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Article

Synchronization of Epidemic Systems with Neumann Boundary Value under Delayed Impulse

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Abstract: This paper reports the construction of synchronization criteria for the delayed impulsive epidemic models with reaction–diffusion under the Neumann boundary value. Different from the previous literature, the reaction–diffusion epidemic model with a delayed impulse brings mathematical difficulties to this paper. In fact, due to the existence of second-order partial derivatives in the reaction–diffusion model with a delayed impulse, the methods of first-order ordinary differential equations from the previous literature cannot be effectively applied in this paper. However, with the help of the variational method and an appropriate boundedness assumption, a new synchronization criterion is derived, and its effectiveness is illustrated by numerical examples.

Keywords: Neumann boundary value; delayed impulse; synchronization; reaction–diffusion epidemic models; variational methods

MSC: 34K24; 34K45

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1. Introduction

The dynamics of epidemic models has always been a hot topic [1,2]. Ordinary differential equation epidemic dynamic models are the most common models, and fractional order models especially have been hot topics in recent research [3–6] whose ideas or methods have been applied to studying epidemic dynamic models. Moreover, the reaction–diffusion epidemic models have become one of the key topics because of the migration behavior of the population [7–10]. Usually, infectious diseases are controlled within a certain range, so we consider the Neumann boundary value, that is, there is no diffusion on the boundary of the infectious area because the disease area is usually isolated from the outside world by some measures, so the Neumann zero boundary value is considered in this paper. To prevent the spread of disease, the government or relevant departments often take impulse measures. This impulse management measure is not only aimed at the epidemic situation, but also considered impulse control measures for economic management, mechanical engineering and other issues [11–20]. Delayed impulse models have also been investigated by many researchers [11,12], for delayed impulse models better simulate the actual situation, that is, the impulse effect usually takes some time to appear. However, the models with a delayed impulse are usually ordinary differential systems, and reaction–diffusion systems with a delayed impulse are rarely seen in the existing literature. This inspired us to write this paper. In fact, due to the existence of second-order partial derivatives in the reaction–diffusion model with a delayed impulse, the methods of first-order ordinary differential equations in the existing literature cannot be effectively applied to partial differential equations. By means of the variational method, differential mean value theorem and convergence of sequence, the synchronization criterion of the delayed impulse reaction–diffusion epidemic model is obtained in this paper. Intuition tells us that the shorter the impulse interval, the greater the impulse frequency and the faster the impulse effect appears, and the greater the impulse intensity, the faster the synchronization between the models must be. These intuitive conclusions are affirmed in the synchronization criterion given in this paper.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries about the reaction–diffusion epidemic model with a delayed impulse. In Section 3, we propose and derive the synchronization criterion for reaction–diffusion epidemic models under a delayed impulse. In Section 4, an illustrative numerical example is provided to show the effectiveness of the newly obtained criterion. Finally, some conclusions are written in Section 5.

The main contributions are as follows:

- Proposing and studying reaction–diffusion epidemic models with a delayed impulse for the first time;
- Deriving for the first time the synchronization criterion of an epidemic system with a Neumann boundary value under a delayed impulse.

2. System Description and Preliminaries

In [1], the following epidemic system was studied:

$$\begin{cases} \dfrac{dS}{dt} = -S\beta(t)I, \\ \dfrac{dI}{dt} = S\beta(t)I - I\gamma(t), \\ \dfrac{dR}{dt} = I\gamma(t), \end{cases} \tag{1}$$

where the function $S(t)$ is the fraction of the susceptible population, $I(t)$ the infected fraction, $R(t)$ the recovered fraction, and $0 < S(t) < 1, 0 < I(t) < 1, 0 < R(t) < 1$. In addition, the disease transmission rate is denoted by $\beta(t)$ and the recovery rate is $\gamma(t)$. In 2020, the authors of [7] considered the epidemic system with inevitable diffusions:

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = \Delta[d_1 S(t,x)] - I(t,x)\beta(t)S(t,x), \\ \dfrac{\partial I(t,x)}{\partial t} = \Delta[d_2 I(t,x)] + I(t,x)\beta(t)S(t,x) - I(t,x)\gamma(t), \\ \dfrac{\partial R(t,x)}{\partial t} = \Delta[d_3 R(t,x)] + I(t,x)\gamma(t), \end{cases} \tag{2}$$

where Δ is the Laplacian operator.

Generally speaking, $\Delta\varphi(x) = \sum\limits_{i=1}^{n} \frac{\partial^2 \varphi}{\partial x_i^2}$, for $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$.

Denote $X = (X_1, X_2, X_3)^T$ with $X_1 = S, X_2 = I, X_3 = R$. The following impulsive epidemic model with a Neumann boundary value is investigated in this paper:

$$\begin{cases} \dfrac{\partial X(t,x)}{\partial t} = D\Delta X(t,x) + A(t)X(t,x) + f(t, X(t,x)), \quad x \in \Omega, \, t \geqslant t_0, \\ X(t_k^+, x) - X(t_k^-, x) = M_k X(t_k - \tau_k, x), \quad k \in \mathbb{N}, \, x \in \Omega, \\ \dfrac{\partial X(t,x)}{\partial \nu} = 0, \quad x \in \partial\Omega, \, t \geqslant 0, \\ X(0,x) = \varphi(x), \quad x \in \Omega, \end{cases} \tag{3}$$

where $\mathbb{N} = \{1, 2, 3, \cdots\}$, $t_0 = 0$ and t_k is the impulse moment for $k = 1, 2, \cdots$, satisfying $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim\limits_{k\to\infty} t_k = +\infty$. For any impulse moment t_k, M_k is a constant parameters matrix that quantifies the impulse strength at the moment t_k. The time delay $\tau_k \in [0, \tau]$ with $(t_k - \tau_k, t_k) \subset (t_{k-1}, t_k)$ for each $k \in \mathbb{N}$, $\tau = \sup\limits_{k\in\mathbb{N}} \tau_k$, and so $t_0 \leqslant t_1 - \tau_1$. $\Omega \subset \mathbb{R}^N (N \leqslant 2)$ is a bounded smooth domain with smooth boundary $\partial\Omega$.

Denote ν the external normal direction of $\partial\Omega$.

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\gamma(t) & 0 \\ 0 & \gamma(t) & 0 \end{pmatrix}, \quad f(t,X) = \begin{pmatrix} -\beta(t)X_1X_2 \\ \beta(t)X_1X_2 \\ 0 \end{pmatrix}. \tag{4}$$

System (3) is the drive system, and its response system can be considered as follows,

$$\begin{cases} \dfrac{\partial Y(t,x)}{\partial t} = f(t,Y(t,x)) + D\Delta Y(t,x) + A(t)Y(t,x), & x \in \Omega,\ t \geqslant t_0, \\ Y(t_k^+,x) = M_kY(t_k - \tau_k,x) + Y(t_k^-,x), & k \in \mathbb{N},\ x \in \Omega, \\ \dfrac{\partial Y(t,x)}{\partial \nu} = 0, & x \in \partial\Omega, t \geqslant 0, \\ \phi(x) = Y(s,x), & x \in \Omega,\ s \in [-\tau, 0], \end{cases} \tag{5}$$

and then the error system is proposed as follows,

$$\begin{cases} \dfrac{\partial e(t,x)}{\partial t} = D\Delta e(t,x) + A(t)e(t,x) + F(t,e(t,x)), & x \in \Omega,\ t \geqslant t_0, \\ e(t_k^+,x) - e(t_k^-,x) = M_ke(t_k - \tau_k,x), & k \in \mathbb{N},\ x \in \Omega, \\ \dfrac{\partial e(t,x)}{\partial \nu} = 0, & t \geqslant 0,\ x \in \partial\Omega, \\ e(0,x) = \varphi(x) - \phi(x), & x \in \Omega, \end{cases} \tag{6}$$

where $e = X - Y$. Moreover,

$$F(e(t,x)) = f(t,X(t,x)) - f(t,Y(t,x)) = \begin{pmatrix} -\beta(t)X_1X_2 + \beta(t)Y_1Y_2 \\ \beta(t)X_1X_2 - \beta(t)Y_1Y_2 \\ 0 \end{pmatrix} \tag{7}$$

We assume in this paper that variables are left continuous at impulse moment t_k, for example, $e(t_k,x) = e(t_k^-,x)$.

Obviously, $-1 < e_i < 1$. Moreover, in consideration of the fact that population resources are limited, we can assume throughout this paper that their regional change rate is also limited, and so the change rate of the change rate is even limited:

H1 For any $i = 1,2,3$, there exists a constant $c_i > 0$ such that $|\Delta e_i(t,x)| < c_i|e_i(t,x)|$;
H2 There is a constant $\beta > 0$ such that $0 \leqslant \beta(t) \leqslant \beta$;
H3 There is a constant $\gamma > 0$ such that $|\gamma(t)| \leqslant \gamma$.

Lemma 1 (See, e.g., [21]). *$\Omega \subset \mathbb{R}^m$ is a bounded domain with its smooth boundary $\partial\Omega$ that is of class $\mathcal{C}^2$. $\xi(x) \in H_0^1(\Omega)$ is a real-valued function and $\frac{\partial \xi(x)}{\partial \nu}|_{\partial\Omega} = 0$. Then,*

$$\int_\Omega |\nabla \xi(x)|^2 dx \geqslant \lambda_1 \int_\Omega |\xi(x)|^2 dx,$$

where λ_1 is defined by the least positive eigenvalue of the problem:

$$\begin{cases} \lambda\xi - \Delta\xi = 0, & x \in \Omega, \\ \dfrac{\partial \xi(x)}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

3. Main Result

Theorem 1. *Assume there exists a positive definite diagonal matrix Q and a constant $q_0 > 0$ such that*

$$Q \leqslant q_0 \mathcal{I} \tag{8}$$

and

$$\sup_{k\in\mathbb{N}}\left[\|\mathcal{I}+M_k\|+\tau_k\|M_k\|\cdot\Big(\|D\|\sqrt{\lambda_{\max}C^2}+\|\widetilde{A}\|+2\beta\Big)\right]\sqrt{\frac{\lambda_{\max}Q}{\lambda_{\min}Q}e^{\lambda\rho}}\leqslant\rho_0<1, \quad (9)$$

then system (3) and system (5) are synchronized, where $\mathcal{I}$ *is the identity matrix,* $C=diag(c_1,c_2,c_3)>0$ *with* $c_i>0$ *defined in (H1),* $\rho=\sup_{k\in\mathbb{N}}(-t_{k-1}+t_k)$, $\zeta=\inf_{k\in\mathbb{N}}(t_k-t_{k-1})>0$,

$$\widetilde{A}=\begin{pmatrix}0&0&0\\0&\gamma&0\\0&\gamma&0\end{pmatrix}, \quad (10)$$

$$\lambda=\left[\frac{1}{\lambda_{\min}Q}\lambda_{\max}\Big(-\lambda_1(QD+DQ)+Q\widetilde{A}+\widetilde{A}^TQ+Q+4q_0\beta^2\mathcal{I}\Big)\right]. \quad (11)$$

Here, inequality (8) indicates that $(q_0\mathcal{I}-Q)$ is a non-negative definite matrix. For any symmetric matrix B, the real numbers $\lambda_{\min}B$ and $\lambda_{\max}B$ represent the minimum and maximum eigenvalue of B, respectively. For a matrix B, $\|B\|$ is its norm with $\|B\|=\sqrt{\lambda_{\max}(B^TB)}$.

Proof. Consider the following Lyapunov function:

$$V(t)=\int_\Omega e^T(t,x)Qe(t,x)dx,$$

where Q is a positive definite symmetric matrix. Denote $\|\eta\|^2=\sum_{i=1}^{3}\int_\Omega|\eta_i(x)|^2dx$ for any vector Lebesgue square-integrable function $\eta(x)=(\eta_1(x),\eta_2(x),\eta_3(x))^T$.

It follows from $0<X_i<1$ and $0<Y_i<1$ $(i=1,2,3)$ that

$$F^T(t,e)QF(t,e)\leqslant 2q_0\beta^2\cdot 2[(X_2-Y_2)^2+(X_1-Y_1)^2]\leqslant 4q_0\beta^2e^Te$$

So,

$$\begin{aligned}D^+V(t)\leqslant&-\lambda_1\int_\Omega e^T(QD+DQ)edx+\int_\Omega e^T\Big(Q\widetilde{A}+\widetilde{A}^TQ\Big)edx+\int_\Omega\Big(e^TQF(t,e)+F^T(t,e)Qe\Big)dx\\\leqslant&\frac{1}{\lambda_{\min}Q}\lambda_{\max}\Big(-\lambda_1(QD+DQ)+Q\widetilde{A}+\widetilde{A}^TQ+Q+4q_0\beta^2\mathcal{I}\Big)\int_\Omega e^T(t,x)Qe(t,x)dx,\ t\in(t_{k-1},t_k],\end{aligned}$$

which means

$$\|e(t)\|^2\leqslant\frac{\lambda_{\max}Q}{\lambda_{\min}Q}e^{\lambda(t-t_{k-1})}\|e(t_{k-1}^+)\|^2,\quad t\in(t_{k-1},t_k].$$

Particularly,

$$\|e(t_k)\|^2=\|e(t_k^-)\|^2\leqslant\frac{\lambda_{\max}Q}{\lambda_{\min}Q}e^{\lambda(t-t_{k-1})}\|e(t_{k-1}^+)\|^2,\quad k\in\mathbb{N}.$$

On the other hand,

$$\|D\Delta e(\varsigma_k,x)\|\leqslant\|D\|\cdot\|\Delta e(\varsigma_k,x)\|\leqslant\|D\|\sqrt{\lambda_{\max}C^2}\cdot\|e(\varsigma_k)\|,$$

and

$$\|F(t,e(t))\|\leqslant\sqrt{2}\beta\cdot\sqrt{2}\sqrt{\int_\Omega[(X_2-Y_2)^2+(X_1-Y_1)^2]dx}\leqslant\|e(t)\|\cdot 2\beta.$$

Now we can see it from the differential mean value theorem and definition of $A(t)$ that there exists $\varsigma_k\in(t_k-\tau_k,t_k)\subset(t_{k-1},t_k)$ such that

$$\begin{aligned}
\|e(t_k^+)\| =&\|e(t_k^-,x)+M_k e(t_k-\tau_k,x)\| \\
\leqslant&\|\mathcal{I}+M_k\|\cdot\|e(t_k,x)\|+\|M_k\|\cdot\|e(t_k,x)-e(t_k-\tau_k,x)\| \\
\leqslant&\|\mathcal{I}+M_k\|\cdot\|e(t_k,x)\|+\tau_k\|M_k\|\cdot\left(\|D\|\sqrt{\lambda_{\max}C^2}\cdot\|e(\varsigma_k)\|+\|\widetilde{A}\|\cdot\|e(\varsigma_k)\|+2\beta\|e(\varsigma_k)\|\right) \\
\leqslant&\left[\|\mathcal{I}+M_k\|+\tau_k\|M_k\|\cdot\left(\|D\|\sqrt{\lambda_{\max}C^2}+\|\widetilde{A}\|+2\beta\right)\right]\sqrt{\frac{\lambda_{\max}Q}{\lambda_{\min}Q}e^{\lambda(t_k-t_{k-1})}}\|e(t_{k-1}^+)\| \\
\leqslant&\rho_0\|e(t_{k-1}^+)\|,\quad \forall k=1,2,3,\cdots
\end{aligned}$$

which means

$$\|e(t_k^+)\leqslant\rho_0^k\|e(0)\|,\quad \forall k=1,2,3,\cdots. \tag{12}$$

Finally, for $t\in(t_{k-1},t_k]$,

$$\|e(t)\|^2\leqslant\frac{\lambda_{\max}Q}{\lambda_{\min}Q}e^{\lambda(t-t_{k-1})}\|e(t_{k-1}^+)\|^2\leqslant\frac{\lambda_{\max}Q}{\lambda_{\min}Q}e^{\lambda\rho}\rho_0^{2(k-1)}\|e(0)\|^2,$$

which, together with $t>t_{k-1}$, implies

$$\|e(t)\|\leqslant\sqrt{\frac{\lambda_{\max}Q}{\lambda_{\min}Q}e^{\lambda\rho}}\|e(0)\|e^{-\lambda_0(t-t_0)}. \tag{13}$$

where $\lambda_0=-\frac{1}{\varsigma}\ln\rho_0>0$. This completes the proof. □

Remark 1. *Contrary to the existing literature related to impulsive reaction–diffusion epidemic models (see, e.g., [13,14]), the delayed impulse is firstly considered in the reaction–diffusion epidemic system in this paper. Indeed, although time delays were introduced in [13], the impulse was not delayed. However, in real life, the effectiveness of many defensive measures usually takes place after a period of time. Therefore, the delayed impulse epidemic model studied in this paper clearly has practical significance.*

Remark 2. *Introducing the delayed impulse into reaction–diffusion epidemic models means bringing new mathematical difficulties to this paper. Therefore, this paper adopts a method different from [13,14] to overcome the mathematical difficulties, and a new synchronization criterion is derived.*

4. Numerical Example

Example 1. *Let $\Omega=(0,1)\times(0,1)\subset\mathbb{R}^2$, then $\lambda_1=\pi^2$. Set $C=\mathcal{I}=diag(1,1,1)$, $\gamma(t)=0.1\sin^2 t$, $\beta(t)=0.1\cos^2 t$, and $\beta=0.1=\gamma$,*

$$A(t)=\begin{pmatrix}0&0&0\\0&-0.1\sin^2 t&0\\0&0.1\sin^2 t&0\end{pmatrix},\ f(t,X)=\begin{pmatrix}-0.1\cos^2 tX_1X_2\\0.1\cos^2 tX_1X_2\\0\end{pmatrix},\ \widetilde{A}=\begin{pmatrix}0&0&0\\0&0.1&0\\0&0.1&0\end{pmatrix}.$$

Case 1: Let $\rho=0.1$, $\tau_k\equiv\tau=0.01$, $M_k\equiv-0.7\mathcal{I}$, and

$$D=\begin{pmatrix}0.045&0&0\\0&0.035&0\\0&0&0.055\end{pmatrix}.$$

Using a computer with Matlab's LMI toolbox results in the following feasibility datum:

$$Q=\begin{pmatrix}0.0155&0&0\\0&0.0135&0\\0&0&0.0161\end{pmatrix}$$

then, $q_0 = \lambda_{\max} Q = 0.0161$, $\lambda_{\min} Q = 0.0135$, and

$$\sup_{k\in\mathbb{N}} \left[\|\mathcal{I} + M_k\| + \tau_k \|M_k\| \cdot \left(\|D\| \sqrt{\lambda_{\max} C^2} + \|\widetilde{A}\| + 2\beta \right) \right] \sqrt{\frac{\lambda_{\max} Q}{\lambda_{\min} Q} e^{\lambda\rho}} = 0.8196 \leqslant \rho_0 < 1,$$

where $\rho_0 = 0.8196$. According to Theorem 1, system (3) and system (5) are synchronized.
Case 2: Let $\rho = 0.05$, $\tau_k \equiv \tau = 0.01$, $M_k \equiv -0.7\mathcal{I}$, and

$$D = \begin{pmatrix} 0.045 & 0 & 0 \\ 0 & 0.035 & 0 \\ 0 & 0 & 0.055 \end{pmatrix},$$

Using Matlab's LMI toolbox results in the following feasibility datum:

$$Q = \begin{pmatrix} 0.0111 & 0 & 0 \\ 0 & 0.0131 & 0 \\ 0 & 0 & 0.0112 \end{pmatrix}$$

then, $q_0 = \lambda_{\max} Q = 0.0131$, $\lambda_{\min} Q = 0.0111$, and

$$\sup_{k\in\mathbb{N}} \left[\|\mathcal{I} + M_k\| + \tau_k \|M_k\| \cdot \left(\|D\| \sqrt{\lambda_{\max} C^2} + \|\widetilde{A}\| + 2\beta \right) \right] \sqrt{\frac{\lambda_{\max} Q}{\lambda_{\min} Q} e^{\lambda\rho}} = 0.7988 \leqslant \rho_0 < 1,$$

where $\rho_0 = 0.7988$. According to Theorem 1, system (3) and system (5) are synchronized.

Remark 3. *Table 1 reveals that the bigger the impulse frequency, the faster the synchronization speed.*

Table 1. Comparison of the influence from different impulse frequencies when other data are unchanged.

	Case 1: $\rho = 0.1$	**Case 2: $\rho = 0.05$**
τ_k	0.01	0.01
M_k	$-0.7\mathcal{I}$	$-0.7\mathcal{I}$
ρ_0	0.8196	0.7988

Case 3: Let $\rho = 0.1$, $\tau_k \equiv \tau = 0.01$, $M_k \equiv -0.8\mathcal{I}$, and

$$D = \begin{pmatrix} 0.045 & 0 & 0 \\ 0 & 0.035 & 0 \\ 0 & 0 & 0.055 \end{pmatrix},$$

Using Matlab's LMI toolbox results in the following feasibility datum:

$$Q = \begin{pmatrix} 0.0166 & 0 & 0 \\ 0 & 0.0163 & 0 \\ 0 & 0 & 0.0165 \end{pmatrix}$$

then, $q_0 = \lambda_{\max} Q = 0.0163$, $\lambda_{\min} Q = 0.0166$, and

$$\sup_{k\in\mathbb{N}} \left[\|\mathcal{I} + M_k\| + \tau_k \|M_k\| \cdot \left(\|D\| \sqrt{\lambda_{\max} C^2} + \|\widetilde{A}\| + 2\beta \right) \right] \sqrt{\frac{\lambda_{\max} Q}{\lambda_{\min} Q} e^{\lambda\rho}} = 0.5466 \leqslant \rho_0 < 1,$$

where $\rho_0 = 0.5466$. According to Theorem 1, system (3) and system (5) are synchronized.

Remark 4. *Table 2 implies that the bigger the impulse intensity, the faster the synchronization speed.*

Table 2. Comparison of the influence from different impulse intensities when other data are unchanged.

	Case 1: $M_k = -0.7\mathcal{I}$	**Case 2:** $M_k = -0.8\mathcal{I}$
ρ	0.1	0.1
τ_k	0.01	0.01
ρ_0	0.8196	0.5466

Case 4: Let $\rho = 0.1$, $\tau_k \equiv \tau = 0.001$, $M_k \equiv -0.7\mathcal{I}$, and

$$D = \begin{pmatrix} 0.045 & 0 & 0 \\ 0 & 0.035 & 0 \\ 0 & 0 & 0.055 \end{pmatrix},$$

Using Matlab's LMI toolbox results in the following feasibility datum:

$$Q = \begin{pmatrix} 0.0155 & 0 & 0 \\ 0 & 0.0135 & 0 \\ 0 & 0 & 0.0161 \end{pmatrix}$$

then, $q_0 = \lambda_{\max} Q = 0.0161$, $\lambda_{\min} Q = 0.0135$, and

$$\sup_{k \in \mathbb{N}} \left[\|\mathcal{I} + M_k\| + \tau_k \|M_k\| \cdot \left(\|D\| \sqrt{\lambda_{\max} C^2} + \|\widetilde{A}\| + 2\beta \right) \right] \sqrt{\frac{\lambda_{\max} Q}{\lambda_{\min} Q} e^{\lambda\rho}} = 0.7188 \leqslant \rho_0 < 1,$$

where $\rho_0 = 0.7188$. According to Theorem 1, system (3) and system (5) are synchronized.

Remark 5. *Table 3 means that the smaller the time delays of the impulse effect, the faster the synchronization speed.*

Table 3. Comparison of the influence of different time delays of the impulse effect when other data are unchanged.

	Case 1: $\tau_k = 0.01$	**Case 2:** $\tau_k = 0.001$
ρ	0.1	0.1
M_k	$-0.7\mathcal{I}$	$-0.7\mathcal{I}$
ρ_0	0.8196	0.7188

Below, another numerical example is presented to show the validity of Theorem 1 via very simple computations.

Example 2. *Set* $C = 15\mathcal{I} = diag(15, 15, 15)$, $M_k \equiv -0.9\mathcal{I}$, $D = 0.1\mathcal{I}$, $Q = \mathcal{I}$, $\tau_k \equiv \tau = 0.01$, $\beta = 0.1$, *and then direct computations lead to*

$$\sqrt{\lambda_{\max} C^2} = 15, \ \|\mathcal{I} + M_k\| \equiv 0.1, \ \|M_k\| \equiv 0.9, \ \|D\| = 0.1, \ \lambda_{\max} Q = \lambda_{\min} Q = 1 = q_0. \tag{14}$$

Let $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$; then, $\lambda_1 = \pi^2$. Set $\gamma = 0.1$; then,

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0.1 & 0 \end{pmatrix} \text{ and } \|\widetilde{A}\| = \sqrt{0.0200} = 0.1414. \tag{15}$$

Hence, it follows from (14) and (15) that

$$\begin{aligned}\lambda =&\Big[\frac{1}{\lambda_{\min}Q}\lambda_{\max}\Big(-\lambda_1(QD+DQ)+Q\widetilde{A}+\widetilde{A}^TQ+Q+4q_0\beta^2\mathcal{I}\Big)\Big]\\ =&\lambda_{\max}\Big(-\lambda_1(QD+DQ)+Q\widetilde{A}+\widetilde{A}^TQ+Q+4q_0\beta^2\mathcal{I}\Big)\\ =&\lambda_{\max}\Big(-\pi^2(0.1\mathcal{I}+0.1\mathcal{I})+\widetilde{A}+\widetilde{A}^T+\mathcal{I}+0.04\mathcal{I}\Big)\\ =&-0.6925,\end{aligned} \tag{16}$$

where

$$\Big(-\pi^2(0.1\mathcal{I}+0.1\mathcal{I})+\widetilde{A}+\widetilde{A}^T+\mathcal{I}+0.04\mathcal{I}\Big)=\begin{pmatrix}-0.9339 & 0 & 0\\ 0 & -0.7339 & 0.1000\\ 0 & 0.1000 & -0.9339\end{pmatrix}.$$

Now, letting $\rho = 0.3$ and $\rho_0 = 0.9$, we can get by (14)–(16) that

$$\begin{aligned}&\sup_{k\in\mathbb{N}}\Big[\|\mathcal{I}+M_k\|+\tau_k\|M_k\|\cdot\Big(\|D\|\sqrt{\lambda_{\max}C^2}+\|\widetilde{A}\|+2\beta\Big)\Big]\sqrt{\frac{\lambda_{\max}Q}{\lambda_{\min}Q}e^{\lambda\rho}}\\ \equiv&\Big[\|\mathcal{I}+M_k\|+\tau_k\|M_k\|\cdot\Big(\|D\|\sqrt{\lambda_{\max}C^2}+\|\widetilde{A}\|+2\beta\Big)\Big]\sqrt{\frac{\lambda_{\max}Q}{\lambda_{\min}Q}e^{\lambda\rho}}\\ =&\Big[0.1+0.01\times0.9\times\Big(0.1\times15+0.1414+0.2\Big)\Big]\sqrt{e^{-0.6925\times0.3}}\\ <&\Big[0.1+0.01\times0.9\times\Big(0.1\times15+0.1414+0.2\Big)\Big]\times1\\ =&0.1166<\rho_0<1,\end{aligned}$$

which implies (9) holds.

According to Theorem 1, system (3) and system (5) are synchronized (see Figures 1–3).

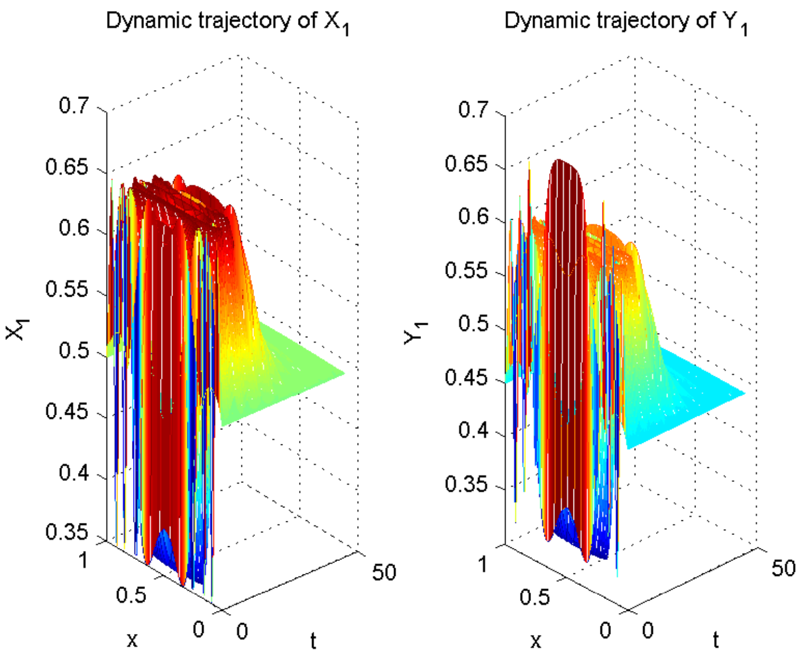

Figure 1. Computer simulation of X_1 in (3) and Y_1 in (5).

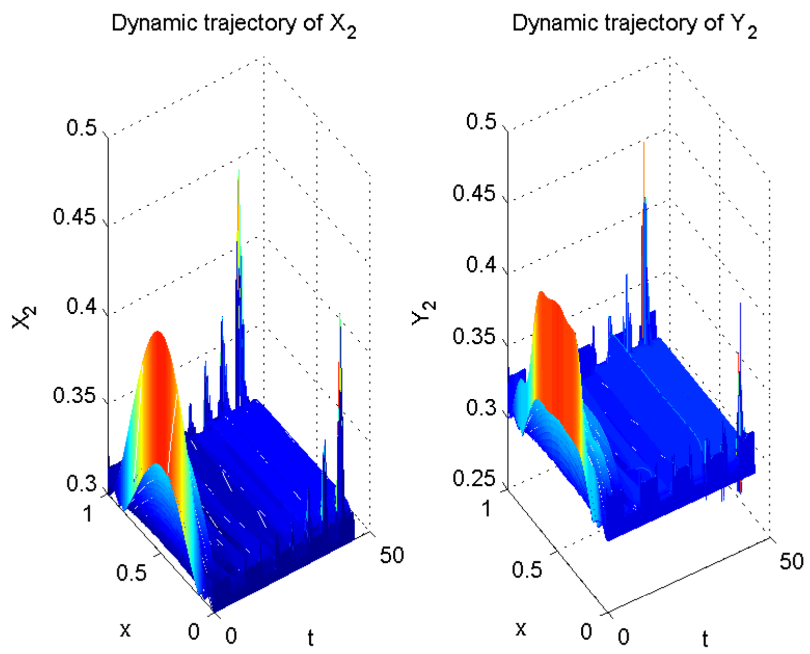

Figure 2. Computer simulation of X_2 in (3) and Y_2 in (5).

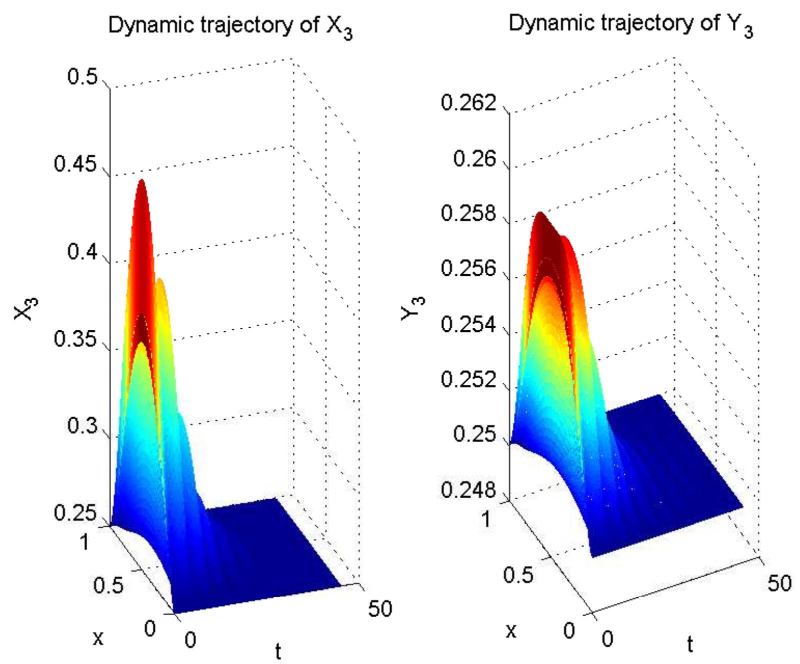

Figure 3. Computer simulation of X_3 in (3) and Y_3 in (5).

Remark 6. *Example 2 illustrates that the validity of Theorem 1 can be easily verified even without using Matlab's LMI toolbox.*

5. Conclusions

This paper reported the synchronization control of two epidemic systems with a Neumann boundary value under a delayed impulse. Different from the previous relevant literature in which the effect of the impulse control was immediate, our impulse effect was delayed, which is in line with the actual situation during an epidemic. At the same time, the newly obtained criterion and numerical examples illustrate that the shorter the time delay of the pulse effect, the faster the synchronization speed. In addition, the smaller the pulse interval, the faster the synchronization. On the other hand, Remarks 1 and 2 illustrated the novelty of this paper by comparing the related literature with this paper.

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