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# Reciprocal Formulae among Pell and Lucas Polynomials

**Mei Bai [1], Wenchang Chu [1,2,*] and Dongwei Guo [1]**

[1] School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China; baimei0418@outlook.com (M.B.); guo.dongwei2018@outlook.com (D.G.)

[2] Department of Mathematics and Physics, University of Salento, 73100 Lecce, Italy

* Correspondence: chu.wenchang@unisalento.it

**Abstract:** Motivated by a problem proposed by Seiffert a quarter of century ago, we explicitly evaluate binomial sums with Pell and Lucas polynomials as weight functions. Their special cases result in several interesting identities concerning Fibonacci and Lucas numbers.

**Keywords:** generating function; Lambert series; bisection series; reciprocal relation; binomial coefficient; Fibonacci and Lucas numbers; Pell and Lucas polynomials

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## 1. Introduction and Motivation

The Pell and Lucas polynomials were introduced by Horadam and Mahon [1]. They can equivalently be defined by the recurrence relations

$$\begin{aligned} P_n(x) &= 2xP_{n-1}(x) + P_{n-2}(x), \\ Q_n(x) &= 2xQ_{n-1}(x) + Q_{n-2}(x); \end{aligned}$$

with different initial conditions

$$\begin{aligned} P_0(x) &= 0 \quad \text{and} \quad P_1(x) = 1, \\ Q_0(x) &= 2 \quad \text{and} \quad Q_1(x) = 2x. \end{aligned}$$

The corresponding ordinary generating functions read as

$$\begin{aligned} \Phi(x,y) &= \sum_{k=0}^{\infty} P_k(x)y^k = \frac{y}{1-2xy-y^2} = \frac{1}{(\alpha-\beta)(1-y\alpha)} - \frac{1}{(\alpha-\beta)(1-y\beta)}, \\ \Psi(x,y) &= \sum_{k=0}^{\infty} Q_k(x)y^k = \frac{2-2xy}{1-2xy-y^2} = \frac{1}{1-y\alpha} + \frac{1}{1-y\beta}; \end{aligned}$$

leading us to the explicit formulae with Binet forms

$$P_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad Q_n(x) = \alpha^n + \beta^n,$$

where, for brevity, we employ the following two notations:

$$\alpha = \alpha(x) = x + \sqrt{x^2+1} \quad \text{and} \quad \beta = \beta(x) = x - \sqrt{x^2+1}.$$

These polynomials have constantly been continuing to amaze the mathematical world for their fascinating properties and wide applications in intertwining various topics of mathematics, such as combinatorics, discrete mathematics and number theory. Some basic properties and power sums are examined by Chu–Li [2,3] and Horadam–Mahon [1]. A comprehensive coverage about more properties and known results can be found in the monographs by Grimaldi [4] and Koshy [5].

In particular, they can be considered as polynomial extensions of the Fibonacci number $P_n(1/2) = F_n$ and the Lucas number $Q_n(1/2) = L_n$ that satisfy the recurrence relations

$$F_n = F_{n-1} + F_{n-2} \quad \text{and} \quad L_n = L_{n-1} + L_{n-2}$$

and the initial conditions

$$F_1 = F_2 = 1 \quad \text{and} \quad L_1 = 1, \; L_2 = 3.$$

A quarter of a century ago, Seiffert [6] proposed, as a problem, an identity that can be equivalently expressed as the following reciprocity:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n+2}{n-2k} P_{2k+1}(x) = (2x)^n P_{n+1}(x^{-1}). \tag{1}$$

This problem was forgotten until very recently when Abel and Kushnirevych [6] found a solution by employing generating functions and the Lagrange inversion formula. When searching for an alternative approach to solve this problem, we found three further reciprocal relations with similar forms. Their particular cases result in several remarkable identities for Fibonacci and Lucas numbers. Eight representatives of them are highlighted:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n+2}{n-2k} F_{2k+1} = \frac{F_{3n+3}}{2}, \qquad \text{(see Corollary 1)}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{n-2k} L_{2k} = \frac{L_{3n}}{2} + \binom{2n}{n}, \qquad \text{(see Corollary 3)}$$

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{k}{n} \binom{2n}{n-2k} F_{2k} = \frac{F_{3n-3}}{4}, \qquad \text{(see Corollary 5)}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1+2k}{1+n} \binom{2n+2}{n-2k} L_{2k+1} = \frac{L_{3n}}{2}, \qquad \text{(see Corollary 7)}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n+2}{n-2k} F_{6k+3} = 2^{2n+1} F_{n+1}, \qquad \text{(see Corollary 1)}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{n-2k} L_{6k} = 2^{2n-1} L_n + \binom{2n}{n}, \qquad \text{(see Corollary 3)}$$

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{k}{n} \binom{2n}{n-2k} F_{6k} = 2^{2n-2} F_{n-1}, \qquad \text{(see Corollary 5)}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1+2k}{1+n} \binom{2n+2}{n-2k} L_{6k+3} = 2^{2n+1} L_n. \qquad \text{(see Corollary 7)}$$

Therefore, the objective of this paper is to systematically examine reciprocal formulae for the Pell and Lucas polynomials as well their applications to the counterparts of Fibonacci and Lucas numbers. Even though we failed to locate them in the encyclopedic monographs of Grimaldi [4] and Koshy [5], it cannot be excluded that some of them may have appeared previously in the vast literature on the related topics.

Throughout the paper, we make use of Lambert's binomial series (see Riordan [7] Section 4.5 and [8,9]), which are well-known in classical analysis. Let $y$ and $\tau$ be the two variables related by the equation $y = \tau/(1+\tau)^b$. Then,

$$\sum_{n=0}^{\infty} \frac{a}{a+bn}\binom{a+bn}{n} y^n = (1+\tau)^a, \tag{2}$$

$$\sum_{n=0}^{\infty} \binom{a+bn}{n} y^n = \frac{(1+\tau)^{a+1}}{1+\tau-b\tau}. \tag{3}$$

The rest of the paper is divided into four sections with each section being dedicated to a binomial sum, like (1). By combining Lambert's series with the bisectional series method, we derive, for each binomial sum, its ordinary generating function and the closed reciprocal formula in terms of $P_n(x)$ or $Q_n(x)$. Then eight summation formulae for the Fibonacci and Lucas numbers follow as consequences.

## 2. Reciprocity for $\mathcal{A}_n(x)$

As a warm up, we begin by examining the first binomial sum of Seiffert [6] given by

$$\mathcal{A}_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n+2}{n-2k} P_{2k+1}(x). \tag{4}$$

**Lemma 1.** *Let $y$ and $T$ be the two variables related by $T = y(1+T)^2$. Then, we have the closed form generating function:*

$$\mathrm{A}(x,y) = \sum_{n=0}^{\infty} \mathcal{A}_n(x) y^n = \frac{(1+T)^4}{(1-T^2)^2 - 4T^2x^2}.$$

**Proof.** By interchanging the summation order and then making the replacement $n \to 2k+i$ on the summation index, we can manipulate its generating function, as follows:

$$\begin{aligned}
\mathrm{A}(x,y) = \sum_{n=0}^{\infty} \mathcal{A}_n(x) y^n &= \sum_{n=0}^{\infty} y^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n+2}{n-2k} P_{2k+1}(x) \\
&= \sum_{k=0}^{\infty} y^{2k} P_{2k+1}(x) \sum_{n=2k}^{\infty} \binom{2n+2}{n-2k} y^{n-2k}.
\end{aligned}$$

The inner sum can be evaluated by (3)

$$\sum_{n=2k}^{\infty} \binom{2n+2}{n-2k} y^{n-2k} = \sum_{i=0}^{\infty} \binom{4k+2i+2}{i} y^i = \frac{(1+T)^{3+4k}}{1-T},$$

where $T$ is an implicit function of $y$ determined by $T = y(1+T)^2$. By substitution, we have

$$\begin{aligned}
\mathrm{A}(x,y) &= \sum_{k=0}^{\infty} y^{2k} P_{2k+1}(x) \frac{(1+T)^{3+4k}}{1-T} \\
&= \frac{1+T}{(1-T)y} \sum_{k=0}^{\infty} P_{2k+1}(x) T^{2k+1}.
\end{aligned}$$

The last sum results in the bisectional series

$$\sum_{k=0}^{\infty} P_{2k+1}(x) T^{2k+1} = \frac{\Phi(x,T) - \Phi(x,-T)}{2} = \frac{T(1-T^2)}{(1-T^2)^2 - 4T^2x^2}.$$

By simplifying

$$\mathrm{A}(x,y) = \frac{(1+T)^3}{(1-T)T} \times \frac{T(1-T^2)}{(1-T^2)^2 - 4T^2x^2},$$

we find the generating function stated in Lemma 1. □

**Theorem 1** ($n \in \mathbb{N}_0$: Seiffert [6]).

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n+2}{n-2k} P_{2k+1}(x) = (2x)^n P_{n+1}(x^{-1}).$$

**Proof.** Observe that the generating function in Lemma 1 almost coincides with

$$\Phi(x^{-1}, 2xy) = \sum_{n=0}^{\infty} P_n(x^{-1})(2xy)^n = \frac{2xy}{1-4y-4x^2y^2} = \frac{2xy(1+T)^4}{(1-T^2)^2 - 4T^2x^2} \tag{5}$$

since $y \to T/(1+T)^2$. By comparing the coefficients of $y^n$ in the two equations, we confirm Seiffert's reciprocal Formula (1) in the theorem. □

Let $\lambda$ be a natural number. It is routine to check the following relations:

$$P_n\left(\frac{L_{2\lambda-1}}{2}\right) = \frac{F_{2\lambda n-n}}{F_{2\lambda-1}}, \qquad P_n\left(\frac{\sqrt{5}}{2}F_{2\lambda}\right) = \begin{cases} \frac{F_{2\lambda n}}{L_{2\lambda}}\sqrt{5}, & n \equiv_2 0; \\ \frac{L_{2\lambda n}}{L_{2\lambda}}, & n \equiv_2 1; \end{cases} \tag{6}$$

$$Q_n\left(\frac{L_{2\lambda-1}}{2}\right) = L_{2\lambda n-n}, \qquad Q_n\left(\frac{\sqrt{5}}{2}F_{2\lambda}\right) = \begin{cases} L_{2\lambda n}, & n \equiv_2 0; \\ F_{2\lambda n}\sqrt{5}, & n \equiv_2 1. \end{cases} \tag{7}$$

As applications, we present four identities for Fibonacci and Lucas numbers.

**Corollary 1** ($n \in \mathbb{N}_0$).

$$\boxed{x = \frac{1}{2} = \frac{L_1}{2}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n+2}{n-2k} F_{2k+1} = \frac{1}{2} F_{3n+3},$$

$$\boxed{x = \frac{\sqrt{5}}{2} = \frac{\sqrt{5}}{2}F_2} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n+2}{n-2k} L_{4k+2} = \frac{5^{n+1} - (-1)^{n+1}}{2},$$

$$\boxed{x = 2 = \frac{L_3}{2}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n+2}{n-2k} F_{6k+3} = 2^{2n+1} F_{n+1},$$

$$\boxed{x = \frac{3\sqrt{5}}{2} = \frac{\sqrt{5}}{2}F_4} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n+2}{n-2k} L_{8k+4} = \frac{9^{n+1} - (-5)^{n+1}}{2}.$$

We remark further that the expressions corresponding to the second and the fourth sums have the following simple rational generating functions:

$$\frac{3}{1-4x-5x^2} \quad \text{and} \quad \frac{7}{1-4x-45x^2}.$$

**Proof.** By assigning four specific values to $x$ in Theorem 1 and then applying (6) and (7), we recover the four summation formulae for the Fibonacci and Lucas numbers in Corollary 1, where the first three were given explicitly by Seiffert [6]. □

Furthermore, denoting the imaginary unit by "$\mathbf{i}$", we also have, for $\lambda \in \mathbb{N}$, four similar relations:

$$P_n\Big(\frac{\mathbf{i}}{2}L_{2\lambda}\Big) = \mathbf{i}^{n-1}\frac{F_{2\lambda n}}{F_{2\lambda}}, \quad P_n\Big(\frac{\mathbf{i}\sqrt{5}}{2}F_{2\lambda-1}\Big) = \mathbf{i}^{n-1}\begin{cases} \dfrac{F_{2\lambda n-n}}{L_{2\lambda-1}}\sqrt{5}, & n \equiv_2 0; \\ \dfrac{L_{2\lambda n-n}}{L_{2\lambda-1}}, & n \equiv_2 1; \end{cases} \tag{8}$$

$$Q_n\Big(\frac{\mathbf{i}}{2}L_{2\lambda}\Big) = \mathbf{i}^{n}L_{2\lambda n}, \qquad Q_n\Big(\frac{\mathbf{i}\sqrt{5}}{2}F_{2\lambda-1}\Big) = \mathbf{i}^{n}\begin{cases} L_{2\lambda n-n}, & n \equiv_2 0; \\ F_{2\lambda n-n}\sqrt{5}, & n \equiv_2 1. \end{cases} \tag{9}$$

Then, we have the following four formulae of alternating sums.

**Corollary 2** ($n \in \mathbb{N}_0$)**.**

$$\boxed{\begin{matrix} x = \frac{\mathbf{i}\sqrt{5}}{2} = \frac{\mathbf{i}\sqrt{5}}{2}F_1 \\ \omega = 2+\mathbf{i} \end{matrix}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{2n+2}{n-2k}L_{2k+1} = \frac{\omega^{n+1}-\bar{\omega}^{n+1}}{\omega-\bar{\omega}},$$

$$\boxed{\begin{matrix} x = \frac{3\mathbf{i}}{2} = \frac{\mathbf{i}}{2}L_2 \\ \omega = 2+\mathbf{i}\sqrt{5} \end{matrix}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{2n+2}{n-2k}F_{4k+2} = \frac{\omega^{n+1}-\bar{\omega}^{n+1}}{\omega-\bar{\omega}},$$

$$\boxed{\begin{matrix} x = \mathbf{i}\sqrt{5} = \frac{\mathbf{i}\sqrt{5}}{2}F_3 \\ \omega = 2+4\mathbf{i} \end{matrix}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{2n+2}{n-2k}L_{6k+3} = 4\frac{\omega^{n+1}-\bar{\omega}^{n+1}}{\omega-\bar{\omega}},$$

$$\boxed{\begin{matrix} x = \frac{7\mathbf{i}}{2} = \frac{\mathbf{i}}{2}L_4 \\ \omega = 2+3\mathbf{i}\sqrt{5} \end{matrix}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{2n+2}{n-2k}F_{8k+4} = 3\frac{\omega^{n+1}-\bar{\omega}^{n+1}}{\omega-\bar{\omega}}.$$

Among the above four closed expressions displayed on the right, the former two correspond to the integer sequences **[A099456]** and **[A190967]** recorded in [10] and generated, respectively, by

$$\frac{1}{1-4x+5x^2} \quad \text{and} \quad \frac{1}{1-4x+9x^2};$$

while the latter two are the integer sequences generated, respectively, by

$$\frac{4}{1-4x+20x^2} \quad \text{and} \quad \frac{3}{1-4x+49x^2}.$$

**Proof.** These identities may serve as counterparts to those in Corollary 1. They can be verified by applying (8) and (9) in conjunction with Theorem 1. □

### 3. Reciprocity for $\mathcal{B}_n(x)$

This section is devoted to the second binomial sum defined by

$$\mathcal{B}_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{2n}{n-2k}Q_{2k}(x). \tag{10}$$

**Lemma 2.** *Let $y$ and $T$ be the two variables related by $T = y(1+T)^2$. Then, we have the closed form generating function*

$$\mathrm{B}(x,y) = \sum_{n=0}^{\infty}\mathcal{B}_n(x)y^n = \frac{1+T}{1-T} + \frac{(1+T^2)(1+T)^2}{(1-T^2)^2-4T^2x^2}.$$

**Proof.** The generating function can similarly be computed as follows:

$$\begin{aligned}
\mathrm{B}(x,y) &= \sum_{n=1}^{\infty} \mathcal{B}_n(x) y^n = \sum_{n=1}^{\infty} y^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{n-2k} Q_{2k}(x) \\
&= \sum_{k=0}^{\infty} y^{2k} Q_{2k}(x) \sum_{n=2k}^{\infty} \binom{2n}{n-2k} y^{n-2k} \\
&= \sum_{k=0}^{\infty} y^{2k} Q_{2k}(x) \frac{(1+T)^{1+4k}}{1-T} \\
&= \frac{1+T}{1-T} \sum_{k=0}^{\infty} Q_{2k}(x) T^{2k}.
\end{aligned}$$

Since the last sum results in the bisectional series

$$\sum_{k=0}^{\infty} Q_{2k}(x) T^{2k} = \frac{\Psi(x,T) + \Psi(x,-T)}{2} = 1 + \frac{1-T^4}{(1-T^2)^2 - 4T^2x^2},$$

we get the expression

$$\mathrm{B}(x,y) = \frac{1+T}{1-T} \times \left\{ 1 + \frac{1-T^4}{(1-T^2)^2 - 4T^2x^2} \right\},$$

which is equivalent to that in Lemma 2. □

From Lemma 2, we can prove the second reciprocal formula.

**Theorem 2** ($n \in \mathbb{N}_0$).

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{n-2k} Q_{2k}(x) = \binom{2n}{n} + 2^{n-1} x^n Q_n(x^{-1}).$$

**Proof.** Keeping in mind that $y \to T/(1+T)^2$, comparing the generating function in Lemma 2 with the function

$$\Psi(x^{-1}, 2xy) = \sum_{n=0}^{\infty} Q_n(x^{-1})(2xy)^n = \frac{2-4y}{1-4y-4x^2y^2} = \frac{2(1+T^2)(1+T)^2}{(1-T^2)^2 - 4T^2x^2} \tag{11}$$

and then extracting the coefficients of $y^n$, we find the formula presented in Theorem 2. □

Theorem 2 contains four summation formulae for Lucas numbers.

**Corollary 3** ($n \in \mathbb{N}_0$).

$$\boxed{x = \frac{1}{2} = \frac{L_1}{2}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{n-2k} L_{2k} = \binom{2n}{n} + \frac{1}{2} L_{3n},$$

$$\boxed{x = \frac{\sqrt{5}}{2} = \frac{\sqrt{5}}{2} F_2} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{n-2k} L_{4k} = \binom{2n}{n} + \frac{5^n + (-1)^n}{2},$$

$$\boxed{x = 2 = \frac{L_3}{2}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{n-2k} L_{6k} = \binom{2n}{n} + 2^{2n-1} L_n,$$

$$\boxed{x = \frac{3\sqrt{5}}{2} = \frac{\sqrt{5}}{2} F_4} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2n}{n-2k} L_{8k} = \binom{2n}{n} + \frac{9^n + (-5)^n}{2}.$$

**Proof.** These identities were derived from Theorem 2 by applying (6) and (7). □

The corresponding alternating sums are as follows.

**Corollary 4** ($n \in \mathbb{N}_0$).

$$\boxed{\begin{matrix} x = \frac{\mathbf{i}\sqrt{5}}{2} = \frac{\mathbf{i}\sqrt{5}}{2}F_1 \\ \omega = 2+\mathbf{i} \end{matrix}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{2n}{n-2k}L_{2k} = \binom{2n}{n} + \frac{1}{2}\Big\{\omega^n + \bar{\omega}^n\Big\},$$

$$\boxed{\begin{matrix} x = \frac{3\mathbf{i}}{2} = \frac{\mathbf{i}}{2}L_2 \\ \omega = 2+\mathbf{i}\sqrt{5} \end{matrix}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{2n}{n-2k}L_{4k} = \binom{2n}{n} + \frac{1}{2}\Big\{\omega^n + \bar{\omega}^n\Big\},$$

$$\boxed{\begin{matrix} x = \mathbf{i}\sqrt{5} = \frac{\mathbf{i}\sqrt{5}}{2}F_3 \\ \omega = 2+4\mathbf{i} \end{matrix}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{2n}{n-2k}L_{6k} = \binom{2n}{n} + \frac{1}{2}\Big\{\omega^n + \bar{\omega}^n\Big\},$$

$$\boxed{\begin{matrix} x = \frac{7\mathbf{i}}{2} = \frac{\mathbf{i}}{2}L_4 \\ \omega = 2+3\mathbf{i}\sqrt{5} \end{matrix}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{2n}{n-2k}L_{8k} = \binom{2n}{n} + \frac{1}{2}\Big\{\omega^n + \bar{\omega}^n\Big\}.$$

**Proof.** They were similarly obtained from Theorem 2 by applying (8) and (9). □

## 4. Reciprocity for $\mathcal{C}_n(x)$

Furthermore, the third binomial sum is defined by

$$\mathcal{C}_n(x) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{2k}{n}\binom{2n}{n-2k}P_{2k}(x). \tag{12}$$

Its generating function can also be determined explicitly.

**Lemma 3.** *Let $y$ and $T$ be the two variables related by $T = y(1+T)^2$. Then, we have the closed form generating function:*

$$\mathrm{C}(x,y) = \sum_{n=1}^{\infty}\mathcal{C}_n(x)y^n = \frac{(1-T)^2(1+T)^2}{2x\big\{(1-T^2)^2 - 4T^2x^2\big\}} - \frac{1}{2x}.$$

**Proof.** In fact, we have

$$\begin{aligned} \mathrm{C}(x,y) = \sum_{n=1}^{\infty}\mathcal{C}_n(x)y^n &= \sum_{n=1}^{\infty}y^n\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}\frac{2k}{n}\binom{2n}{n-2k}P_{2k}(x) \\ &= \sum_{k=1}^{\infty}y^{2k}P_{2k}(x)\sum_{n=2k}^{\infty}\frac{2k}{n}\binom{2n}{n-2k}y^{n-2k} \\ &= \sum_{k=1}^{\infty}y^{2k}P_{2k}(x)\sum_{i=0}^{\infty}\frac{2k}{2k+i}\binom{4k+2i}{i}y^i. \end{aligned}$$

Then, we can evaluate the inner sum by (2)

$$\sum_{n=2k}^{\infty}\frac{2k}{n}\binom{2n}{n-2k}y^{n-2k} = \sum_{i=0}^{\infty}\frac{4k}{4k+2i}\binom{4k+2i}{i}y^i = (1+T)^{4k}.$$

By substitution, we have

$$\mathrm{C}(x,y) = \sum_{k=1}^{\infty}y^{2k}P_{2k}(x)(1+T)^{4k} = \sum_{k=1}^{\infty}P_{2k}(x)T^{2k}.$$

Rewriting the bisectional series

$$\sum_{k=1}^{\infty} P_{2k}(x)T^{2k} = \frac{\Phi(x,T)+\Phi(x,-T)}{2} = \frac{T(1-T^2)}{(1-T^2)^2-4T^2x^2}$$
$$= \frac{(1-T)^2(1+T)^2}{2x\{(1-T^2)^2-4T^2x^2\}} - \frac{1}{2x},$$

we get the generating function for $\mathcal{C}_n(x,y)$. □

Now, we have the following third reciprocal formula.

**Theorem 3** ($n \in \mathbb{N}$)**.**

$$\mathcal{C}_n(x) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{k}{n}\binom{2n}{n-2k} P_{2k}(x) = 2^{n-2}x^{n-1}P_{n-1}(x^{-1}).$$

**Proof.** Under the setting of $P_{-1}(x) = 1$, the generating function $\Phi(x^{-1}, 2xy)$ in (5) can be slightly modified as follows:

$$1 + 2xy\Phi(x^{-1}, 2xy) = \sum_{n=0}^{\infty} P_{n-1}(x^{-1})(2xy)^n = \frac{1-4y}{1-4y-4x^2y^2}$$
$$= \frac{(1-T)^2(1+T)^2}{(1-T^2)^2-4T^2x^2} \text{ where } y \to T/(1+T)^2.$$

Taking into account Lemma 3 and then comparing the coefficients of $y^n$ in the two equations displayed above, we find the formula stated in Theorem 3. □

As applications, we give four summation formulae for Fibonacci numbers.

**Corollary 5** ($n \in \mathbb{N}$)**.**

$$\boxed{x = \frac{1}{2} = \frac{L_1}{2}} \qquad \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{k}{n}\binom{2n}{n-2k} F_{2k} = \frac{F_{3n-3}}{4},$$
$$\boxed{x = \frac{\sqrt{5}}{2} = \frac{\sqrt{5}}{2}F_2} \qquad \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{k}{n}\binom{2n}{n-2k} F_{4k} = \frac{5^{n-1}-(-1)^{n-1}}{4},$$
$$\boxed{x = 2 = \frac{L_3}{2}} \qquad \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{k}{n}\binom{2n}{n-2k} F_{6k} = 2^{2n-2}F_{n-1},$$
$$\boxed{x = \frac{3\sqrt{5}}{2} = \frac{\sqrt{5}}{2}F_4} \qquad \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{k}{n}\binom{2n}{n-2k} F_{8k} = \frac{3}{4}\Big\{9^{n-1}-(-5)^{n-1}\Big\}.$$

**Proof.** These identities follow directly by applying (6) and (7) to Theorem 3. □

Analogously, the counterparts of the alternating sums are stated below.

**Corollary 6** ($n \in \mathbb{N}$)**.**

$$\boxed{\begin{gathered} x = \tfrac{\mathbf{i}\sqrt{5}}{2} = \tfrac{\mathbf{i}\sqrt{5}}{2}F_1 \\ \omega = 2+\mathbf{i} \end{gathered}} \qquad \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}(-1)^k \frac{k}{n}\binom{2n}{n-2k}F_{2k} = \frac{1}{2}\frac{\left\{\omega^{n-1}-\bar{\omega}^{n-1}\right\}}{\bar{\omega}-\omega},$$

$$\boxed{\begin{gathered} x = \tfrac{3\mathbf{i}}{2} = \tfrac{\mathbf{i}}{2}L_2 \\ \omega = 2+\mathbf{i}\sqrt{5} \end{gathered}} \qquad \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}(-1)^k \frac{k}{n}\binom{2n}{n-2k}F_{4k} = \frac{3}{2}\frac{\left\{\omega^{n-1}-\bar{\omega}^{n-1}\right\}}{\bar{\omega}-\omega},$$

$$\boxed{\begin{gathered} x = \mathbf{i}\sqrt{5} = \tfrac{\mathbf{i}\sqrt{5}}{2}F_3 \\ \omega = 2+4\mathbf{i} \end{gathered}} \qquad \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}(-1)^k \frac{k}{n}\binom{2n}{n-2k}F_{6k} = 4\frac{\left\{\omega^{n-1}-\bar{\omega}^{n-1}\right\}}{\bar{\omega}-\omega},$$

$$\boxed{\begin{gathered} x = \tfrac{7\mathbf{i}}{2} = \tfrac{\mathbf{i}}{2}L_4 \\ \omega = 2+3\mathbf{i}\sqrt{5} \end{gathered}} \qquad \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}(-1)^k \frac{k}{n}\binom{2n}{n-2k}F_{8k} = \frac{21}{2}\frac{\left\{\omega^{n-1}-\bar{\omega}^{n-1}\right\}}{\bar{\omega}-\omega}.$$

**Proof.** They are confirmed by Theorem 3 under parameter settings (8) and (9). □

### 5. Reciprocity for $\mathcal{D}_n(x)$

Finally, we examine the fourth binomial sum defined by

$$\mathcal{D}_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1+2k}{1+n}\binom{2n+2}{n-2k} Q_{2k+1}(x). \tag{13}$$

The corresponding generating function is given as in the lemma below.

**Lemma 4.** *Let $y$ and $T$ be the two variables related by $T = y(1+T)^2$. Then, we have the closed form generating function:*

$$\mathrm{D}(x,y) = \sum_{n=0}^{\infty} \mathcal{D}_n(x) y^n = \frac{2x(1+T^2)(1+T)^2}{(1-T^2)^2 - 4T^2x^2}.$$

**Proof.** This can be treated in a similar manner:

$$\begin{aligned}
\mathrm{D}(x,y) = \sum_{n=0}^{\infty} \mathcal{D}_n(x) y^n &= \sum_{n=0}^{\infty} y^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1+2k}{1+n}\binom{2n+2}{n-2k} Q_{2k+1}(x) \\
&= \sum_{k=0}^{\infty} y^{2k} Q_{2k+1}(x) \sum_{n=2k}^{\infty} \frac{4k+2}{2n+2}\binom{2n+2}{n-2k} y^{n-2k} \\
&= \sum_{k=0}^{\infty} y^{2k} Q_{2k+1}(x) \frac{(1+T)^{2+4k}}{1-T} \\
&= y^{-1} \sum_{k=0}^{\infty} Q_{2k+1}(x) T^{2k+1}.
\end{aligned}$$

By evaluating the last sum by the bisectional series

$$\sum_{k=0}^{\infty} Q_{2k+1}(x) T^{2k+1} = \frac{\Psi(x,T) - \Psi(x,-T)}{2} = \frac{2xT(1+T^2)}{(1-T^2)^2 - 4T^2x^2},$$

we obtain the generating function in Lemma 4. □

Then, we have the fourth reciprocal formula.

**Theorem 4** ($n \in \mathbb{N}_0$)**.**

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1+2k}{1+n}\binom{2n+2}{n-2k} Q_{2k+1}(x) = 2^n x^{n+1} Q_n(x^{-1}).$$

**Proof.** For $y \to T/(1+T)^2$, the generating function in Lemma 4 is almost the same as the function $\Psi(x^{-1}, 2xy)$ displayed in (11). By extracting the coefficients of $y^n$, we can obtain the formula stated exactly in Theorem 4. □

The above formula contains four identities for Fibonacci and Lucas numbers.

**Corollary 7** ($n \in \mathbb{N}_0$)**.**

$$\boxed{x = \frac{1}{2} = \frac{L_1}{2}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1+2k}{1+n}\binom{2n+2}{n-2k} L_{2k+1} = \frac{L_{3n}}{2},$$

$$\boxed{x = \frac{\sqrt{5}}{2} = \frac{\sqrt{5}}{2}F_2} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1+2k}{1+n}\binom{2n+2}{n-2k} F_{4k+2} = \frac{5^n + (-1)^n}{2},$$

$$\boxed{x = 2 = \frac{L_3}{2}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1+2k}{1+n}\binom{2n+2}{n-2k} L_{6k+3} = 2^{2n+1} L_n,$$

$$\boxed{x = \frac{3\sqrt{5}}{2} = \frac{\sqrt{5}}{2}F_4} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1+2k}{1+n}\binom{2n+2}{n-2k} F_{8k+4} = \frac{3}{2}\Big\{9^n + (-5)^n\Big\}.$$

**Proof.** These identities were obtained from Theorem 4 by applying (6) and (7). □

We also have four analogous identities for the alternating sums.

**Corollary 8** ($n \in \mathbb{N}_0$)**.**

$$\boxed{\begin{matrix} x = \frac{\mathbf{i}\sqrt{5}}{2} = \frac{\mathbf{i}\sqrt{5}}{2}F_1 \\ \omega = 2 + \mathbf{i} \end{matrix}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{1+2k}{1+n}\binom{2n+2}{n-2k} F_{2k+1} = \frac{1}{2}\Big\{\omega^n + \bar{\omega}^n\Big\},$$

$$\boxed{\begin{matrix} x = \frac{3\mathbf{i}}{2} = \frac{\mathbf{i}}{2}L_2 \\ \omega = 2 + \mathbf{i}\sqrt{5} \end{matrix}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{1+2k}{1+n}\binom{2n+2}{n-2k} L_{4k+2} = \frac{3}{2}\Big\{\omega^n + \bar{\omega}^n\Big\},$$

$$\boxed{\begin{matrix} x = \mathbf{i}\sqrt{5} = \frac{\mathbf{i}\sqrt{5}}{2}F_3 \\ \omega = 2 + 4\mathbf{i} \end{matrix}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{1+2k}{1+n}\binom{2n+2}{n-2k} F_{6k+3} = \omega^n + \bar{\omega}^n,$$

$$\boxed{\begin{matrix} x = \frac{7\mathbf{i}}{2} = \frac{\mathbf{i}}{2}L_4 \\ \omega = 2 + 3\mathbf{i}\sqrt{5} \end{matrix}} \qquad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{1+2k}{1+n}\binom{2n+2}{n-2k} L_{8k+4} = \frac{7}{2}\Big\{\omega^n + \bar{\omega}^n\Big\}.$$

**Proof.** They can be analogously formed by applying (8) and (9) to Theorem 4. □

*Concluding Comments*

By means of the generating function approach, we conducted a systematic investigation of reciprocal formulae for Pell and Lucas polynomials as well Fibonacci and Lucas numbers. However, the findings presented in this paper are only the tip of the iceberg with respect to numerous existing results regarding these polynomials. Here, we present a brief summary of the status of four closely related topics that the interested reader is enthusiastically encouraged to further explore.

- Computations of binomial sums (see Chu–Li [2,3] and Martinjak–Vrsaljko [11]), convolution sums (see Djordjevic [12]), and square sums (see Cerin–Gianella [13]).

- Evaluations of reciprocal sums of Pell and Lucas polynomials as well as their extensions (see Wu–Zhang [14] and Trojovský [15]).
- Combinatorial interpretations by arranging the Pell numbers on the vertices of polygons (see Celik–Durukan–Özkan [16]) and by counting restricted set partitions (see Mansour and Shattuck [17]).
- Variants and extensions (see Trojnar-Spelina–Wloch [18], and Tasci–Yalcin [19]) as well as connection coefficients between Pell and Lucas polynomials (see Abd-Elhameed–Philippou–Zeyada [20]).

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