac{t^n}{n!}, \quad (10)$$

where, as in the classic case, $A(t) = \sum_{k=0}^{\infty} a_k t^k$ is a formal power series with complex coefficients a_k , ($k = 0, 1, \dots$) and $a_0 \neq 0$.

By applying the Laguerre operator to both sides of Equation (10) and recalling the eigenvalue property (2) of this operator, one finds

$$D_L[{}_L a_n(x)] = n[{}_L a_{n-1}(x)]. \quad (11)$$

We recall that the Appell polynomials are used in the field of differential operator, because, considering, in the complex field, the expansion $f(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$ and the differential operator $A(D) := \sum_{k=0}^{\infty} a_k D^k$ associated with $A(t)$, the formal solution of the equation is

$$A(D)y(z) = f(z), \quad \text{is given by } y(z) = \frac{f(z)}{A(D)},$$

and it turns out that can be represented as [35]

$$y(z) = \sum_{n=0}^{\infty} c_n a_n^*(z),$$

where $a_n^*(z)$ are the Appell polynomials defined by the generating function

$$\frac{e^{zt}}{A(t)} = \sum_{n=0}^{\infty} a_n^*(z) \frac{t^n}{n!}.$$

Now, the same technique can be used dealing with the Laguerre-types Appell polynomials, since by exploiting the differential isomorphism defined in Equation (8), we can consider the operator $A({}_L D) := \sum_{k=0}^{\infty} a_k {}_L D^k$ and the equation

$$A({}_L D)y(z) = f(z), \quad \text{so that } y(z) = \frac{f(z)}{A({}_L D)},$$

so that the formal solution is given by

$$y(z) = \sum_{n=0}^{\infty} c_n {}_L a_n^*(z),$$

where $a_n^*(z)$ are the Appell polynomials defined by the generating function

$$\frac{e_1(zt)}{A(t)} = \sum_{n=0}^{\infty} a_n^*(z) \frac{t^n}{n!}.$$

3.1. Basic Definitions

Definition 2. The Laguerre-type Bernoulli polynomials (shortly L -Bernoulli polynomials) are defined by the exponential generating function

$$\frac{t e_1(xt)}{e^t - 1} = \sum_{n=0}^{\infty} {}_L B_n(x) \frac{t^n}{n!},$$

and the Laguerre-type Euler polynomials (shortly L-Euler polynomials) by

$$\frac{2e_1(xt)}{e^t + 1} = \sum_{n=0}^{\infty} {}_L E_n(x) \frac{t^n}{n!}.$$

3.2. Laguerre-Type Bernoulli Polynomials

The first few L-Bernoulli polynomials are as follows (Figure 1).

$${}_L B_0(x) = 1$$

$${}_L B_1(x) = x - \frac{1}{2}$$

$${}_L B_2(x) = \frac{x^2}{2} - x + \frac{1}{6}$$

$${}_L B_3(x) = \frac{x^3}{6} - \frac{3}{4}x^2 + \frac{1}{2}x$$

$${}_L B_4(x) = \frac{x^4}{24} - \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{30}$$

$${}_L B_5(x) = \frac{x^5}{120} - \frac{5}{48}x^4 + \frac{5}{18}x^3 - \frac{1}{6}x$$

$${}_L B_6(x) = \frac{x^6}{720} - \frac{1}{40}x^5 + \frac{5}{48}x^4 - \frac{1}{4}x^2 + \frac{1}{42}$$

$${}_L B_7(x) = \frac{x^7}{5040} - \frac{7}{1440}x^6 + \frac{7}{240}x^5 - \frac{7}{36}x^3 + \frac{1}{6}x$$

$${}_L B_8(x) = \frac{x^8}{40320} - \frac{1}{1260}x^7 + \frac{7}{1080}x^6 - \frac{7}{72}x^4 + \frac{1}{3}x^2 - \frac{1}{30}$$

$${}_L B_9(x) = \frac{x^9}{362880} - \frac{1}{8960}x^8 + \frac{1}{840}x^7 - \frac{7}{200}x^5 + \frac{1}{3}x^3 - \frac{3}{10}x$$

Note that, as the operator $DxD^0 = Id$, the identity operator, we have $\forall n \in \mathbb{N} - \{0\}$, ${}_L B_n(0) = {}_L B_n = B_n$, that is.

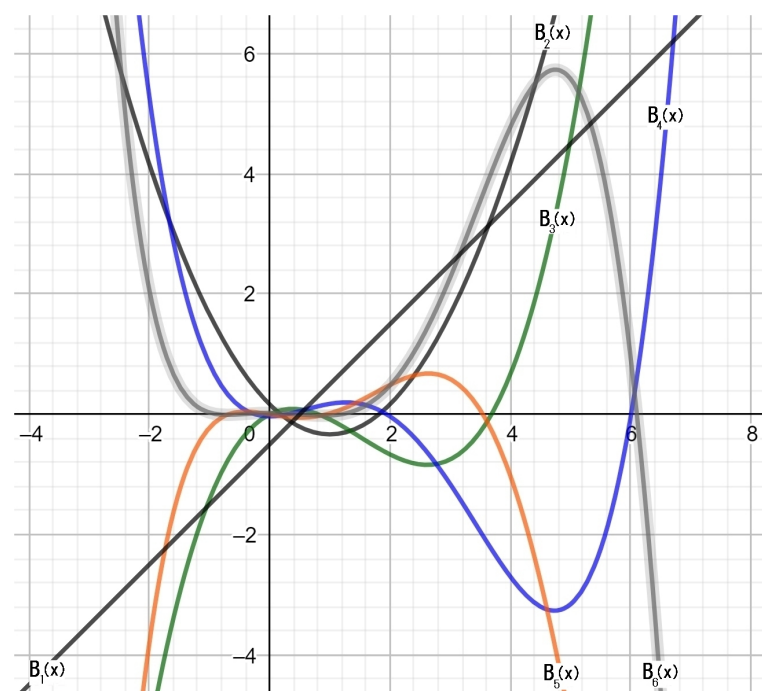


Figure 1. Graphs of polynomials ${}_L B_n(x)$ for $n = 1, 2, \dots, 6$.

The L-Bernoulli numbers are the same as the ordinary Bernoulli numbers.

Although this article is mainly focused on Bernoulli-type polynomials, it should be noted that many extensions can be made, for example, by considering the Genocchi polynomials or the polynomials of Apostol-Bernoulli polynomials of order α , studied in [36], which are defined by means of the exponential generating function

$$\left(\frac{t e^{xt}}{\lambda e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^\alpha(x; \lambda) \frac{t^n}{n!}, \quad (|t + \log \lambda| < 2\pi; \quad 1^\alpha := 1),$$

where, denoting by $B_n^\alpha(x)$ the Bernoulli-Apostol polynomials [37], it results in

$$B_n^\alpha(x) = \mathcal{B}_n^\alpha(x; 1), \quad \text{and} \quad \mathcal{B}_n^\alpha(\lambda) := \mathcal{B}_n^\alpha(0; \lambda),$$

and $\mathcal{B}_n^\alpha(\lambda)$ are the Apostol-Bernoulli numbers of order α , generalizing the classical ones.

The Laguerre-type Apostol-Bernoulli polynomials of order α are obtained by means of the exponential generating function

$$\left(\frac{t e_1(xt)}{\lambda e_1(t) - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} {}_L\mathcal{B}_n^\alpha(x; \lambda) \frac{t^n}{n!}, \quad (|t + \log \lambda| < 2\pi; \quad 1^\alpha := 1),$$

Similar definitions for the Apostol-Euler numbers of order α can be found in the same article [36].

3.3. Laguerre-Type Euler Polynomials

The first few L-Euler polynomials are as follows (Figure 2).

$${}_L E_0(x) = 1$$

$${}_L E_1(x) = x - \frac{1}{2}$$

$${}_L E_2(x) = \frac{x^2}{2} - x$$

$${}_L E_3(x) = \frac{x^3}{6} - \frac{3}{4}x^2 + \frac{1}{4}$$

$${}_L E_4(x) = \frac{x^4}{24} - \frac{1}{3}x^3 + x$$

$${}_L E_5(x) = \frac{x^5}{120} - \frac{5}{48}x^4 + \frac{5}{4}x^2 - \frac{1}{2}$$

$${}_L E_6(x) = \frac{x^6}{720} - \frac{1}{40}x^5 + \frac{5}{6}x^3 - 3x$$

$${}_L E_7(x) = \frac{x^7}{5040} - \frac{7}{1440}x^6 + \frac{35}{96}x^4 - \frac{21}{4}x^2 + \frac{17}{8}$$

$${}_L E_8(x) = \frac{x^8}{40320} - \frac{1}{1260}x^7 + \frac{7}{60}x^5 - \frac{14}{3}x^3 + 17x$$

$${}_L E_9(x) = \frac{x^9}{362880} - \frac{1}{8960}x^8 + \frac{7}{240}x^6 - \frac{21}{8}x^4 + \frac{135}{4}x^2 - \frac{31}{2}$$

In addition, in this case we have $\forall n \in \mathbf{N} \text{---} {}_L E_n(0) = {}_L E_n = E_n$, that is.

The L-Euler numbers are the same as the ordinary Euler numbers.

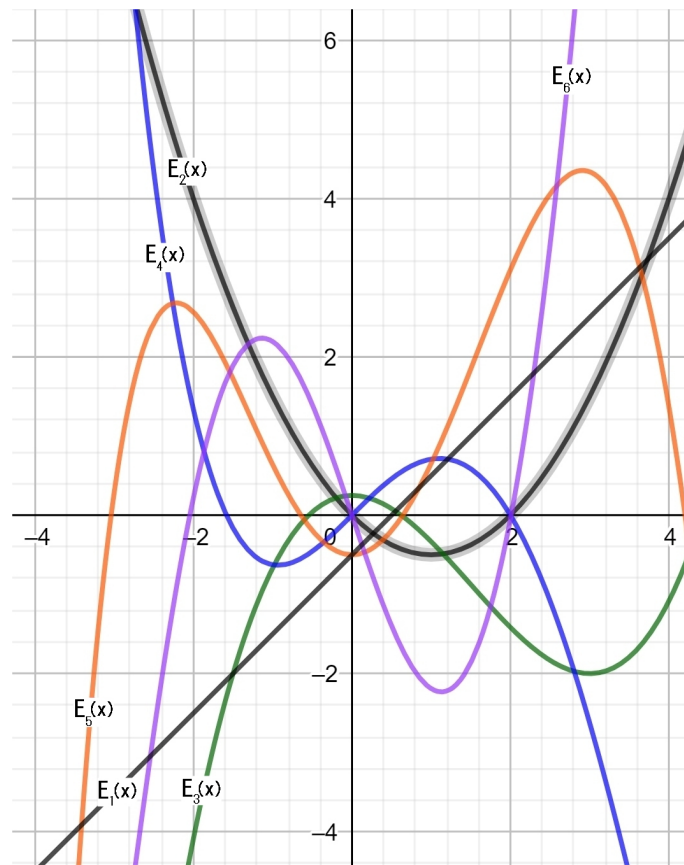


Figure 2. Graphs of polynomials ${}_L E_n(x)$ for $n = 1, 2, \dots, 6$.

3.4. Main Properties

Being a particular case of the L-Appell polynomials, the L-Bernoulli and L-Euler polynomials also satisfy the equations

$$\begin{aligned} D_L[{}_L B_n(x)] &= n[{}_L B_{n-1}(x)], \\ D_L[{}_L E_n(x)] &= n[{}_L E_{n-1}(x)]. \end{aligned} \quad (12)$$

Equations (12) are iterated as

$$\begin{aligned} (D_L)^m[{}_L B_n(x)] &= D^m x^m D^m[{}_L B_n(x)] = \frac{n!}{(n-m)!} [{}_L B_{n-m}(x)], \\ (D_L)^m[{}_L E_n(x)] &= D^m x^m D^m[{}_L E_n(x)] = \frac{n!}{(n-m)!} [{}_L E_{n-m}(x)], \end{aligned}$$

Theorem 2. The following properties hold:

- By introducing the 2nd kind L-Stirling numbers ${}_L S(n, k)$ defined as

$${}_L S(n, k) = S(n, k) n!, \quad (n \in \mathbf{N}, k = 0, 1, \dots, n-1), \quad (13)$$

we find

$$(x)_n = x(x-1) \cdots (x-n+1) = \sum_{k=0}^n {}_L S(n, k) \frac{x^n}{n!}. \quad (14)$$

- This results in

$$\frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} {}_L B_k(x) = \frac{x^n}{n!}, \quad (15)$$

- The integral relations hold:

$$\int_a^y \frac{1}{x} \int_0^x {}_L B_n(t) dt dy = \frac{{}_L B_{n+1}(y) - {}_L B_{n+1}(a)}{n+1},$$

$$\int_a^y \frac{1}{x} \int_0^x {}_L E_n(t) dt dy = \frac{{}_L E_{n+1}(y) - {}_L E_{n+1}(a)}{n+1}.$$
(16)

Proof. The three preceding results follow from a straightforward application of the isomorphism T_x to both sides of the corresponding equations valid for the ordinary Bernoulli polynomials. Formulas (16) are obtained by inverting the Laguerre derivative operator. \square

A general result on differential equations satisfied by Appell polynomials is reported in [38]. This result could be applied even in the Laguerre-type case, substituting the ordinary with the Laguerre derivative, but we do not go further in this direction because of the Ismail's remark states that it is not clear to introduce differential equations of this kind which do not have a finite order.

4. Hypergeometric L-Bernoulli Polynomials

Putting

$${}_L T_r(x) = \sum_{k=0}^r \frac{x^k}{(k!)^2}$$
(17)

and

$${}_1F_2(1; r+1, r+1; x) = \sum_{h=0}^{\infty} \frac{x^h}{[(r+1)_h]^2} = \sum_{h=0}^{\infty} \frac{(1)_h}{[(r+1)_h]^2} \frac{x^h}{h!},$$
(18)

since

$$e_1(x) - {}_L T_r(x) = \sum_{n=r}^{\infty} \frac{x^n}{(n!)^2} = x^r \sum_{n=0}^{\infty} \frac{1}{[(n+r)!]^2} x^n,$$
(19)

and

$$\frac{\frac{x^r}{(r!)^2} e_1(tx)}{e_1(x) - {}_L T_r(x)} = \frac{e_1(tx)}{(r!)^2 \sum_{n=0}^{\infty} \frac{x^n}{[(n+r)!]^2}},$$
(20)

We have the exponential generating function of the hypergeometric Bernoulli polynomials ${}_L B_n^{[r-1,1]}(t)$, where the classical gamma function is used:

$$\frac{e_1(tx)}{[\Gamma(r+1)]^2 \sum_{n=0}^{\infty} \frac{1}{[\Gamma(n+r+1)]^2} x^n} = \sum_{n=0}^{\infty} {}_L B_n^{[r-1,1]}(t) \frac{x^n}{n!}.$$
(21)

Remark 4. Note that, since the isomorphism T_x does not preserve multiplication, the ${}_L B_n^{[r-1,1]}$ polynomials, as the more general ones presented in subsequent Sections, are independent of those presented in [24].

4.1. Computing the Hypergeometric L-Bernoulli Numbers by Recursion

In (21), making $t = 0$ gives the exponential generating function [1] of the generalized hypergeometric L-Bernoulli numbers ${}_L B_n^{[r-1,1]} := {}_L B_n^{[r-1,1]}(0)$:

$$\sum_{n=0}^{\infty} {}_L B_n^{[r-1,1]} \frac{x^n}{n!} = \frac{1}{[\Gamma(r+1)]^2 \sum_{n=0}^{\infty} \frac{n!}{[\Gamma(n+r+1)]^2} \frac{x^n}{n!}},$$
(22)

which is valid even for non integer (in particular for fractional) values of the parameter r .
From Equation (22), we find

$$\sum_{n=0}^{\infty} \sum_{h=0}^n \binom{n}{h} {}_L B_{n-h}^{[r-1,1]} \frac{h! [\Gamma(r+1)]^2}{[\Gamma(h+r+1)]^2} \frac{x^n}{n!} = 1,$$

and therefore

$${}_L B_0^{[r-1,1]} = 1,$$

and for $n = 1, 2, 3, \dots$, we find the ${}_L B_n^{[r-1,1]}$ numbers by solving recursively the triangular system:

$$\sum_{h=0}^n \binom{n}{h} {}_L B_{n-h}^{[r-1,1]} \frac{h! [\Gamma(r+1)]^2}{[\Gamma(h+r+1)]^2} = 0.$$

4.2. Hypergeometric L-Bernoulli Polynomials of Order 2

The results of the preceding section can be generalized to the hypergeometric L-Bernoulli polynomials of order 2, which are defined by the generating function

$$\frac{\left[\frac{x^r}{(r!)^2}\right]^2 e_1(tx)}{[e_1(x) - {}_L T_r(x)]^2} = \frac{e_1(tx)}{[{}_1F_2(1; r+1, r+1; x)]^2} = \sum_{n=0}^{\infty} {}_L B_n^{[r-1,2]}(t) \frac{x^n}{n!}. \quad (23)$$

We find

$$\begin{aligned} [{}_1F_2]^2 &= \sum_{n=0}^{\infty} \sum_{h=0}^n \binom{n}{h} \frac{(1)_h (1)_{n-h}}{[(r+1)_h]^2 [(r+1)_{n-h}]^2} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{h=0}^n \frac{x^n}{[(r+1)_h]^2 [(r+1)_{n-h}]^2}, \end{aligned} \quad (24)$$

so that

$$\frac{e_1(tx)}{\sum_{n=0}^{\infty} \sum_{h=0}^n \frac{x^n}{[(r+1)_h]^2 [(r+1)_{n-h}]^2}} = \sum_{n=0}^{\infty} {}_L B_n^{[r-1,2]}(t) \frac{x^n}{n!}, \quad (25)$$

and introducing the hypergeometric L-Bernoulli numbers ${}_L B_n^{[r-1,2]} := {}_L B_n^{[r-1,2]}(0)$, we find the generating function

$$\frac{1}{\sum_{n=0}^{\infty} \sum_{h=0}^n \frac{x^n}{[(r+1)_h]^2 [(r+1)_{n-h}]^2}} = \sum_{n=0}^{\infty} {}_L B_n^{[r-1,2]} \frac{x^n}{n!}, \quad (26)$$

which gives the generating function of the hypergeometric L-Bernoulli numbers ${}_L B_n^{[r-1,2]}$

$$\frac{1}{\sum_{n=0}^{\infty} \sum_{h=0}^n \frac{[\Gamma(r+1)]^4}{[\Gamma(r+h+1)]^2 [\Gamma(r+n-h+1)]^2} x^n} = \sum_{n=0}^{\infty} {}_L B_n^{[r-1,2]} \frac{x^n}{n!}. \quad (27)$$

5. Conclusions

We have introduced the Laguerre-type versions of some special polynomials as the Appell, Bernoulli and Euler versions. The results followed from the application of the

differential isomorphism \mathcal{T}_x introduced in [26]. Generalization can be done by exploiting the iterated isomorphism $\mathcal{T}_x^s = \mathcal{T}_x[\mathcal{T}_x^{s-1}]_x$, recalled in Section 2. The considered isomorphism can be iterated as many times as we wish, and the corresponding derivative operators are reported in Equation (9). Then, by using the same technique, it is possible to define higher order L-Appell type polynomials

$${}_L^2 a_n(x), {}_L^3 a_n(x), \dots, {}_L^s a_n(x), \dots$$

and in particular, those of Bernoulli or Euler type.

Furthermore, we have considered the hypergeometric-type L-Bernoulli polynomials of order 1 and 2, starting from the exponential generating functions considered in [24].

The higher-order, hypergeometric-type L-Bernoulli polynomials of order k , with $k = 3, 4, \dots$ and $s \geq 2$, could be defined through the generating function

$$\frac{\left[\frac{x^r}{(r!)^{s+1}}\right]^k e_s(tx)}{[e_s(x) - {}_L^s T_r(x)]^k} = \frac{e_s(tx)}{[{}_1F_s(1; r+1, r+1; \dots; r+1; x)]^k} = \sum_{n=0}^{\infty} {}_L^s B_n^{[r-1, k]}(t) \frac{x^n}{n!}, \quad (28)$$

and the corresponding numbers ${}_L^s B_n^{[r-1, k]}$ by

$$\frac{\left[\frac{x^r}{(r!)^{s+1}}\right]^k}{[e_s(x) - {}_L^s T_r(x)]^k} = \frac{1}{[{}_1F_s(1; r+1, r+1; \dots; r+1; x)]^k} = \sum_{n=0}^{\infty} {}_L^s B_n^{[r-1, k]} \frac{x^n}{n!}. \quad (29)$$

However, as mentioned earlier, the construction of these mathematical items does not present difficulties, since the method is essentially the same.

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