mathematics

Article

General Relativistic Space-Time with η_1-Einstein Metrics

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Abstract: The present research paper consists of the study of an η_1-Einstein soliton in general relativistic space-time with a torse-forming potential vector field. Besides this, we try to evaluate the characterization of the metrics when the space-time with a semi-symmetric energy-momentum tensor admits an η_1-Einstein soliton, whose potential vector field is torse-forming. In adition, certain curvature conditions on the space-time that admit an η_1-Einstein soliton are explored and build up the importance of the Laplace equation on the space-time in terms of η_1-Einstein soliton. Lastly, we have given some physical accomplishment with the connection of dust fluid, dark fluid and radiation era in general relativistic space-time admitting an η_1-Einstein soliton.

Keywords: general relativistic space-time; torse-forming vector fields; η_1-Einstein soliton; Einstein's field equation; dust fluid; dark fluid; radiation era; Laplacian equation

MSC: 53C44; 53C50; 53B50

Citation: Li, Y.; Mofarreh, F.; Dey, S.; Roy, S.; Ali, A. General Relativistic Space-Time with η_1-Einstein Metrics. *Mathematics* **2022**, *10*, 2530. https://doi.org/10.3390/math10142530

Academic Editors: Constantin Udriste and Vladimir Rovenski

Received: 1 June 2022
Accepted: 14 July 2022
Published: 21 July 2022

1. Background and Motivations

Throughout the article, we shall utilize the following acronyms: GRS—general relativistic space-time, TFVF—torse-forming vector field, and EMT—energy-momentum tensor. Ricci's soliton is well known among theoretical physicists because it is linked to string theory. It is well known that the theoretical physicists are interested in the Ricci soliton due to its association with string theory. In recent times, Ricci solitons are quite interesting in the field of differential geometry and geometric analysis as they characteristically present the Einstein metric. As a result, Ricci solitons in pseudo-Riemannian settings are extensively studied, and Hamilton introduced the concept of Ricci flow and extended it to address Thurston's geometric hypothesis. A Ricci soliton is a location in Hamilton's Ricci flow that is fixed (see details [1,2]) and an obvious extension of Einstein's metric is defined on a pseudo-Riemannian manifold (M, g) by

$$\frac{1}{2}\pounds_V g + Ric = \Lambda_1 g, \tag{1}$$

where $\pounds_V$ stands for the Lie-derivative in the way of $V \in \chi(M)$, Λ_1 is a constant and the Ricci tensor of g is presented by Ric. The Ricci soliton is classified as follows:

(i) If $\Lambda_1 < 0$, then the Ricci soliton is said to be shrinking.
(ii) for $\Lambda_1 > 0$, then it is said to be expanding.
(iii) If $\Lambda_1 = 0$, then it is implied to be steady.

Pigoli et al. [3] began by assuming the soliton constant Λ_1 becomes a smooth function on M and denoted as a Ricci almost soliton. Besides, Barros et al. proved a Ricci almost soliton which belongs to the details in [4,5].

Cho and Kimura [6] introduced the concept of an η_1-Ricci soliton as a generalization of Ricci soliton. An η_1-Ricci soliton equation is given by:

$$\pounds_V g + 2S + 2\Lambda_1 g + 2\mu_1 \eta_1 \otimes \eta_1 = 0 \tag{2}$$

for real constants Λ_1 and μ_1.

Now, the assumption of Einstein soliton was brought to light by G. Catino and L. Mazzieri [7] in 2016, which set up self-similar solutions to Einstein flow,

$$\frac{\partial}{\partial t} g(t) = -2S(g(t)), \;\; t \in [0, I]$$

where S, g stand for Ricci tensor, Riemannian metric. The equation of the η_1-Einstein soliton [8] is introduced by

$$\pounds_{\xi_1} g + 2S + (2\Lambda_1 - r)g + 2\mu_1 \eta_1 \otimes \eta_1 = 0, \tag{3}$$

where $\pounds_{\xi_1}$ is the Lie derivative endowed with the vector field ξ_1, Λ_1 and μ_1 are real constants and r stands for scalar curvature. For $\mu_1 = 0$, the data (g, ξ_1, Λ_1) are termed an Einstein soliton [7].

In [9], authors proved the space-time admitting Ricci soliton. Later, Blaga [10] evolved a depiction of the perfect fluid space-time admitting η_1-Ricci soliton and η_1-Einstein solitons. Ricci solitons associated with perfect fluid space time were synthesized by Venkatesha et al. [11]. Some Ricci soliton endowed space-time has been explored by several authors (see [12–14]) extensively in different ways. The setting of contact and complex manifolds that contain Ricci solitons and Einstein solitons has been investigated very recently in [15–24]; see their generalizations. We can find more motivations of our work from some papers (see [25–35]). The enchantment of this universe is its symmetry, i.e., the symmetries of the universe force objects to keep their movement. However, each symmetry imposes the conservation of a quantity over time. For translational symmetry, this quantity is the momentum. For rotational symmetry, this quantity is the angular momentum. For temporal symmetry, this quantity is energy. It is also one of the scientific essences that may be utilized to explain anything from natural laws to other physical phenomena such as general relativity. In the early 19th century, Albert Einstein established the "Theory of General Relativity" (GR).

The EMT $\mathcal{T}_1$ of type $(0, 2)$ is of the form [36] for a perfect fluid space-time,

$$\mathcal{T}_1(V_2, V_3) = \rho g(V_2, V_3) + (\sigma + \rho)\eta_1(V_2)\eta_1(V_3), \tag{4}$$

where the energy density and isotropic pressure, respectively, are denoted by σ and ρ. Moreover, $\eta_1(V_2) = g(X, \xi_1)$ is 1-form, which corresponds to the unit vector ξ_1 and $g(\xi_1, \xi_1) = -1$.

Furthermore, if $\rho = \sigma$, the ideal fluid is considered stiff matter [37]. Zel'dovich [38] initially established a stiff matter equation of state, which he employed in his cosmological model in that the primeval cosmos is considered to be a cold gas of baryons [38]. According to Zeldovich, the sound velocity of a stiff matter fluid is equivalent to the velocity of light. The radiation era was preceded by the stiff matter era with $\rho = \frac{\sigma}{3}$, the dark matter era with $\rho = -\sigma$, and the dust matter era with $\rho = 0$, according to [37,39]. It also emerged in certain cosmological theories in which dark matter is comprised of relativistic self-gravitating Bose–Einstein condensate, as cited by [40].

2. GRS with TFVF

Without the cosmological constant, Einstein's field equation is as follows:

$$S(V_2, V_3) - \frac{r}{2}g(V_2, V_3) = \kappa_1 \mathcal{T}_1(V_2, V_3), \tag{5}$$

where the EMT is denoted by $\mathcal{T}_1$, and the gravitational constant is $\kappa_1 \neq 0$. The Equation (5) suggests that matter dictates the geometry of space-time and that matter's velocity is dictated by the metric tensor of the non-flat space. Let (M^4, g) be a GRS that fulfills (5). Then, contracting the Equation (5) and seeing $g(\xi_1, \xi_1) = -1$ to yield

$$r = -\kappa_1 \tau_1, \tag{6}$$

where $\tau_1 = \mathrm{Tr}(\mathcal{T}_1)$. Now consider a specific scenario in which ξ_1 denotes a TFVF of the type [8,41].

$$\nabla_{V_2}\xi_1 = V_2 + \eta_1(V_2)\xi_1. \tag{7}$$

We may also prove the following relations in a GRS if the vector field ξ_1 is torse-forming.

$$\nabla_{\xi_1}\xi_1 = 0, \tag{8}$$

$$(\nabla_X \eta_1)(V_3) = g(V_2, V_3) + \eta_1(V_2)\eta_1(V_3), \tag{9}$$

$$R(V_2, V_3)\xi_1 = \eta_1(V_3)V_2 - \eta_1(V_2)V_3, \tag{10}$$

$$\eta_1(R(V_2, V_3)Z_1) = \eta_1(V_2)g(V_3, Z_1) - \eta_1(V_3)g(V_2, Z_1) \tag{11}$$

$\forall\ V_2, V_3, Z_1$. Utilizing (7), we conclude the following:

$$\begin{aligned}(\pounds_{\xi_1} g)(V_2, V_3) =& g(\nabla_{V_2}\xi_1, V_3) + g(V_2, \nabla_{V_3}\xi_1) \\ =& 2[g(V_2, V_3) + \eta_1(V_2)\eta_1(V_3)]\end{aligned} \tag{12}$$

$\forall\ V_2, V_3$.

3. Emergence of η_1-Einstein Solitons on GRS

Let the metric of a GRS (M^4, g) satisfy (3) for the η_1-Einstein soliton equation that the vector field V potential replaces with ξ_1 for torse-forming. Then (12) and (3) identities give the following:

$$S(V_2, V_3) = -[\Lambda_1 + 1 - \frac{r}{2}]g(V_2, V_3) - (\mu_1 + 1)\eta_1(V_2)\eta_1(V_3) \tag{13}$$

$\forall\ V_2, V_3$. Now, we use the contract property in the above equation to find

$$r = 4\Lambda_1 - \mu_1 + 3. \tag{14}$$

We scrutinize r in (14) with (6) to obtain

$$\mu_1 = 4\Lambda_1 + 3 + \kappa_1 \tau_1. \tag{15}$$

Let a semi-symmetric EMT $\mathcal{T}_1$ be given as

$$R(V_2, V_3) \cdot \mathcal{T}_1 = 0, \tag{16}$$

where the derivation on the tensor $\mathcal{T}_1$ delas with $R(V_2, V_3)$. From Equation (16), we imply the following

$$(R(V_2, V_3) \cdot \mathcal{T}_1)(Z_1, U_1) = 0, \tag{17}$$

which implies that

$$\mathcal{T}_1(R(V_2, V_3)Z_1, U_1) + \mathcal{T}_1(Z_1, R(V_2, V_3)U_1) = 0. \tag{18}$$

Now using (5), then (18), we have the following form

$$S(R(V_2,V_3)Z_1,U_1)+S(Z_1,R(V_2,V_3)U_1)=0, \tag{19}$$

that gives $R(V_2,V_3)\cdot S=0$, which means the space-time is Ricci semi-symmetric [42]. In view of (13) and (19), we find

$$(\mu_1+1)[\eta_1(R(V_2,V_3)Z_1)\eta_1(U_1)+\eta_1(Z_1)\eta_1(R(V_2,V_3)U_1)]=0. \tag{20}$$

Then we plug $V_2=U_1=\xi_1$ in (20) and employing (11) to construct $\mu_1=-1$. Putting $\mu_1=-1$ in (15), we have

$$\Lambda_1=-\frac{\kappa_1\tau_1}{4}-1.$$

This encourages the following:

Theorem 1. *Let semi-symmetric EMT endowed with GRS (M^4,g) contain an η_1-Einstein soliton $(g,\xi_1,\Lambda_1,\mu_1)$, such that ξ_1 is a TFVF. Then $\mu_1=-1$ and $\Lambda_1=-\frac{\kappa_1\tau_1}{4}-1$, where τ_1 is the trace of the EMT.*

Definition 1. *A space-time is present to be $\mathcal{W}_2$-flat if its $\mathcal{W}_2$-curvature tensor on n-dimensional manifold [43]*

$$\begin{aligned} &\mathcal{W}_2(V_2,V_3,Z_1,U_1)\\ =\quad &\grave{R}(V_2,V_3,Z_1,U_1)+\frac{1}{n-1}[g(V_2,Z_1)S(V_3,U_1)-g(V_3,Z_1)S(V_2,U_1)] \end{aligned} \tag{21}$$

$\forall$ V_2,V_3,Z_1 and U_1, identically zero.

Consider (M^4,g) to be a GRS that is $\mathcal{W}_2$-flat. Then from (21), we have

$$\grave{R}(V_2,V_3,Z_1,U_1)=-\frac{1}{3}[g(V_2,Z_1)S(V_3,U_1)-g(V_3,Z_1)S(V_2,U_1)]. \tag{22}$$

We set $V_2=U_1=e_i$ in (22), then tracing over $1\le i\le 4$ and then restoring the formulation of S from (13) to derive

$$\frac{4}{3}\Big[\Big(\Lambda_1+1-\frac{r}{2}\Big)g(V_3,Z_1)+(\mu_1+1)\eta_1(V_3)\eta_1(Z_1)\Big]+\frac{r}{3}g(V_3,Z_1)=0. \tag{23}$$

Using (6), the above equation becomes

$$\Big[4\Big(\Lambda_1+1+\frac{\kappa_1\tau_1}{2})\Big)-\kappa_1\tau_1\Big]g(V_3,Z_1)+4(\mu_1+1)\eta_1(V_3)\eta_1(Z_1)=0. \tag{24}$$

We take $V_3=Z_1=\xi_1$ in (24) to yield

$$\Lambda_1-\mu_1=-\frac{\kappa_1\tau_1}{4}. \tag{25}$$

Inserting the value of μ_1 given in (15), the preceding equation has the following form:

$$\Lambda_1=-\frac{\kappa_1\tau_1}{4}-1. \tag{26}$$

So, we established the following theorem:

Theorem 2. *Let (M^4,g) be a GRS, which is $\mathcal{W}_2$-flat and admits an η_1-Einstein soliton $(g,\xi_1,\Lambda_1,\mu_1)$, where ξ_1 is a TFVF. Then $\Lambda_1=-\frac{\kappa_1\tau_1}{4}-1$.*

Definition 2. *If its pseudo-projective curvature tensor $\mathcal{P}$ accordingly [44] in space-time is equal to zero*

$$\begin{aligned}\overline{\mathcal{P}}(V_2,V_3)Z_1 =& aR(V_2,V_3)Z_1 + b[S(V_3,Z_1)V_2 - S(V_2,Z_1)V_3] \\ & -\frac{r}{n}\Big(\frac{a}{n-1}+b\Big)[g(V_3,Z_1)V_2 - g(V_2,Z_1)V_3] \end{aligned} \tag{27}$$

$\forall$ V_2, V_3, Z_1 and a, $b \neq 0$ are constants, then space-time is preseted as pseudo-projectively flat.

Equation (27) reduces the following for exceptional case $a = 1, b = -\frac{1}{n-1}$

$$\begin{aligned}\overline{\mathcal{P}}(V_2,V_3)Z_1 &= R(V_2,V_3)Z_1 - \frac{1}{(n-1)}\Big[S(V_3,Z_1)V_2 - S(V_2,Z_1)V_3\Big] \\ &= \mathcal{P}(V_2,V_3)Z_1. \end{aligned} \tag{28}$$

Applying the inner product with W in (27) for pseudo-projective flat GRS (M^4, g), we have

$$\begin{aligned} a\grave{R}(V_2,V_3,Z_1,U_1) =& \frac{r}{4}\Big[\frac{a}{3}+b\Big][g(V_3,Z_1)g(V_2,U_1) - g(V_2,Z_1)g(V_3,U_1)] \\ & -b[S(V_3,Z_1)g(V_2,U_1) - S(V_2,Z_1)g(V_3,U_1)]. \end{aligned} \tag{29}$$

Setting $V_2 = U_1 = e_i$ in (29), tracing accordingly i, $1 \leq i \leq 4$ and reconstitution, the formula of S in (13) gives

$$\Big[\frac{r}{4} + \Lambda_1 - \frac{r}{2} + 1\Big](a+3b)g(V_3,Z_1) + (a+3b)(\mu_1+1)\eta_1(Y)\eta_1(Z_1) = 0. \tag{30}$$

We substitute $Y = Z = \xi_1$ into Equation (30) to give

$$\mu_1 - \Lambda_1 = -\frac{r}{4}, \tag{31}$$

provided $a + 3b \neq 0$. Now, we utilize identity (6) and locum the value of μ_1 from the identity (15) to yield

$$\Lambda_1 = -\frac{\kappa_1\tau_1}{4} - 1. \tag{32}$$

Hence, we find the following theorem:

Theorem 3. *Let pseudo-projectively flat GRS (M^4, g) contain an η_1-Einstein soliton $(g, \xi_1, \Lambda_1, \mu_1)$ such that ξ_1 is a TFVF. Then $\Lambda_1 = -\frac{\kappa_1\tau_1}{4} - 1$, provided $a + 3b \neq 0$.*

Definition 3. *A space-time is presented to be con-harmonically flat on n-dimensional manifold if its con-harmonic curvature tensor $\mathcal{H}$ [45]*

$$\begin{aligned}\mathcal{H}(V_2,V_3)Z_1 =& R(V_2,V_3)Z_1 - \frac{1}{(n-2)}[g(V_3,Z_1)QV_2 - g(V_2,Z_1)QV_3 \\ & + S(V_3,Z_1)V_2 - S(V_2,Z_1)V_3] \end{aligned} \tag{33}$$

for all fields V_2, V_3, Z_1 identically vanishes.

For con-harmonically flat GRS (M^4, g) and implementation inner product with U_1 in (33), we have

$$\begin{aligned}\grave{R}(V_2,V_3,Z_1,U_1) =& \frac{1}{2}[g(V_3,Z_1)S(V_2,U_1) - g(V_2,Z_1)S(V_3,U_1) \\ & + S(V_3,Z_1)g(V_2,U_1) - S(V_2,Z_1)g(V_3,U_1)]. \end{aligned} \tag{34}$$

Applying summation over $1 \leq i \leq 4$ after inserting $V_2 = U_1 = e_i$ in (34) and obtained formula for S from (13), we acquire

$$r = 0, \tag{35}$$

which implies that the space-time is flat. Next, by using (35), Equation (14) becomes

$$\Lambda_1 = \frac{\mu_1}{4} - \frac{3}{4}. \tag{36}$$

Therefore, we generate the following statement of the theorem:

Theorem 4. *Let conharmonically flat GRS (M^4, g) consist of an η_1-Einstein soliton $(g, \xi_1, \Lambda_1, \mu_1)$ such that ξ_1 is a TFVF. Then the space-time becomes flat and $\Lambda_1 = \frac{\mu_1}{4} - \frac{3}{4}$.*

The Q-curvature tensor formula for n-dimensional Riemannian manifold was initiated by Mantica and Suh [46] and presented notation $\mathcal{Q}$ is derived as

$$\mathcal{Q}(V_2, V_3)Z_1 = R(V_2, V_3)Z_1 - \frac{\psi}{(n-1)}[g(V_3, Z_1)V_2 - g(V_2, Z_1)V_3], \tag{37}$$

where ψ is an arbitrary scalar function.

Definition 4. *If $\mathcal{Q}$-curvature tensor is zero identically, then a space-time is $\mathcal{Q}$-flat.*

The following formula derived by considering GRS (M^4, g) is $\mathcal{Q}$-flat and exploring inner product with U_1 in (37)

$$\grave{R}(V_2, V_3, Z_1, U_1) = \frac{\psi}{3}[g(V_3, Z_1)g(V_2, U_1) - g(V_2, Z_1)g(V_3, U_1)]. \tag{38}$$

Tracing (38) by setting $V_2 = U_1 = e_i$, we arrive at

$$S(V_3, Z_1) = \psi g(V_3, Z_1). \tag{39}$$

The above conclusion provides that the space-time is Einstein. Entering $V_3 = Z_1 = \xi_1$ in (39) and utilizing Equation (13) for S, it provides

$$\mu_1 - \Lambda_1 + \frac{r}{2} = \psi. \tag{40}$$

Now, we exchange the value of μ_1 from the identity (15) to acquire

$$\Lambda_1 = \frac{2\psi - \kappa_1\tau_1}{6} - 1. \tag{41}$$

By previous conclusion, we obtain the following.

Theorem 5. *Let a $\mathcal{Q}$-flat GRS (M^4, g) consisting η_1-Einstein soliton $(g, \xi_1, \Lambda_1, \mu_1)$ that ξ_1 is a TFVF. Then the space-time converts into Einstein and provides $\Lambda_1 = \frac{2\psi - \kappa_1\tau_1}{6} - 1$.*

In (3), the metric of a GRS (M^4, g) satisfies the η_1-Einstein soliton (g, V, Λ_1, μ_1), then the Lie derivative $(\pounds_V g)$ as

$$(\pounds_V g)(V_2, V_3) = g(\nabla_{V_2} V, V_3) + g(V_2, \nabla_{V_3} V),$$

and with implementation (3), we have

$$S(V_2, V_3) = -\frac{1}{2}[g(\nabla_{V_2} V, V_3) + g(V_2, \nabla_{V_3} V)] - \left[\Lambda_1 - \frac{r}{2}\right] g(V_2, V_3) \\ -\mu_1 \eta_1(V_2)\eta_1(V_3). \tag{42}$$

Substituting $V_2 = V_3 = e_i$ (42) implies that

$$r = -div(V) - 4\left[\Lambda_1 - \frac{r}{2}\right] + \mu_1, \tag{43}$$

such that $div(V)$ stands for the divergence of V. Now, in light of (6) and making use of μ_1 from the identity (15), the previous equation reads

$$div(V) = 3. \tag{44}$$

If we consider $V = grad(f)$, for a smooth function f, the identity (44) turns into

$$\Delta(f) = 3, \tag{45}$$

where $\Delta(f)$ is the Laplacian equation confirmed by f. This leads to the following:

Theorem 6. *Assuming that (M^4, g) is a GRS that admits an η_1-Einstein soliton (g, V, Λ_1, μ_1), then the Laplacian Equation (45) is satisfied for the Laplacian, where a smooth function $V = f$.*

4. η_1-Einstein Soliton with Dust Fluid GRS

For the EMT defined in [47] and pressure-less fluid space-time, we have

$$\mathcal{T}_1(V_2, V_3) = \sigma \eta_1(V_2)\eta_1(V_3). \tag{46}$$

Now, with the help of the identities (5) and (46), we obtain

$$S(V_2, V_3) = \frac{r}{2} g(V_2, V_3) + \kappa_1 \sigma \eta_1(V_2)\eta_1(V_3). \tag{47}$$

Taking into account (3), Equation (47) turns into the following

$$(\pounds_V g)(V_2, V_3) + 2\Lambda_1 g(V_2, V_3) + 2(\kappa_1 \sigma + \mu_1)\eta_1(V_2)\eta_1(V_3) = 0. \tag{48}$$

Tracing after putting $V_2 = V_3 = e_i$ in (48), we have

$$\Lambda_1 = \frac{\mu_1 + \kappa_1 \sigma}{4} - \frac{div(V)}{4}. \tag{49}$$

So, from the previous identity, we obtain

Theorem 7. *If a dust fluid GRS contains an η_1-Einstein soliton (g, V, Λ_1, μ_1), then $\Lambda_1 = \frac{\mu_1 + \kappa_1 \sigma}{4} - \frac{div(V)}{4}$.*

Utilizing (49), we can give the following remark:

Remark 1. *If a dust fluid GRS contains an η_1-Einstein soliton (g, V, Λ_1, μ_1), then $\Lambda_1 = \frac{\mu_1 + \kappa_1 \sigma}{4}$ iff the vector field V is solenoidal.*

5. η_1-Einstein Soliton on Dark Fluid GRS

In this space-time, ρ is organized by σ. Then, the structure of EMT (4) is

$$\mathcal{T}_1(V_2, V_3) = \rho g(V_2, V_3). \tag{50}$$

Combining (5) and (50), we derive

$$S(V_2, V_3) = \Big[\kappa_1 \rho + \frac{r}{2}\Big] g(V_2, V_3). \tag{51}$$

In view of (3), the above equation takes the form

$$(\pounds_V g)(V_2, V_3) + (2\Lambda_1 + 2\kappa_1\rho) g(V_2, V_3) + 2\mu_1 \eta_1(V_2)\eta_1(V_3) = 0. \tag{52}$$

Tracing Equation (52) after invoking $V_2 = V_3 = e_i$, we have

$$\Lambda_1 = \frac{\mu_1}{4} - \kappa_1\rho - \frac{div(V)}{4}. \tag{53}$$

So, we have finalized the following result:

Theorem 8. *If an η_1-Einstein soliton (g, V, Λ_1, μ_1) is associated with dark fluid GRS, then the scalar curvature turns into $\Lambda_1 = \frac{\mu_1}{4} - \kappa_1\rho - \frac{div(V)}{4}$.*

In view of (53), we achieve

Remark 2. *If a dark fluid GRS satisfies an η_1-Einstein soliton (g, V, Λ_1, μ_1), then the scalar curvature develops into $\Lambda_1 = \frac{\mu_1}{4} - \kappa_1\rho$ iff the vector field V is solenoidal.*

6. η_1-Einstein Soliton Admitting Radiation Era in GRS

Now, characterization of radiation era is denoted by $\rho = \frac{\sigma}{3}$ in the perfect fluid space-time. So, the feature of EMT (4) develops into

$$\mathcal{T}_1(V_2, V_3) = \rho[g(V_2, V_3) + 4\eta_1(V_2)\eta_1(V_3)]. \tag{54}$$

Using (5) and (54), we obtain

$$S(V_2, V_3) = \Big[\kappa_1\rho + \frac{r}{2}\Big] g(V_2, V_3) + 4\kappa_1\rho\eta_1(V_2)\eta_1(V_3). \tag{55}$$

Equation (3) provides the following after combining with (55):

$$(\pounds_V g)(V_2, V_3) + (2\Lambda_1 + 2\kappa_1\rho) g(V_2, V_3) + (8\kappa_1\rho + 2\mu_1)\eta_1(V_2)\eta_1(V_3) = 0. \tag{56}$$

Tracing Equation (56) after replacing $V_2 = V_3 = e_i$ provides

$$\Lambda_1 = \frac{\mu_1}{4} - \frac{div(V)}{4}. \tag{57}$$

So, we obtain the next theorem as:

Theorem 9. *If a radiation era GRS contains an η_1-Einstein soliton (g, V, Λ_1, μ_1), then $\Lambda_1 = \frac{\mu_1}{4} - \frac{div(V)}{4}$.*

Also using the identity (57), we obtain

Remark 3. *If a radiation era GRS admits an η_1-Einstein soliton (g, V, Λ_1, μ_1), then V is solenoidal iff $\Lambda_1 = \frac{\mu_1}{4}$.*

7. Conclusions Remark

We investigated the η_1-Einstein soliton which is revealed by the space-time of general relativity with the semi-symmetrical tensor energy-momentum and determined the nature of the metrics, such that the potential vector field is twisted. Next, we established some interesting and needful results for $\mathcal{W}_2-$flat space-time, pseudo-projectively flat and $\mathcal{Q}$-flat, admitting the η_1-Einstein soliton. We have also shown that if the space-time is conharmonically flat and admits a η_1-Einstein soliton, whose potential vector field is torse-forming, then the space-time becomes flat. We assumed the potential vector fields are of the gradient type of η_1-Einstein soliton, thus the Laplace equation has been constructed.

The gravitational field contains the space-time curvature with the origin as an EMT in General Theory of Relativity. In mathematical language, the most effective tools for understanding general relativity are the relativistic fluids models and differential geometry. The geometry of the Lorentzian manifold starts with the investigation of the causal character of the manifold's vectors; as a result of this causality, the Lorentzian manifold becomes a convenient choice for the study of general relativity. As a matter of the substance of space-time, the EMT plays a crucial role; the matter is considered to be fluid with density and pressure, as well as kinematic and dynamical characteristics such as velocity, vorticity, shear and expansion [44,46,48–52]. The η_1-Einstein soliton is important as it can help in understanding the concepts of energy and entropy in general relativity. This property is the same as that of the heat equation due to which an isolated system loses the heat for a thermal equilibrium.

Author Contributions: Conceptualization, F.M. and Y.L.; methodology, A.A.; software, S.D.; validation, Y.L., F.M. and S.R.; formal analysis, A.A.; investigation, Y.L.; resources, S.D.; data curation, S.R.; writing—original draft preparation, S.D.; writing—review and editing, Y.L.; visualization, F.M.; supervision, A.A.; project administration, F.M.; funding acquisition, Y.L. All authors have read and agreed to the published version of the manuscript.

Funding: The last author extends his appreciation to the deanship of scientific research at King Khalid University for funding this work through the research groups program under grant number R.G.P.2/130/43. The authors also express their gratitude to Princess Nourah bint Abdulrahman University Researchers supporting project number (PNURSP2022R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. This research was funded by National Natural Science Foundation of China (Grant No. 12101168) and Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ22A010014).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We gratefully acknowledge the constructive comments from the editor and the anonymous referees. The last author extends his appreciation to the deanship of scientific research at King Khalid University for funding this work through the research groups program under grant number R.G.P.2/130/43. The authors also express their gratitude to Princess Nourah bint Abdulrahman University Researchers supporting project number (PNURSP2022R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. This research was funded by National Natural Science Foundation of China (Grant No. 12101168) and Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ22A010014).

Conflicts of Interest: The authors declare no conflict of interest.

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