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# Remarks on Sugeno Integrals on Bounded Lattices

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**Abstract:** A discrete Sugeno integral on a bounded distributive lattice $L$ is defined as an idempotent weighted lattice polynomial. Another possibility for axiomatization of Sugeno integrals is to consider compatible aggregation functions, uniquely extending the $L$-valued fuzzy measures. This paper aims to study the mentioned unique extension property concerning the possible extension of a Sugeno integral to non-distributive lattices. We show that this property is equivalent to the distributivity of the underlying bounded lattice. As a byproduct, an alternative proof of Iseki's result, stating that a lattice having prime ideal separation property for every pair of distinct elements is distributive, is provided.

**Keywords:** Sugeno integral; compatibility; distributive lattice; uniqueness of extension

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## 1. Introduction

The Sugeno integral was introduced in 1974 by Sugeno [1] on real numbers. This type of integral has been widely investigated in aggregation theory due to its many applications in fuzzy set theory, decision-making, data fusion, pattern recognition, etc. An interesting feature of the Sugeno integral is that it can be well defined on ordered domains (not necessarily linearly ordered), where the usual arithmetic operations are unavailable. Marichal in [2] extended the original definition of the Sugeno integral to bounded distributive lattices. In this case, the Sugeno integral is an idempotent weighted lattice polynomial function. Clearly, these functions can be defined on an arbitrary lattice. Following this definition of the Sugeno integral on bounded distributive lattices, several characterizations were presented in [3].

With respect to the study of Sugeno integrals on bounded distributive lattices, the so-called compatibility property of aggregation functions has been shown to be very important. Recall that the compatibility just means that the function preserves congruences on $L$. Especially, in a classical case of aggregation functions on intervals of reals, the compatibility property has an interesting application. The blocks of congruences correspond to convex subsets of the interval. The choice of a representative of a certain block then can be viewed as rounding in this interval. Hence, a natural question arises regarding the rounding of values: first aggregate and then round, or, conversely, first round and then aggregate. The compatibility property ensures that both of these procedures lead to the same value. In [4], we showed that compatible aggregation functions on intervals of reals correspond just to Sugeno integrals. The compatibility property as a characteristic of discrete Sugeno integrals on bounded distributive lattices was discussed in [5]. One of the crucial observations was the fact that compatible functions on bounded distributive lattices are completely determined by their values in the so-called Boolean elements.

The aim of this paper is to study the above-mentioned properties with respect to characterizations of distributive lattices and the definition of the Sugeno integral. Compatible functions on distributive lattices have been studied by several authors; see, e.g., [6–8].

Although some of the presented results are well known, they were obtained by different approaches based on basic lattice theory results. The only deep result from lattice theory used in the paper is the famous Stone theorem stating that every two elements in a distributive lattice can be separated by a prime ideal. We show that if a bounded lattice $L$ has this prime ideal separation property, then every compatible function on $L$ is uniquely determined by its values at Boolean elements. Further, it is proven that this uniqueness property implies the distributivity of the underlying bounded lattice; hence, a new characterization of bounded distributive lattices is obtained. As a byproduct, using this characterization, we are able to provide an alternative proof of Iséki's statement [9], i.e., a lattice having the prime ideal separation property for every pair of distinct elements is distributive.

The paper is divided into three parts. After the Introduction, we recall basic facts and definitions necessary for the paper. In the last part, we present the main results concerning the uniqueness of the extension of particular types of aggregations functions from Boolean elements to the whole lattice.

## 2. Preliminaries

In this section, we briefly recall basic algebraic notions needed for our purposes. We assume that the reader is familiar with the basic notions of lattice theory, and we refer the reader to the standard monograph [10].

Recall that an equivalence relation on a set $A$ is a binary relation $\theta \subseteq A \times A$, which is reflexive, symmetric, and transitive. We use the notation $a \equiv b \pmod{\theta}$ to indicate that $a$ and $b$ are related under the relation $\theta$. Let $L$ be a lattice. An equivalence relation $\theta \subseteq L \times L$ is a *congruence* on $L$ if it is compatible with the meet and join, i.e., if for all $a, b, c, d \in L$:

$$a \equiv b \pmod{\theta} \quad \text{and} \quad c \equiv d \pmod{\theta},$$

imply

$$a \vee c \equiv b \vee d \pmod{\theta} \quad \text{and} \quad a \wedge c \equiv b \wedge d \pmod{\theta}.$$

Observe that the compatibility condition says that any congruence formally forms a sublattice of $L^2$. We explicitly use congruences determined by prime ideals. A proper ideal $P$ of a lattice $L$ is prime if $a, b \in L$ and $a \wedge b \in P$ imply that $a \in P$ or $b \in P$. It can be easily verified that an ideal $P$ of $L$ is prime if and only if $L \setminus P$ is a filter of $L$.

Given a prime ideal $P$, we define a binary relation $\theta_P$ by $a \equiv b \pmod{\theta_P}$ iff $a, b \in P$ or $a, b \in L \setminus P$. Obviously, $\theta_P$ is compatible with the lattice operations. Thus, it forms a congruence of $L$. In this case, the factor set $L/\theta_P$ has exactly two blocks, namely $P$ and $L \setminus P$.

Let $n \geq 1$ be a positive integer. A function $f \colon L^n \to L$ is *compatible* if, for any congruence relation $\theta$ of $L$, the validity of $a_i \equiv b_i \pmod{\theta}$ for all $i = 1, \ldots, n$ implies

$$f(a_1, \ldots, a_n) \equiv f(b_1, \ldots, b_n) \pmod{\theta}.$$

Typical examples of compatible functions are constant functions or the lattice operations join and meet, respectively. It can be easily verified that the composition of compatible functions is again compatible. The family of all compatible functions defined on a lattice $L$ is denoted by $\mathcal{C}(L)$. The family of all monotone compatible functions is denoted by $m\mathcal{C}(L)$. Note that not all compatible functions are monotone, e.g., if $L$ is a Boolean lattice, then the unary operation of complementation is compatible, but not monotone.

Further, we consider lattice polynomials (lattice term functions) and the so-called weighted lattice polynomials. Formally, the set $\mathrm{Term}(n)$ of $n$-ary lattice terms is the smallest set satisfying (i) and (ii):

(i) $x_i \in \mathrm{Term}(n)$ for $i = 1, \ldots, n$,
(ii) if $\mathsf{p}, \mathsf{q} \in \mathrm{Term}(n)$ then $(\mathsf{p} \vee \mathsf{q}), (\mathsf{p} \wedge \mathsf{q}) \in \mathrm{Term}(n)$.

A term is a sequence of symbols. Using this sequence of symbols, one can naturally define a function on any lattice $L$. An $n$-ary term $\mathsf{p}$ defines a function $p$ in $n$ variables, called a polynomial (term function), on a lattice $L$ by the following rules: let $a_1, \ldots, a_n \in L$:

(i) If $\mathsf{p} = x_i$, then $p(a_1, \ldots, a_n) = a_i$ for any $i = 1, \ldots, n$.
(ii) If $p(a_1, \ldots, a_n) = a$, $q(a_1, \ldots, a_n) = b$ and $\mathsf{r} = \mathsf{p} \vee \mathsf{q}$, $\mathsf{t} = \mathsf{p} \wedge \mathsf{q}$, then $r(a_1, \ldots, a_n) = a \vee b$ and $t(a_1, \ldots, a_n) = a \wedge b$.

A larger class of functions on a lattice $L$ is obtained by substituting elements of $L$ for variables by term functions. According to [2], we call these functions weighted polynomials over $L$. The symbols $\mathcal{P}(L)$ and $w\mathcal{P}(L)$ denote the family of all polynomial functions on $L$ and the family of all weighted polynomial functions on $L$, respectively.

Obviously, polynomials, as well as weighted polynomials are monotone functions. Moreover, these are also compatible since they are composed of the projections, constants, and lattice operations. Thus, we have the following chain of inclusions:

$$\mathcal{P}(L) \subseteq w\mathcal{P}(L) \subseteq m\mathcal{C}(L) \subseteq \mathcal{C}(L).$$

Recall that a lattice $L$ is distributive if it satisfies one of the distributive identities, i.e., for all $a, b, c \in L$.

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \text{and} \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Let us note that if a lattice $L$ satisfies one of these identities, then the second one is also valid in $L$.

One of the most important results concerning distributive lattices is the existence of sufficiently many prime ideals. The fundamental result of M. H. Stone says that if $I \subseteq L$ is an ideal and $D \subseteq L$ is a filter of a distributive lattice $L$, where $I$ and $D$ are disjoint, then there is a prime ideal $P$ such that $I \subseteq P$ and $P \cap D = \emptyset$ (see [10] for the proof of this theorem). In this paper, we will use the following proposition, which is an easy consequence of Stone's theorem.

**Proposition 1.** *Let $L$ be a distributive lattice, $a, b \in L$ such that $a \neq b$. Then there is a prime ideal containing exactly one of $a$ and $b$.*

Finally, we recall some basic facts on discrete Sugeno integrals. For a positive integer $n \geq 1$, we put $[n] = \{1, \ldots, n\}$. Let $L$ be a bounded distributive lattice. An $L$-valued fuzzy measure, also known as a capacity on $L$, is a monotone set function $\mu\colon 2^{[n]} \to L$ such that $\mu(\emptyset) = 0$ and $\mu([n]) = 1$.

Let $\mu\colon 2^{[n]} \to L$ be an $L$-valued fuzzy measure. The Sugeno integral of an $n$-tuple $\mathbf{x} = (x_1, \ldots, x_n) \in L^n$ with respect to $\mu$ is defined by

$$\mathsf{S}_\mu(\mathbf{x}) = \bigvee_{X \subseteq [n]} \big(\mu(X) \wedge \bigwedge_{i \in X} x_i\big). \tag{1}$$

Formula (1) is the disjunctive normal representation, and due to the distributivity of $L$, it is equivalent to the following so-called conjunctive normal representation of the Sugeno integral:

$$\mathsf{S}_\mu(\mathbf{x}) = \bigwedge_{X \subseteq [n]} \big(\mu([n] \setminus X) \vee \bigvee_{i \in X} x_i\big). \tag{2}$$

One of the important properties of the Sugeno integral is that it extends the $L$-fuzzy measure $\mu$ in the sense that $\mathsf{S}_\mu(\mathbf{e}_S) = \mu(S)$ for all $S \subseteq [n]$, where $\mathbf{e}_S$ is the characteristic function of $S$, i.e., $(\mathbf{e}_S)_i = 1$ if $i \in S$ and $(\mathbf{e}_S)_i = 0$ otherwise. Indeed, for any $X \subseteq S$, we have $\bigwedge_{i \in X}(\mathbf{e}_S)_i = 1$, while $\bigwedge_{i \in X}(\mathbf{e}_S)_i = 0$ for all $X \nsubseteq S$. Consequently, all factors corresponding to the subsets $X \nsubseteq S$ can be omitted from the join in Formula (1), and since $\mu$ is a monotone set function, we obtain

$$\mathsf{S}_\mu(\mathbf{e}_S) = \bigvee_{X \subseteq S} \big(\mu(X) \wedge \bigwedge_{i \in X} (\mathbf{e}_S)_i\big) = \bigvee_{X \subseteq S} \mu(X) = \mu(S).$$

According to [3], the Sugeno integral $\mathsf{S}_\mu$ defined on a bounded distributive lattice $L$ can be characterized as the unique function extending the $L$-valued fuzzy measure $\mu$ such that $\mathsf{S}_\mu \in w\mathcal{P}(L)$. Similarly, in [5], it was shown that $\mathsf{S}_\mu$ is the unique function extending $\mu$ with $\mathsf{S}_\mu \in m\mathcal{C}(L)$. Motivated by these characterizations of the Sugeno integral, our aim is to study the unique extension property with respect to several classes of functions and its impact on the structure of the underlying lattice.

## 3. Results

Let $L$ be a lattice, $M \subseteq L$ be a non-void subset of $L$, and $\mathcal{K} \in \{\mathcal{P}, w\mathcal{P}, m\mathcal{C}, \mathcal{C}\}$ be a symbol denoting the family of functions. Note that $\mathcal{K}(L)$ denotes the corresponding family of functions defined on $L$. Related to $L$, $M$ and $\mathcal{K}$ we consider the following property:

($\mathsf{D}_M^{\mathcal{K}}$) For any two $n$-ary functions $f, g \in \mathcal{K}(L)$, the property $f(\mathbf{y}) = g(\mathbf{y})$ for all $\mathbf{y} \in M^n$ implies $f(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in L^n$.

In other words, a lattice $L$ has the property ($\mathsf{D}_M^{\mathcal{K}}$) provided any function defined on $L$ belonging to $\mathcal{K}(L)$ is completely determined by its values attained at $M$. Equivalently, this condition can be stated as follows: if two $n$-ary functions $f, g \in \mathcal{K}(L)$ are different, then there is an element $\mathbf{y} \in M^n$ such that $f(\mathbf{y}) \neq g(\mathbf{y})$. Obviously, the larger $M$ is, then it is more likely that we will find an element in $M^n$ distinguishing a pair of functions belonging to $\mathcal{K}(L)$. Therefore, the most interesting cases appear when $M$ is relatively small compared to $L$. However, in what follows, we show that if $M$ forms a sublattice of $L$ satisfying ($\mathsf{D}_M^{\mathcal{K}}$), then these two structures must be connected in a certain algebraic sense. For this, it suffices to consider the weakest condition ($\mathsf{D}_M^{\mathcal{P}}$).

**Lemma 1.** *Let $M \subseteq L$ be a sublattice of a lattice $L$ satisfying the condition ($\mathsf{D}_M^{\mathcal{P}}$). Then $L$ and $M$ satisfy exactly the same lattice identities.*

**Proof.** Obviously, if a lattice identity $p \approx q$ holds in $L$, then it also holds in the sublattice $M$.

Conversely, assume that $p \approx q$ is valid in $M$. Let $p, q \in \mathcal{P}(L)$ be the corresponding $n$-ary lattice polynomial functions occurring in the considered identity. Since $p \approx q$ holds in $M$, it follows that $p(\mathbf{y}) = q(\mathbf{y})$ for all $\mathbf{y} \in M^n$. Applying the condition ($\mathsf{D}_M^{\mathcal{P}}$), we obtain that $p(\mathbf{x}) = q(\mathbf{x})$ for all $\mathbf{x} \in L^n$.

This shows that $L$ and $M$ satisfy exactly the same lattice identities. □

The previous lemma can also be stated in terms of varieties (equational classes) of lattices. If $M$ is a sublattice of $L$ satisfying ($\mathsf{D}_M^{\mathcal{P}}$), then $\mathbf{HSP}(L) = \mathbf{HSP}(M)$. This means that these two lattices cannot be separated by any variety of lattices, i.e., there is no variety $\mathcal{V}$ of lattices such that $M \in \mathcal{V}$ and $L \notin \mathcal{V}$.

Further, as $\mathcal{P}(L)$ is the smallest of the four considered families of functions on $L$, it follows that Lemma 1 is also valid when any of the conditions ($\mathsf{D}_M^{w\mathcal{P}}$), ($\mathsf{D}_M^{m\mathcal{C}}$) or ($\mathsf{D}_M^{\mathcal{C}}$) is considered.

In the sequel, we focus on $L$-valued fuzzy measure extensions, i.e., we assume that $L$ is a bounded lattice and $M = \{0,1\}$ consists of its universal bounds 0 and 1, respectively. Obviously, $\{0,1\}$ forms a distributive sublattice of any bounded lattice $L$. We obtain the following corollary as an immediate consequence of Lemma 1.

**Corollary 1.** *Let $L$ be a bounded lattice satisfying the condition ($\mathsf{D}_{\{0,1\}}^{\mathcal{K}}$) for $\mathcal{K} \in \{\mathcal{P}, w\mathcal{P}, m\mathcal{C}, \mathcal{C}\}$. Then $L$ is distributive.*

In what follows, we describe the unique extension of a monotone $L$-valued set function to a monotone compatible function when a lattice $L$ satisfies ($\mathsf{D}_{\{0,1\}}^{m\mathcal{C}}$). Let $L$ be a bounded lattice with $0,1$ and $n \geq 1$ be a positive integer. For a Boolean element $\mathbf{b} = (b_1, \dots, b_n) \in \{0,1\}^n$, denote $\mathbf{b}^{-1}(0) = \{i \mid b_i = 0\}$ and similarly $\mathbf{b}^{-1}(1) = \{i \mid b_i = 1\}$. Obviously, $\mathbf{b}^{-1}(0)$ and $\mathbf{b}^{-1}(1)$ form a pair of complementary subsets of the set $[n] = \{1, \dots, n\}$. Given a monotone function $f \colon L^n \to L$ and $\mathbf{b} \in \{0,1\}^n$, we define for all $\mathbf{x} \in L^n$ the following functions:

$$G_{\mathbf{b}}(\mathbf{x}) := f(\mathbf{b}) \wedge \bigwedge \{x_i \mid i \in \mathbf{b}^{-1}(1)\}, \tag{3}$$

$$H_{\mathbf{b}}(\mathbf{x}) := f(\mathbf{b}) \vee \bigvee \{x_i \mid i \in \mathbf{b}^{-1}(0)\}. \tag{4}$$

Let us note that we formally put $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$, hence $G_{(0,\dots,0)}$ and $H_{(1,\dots,1)}$ represent the constant functions with values $f(0,\dots,0)$ and $f(1,\dots,1)$, respectively.

The next theorem shows that the two families of functions $w\mathcal{P}(L)$ and $m\mathcal{C}(L)$ coincide on lattices fulfilling the property ($\mathsf{D}_{\{0,1\}}^{m\mathcal{C}}$).

**Theorem 1.** *Let $L$ be a bounded lattice with the property ($\mathsf{D}_{\{0,1\}}^{m\mathcal{C}}$) and $f \colon L^n \to L$ be a monotone function. Then $f$ is compatible on $L$ if and only if it can be expressed in the following two equivalent forms:*

$$f(\mathbf{x}) = \bigvee \{G_{\mathbf{b}}(\mathbf{x}) \mid \mathbf{b} \in \{0,1\}^n\}, \tag{5}$$

$$f(\mathbf{x}) = \bigwedge \{H_{\mathbf{b}}(\mathbf{x}) \mid \mathbf{b} \in \{0,1\}^n\}. \tag{6}$$

**Proof.** For all $\mathbf{x} \in L^n$, denote the expressions (5) and (6) by

$$\bigvee \{G_{\mathbf{b}}(\mathbf{x}) \mid \mathbf{b} \in \{0,1\}^n\} = g(\mathbf{x}) \text{ and } \bigwedge \{H_{\mathbf{b}}(\mathbf{x}) \mid \mathbf{b} \in \{0,1\}^n\} = h(\mathbf{x}).$$

Obviously, $g$ and $h$ are compatible functions on $L$. Thus, if $f$ is equal to one of them, it has to be compatible as well.

Conversely, assume that $f$ is compatible. Since all three functions $f, g$, and $h$ are compatible, due to the property ($\mathsf{D}_{\{0,1\}}^{m\mathcal{C}}$) being valid for $L$, it is sufficient to prove $f(\mathbf{a}) = g(\mathbf{a}) = h(\mathbf{a})$ for all Boolean inputs $\mathbf{a} \in \{0,1\}^n$.

Let $\mathbf{a} \in \{0,1\}^n$ be fixed. Then for the value $g(\mathbf{a})$ we obtain

$$g(\mathbf{a}) = \bigvee \{G_{\mathbf{b}}(\mathbf{a}) \mid \mathbf{b} \not\leq \mathbf{a}\} \vee \bigvee \{G_{\mathbf{b}}(\mathbf{a}) \mid \mathbf{b} = \mathbf{a}\} \vee \bigvee \{G_{\mathbf{b}}(\mathbf{a}) \mid \mathbf{b} < \mathbf{a}\}.$$

Now, consider the following three possibilities:

(i) If $\mathbf{b} \not\leq \mathbf{a}$, then there is $j \in \{1, \dots, n\}$ with $b_j = 1$ and $a_j = 0$. In this case $j \in \mathbf{b}^{-1}(1)$, which yields $\bigwedge\{a_i \mid i \in \mathbf{b}^{-1}(1)\} = 0$. Consequently, from (3), we obtain $G_{\mathbf{b}}(\mathbf{a}) = f(\mathbf{b}) \wedge 0 = 0$.

(ii) Let $\mathbf{b} = \mathbf{a}$. Then evidently $\bigwedge\{a_i \mid i \in \mathbf{a}^{-1}(1)\} = 1$, and we obtain $G_{\mathbf{a}}(\mathbf{a}) = f(\mathbf{a}) \wedge 1 = f(\mathbf{a})$.

(iii) Assume $\mathbf{b} < \mathbf{a}$. Since the function $f$ is monotone and $\mathbf{b} < \mathbf{a}$, from (3), it follows that $G_{\mathbf{b}}(\mathbf{a}) \leq f(\mathbf{b}) \leq f(\mathbf{a})$.

The above-considered three cases lead to the equality

$$g(\mathbf{a}) = \bigvee 0 \vee f(\mathbf{a}) \vee \bigvee \{G_{\mathbf{b}}(\mathbf{a}) \mid \mathbf{b} < \mathbf{a}\} = f(\mathbf{a}).$$

Similarly, for $h(\mathbf{a})$ we have

$$h(\mathbf{a}) = \bigwedge \{H_{\mathbf{b}}(\mathbf{a}) \mid \mathbf{a} \not\leq \mathbf{b}\} \ \wedge \ \bigwedge \{H_{\mathbf{b}}(\mathbf{a}) \mid \mathbf{a} = \mathbf{b}\} \ \wedge \ \bigwedge \{H_{\mathbf{b}}(\mathbf{a}) \mid \mathbf{a} < \mathbf{b}\}.$$

If $\mathbf{a} \not\leq \mathbf{b}$, then there is $j \in \{1, \ldots, n\}$ with $a_j = 1$ and $b_j = 0$. Then, $j \in \mathbf{b}^{-1}(0)$ and $\bigvee\{a_i \mid i \in \mathbf{b}^{-1}(0)\} = 1$. Consequently, $H_{\mathbf{b}}(\mathbf{a}) = 1$. For $\mathbf{a} = \mathbf{b}$, we obtain $H_{\mathbf{b}}(\mathbf{a}) = f(\mathbf{a})$ and $\mathbf{a} < \mathbf{b}$, yielding $f(\mathbf{a}) \leq f(\mathbf{b}) \leq H_{\mathbf{b}}(\mathbf{a})$, since $f$ is monotone. Again, we obtain

$$h(\mathbf{a}) = \bigwedge 1 \wedge f(\mathbf{a}) \wedge \bigwedge \{H_{\mathbf{b}}(\mathbf{a}) \mid \mathbf{a} < \mathbf{b}\} = f(\mathbf{a}).$$

□

Observe that the previous theorem enables us to characterize Sugeno integrals in terms of compatibility; cf. [5], where the discrete Sugeno integral was characterized as a compatible aggregation function.

Let $L$ be a bounded lattice with the property $(\mathsf{D}^{m\mathcal{C}}_{\{0,1\}})$ and $\mu\colon 2^{[n]} \to L$ be an $L$-valued fuzzy measure. In this case, $\mathsf{S}_\mu$ is the unique monotone compatible function on $L$ extending $\mu$. Indeed, if $f$ extends $\mu$, then for any subset $X \subseteq [n]$, we have $f(\mathbf{e}_X) = \mu(X)$ and $\mathbf{e}_X^{-1}(1) = X$. Thus the expressions (3) and (5) for all $\mathbf{x} \in L^n$ yield the conjunctive normal form of $f$:

$$f(\mathbf{x}) = \bigvee_{X \subseteq [n]} \Big(f(\mathbf{e}_X) \wedge \bigwedge_{i \in \mathbf{e}_X^{-1}(1)} x_i\Big) = \bigvee_{X \subseteq [n]} \Big(\mu(X) \wedge \bigwedge_{i \in X} x_i\Big) = \mathsf{S}_\mu(\mathbf{x}).$$

Similarly, as $\mathbf{e}_Y^{-1}(0) = [n] \smallsetminus Y$, using the substitution $X = [n] \smallsetminus Y$, the expressions (4) and (6) yield for all $\mathbf{x} \in L^n$

$$f(\mathbf{x}) = \bigwedge_{Y \subseteq [n]} \Big(f(\mathbf{e}_Y) \vee \bigvee_{i \notin Y} x_i\Big) = \bigwedge_{X \subseteq [n]} \Big(\mu([n] \smallsetminus X) \vee \bigvee_{i \in X} x_i\Big) = \mathsf{S}_\mu(\mathbf{x}).$$

In Corollary 1 it was shown that a bounded lattice $L$ fulfilling $(\mathsf{D}^{\mathcal{K}}_{\{0,1\}})$ has to be distributive. To show that the converse is also true, we first prove the next theorem, which states that distinct compatible functions defined on a lattice $L$ with the separation property must differ on a relatively "large" subset of $L^n$.

For the sake of brevity, we define the notion of a $P$-set of $L^n$. Let $n \geq 1$ be a positive integer and $P$ be a proper prime ideal of $L$. We say that $S \subseteq L^n$ is a $P$-set of $L^n$ if there is a subset $J \subseteq [n]$ such that

$$S = \{\mathbf{x} = (x_1, \ldots, x_n) \in L^n \mid x_i \in P \text{ if } i \in J,\ x_i \notin P \text{ if } i \notin J\}.$$

Hence, given a fixed prime ideal $P$, there exist $2^n$ $P$-sets of $L^n$, each of them corresponding to some subset $J \subseteq [n]$. Observe that if $S$ is a $P$-set of $L$, then either $S = P$ or $S = L \smallsetminus P$. Further, it is said that a $P$-set $S$ of $L^n$ is distinguishing functions $f, g\colon L^n \to L$ provided $f(\mathbf{x}) \neq g(\mathbf{x})$ for all $\mathbf{x} \in S$.

**Theorem 2.** *Let $L$ be a lattice such that any two distinct elements of $L$ can be separated by a prime ideal. Then for any two distinct $n$-ary compatible functions $f, g\colon L^n \to L$ there is a $P$-set of $L^n$ distinguishing $f$ and $g$.*

**Proof.** Assume that $L$ has the separation property and $f(\mathbf{a}) \neq g(\mathbf{a})$ for some $\mathbf{a} = (a_1, \ldots, a_n) \in L^n$. There is a proper prime ideal $P \subseteq L$ separating the values $f(\mathbf{a})$ and $g(\mathbf{a})$. Suppose that $f(\mathbf{a}) \in P$, while $g(\mathbf{a}) \notin P$. Define the set $J = \{i \in [n] \mid a_i \in P\}$, i.e., the index $i \in [n]$ belongs to $J$ if and only if the corresponding $i$-th component $a_i$ of $\mathbf{a}$ belongs to $P$.

Let $S$ be the $P$-set corresponding to the set of indices $J$, i.e., $\mathbf{x} \in S$ iff $x_i \in P$ for $i \in J$ and $x_i \notin P$ for $i \notin J$. Considering the congruence $\theta_P$ corresponding to the prime ideal $P$, for any $\mathbf{x} \in S$, it follows that $x_i \equiv a_i (\mathrm{mod}\ \theta_P)$ for all indices $i \in [n]$. Using the compatibility of $f$ and $g$, respectively, we obtain

$$f(x_1, \ldots, x_n) \equiv f(a_1, \ldots, a_n)(\mathrm{mod}\ \theta_P), \quad g(x_1, \ldots, x_n) \equiv g(a_1, \ldots, a_n)(\mathrm{mod}\ \theta_P).$$

However, $f(\mathbf{x}) = g(\mathbf{x})$ for some $\mathbf{x} \in S$ yields $f(\mathbf{a}) \equiv f(\mathbf{x}) \equiv g(\mathbf{x}) \equiv g(\mathbf{a})(\mathrm{mod}\ \theta_P)$, which is a contradiction, since $P$ separates $f(\mathbf{a})$ and $g(\mathbf{a})$. Thus the $P$-set $S$ distinguishes $f$ and $g$. □

**Remark 1.** *According to Proposition 1, the previous theorem is valid for distributive lattices. In the proof, only the assumption that functions preserve congruences of the form $\theta_P$, $P$ prime ideal, was used. However, it turns out that in distributive lattices, this assumption is equivalent to the compatibility. This follows from the fact that in any distributive lattice, for every congruence $\theta$, there is a family of prime ideals $\mathcal{S}$ such that*

$$\theta = \bigcap_{P \in \mathcal{S}} \theta_P$$

*holds. It can be easily verified that if a function preserves a family of congruences, then it preserves their intersection as well.*

**Corollary 2.** *If in a bounded lattice any two elements can be separated by a prime ideal, then it has the property* $(\mathsf{D}^{\mathcal{C}}_{\{0,1\}})$.

**Proof.** Let $L$ be a bounded lattice with $0, 1$ satisfying the given assumption and $f, g\colon L^n \to L$ be two compatible functions such that $f \neq g$.

Due to the previous theorem, there is a proper prime ideal $P \subseteq L$, a subset $J \subseteq [n]$ of indices, and the corresponding $P$-set $S$ distinguishing $f$ and $g$. Since $0 \in P$ and $1 \in L \setminus P$, there is $\mathbf{b} = (b_1, \ldots, b_n) \in \{0, 1\}^n$ such that $\mathbf{b} \in S$. Hence the Boolean element $\mathbf{b}$ is given by $b_i = 0$ for $i \in J$, $b_i = 1$ otherwise. Obviously, $f(\mathbf{b}) \neq g(\mathbf{b})$ as $\mathbf{b} \in S$. □

With respect to Proposition 1, we obtain the following corollary.

**Corollary 3.** *Any bounded distributive lattice has the property* $(\mathsf{D}^{\mathcal{C}}_{\{0,1\}})$.

As $\mathcal{C}(L)$ is the largest family of functions that we consider, the previous statement is obviously valid also for $(\mathsf{D}^{\mathcal{K}}_{\{0,1\}})$, $\mathcal{K} \in \{\mathcal{P}, w\mathcal{P}, m\mathcal{C}\}$. Consequently, from Corollaries 1–3, we obtain the following characterization theorem for bounded distributive lattices. Let us note that the implication (i)⟹(ii) is the famous Stone theorem, (i)⟹(vi) is due to Grätzer [7], and (ii)⟹(i) is a result of Iséki [9], who proved it without the boundedness assumption.

**Theorem 3.** *Let $L$ be a bounded lattice $L$. Then the following are equivalent:*

*(i) $L$ is distributive.*

*(ii)* *Any two distinct elements of L can be separated by a prime ideal.*
*(iii)* *L fulfills the condition* $(\mathsf{D}^{\mathcal{P}}_{\{0,1\}})$.
*(iv)* *L fulfills the condition* $(\mathsf{D}^{w\mathcal{P}}_{\{0,1\}})$.
*(v)* *L fulfills the condition* $(\mathsf{D}^{m\mathcal{C}}_{\{0,1\}})$.
*(vi)* *L fulfills the condition* $(\mathsf{D}^{\mathcal{C}}_{\{0,1\}})$.

**Remark 2.** *Let us note that our methods also allow us to prove Iséki's result in its full generality.*

*Indeed, suppose that L is a lattice such that any two elements can be separated by a prime ideal. Denote by* $L^1_0 = L \cup \{0,1\}$ *the lattice with newly added universal bounds* $\{0,1\}$, *regardless of whether L has them or not. If* $P \subseteq L$ *is a prime ideal of L, then* $P \cup \{0\}$ *is a prime ideal of* $L^1_0$. *As* $0$ *is a meet-irreducible element of* $L^1_0$, $\{0\}$ *forms a prime ideal of* $L^1_0$. *The same holds for the set* $L \cup \{0\}$. *Thus* $L^1_0$ *fulfills the same assumptions as the lattice L.*

*However, in this case* $L^1_0$ *is distributive, and the same is valid for the sublattice* $L \subseteq L^1_0$.

## 4. Conclusions

Aggregation functions on a bounded lattice $L$ can be considered as extensions of $L$-valued fuzzy measures defined on Boolean elements of the respective power of the lattice $L$. A natural question arises regarding under which conditions these extensions unique are. It was shown that one of the crucial conditions is the compatibility. When the lattice $L$ is distributive, we obtain a discrete Sugeno integral. One of the main results of the paper shows that the uniqueness of the extension of compatible aggregation functions from Boolean elements to the whole lattice is already equivalent to its distributivity. This observation has a consequence concerning the axiomatization of Sugeno integrals.

In particular, our result shows that the notion of the Sugeno integral as a monotone compatible function, the values of which are completely determined in its Boolean inputs, is sound only for distributive lattices. This conclusion is valid not only for the class of compatible functions, but also for any class containing lattice polynomials.

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