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Freezing Sets for Arbitrary Digital Dimension

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Abstract: The study of freezing sets is part of the theory of fixed points in digital topology. Most of the previous work on freezing sets is for digital images in the digital plane $\mathbb{Z}^2$. In this paper, we show how to obtain freezing sets for digital images in $\mathbb{Z}^n$ for arbitrary n, using the c_1 and c_n adjacencies.

Keywords: digital image; adjacency; digitally continuous function; freezing set

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1. Introduction

The study of freezing sets is part of the fixed point theory of digital topology. Freezing sets were introduced in [1] and studied in subsequent papers including [2–5]. These papers focus mostly on digital images in $\mathbb{Z}^2$.

In the current paper, we obtain results for freezing sets in $\mathbb{Z}^n$, for arbitrary n. We show that given a finite connected digital image $X \subset \mathbb{Z}^n$, if we use the c_1 or c_n adjacency and X is decomposed into a union of cubes K_i, then we can construct a freezing set for X from those of the K_i.

2. Preliminaries

Researchers have taken several different approaches to the study of digital topology, including the *Khalimsky topology* [6–8], the *Marcus–Wyse topology* [9,10], and Rosenfeld's graph-based approach [11,12]. We use the latter in this paper.

For Rosenfeld's graph-based approach, we present foundational material in this section on adjacencies, digitally continuous functions, and terminology.

2.1. Adjacencies

Much of this section is quoted or paraphrased from [13].

A digital image is a pair (X, κ) where $X \subset \mathbb{Z}^n$ for some n and κ is an adjacency on X. Thus, (X, κ) is a graph with X for the vertex set and κ determining the edge set. Usually, X is finite, although there are papers that consider infinite X. Usually, adjacency reflects some type of "closeness" in $\mathbb{Z}^n$ of the adjacent points. When these "usual" conditions are satisfied, one may consider the digital image as a model of a black-and-white "real world" digital image in which the black points (foreground) are the members of X and the white points (background) are members of $\mathbb{Z}^n \setminus X$.

We write $x \leftrightarrow_\kappa y$, or $x \leftrightarrow y$ when κ is understood or when it is unnecessary to mention κ, to indicate that x and y are κ-adjacent. Notations $x \leftrightarroweq_\kappa y$, or $x \leftrightarroweq y$ when κ is understood, indicate that x and y are κ-adjacent or are equal.

The most commonly used adjacencies are the c_u adjacencies, defined as follows. Let $X \subset \mathbb{Z}^n$ and let $u \in \mathbb{Z}$, $1 \le u \le n$. Then, for points

$$x = (x_1, \ldots, x_n) \neq (y_1, \ldots, y_n) = y$$

we have $x \leftrightarrow_{c_u} y$ if and only if

- for at most u indices i we have $|x_i - y_i| = 1$, and
- for all indices j, $|x_j - y_j| \neq 1$ implies $x_j = y_j$.

The c_u-adjacencies are often denoted by the number of adjacent points a point can have in the adjacency. For example,

- in $\mathbb{Z}$, c_1-adjacency is 2-adjacency;
- in $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency;
- in $\mathbb{Z}^3$, c_1-adjacency is 8-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

In this paper, we mostly use the c_1- and c_n-adjacencies.

When (X, κ) is understood to be a digital image under discussion, we use the following notations. For $x \in X$,

$$N(x) = \{y \in X \mid y \leftrightarrow_\kappa x\},$$

$$N^*(x) = \{y \in X \mid y \leftrightarrows_\kappa x\} = N(x) \cup \{x\}.$$

Definition 1 ([11])**.** *Let $X \subset \mathbb{Z}^n$. The boundary of X is*

$$Bd(X) = \{\, x \in X \mid \text{there exists } y \in \mathbb{Z}^n \setminus X \text{ such that } x \leftrightarrow_{c_1} y \,\}.$$

2.2. Digitally Continuous Functions

Much of this section is quoted or paraphrased from [13].

We denote by id or id_X the identity map $\mathrm{id}(x) = x$ for all $x \in X$.

Definition 2 ([12,14])**.** *Let (X, κ) and (Y, λ) be digital images. A function $f\colon X \to Y$ is (κ, λ)-continuous, or digitally continuous or just* continuous*, when κ and λ are understood, if for every κ-connected subset X' of X, $f(X')$ is a λ-connected subset of Y. If $(X, \kappa) = (Y, \lambda)$, we say a function is κ-continuous to abbreviate "(κ, κ)-continuous."*

Theorem 1 ([14])**.** *A function $f\colon X \to Y$ between digital images (X, κ) and (Y, λ) is (κ, λ)-continuous if and only if for every $x, y \in X$, if $x \leftrightarrow_\kappa y$ then $f(x) \leftrightarrows_\lambda f(y)$.*

Similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings* in [15,16].

Theorem 2 ([14])**.** *Let $f : (X, \kappa) \to (Y, \lambda)$ and $g\colon (Y, \lambda) \to (Z, \mu)$ be continuous functions between digital images. Then, $g \circ f : (X, \kappa) \to (Z, \mu)$ is continuous.*

A *κ-path* is a continuous function $r\colon ([0, m]_\mathbb{Z}, c_1) \to (X, \kappa)$.

For a digital image (X, κ), we use the notation

$$C(X, \kappa) = \{f : X \to X \mid f \text{ is } \kappa\text{-continuous}\}.$$

A function $f\colon (X, \kappa) \to (Y, \lambda)$ is an *isomorphism* (called a *homeomorphism* in [17]) if f is a continuous bijection such that f^{-1} is continuous.

For $X \in \mathbb{Z}^n$, the *projection to the* ith *coordinate* is the function $p_i\colon X \to \mathbb{Z}$ defined by

$$p_i(x_1, \ldots, x_n) = x_i.$$

A *(digital) line segment* in (X, κ) is a set $S = f([0, m]_\mathbb{Z})$, where f is a digital path, such that the points of S are collinear; S is *axis parallel* if for all but one of the indices i, $p_i \circ f$ is a constant function.

2.3. Cube Terminology

Let $Y = \Pi_{i=1}^n [a_i, b_i]_\mathbb{Z}$, where $b_i \geq a_i$.

If for $1 \le j \le n$ there are exactly j indices i such that $b_i > a_i$ (equivalently, exactly $n - j$ indices i such that $b_i = a_i$), we call Y a *j-dimensional cube* or a *j-cube*.

A j-cube K in Y, such that

- for j indices i, $p_i(K) = [a_i, b_i]_{\mathbb{Z}}$ and
- for all other indices i, $p_i(K) = \{a_i\}$ or $p_i(K) = \{b_i\}$,

is a *face* or a *j-face* of Y.

A *corner* of Y is any of the points of $\Pi_{i=1}^{n}\{a_i, b_i\}$. An *edge* of Y is an axis-parallel digital line segment joining two corners of Y.

3. Tools for Determining Fixed Point Sets

Definition 3 ([1])**.** *Let (X, κ) be a digital image. We say $A \subset X$ is a* freezing set for X *if given $g \in C(X, \kappa)$, $A \subset \mathrm{Fix}(g)$ implies $g = \mathrm{id}_X$.*

Theorem 3 ([1])**.** *Let A be a freezing set for the digital image (X, κ) and let $F : (X, \kappa) \to (Y, \lambda)$ be an isomorphism. Then, $F(A)$ is a freezing set for (Y, λ).*

The following are useful for determining fixed point and freezing sets.

Proposition 1 (Corollary 8.4 of [13])**.** *Let (X, κ) be a digital image and $f \in C(X, \kappa)$. Suppose $x, x' \in \mathrm{Fix}(f)$ are such that there is a unique shortest κ-path P in X from x to x'. Then, $P \subseteq \mathrm{Fix}(f)$.*

Lemma 1, below,

> "... can be interpreted to say that in a c_u-adjacency, a continuous function that moves a point p also moves a point that is "behind" p. E.g., in $\mathbb{Z}^2$, if q and q' are c_1- or c_2-adjacent with q left, right, above, or below q', and a continuous function f moves q to the left, right, higher, or lower, respectively, then f also moves q' to the left, right, higher, or lower, respectively [1]."

Lemma 1 ([1])**.** *Let $(X, c_u) \subset \mathbb{Z}^n$ be a digital image, $1 \le u \le n$. Let $q, q' \in X$ be such that $q \leftrightarrow_{c_u} q'$. Let $f \in C(X, c_u)$.*

1. *If $p_i(f(q)) > p_i(q) > p_i(q')$, then $p_i(f(q')) > p_i(q')$.*
2. *If $p_i(f(q)) < p_i(q) < p_i(q')$, then $p_i(f(q')) < p_i(q')$.*

Definition 4 ([2])**.** *Let (X, κ) be a digital image. Let $p, q \in X$ such that*

$$N(X, p, \kappa) \subset N^*(X, q, \kappa).$$

Then, q is a close κ-neighbor *of p.*

Lemma 2 ([2,13])**.** *Let (X, κ) be a digital image. Let $p, q \in X$ such that q is a close κ-neighbor of p. Then, p belongs to every freezing set of (X, κ).*

Theorem 4. *Let $Y = \Pi_{i=1}^{3}[a_i, b_i]_{\mathbb{Z}}$ be such that $b_i > a_i + 1$ for all i. Let $A = \Pi_{i=1}^{3}\{a_i, b_i\}$. Then, A is a subset of every freezing set for (X, c_3).*

Proof. By Theorem 3, we may assume $a_i = 0$ for all i, so $A = \Pi_{i=1}^{3}\{0, b_i\}$. It is easily seen that every $a \in A$ has a close neighbor in X, namely the unique member of X that differs from a by 1 in every coordinate. Therefore, by Lemma 2, A is a subset of every freezing set for (X, c_3). $\square$

4. c_1-Freezing Sets for Cubes

The following is presented as Theorem 5.11 of [1]. However, there is an error in the argument given in [1] for the proof of the first assertion. We give a correct proof below.

Theorem 5. *Let* $X = \Pi_{i=1}^{n}[r_i, s_i]_{\mathbb{Z}}$. *Let* $A = \Pi_{i=1}^{n}\{r_i, s_i\}$.

- *Let* $Y = \Pi_{i=1}^{n}[a_i, b_i]_{\mathbb{Z}}$ *be such that* $[r_i, s_i] \subset [a_i, b_i]_{\mathbb{Z}}$ *for all* i. *Let* $f : X \to Y$ *be* (c_1, c_1)*-continuous. If* $A \subset \operatorname{Fix}(f)$, *then* $X \subset \operatorname{Fix}(f)$.
- A *is a freezing set for* (X, c_1) *that is minimal for* $n \in \{1, 2\}$.

The argument given in [1] is based on induction. We quote the beginning of the argument's inductive step:

"Now suppose $n = k + 1$ and $f : X \to Y$ is c_1-continuous with $A \subset \operatorname{Fix}(f)$. Let

$$X_0 = \Pi_{i=1}^{k}[0, m_i]_{\mathbb{Z}} \times \{0\}, \quad X_1 = \Pi_{i=1}^{k}[0, m_i]_{\mathbb{Z}} \times \{m_{k+1}\}.$$

We have that $f|_{X_0}$ and $f|_{X_1}$ are c_1-continuous, $A \cap X_0 \subset \operatorname{Fix}(f|_{X_0})$, and $A \cap X_1 \subset \operatorname{Fix}(f|_{X_1})$. Since X_0 and X_1 are isomorphic to k-dimensional digital cubes, by Theorem 3 [of the current paper; it's Theorem 5.2 of [1]] and the inductive hypothesis, we have

$$\left(\Pi_{i=1}^{k}[0, m_i]_{\mathbb{Z}} \times \{0\}\right) \cup \left(\Pi_{i=1}^{k}[0, m_i]_{\mathbb{Z}} \times \{m_n\}\right) \subset \operatorname{Fix}(f)."$$

Note that the above fails to show that $f(X_0) \subset X_0$ and $f(X_1) \subset X_1$; hence, if X is a proper subset of Y it does not follow from the above that $A \cap X_0 \subset \operatorname{Fix}(f|_{X_0})$ and $A \cap X_1 \subset \operatorname{Fix}(f|_{X_1})$. In the following, we give a correct proof of the first assertion of Theorem 5, using a rather different approach than was employed in [1].

Proof. By Theorem 3, we may assume

$$X = \prod_{i=1}^{n}[0, m_i]_{\mathbb{Z}}, \qquad A = \prod_{i=1}^{n}\{\, 0, m_i \,\}.$$

Let $f : X \to Y$ be (c_1, c_1)-continuous such that $f|_A = \operatorname{id}_A$. Observe that by Proposition 1,

$$\begin{array}{l}\text{if } b_1 \text{ and } b_2 \text{ are members of } \operatorname{Fix}(f) \text{ that differ in exactly one coordinate,}\\ \text{then the digital segment from } b_1 \text{ to } b_2 \text{ is a subset of } \operatorname{Fix}(f).\end{array} \tag{1}$$

In particular, let

$$X_1 = \left\{ \begin{array}{l|l} x \in X & \begin{array}{l} x \text{ belongs to a 1-cube (an axis-parallel segment)} \\ \text{with endpoints in } A \end{array} \end{array} \right\}.$$

By (1), $X_1 \subset \operatorname{Fix}(f)$.

We proceed inductively. For $j \in \{\, 1, \ldots, n \,\}$, let

$$X_j = \{\, x \in X \mid x \text{ belongs to a } j\text{-face of } X \,\}.$$

Note a j-face of X is a j-cube with corners in A. Suppose $X_\ell \subset \operatorname{Fix}(f)$ for some $\ell \geq 1$. Given $x \in X_{\ell+1}$, let K be an $(\ell + 1)$-face of Y such that $x \in K$. Let F_1 and F_2 be opposite ℓ-faces of K, i.e., for some index d, $x_i \in F_i$ implies, without loss of generality, $p_d(x_1) = 0$ and $p_d(x_2) = m_d$.

Let $x \in K$. Then, x is a point of an axis-parallel segment from a point of F_1 to a point of F_2. By (1), $x \in \operatorname{Fix}(f)$. Thus, $K \subset \operatorname{Fix}(f)$; therefore, $X_{\ell+1} \subset \operatorname{Fix}(f)$. This completes our induction. In particular, $X = X_n \subset \operatorname{Fix}(f)$.

Thus, for $Y = X$, it follows that A is a freezing set for (X, c_1). That A is minimal for $n \in \{\, 1, 2 \,\}$ follows as in [1]. □

The set of corners of a cube is not always a minimal c_1-freezing set, as shown by the following example in which the set A is a proper subset of the set of corners.

Example 1 ([1]). *Let* $X = [0,1]^3_{\mathbb{Z}}$. *Let*

$$A = \{ (0,0,0), (0,1,1), (1,0,1), (1,1,0) \}.$$

Then, A is a freezing set for (X, c_1).

5. c_1-Freezing Sets for Unions of Cubes

In this section, we show how to obtain c_1-freezing sets for finite subsets of $\mathbb{Z}^n$.

Theorem 6. *Let* $X = \bigcup_{i=1}^m K_i$ *where* $m \geq 1$,

$$K_i = \prod_{j=1}^n [a_{ij}, b_{ij}]_{\mathbb{Z}} \subset \mathbb{Z}^n,$$

and X is c_1*-connected. Let*

$$A_i = \prod_{j=1}^n \{ a_{ij}, b_{ij} \}.$$

Let $A = \bigcup_{i=1}^n A_i$. *Then, A is a freezing set for* (X, c_1).

Proof. Let $f \in C(X, c_1)$ be such that $A \subset \text{Fix}(f)$.

Given $x \in X$, we have $x \in K_i$ for some i. Since $A_i \subset \text{Fix}(f)$, it follows from Theorem 5 that $x \in \text{Fix}(f)$. Thus, $X = \text{Fix}(f)$, so A is a freezing set. $\square$

Corollary 1. *The wedge* $K_1 \vee K_2$ *of two digital cubes in* $\mathbb{Z}^n$ *with axis-parallel edges has for a* c_1*-freezing set* $K_1' \cup K_2'$, *where* K_i' *is the set of corners of* K_i.

Proof. This follows immediately from Theorem 6. $\square$

Remark 1. *Theorem 6 can be used to obtain a freezing set for any finite* c_1*-connected digital image* $X \subset \mathbb{Z}^3$, *since X is trivially a union of 1-point cubes* $[a,a]_{\mathbb{Z}} \times [b,b]_{\mathbb{Z}} \times [c,c]_{\mathbb{Z}}$. *More usefully, if a subset H of X is a union of cubes,*

$$H = \bigcup_{i=1}^m \prod_{j=1}^n [a_{ij}, b_{ij}]_{\mathbb{Z}},$$

then a freezing set A for (X, c_1) *is*

$$A = (X \setminus H) \cup \bigcup_{i=1}^m \prod_{j=1}^n \{ a_{ij}, b_{ij} \}.$$

Remark 2. *Often, the freezing set of Theorem 6 is not minimal. However, the theorem is valuable in that it often gives a much smaller subset of X than X itself as a freezing set. As a simple example of the non-minimal assertion, consider*

$$X = [0,4]^2_{\mathbb{Z}} \times [0,2]_{\mathbb{Z}} \cup [0,4]^2_{\mathbb{Z}} \times [2,4]_{\mathbb{Z}}.$$

For this description of X, Theorem 6 gives the c_1*-freezing set*

$$A = \{ 0,4 \}^2 \times \{ 0,2 \} \cup \{ 0,4 \}^2 \times \{ 2,4 \} = \{ 0,4 \}^2 \times \{ 0,2,4 \},$$

a set of 12 points. However, by observing that X can be described as $X = [0,4]^3_{\mathbb{Z}}$, *we obtain from Theorem 5 the* c_1*-freezing set* $A' = \{ 0,4 \}^3$, *a set of 8 points.*

The following example shows that a cubical "cavity" (see Figure 1) need not affect determination of a freezing set.

Example 2. *Let $X = [0,6]_{\mathbb{Z}}^3 \setminus [2,4]_{\mathbb{Z}}^3$. Then, $A = \{0,6\}^3$ is a freezing set for (X, c_1).*

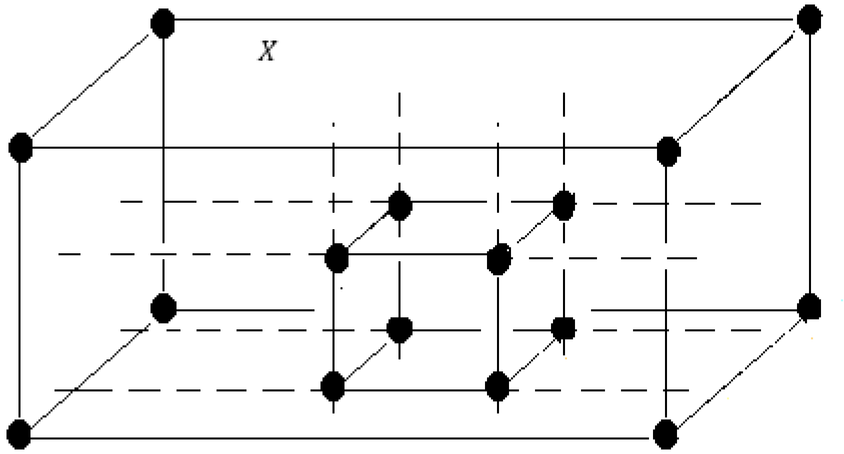

Figure 1. A cube with a cubical cavity.

Proof. Note that by Theorem 5, A is a freezing set for $K = ([0,6]_{\mathbb{Z}}^3, c_1)$. We show that removing $[2,4]_{\mathbb{Z}}^3$ need not change the freezing set.

Observe that we can decompose X as a union of cubes as follows: Let

$$L_X \text{ (left)} = [0,6]_Z \times [0,1]_{\mathbb{Z}} \times [0,6]_Z, \quad R_X \text{ (right)} = [0,6]_Z \times [5,6]_{\mathbb{Z}} \times [0,6]_Z,$$

$$F_X \text{ (front)} = [5,6]_{\mathbb{Z}} \times [0,6]_Z^2, \quad Ba_X \text{ (back)} = [0,1]_{\mathbb{Z}} \times [0,6]_Z^2,$$

$$Bo_X \text{ (bottom)} = [0,6]_Z^2 \times [0,1]_{\mathbb{Z}}, \quad T_X \text{ (top)} = [0,6]_Z^2 \times [5,6]_{\mathbb{Z}}.$$

Then, $X = L_X \cup R_X \cup F_X \cup Ba_X \cup Bo_X \cup T_X$. Theorem 5 gives us a freezing set B for (X, c_1) consisting of the corners of each of $L_X, R_X, F_X, Ba_X, Bo_X, T_X$.

However, suppose $f \in C(X, c_1)$ is such that $f|_A = \mathrm{id}_A$. As in the proof of Theorem 6, each of the faces L, R, F, Ba, Bo, T of K is a subset of $\mathrm{Fix}(f)$. Therefore, each $x \in B \setminus A$ is on an axis-parallel digital segment that joins two points of one of L, R, F, Ba, Bo, T, so by Proposition 1, $x \in \mathrm{Fix}(f)$. Therefore, A is a freezing set for (X, c_1). $\square$

6. c_n-Freezing Sets in $\mathbb{Z}^n$

We have the following.

Proposition 2 ([1]). *Let X be a finite digital image in $\mathbb{Z}^n$. Let $A \subset X$. Let $f \in C(X, c_u)$, where $1 \le u \le n$. If $Bd(A) \subset \mathrm{Fix}(f)$, then $A \subset \mathrm{Fix}(f)$.*

Theorem 7 ([1]). *Let X be a finite digital image in $\mathbb{Z}^n$. For $1 \le u \le n$, $Bd(X)$ is a freezing set for (X, c_u).*

The following is inspired by Theorem 7.

Theorem 8. *Let $X = \prod_{i=1}^n [a_i, b_i]_{\mathbb{Z}}$, where $n > 1$ and for all i, $b_i > a_i$. Then, $Bd(X)$ is a minimal freezing set for (X, c_n).*

Proof. By Theorem 7, $Bd(X)$ is a freezing set for (X, c_n). We must show its minimality.

Consider a point $x_0 = (x_1, \ldots, x_n) \in Bd(X)$. For some index i, $p_i(x_0) \in \{a_i, b_i\}$.

- If $p_i(x_0) = a_i$, the point $(x_1, \ldots, x_{i-1}, a_i + 1, x_i, \ldots, x_n)$ is a close neighbor of x_0.
- If $p_i(x_0) = b_i$, the point $(x_1, \ldots, x_{i-1}, b_i - 1, x_i, \ldots, x_n)$ is a close neighbor of x_0.

In either case, we must have x_0 as a member of every freezing set for (X, c_1), by Lemma 2. Thus, $Bd(X)$ is a minimal freezing set. $\square$

Theorem 9. *Let $X = \bigcup_{i=1}^{m} K_i$ where $m \geq 1$,*

$$K_i = \prod_{j=1}^{n} [a_{ij}, b_{ij}]_{\mathbb{Z}} \subset \mathbb{Z}^n,$$

and X is c_n-connected. Let $A_i = Bd(K_i)$ and let $A = \bigcup_{i=1}^{m} A_i$. Then, A is a freezing set for (X, c_n).

Proof. By Theorem 7, A_i is a freezing set for (K_i, c_n). Let $f \in C(X, c_n)$ be such that $f|_A = \mathrm{id}_A$. It follows from Proposition 2 that each $K_i \subset \mathrm{Fix}(f)$. Thus, $f = \mathrm{id}_X$, and the assertion follows. □

7. Conclusions and Future Work

We have studied freezing sets for finite digital images in $\mathbb{Z}^n$ with respect to the c_1- and c_n-adjacencies. For both of these adjacencies, we have shown that a decomposition of an image X as a finite union of cubes lets us find a freezing set for X as a union of freezing sets for the cubes of the decomposition. Such a freezing set is not generally minimal, but often is useful in having cardinality much smaller than the cardinality of X.

More general restrictions on $f|_A$, where A is a freezing set for (X, κ) and $f \in C(X, \kappa)$, restrict f on all of X in interesting ways. This will be shown in future work.

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References

1. Boxer, L. Fixed point sets in digital topology, 2. *Appl. Gen. Topol.* **2020**, *21*, 111–133. [CrossRef]
2. Boxer, L. Convexity and freezing sets in digital topology. *Appl. Gen. Topol.* **2021**, *22*, 121–137. [CrossRef]
3. Boxer, L. Subsets and freezing sets in the digital plane. *Hacet. J. Math. Stat.* **2021**, *50*, 991–1001. [CrossRef]
4. Boxer, L. Cold and freezing sets in the digital plane. *Topology Proc.* **2023**, *61*, 155–182.
5. Boxer, L. Some consequences of restrictions on digitally continuous functions. *Note Mat. accepted*.
6. Khalimsky, E.D.; Halimskii, E. On topologies of generalized segments. *Soviet Math. Dokl.* **1969**, *10*, 1508–1511.
7. Khalimsky, E.D.; Halimskii, E. Applications of connected ordered topological spaces in topology. In Proceedings of the Conference Math, College Park, MD, USA, 6–17 April 1970.
8. Kong, T.Y.; Kopperman, R.; Meyer, P.R. A topological approach to digital topology. *Am. Math. Mon.* **1991**, *98*, 901–917. [CrossRef]
9. Eckhardt, U.; Latecki, L.J. Topologies for the digital spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$. *Comput. Vision Image Underst.* **2003**, *90*, 295–312. [CrossRef]
10. Wyse, F.; Marcus, D. Solution to problem 5712. *Am. Math. Mon.* **1970**, *77*, 1119.
11. Rosenfeld, A. Digital topology. *Am. Math. Mon.* **1979**, *86*, 621–630. [CrossRef]
12. Rosenfeld, A. 'Continuous' functions on digital pictures. *Pattern Recognit. Lett.* **1986**, *4*, 177–184. [CrossRef]
13. Boxer, L.; Staecker, P.C. Fixed point sets in digital topology, 1. *Appl. Gen. Topol.* **2020**, *21*, 87–110. [CrossRef]
14. Boxer, L. A classical construction for the digital fundamental group. *J. Math. Imaging Vis.* **1999**, *10*, 51–62. [CrossRef]
15. Chen, L. Gradually varied surfaces and its optimal uniform approximation. *SPIE Proc.* **1994**, *2182*, 300–307.
16. Chen, L. *Discrete Surfaces and Manifolds*; Scientific Practical Computing: Rockville, MD, USA, 2004.
17. Boxer, L. Digitally continuous functions. *Pattern Recognit. Lett.* **1994**, *15*, 833–839. [CrossRef]