mathematics

MDPI

Article

Riemann-Liouville Fractional Inclusions for Convex Functions Using Interval Valued Setting

Vuk Stojiljković [1,*,†], Rajagopalan Ramaswamy [2,†], Ola A. Ashour Abdelnaby [2,†] and Stojan Radenović [3]

[1] Faculty of Science, University of Novi Sad, Trg Dositeja Obradovića 3, 21000 Novi Sad, Serbia
[2] Department of Mathematics, College of Science and Humanities, Prince Sattam bin Abdulaziz Univeristy, Al-Kharj 16278, Saudi Arabia
[3] Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia
* Correspondence: vuk.stojiljkovic999@gmail.com; Tel.: +381-640123911
† These authors contributed equally to this work.

Abstract: In this work, various fractional convex inequalities of the Hermite–Hadamard type in the interval analysis setting have been established, and new inequalities have been derived thereon. Recently defined p interval-valued convexity is utilized to obtain many new fractional Hermite–Hadamard type convex inequalities. The derived results have been supplemented with suitable numerical examples. Our results generalize some recently reported results in the literature.

Keywords: convex interval-valued functions; pseudo-order relations; Hermite–Hadamard inequality; Riemann–Liouville fractional integral operators; fuzzy interval-valued analysis

MSC: 26D10; 26A33

check for updates

Citation: Stojiljković, V.; Ramaswamy, R.; Ashour Abdelnaby, O.A.; Radenović, S. Riemann-Liouville Fractional Inclusions for Convex Functions Using Interval Valued Setting. *Mathematics* **2022**, *10*, 3491. https://doi.org/10.3390/math10193491

Academic Editors: Juan Eduardo Nápoles Valdes and Miguel Vivas-Cortez

Received: 29 August 2022
Accepted: 21 September 2022
Published: 24 September 2022

1. Introduction

Convex inequalities have been an ongoing topic of research since the discovery of the first convex inequality by Jensen. Various inequalities were derived as consequences of the famous Jensen's inequality; see [1,2]. Convex inequalities have many applications, for example, in probability theory, analysis, and optimization problems [3–9]. See the following books for further information [10,11]. The most famous inequality, namely the Hermite–Hadamard inequality, is given in [12].

Let $\mho : \mathbb{I} \to \mathbb{R}$ be a convex function on $\mathbb{I}$ in $\mathbb{R}$ and $\rho_1, \rho_2 \in \mathbb{I}$ with $\rho_1 < \rho_2$, then

$$\mho\left(\frac{\rho_1+\rho_2}{2}\right) \leqslant \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} \mho(t)dt \leqslant \frac{\mho(\rho_1)+\mho(\rho_2)}{2}.$$

Many mathematicians have applied this inequality for fractional estimates of Hermite–Hadamard (H-H) inequalities using different kinds of convexity [13–15]. The H-H inequality has been derived using various convex generalizations of Jensen's inequality; see [16–18]. In this paper, we employ the interval valued function setting (IVF) together with convexity properties to derive various convex inequalities together with fractional integral operators. The novelty in this paper is that it generalizes the recently obtained results of Srivastava.

It is interesting to note that the first idea of fractional calculus was presented by Leibniz and L'Hospital (1695). The idea was explored further by Riemann, Liouville, Grünwald, Letnikov, Erdéli, and Kober. They also have made valuable contributions to the field of fractional calculus and its widespread application. Today, fractional calculus is widely used in describing various phenomena, such as fractional conservation of mass as described by Wheatcraft and Meerschaert (2008), fractional Schrödinger equation in quantum theory, and many others. For more information about fractional calculus, see [19–22].

Motivation behind this paper is to derive various fractional IVF inequalities, see [23–28], which find application in numerical analysis and related fields.

We give the very first definition of the convex function given by Jensen [29].

Definition 1. *For an interval $\mathcal{I}$ in $\mathbb{R}$, a function $f : \mathcal{I} \to \mathbb{R}$ is said to be convex on $\mathcal{I}$ if*

$$\mho(\zeta x + (1-\zeta)y) \leqslant \zeta\mho(x) + (1-\zeta)\mho(y)$$

for all $x, y \in \mathcal{I}$ and $\zeta \in [0,1]$ holds and is said to be a concave function if the inequality is reversed.

2. Preliminaries

Let the collection of all closed and bounded intervals of $\mathbb{R}$ be defined as follows:

$$\mathbb{K}_c = \{[\Omega_*, \Omega^*] : \Omega_*, \Omega^* \in \mathbb{R} \textit{ and } \Omega_* \leqslant \Omega^*\}.$$

We say that the interval $[\Omega_*, \Omega^*]$ is a positive interval if $\Omega_* \geqslant 0$, and it is defined as follows:

$$\mathbb{K}_c^+ = \{[\Omega_*, \Omega^*] : \Omega_*, \Omega^* \in \mathbb{R} \textit{ and } \Omega_* \geqslant 0\}.$$

The algebraic addition, the algebraic multiplication, and the scalar multiplication for $[\Phi_*, \Phi^*], [\Omega_*, \Omega^*] \in K_c$ and $\zeta \in \mathbb{R}$ are defined as follows:

$$[\Phi_*, \Phi^*] + [\Omega_*, \Omega^*] = [\Phi_* + \Omega_*, \Phi^* + \Omega^*],$$

$$[\Phi_*, \Phi^*] \cdot [\Omega_*, \Omega^*] = [\min\{\Phi_*\Omega_*, \Phi^*\Omega_*, \Phi_*\Omega^*, \Phi^*\Omega^*\}, \max\{\Phi_*, \Omega_*, \Phi^*\Omega_*, \Phi_*\Omega^*, \Phi^*\Omega^*\}]$$

and

$$\zeta \cdot [\Phi_*, \Phi^*] = \begin{cases} [\zeta\Phi_*, \zeta\Phi^*], (\zeta > 0), \\ \{0\}, (\zeta = 0), \\ [\zeta\Phi^*, \zeta\Phi_*], (\zeta < 0) \end{cases}$$

respectively. The difference is defined as

$$\Phi - \Omega = [\Phi_*, \Phi^*] - [\Omega_*, \Omega^*] = [\Phi_* - \Omega^*, \Phi^* - \Omega_*].$$

The inclusion relation $\Omega \supseteq \Phi$ means that

$$\Omega \supseteq \Phi \; if \; and \; only \; if \; [\Omega_*, \Omega^*] \supseteq [\Phi_*, \Phi^*] \; if \; and \; only \; if \; [\Phi_* \geqslant \Omega_*, \Omega^* \geqslant \Phi^*]$$

The following relation was defined in the following paper by Moore [30].

Remark 1.

(i) *The relation "$\leqslant_p$" defined on K_c by*

$$[\Phi_*, \Phi^*] \leqslant_p [\Omega_*, \Omega^*]$$

if and only if $\Phi_* \leqslant \Omega_*, \Phi^* \leqslant \Omega^*$ for all $[\Phi_*, \Phi^*], [\Omega_*, \Omega^*] \in \mathbb{R}$ is a pseudo order relation. In the interval analysis case, both the pseudo order relation ($\leqslant_p$) and partial order relation ($\leqslant$) behave alike, thus the relation $[\Phi_*, \Phi^*] \leqslant_p [\Omega_*, \Omega^*]$ is coincident to $[\Phi_*, \Phi^*] \leqslant [\Omega_*, \Omega^*]$ on K_c, for more details see [30].

(ii) *It can be easily seen that "$\leqslant_p$" looks similar to "left and right" on the real line $\mathbb{R}$, so we call "$\leqslant_p$" "left and right" (or "LR" order, in short).*

The concept of the Riemann integral for IVF was first introduced by Moore [31] and is defined as follows:

Theorem 1. *Let $\mho : [\rho_1, \rho_2] \subset \mathbb{R} \to K_c$ be an interval valued function such that*

$$\mho(x) = [\mho_*(x), \mho^*(x)].$$

Then, F is Riemann-integrable over $[\rho_1, \rho_2]$ *if and only if* $\mho_*$ *and* $\mho^*$ *are both Riemann integrable over* $[\rho_1, \rho_2]$*.*

$$(IR)\int_{\rho_1}^{\rho_2} \mho(x)dx = \left[(R)\int_{\rho_1}^{\rho_2} \mho_*(x)dx, (R)\int_{\rho_1}^{\rho_2} \mho^*(x)dx\right]$$

In the following, we give a definition of the IVF convex function [31].

Definition 2. *The interval-valued function* $\mathscr{F} : \mathscr{X} \to K_c^+$ *is said to be LR-convex interval-valued on a convex set* $\mathscr{X}$ *if, for all* $\rho_1, \rho_2 \in \mathscr{X}$*, and* $\zeta \in [0,1]$*, we have*

$$\mathscr{F}(\zeta\rho_1 + (1-\zeta)\rho_2) \leqslant_p \zeta\mathscr{F}(\rho_1) + (1-\zeta)\mathscr{F}(\rho_2).$$

If the inequality is reversed, then $\mathscr{F}$ *is said to be LR-concave on* $\mathscr{X}$*. Moreover,* $\mathscr{F}$ *is affine on* $\mathscr{X}$ *if and only if it is both LR-convex and LR-concave on* $\mathscr{X}$*.*

Now we define IVF fractional integrals.
The first fractional integral is due to Katugampola [32].

Definition 3. *Let* $p, \alpha > 0$ *and* $f \in L[u,v]$ *be the collection of all complex-valued Lebesgue integrable IVFs on* $[u,v]$*. Then, the interval left and right Katugampol fractional integrals of* $f \in L[u,v]$ *with order* $\alpha > 0$ *are defined by*

$$J_{u^+}^{p,\alpha}\mho(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_u^x (x^p - \zeta^p)^{\alpha-1}\zeta^{p-1}\mho(\zeta)d\zeta \quad (x > u),$$

$$J_{v^-}^{p,\alpha}\mho(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_x^v (\zeta^p - x^p)^{\alpha-1}\zeta^{p-1}\mho(\zeta)d\zeta \quad (x < v)$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-x}x^{\alpha-1}dx$ *is the Euler Gamma function; see [33].*

The concept of p-convex functions were established by Zhang and Wang [34], and a number of properties of the functions were introduced.

Definition 4. *Let* $p \in \mathbb{R}$ *with* $p \neq 0$*. Then, the interval* $\mathcal{I}$ *is said to be p-convex if*

$$[\rho x^p + (1-\rho)y^p]^{\frac{1}{p}} \in \mathcal{I}.$$

for all $x, y \in \mathcal{I}, \rho \in [0,1]$*, where* $p = 2n+1$ *and* $n \in \mathcal{N}$ *or p is an odd number.*

Definition 5. *Let* $p \in \mathbb{R}$ *with* $p \neq 0$*. Then, the interval* $\mathbb{I}$ *is said to be p-convex if*

$$\mho\left((\rho x^p + (1-\rho)y^p)^{\frac{1}{p}}\right) \leqslant \rho\mho(x) + (1-\rho)\mho(y)$$

or all $x, y \in [u,v]$, $\rho \in [0,1]$*. If the inequality is reversed, then f is called p-concave function. The set of all p-convex (LR-p-concave, LR-p-affine) functions is denoted by*

$$SX([u,v], \mathbb{R}^+, p) \qquad (SV([u,v], \mathbb{R}^+, p),).$$

Now we define the class of functions that will be used in this paper, defined by Khan et al. [35].

Definition 6. *The IVF $f : [u,v] \to K_c^+$ is said to be LR-p-convex-IVF if for all $x, y \in [u,v]$ and $\rho \in [0,1]$ we have*

$$\mho\left([\rho x^p + (1-\rho)y^p]^{\frac{1}{p}}\right) \leqslant_p \rho\mho(x) + (1-\rho)\mho(y).$$

If inequality is reversed, then f is said to be LR-p-concave on $[u,v]$. The set of all LR-p-convex (LR-p-concave) IVFs is denoted by

$$LRSX([u,v], K_c^+, p) \quad (LRSV([u,v], K_c^+, p)).$$

3. Main Results

We present our first theorem that generalizes the theorem from the paper by Srivastava et al. [36].

Theorem 2. *Let $f : [a,b] \to \mathbb{K}_c^+$ be an IVF that is LR-p convex. Then the following inequality holds:*

$$\mathscr{F}\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \frac{2^{\alpha-1}p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}(b) + J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}(a)\right)$$

$$\leqslant_p \frac{p\alpha}{4}(\mathscr{F}(a)+\mathscr{F}(b))\int_0^1 t^p t^{\alpha p-1}dt + \frac{p\alpha}{2}(\mathscr{F}(a)+\mathscr{F}(b))\int_0^1 (1-\frac{t^p}{2})t^{\alpha p-1}dt.$$

Proof. From the definition of the LR-p convex IVF, we have

$$\mathscr{F}\left([\rho x^p + (1-\rho)y^p]^{\frac{1}{p}}\right) \leqslant_p \rho\mathscr{F}(x) + (1-\rho)\mathscr{F}(y).$$

Setting $\rho = \frac{1}{2}$, we obtain

$$2\mathscr{F}\left([\frac{x^p+y^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \mathscr{F}(x) + \mathscr{F}(y).$$

Setting $x^p = \frac{t^p a^p}{2} + \frac{2-t^p}{2}b^p, y^p = \frac{2-t^p}{2}a^p + \frac{t^p b^p}{2}$, we obtain

$$2\mathscr{F}\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \mathscr{F}\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right) + \mathscr{F}\left([\frac{t^p b^p}{2} + \frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right).$$

Therefore, from the definition of the IVF, we have

$$2\mathscr{F}_*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant \mathscr{F}_*\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right) + \mathscr{F}_*\left([\frac{t^p b^p}{2} + \frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right)$$

and

$$2\mathscr{F}^*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant \mathscr{F}^*\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right) + \mathscr{F}^*\left([\frac{t^p b^p}{2} + \frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right).$$

Multiplying both inequalities with $t^{\alpha p-p}t^{p-1}$ and integrating with respect to t from 0 to 1, we get

$$2\int_0^1 t^{\alpha p-1}\mathscr{F}_*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right)dt \leqslant$$

$$\int_0^1 t^{\alpha p-1}\mathscr{F}_*\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)dt + \int_0^1 t^{\alpha p-1}\mathscr{F}_*\left([\frac{t^p b^p}{2} + \frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right)dt.$$

and

$$2\int_0^1 t^{\alpha p-1}\mathscr{F}^*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right)dt \leqslant$$

$$\int_0^1 t^{\alpha p-1}\mathscr{F}^*\left([\frac{t^pa^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)dt+\int_0^1 t^{\alpha p-1}\mathscr{F}^*\left([\frac{t^pb^p}{2}+\frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right)dt.$$

Now setting $\frac{t^pa^p}{2}+\frac{2-t^p}{2}b^p=\omega^p$, $\frac{t^pb^p}{2}+\frac{2-t^p}{2}a^p=v^p$, we obtain

$$\frac{2}{\alpha p}\mathscr{F}_*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant$$

$$\frac{2^\alpha}{(b^p-a^p)^\alpha}\left(\int_{(\frac{a^p+b^p}{2})^{\frac{1}{p}}}^{b}(b^p-\omega^p)^{\alpha-1}\omega^{p-1}\mathscr{F}_*(\omega)d\omega+\int_a^{(\frac{a^p+b^p}{2})^{\frac{1}{p}}}(v^p-a^p)^{\alpha-1}v^{p-1}\mathscr{F}_*(v)dv.\right)$$

and

$$\frac{2}{\alpha p}\mathscr{F}^*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant$$

$$\frac{2^\alpha}{(b^p-a^p)^\alpha}\left(\int_{(\frac{a^p+b^p}{2})^{\frac{1}{p}}}^{b}(b^p-\omega^p)^{\alpha-1}\omega^{p-1}\mathscr{F}^*(\omega)d\omega+\int_a^{(\frac{a^p+b^p}{2})^{\frac{1}{p}}}(v^p-a^p)^{\alpha-1}v^{p-1}\mathscr{F}^*(v)dv.\right)$$

which, when identified in terms of the Katagumpola fractional integral, we obtain

$$\frac{2}{\alpha p}\mathscr{F}_*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant$$

$$\frac{2^\alpha\Gamma(\alpha)}{p^{1-\alpha}(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}_*(b)+J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}_*(a)\right)$$

and

$$\frac{2}{\alpha p}\mathscr{F}^*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant$$

$$\frac{2^\alpha\Gamma(\alpha)}{p^{1-\alpha}(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}^*(b)+J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}^*(a)\right).$$

Consequently, we have

$$\frac{2}{\alpha p}[\mathscr{F}^*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right),\mathscr{F}_*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right)]$$

$$\leqslant_p \frac{2^\alpha\Gamma(\alpha)}{p^{1-\alpha}(b^p-a^p)^\alpha}\left(\left(J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}^*(b)+J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}^*(a)\right),\right.$$

$$\left.\left(J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}_*(b)+J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}_*(a)\right)\right).$$

From which we obtain the following:

$$\mathscr{F}\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \frac{2^{\alpha-1}p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}(b)+J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}(a)\right).$$

Now to obtain the right hand side inequality, we apply the definition of the LR-p convex IVF on the expression

$$\mathscr{F}\left([\frac{t^pa^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)+\mathscr{F}\left([\frac{t^pb^p}{2}+\frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right).$$

Multiplying everything by $t^{\alpha p-1}$ and integrating with respect to t from 0 to 1, we get the original inequality

$$\mathscr{F}\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \frac{2^{\alpha-1}p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}(b)+J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}(a)\right)$$

$$\leqslant_p \frac{p\alpha}{4}(\mathscr{F}(a)+\mathscr{F}(b))\int_0^1 t^pt^{\alpha p-1}dt+\frac{p\alpha}{2}(\mathscr{F}(a)+\mathscr{F}(b))\int_0^1(1-\frac{t^p}{2})t^{\alpha p-1}dt.$$

□

Corollary 1. *If $\mathscr{F}_*=\mathscr{F}^*$, then p-convex-IVF reduces to the classical fractional p-convex inequality.*

Corollary 2. *Setting $p=1$ in the derived theorem, we obtain Theorem 6 from Srivastava et al. [36]:*

$$\mathscr{F}\left(\frac{a+b}{2}\right) \leqslant_p \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J^{1,\alpha}_{((\frac{a+b}{2}))^+}\mathscr{F}(b)+J^{1,\alpha}_{((\frac{a+b}{2}))^-}\mathscr{F}(a)\right)$$

$$\leqslant_p \frac{\mathscr{F}(a)+\mathscr{F}(b)}{2}.$$

Example 1. *Let p be an odd number and the I-V-F $\mathscr{F}:[a,b]=[-1,1]\to\mathbb{R}_l^+$ defined by $\mathscr{F}(\omega)=[\omega^p,e^{\omega^p}]$. End point functions $\mathscr{F}_*(\omega)=\omega^p$ and $\mathscr{F}^*(\omega)=e^{\omega^p}$ are both p-convex functions, hence $\mathscr{F}$ is LR-p-convex-I-V-F. We will compute the following while setting $p=1,\alpha=2$:*

$$\mathscr{F}\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \frac{2^{\alpha-1}p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}(b)+J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}(a)\right)$$

$$\leqslant_p \frac{p\alpha}{4}(\mathscr{F}(a)+\mathscr{F}(b))\int_0^1 t^pt^{\alpha p-1}dt+\frac{p\alpha}{2}(\mathscr{F}(a)+\mathscr{F}(b))\int_0^1(1-\frac{t^p}{2})t^{\alpha p-1}dt.$$

$$\mathscr{F}_*\left((\frac{a^p+b^p}{2})^{\frac{1}{p}}\right)=\mathscr{F}_*(0)=0,$$

$$\frac{2^{\alpha-1}p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}_*(b)+J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}_*(a)\right)=0,$$

and

$$\frac{p\alpha}{4}(\mathscr{F}_*(a)+\mathscr{F}_*(b))\int_0^1 t^pt^{\alpha p-1}dt+\frac{p\alpha}{2}(\mathscr{F}_*(a)+\mathscr{F}_*(b))\int_0^1(1-\frac{t^p}{2})t^{\alpha p-1}dt=0.$$

This means $0 \leqslant 0 \leqslant 0$.
Computing the upper end point function, we get

$$\mathscr{F}^*\left((\frac{a^p + b^p}{2})^{\frac{1}{p}} \right) = \mathscr{F}^*(0) = 1,$$

$$\frac{2^{\alpha-1}p^{\alpha}\Gamma(\alpha+1)}{(b^p - a^p)^{\alpha}} \left(J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2} \right)^{\frac{1}{p}} \right)^+} \mathscr{F}^*(b) + J^{p,\alpha}_{\left(\left(\frac{a^p+b^p}{2} \right)^{\frac{1}{p}} \right)^-} \mathscr{F}^*(a) \right)$$

$$= \frac{1}{3}\left(E_{\frac{1}{3}}(-1) + \sqrt[3]{-1}\Gamma\left(\frac{2}{3}\right) + (-1)^{2/3}\left(\Gamma\left(\frac{1}{3}, -1\right) - \Gamma\left(\frac{1}{3}\right) \right) \right)$$

$$+\frac{1}{3}\left(E_{\frac{1}{3}}(1) - \Gamma\left(\frac{2}{3}\right) + 3\Gamma\left(\frac{4}{3}\right) - \Gamma\left(\frac{1}{3}, 1\right) \right) \sim 1.01832,$$

Here E denotes the exponential integral and $\Gamma(a, b)$ *denotes the incomplete Gamma function.*

$$\frac{p\alpha}{4}(\mathscr{F}^*(a) + \mathscr{F}^*(b)) \int_0^1 t^p t^{\alpha p - 1} dt + \frac{p\alpha}{2}(\mathscr{F}^*(a) + \mathscr{F}^*(b)) \int_0^1 (1 - \frac{t^p}{2}) t^{\alpha p - 1} dt$$

$$= \frac{1}{2}\left(e + \frac{1}{e} \right) \sim 1.54308$$

From which, it follows that

$$1 \leqslant 1.01832 \leqslant 1.54308.$$

Thus, we have,

$$[0, 1] \leqslant_p [0, 1.01832] \leqslant_p [0, 1.54308].$$

Theorem 3. *Let* $f : [a, b] \to \mathbb{K}_c^+$ *be an IVF that is LR-p convex. Then the following inequality holds:*

$$\mathscr{F}\left([\frac{a^p + b^p}{2}]^{\frac{1}{p}} \right) \leqslant_p \frac{2^{\alpha-1}p^{\alpha}\Gamma(\alpha+1)}{(b^p - a^p)^{\alpha}} \left(J^{p,\alpha}_{\left(\left(\frac{q}{2}(a^p - b^p) + \frac{b^p}{2} + \frac{a^p}{2} \right)^{\frac{1}{p}} \right)^+} \mathscr{F}\left(\frac{q}{2}(a^p - b^p) + b^p \right) \right.$$

$$\left. + J^{p,\alpha}_{\left(\left(\frac{q}{2}(b^p - a^p) + \frac{a^p}{2} + \frac{b^p}{2} \right)^{\frac{1}{p}} \right)^-} \mathscr{F}\left(\frac{q}{2}(b^p - a^p) + a^p \right) \right)$$

$$\leqslant_p \frac{p\alpha}{2}(\mathscr{F}(a) + \mathscr{F}(b)) \int_0^1 \frac{q + t^p}{2} t^{\alpha p - 1} dt$$

$$+ \frac{p\alpha}{2}(\mathscr{F}(a) + \mathscr{F}(b)) \int_0^1 \left(1 - \frac{q + t^p}{2} \right) t^{\alpha p - 1} dt.$$

Proof. From the definition of the LR-p convex IVF, we have

$$\mathscr{F}\left([\rho x^p + (1-\rho) y^p]^{\frac{1}{p}} \right) \leqslant_p \rho \mathscr{F}(x) + (1-\rho)\mathscr{F}(y).$$

Setting $\rho = \frac{1}{2}$, we obtain

$$2\mathscr{F}\left([\frac{x^p + y^p}{2}]^{\frac{1}{p}} \right) \leqslant_p \mathscr{F}(x) + \mathscr{F}(y).$$

Setting $x^p = \frac{q+t^p}{2}a^p + (1-\frac{q+t^p}{2})b^p, y^p = \frac{q+t^p}{2}b^p + (1-\frac{q+t^p}{2})a^p$, we obtain

$$2\mathscr{F}\bigg([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\bigg) \leqslant_p \mathscr{F}\bigg([\frac{q+t^p}{2}a^p + (1-\frac{q+t^p}{2})b^p]^{\frac{1}{p}}\bigg) + \mathscr{F}\bigg([\frac{q+t^p}{2}b^p + (1-\frac{q+t^p}{2})a^p]^{\frac{1}{p}}\bigg).$$

Therefore, from the definition of the IVF, we have

$$2\mathscr{F}_*\bigg([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\bigg) \leqslant \mathscr{F}_*\bigg([\frac{q+t^p}{2}a^p + (1-\frac{q+t^p}{2})b^p]^{\frac{1}{p}}\bigg) + \mathscr{F}_*\bigg([\frac{q+t^p}{2}b^p + (1-\frac{q+t^p}{2})a^p]^{\frac{1}{p}}\bigg)$$

and

$$2\mathscr{F}^*\bigg([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\bigg) \leqslant \mathscr{F}^*\bigg([\frac{q+t^p}{2}a^p + (1-\frac{q+t^p}{2})b^p]^{\frac{1}{p}}\bigg) + \mathscr{F}^*\bigg([\frac{q+t^p}{2}b^p + (1-\frac{q+t^p}{2})a^p]^{\frac{1}{p}}\bigg).$$

Multiplying both inequalities with $t^{\alpha p-p}t^{p-1}$ and integrating with respect to t from 0 to 1, we get

$$2\int_0^1 t^{\alpha p-1}\mathscr{F}_*\bigg([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\bigg)dt \leqslant$$

$$\int_0^1 t^{\alpha p-1}\mathscr{F}_*\bigg([\frac{q+t^p}{2}a^p + (1-\frac{q+t^p}{2})b^p]^{\frac{1}{p}}\bigg)dt + \int_0^1 t^{\alpha p-1}\mathscr{F}_*\bigg([\frac{q+t^p}{2}b^p + (1-\frac{q+t^p}{2})a^p]^{\frac{1}{p}}\bigg)dt$$

and

$$2\int_0^1 t^{\alpha p-1}\mathscr{F}^*\bigg([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\bigg)dt \leqslant$$

$$\int_0^1 t^{\alpha p-1}\mathscr{F}^*\bigg([\frac{q+t^p}{2}a^p + (1-\frac{q+t^p}{2})b^p]^{\frac{1}{p}}\bigg)dt + \int_0^1 t^{\alpha p-1}\mathscr{F}^*\bigg([\frac{q+t^p}{2}b^p + (1-\frac{q+t^p}{2})a^p]^{\frac{1}{p}}\bigg)dt.$$

Now setting $\frac{q+t^p}{2}a^p + (1-\frac{q+t^p}{2})b^p = \omega^p$, $\frac{q+t^p}{2}b^p + (1-\frac{q+t^p}{2})a^p = v^p$, we obtain

$$\frac{2}{\alpha p}\mathscr{F}_*\bigg([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\bigg) \leqslant$$

$$\frac{2^\alpha}{(b^p-a^p)^\alpha}\bigg(\int_{(\frac{q}{2}(a^p-b^p)+\frac{b^p}{2}+\frac{a^p}{2})^{\frac{1}{p}}}^{(\frac{q}{2}(a^p-b^p)+b^p)^{\frac{1}{p}}} (\frac{q}{2}(a^p-b^p)+b^p-\omega^p)^{\alpha-1}\omega^{p-1}\mathscr{F}_*(\omega)d\omega$$

$$+\int_{(\frac{q}{2}(b^p-a^p)+a^p)^{\frac{1}{p}}}^{(\frac{q}{2}(b^p-a^p)+\frac{a^p}{2}+\frac{b^p}{2})^{\frac{1}{p}}} (v^p-\frac{q}{2}(b^p-a^p)-a^p)^{\alpha-1}v^{p-1}\mathscr{F}_*(v)dv\bigg)$$

and

$$\frac{2}{\alpha p}\mathscr{F}^*\bigg([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\bigg) \leqslant$$

$$\frac{2^\alpha}{(b^p-a^p)^\alpha}\bigg(\int_{(\frac{q}{2}(a^p-b^p)+\frac{b^p}{2}+\frac{a^p}{2})^{\frac{1}{p}}}^{(\frac{q}{2}(a^p-b^p)+b^p)^{\frac{1}{p}}} (\frac{q}{2}(a^p-b^p)+b^p-\omega^p)^{\alpha-1}\omega^{p-1}\mathscr{F}^*(\omega)d\omega$$

$$+\int_{(\frac{q}{2}(b^p-a^p)+a^p)^{\frac{1}{p}}}^{(\frac{q}{2}(b^p-a^p)+\frac{a^p}{2}+\frac{b^p}{2})^{\frac{1}{p}}} (v^p-\frac{q}{2}(b^p-a^p)-a^p)^{\alpha-1}v^{p-1}\mathscr{F}^*(v)dv\bigg).$$

When identified in terms of the Katagumpola fractional integral, we obtain

$$\frac{2}{\alpha p}\mathscr{F}_*\bigg([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\bigg) \leqslant$$

$$\frac{2^\alpha\Gamma(\alpha)}{p^{1-\alpha}(b^p-a^p)^\alpha}\bigg(J^{p,\alpha}_{\left(\left(\frac{q}{2}(a^p-b^p)+\frac{b^p}{2}+\frac{a^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}_*((\frac{q}{2}(a^p-b^p)+b^p))+\frac{b^p}{2}+\frac{a^p}{2})$$

$$+J^{p,\alpha}_{\left(\left(\frac{q}{2}(b^p-a^p)+\frac{a^p}{2}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}_*(\frac{q}{2}(b^p-a^p)+a^p)\bigg)$$

and

$$\frac{2}{\alpha p}\mathscr{F}^*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right)\leqslant$$

$$\frac{2^\alpha\Gamma(\alpha)}{p^{1-\alpha}(b^p-a^p)^\alpha}\bigg(J^{p,\alpha}_{\left(\left(\frac{q}{2}(a^p-b^p)+\frac{b^p}{2}+\frac{a^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}^*((\frac{q}{2}(a^p-b^p)+b^p))+\frac{b^p}{2}+\frac{a^p}{2})$$

$$+J^{p,\alpha}_{\left(\left(\frac{q}{2}(b^p-a^p)+\frac{a^p}{2}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}^*(\frac{q}{2}(b^p-a^p)+a^p)\bigg).$$

Consequently, we have

$$\frac{2}{\alpha p}[\mathscr{F}^*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right),\mathscr{F}_*\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right)]$$

$$\leqslant_p\frac{2^\alpha\Gamma(\alpha)}{p^{1-\alpha}(b^p-a^p)^\alpha}\bigg(J^{p,\alpha}_{\left(\left(\frac{q}{2}(a^p-b^p)+\frac{b^p}{2}+\frac{a^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}^*((\frac{q}{2}(a^p-b^p)+b^p))+\frac{b^p}{2}+\frac{a^p}{2})$$

$$+J^{p,\alpha}_{\left(\left(\frac{q}{2}(b^p-a^p)+\frac{a^p}{2}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}^*(\frac{q}{2}(b^p-a^p)+a^p),$$

$$J^{p,\alpha}_{\left(\left(\frac{q}{2}(a^p-b^p)+\frac{b^p}{2}+\frac{a^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}_*((\frac{q}{2}(a^p-b^p)+b^p))+\frac{b^p}{2}+\frac{a^p}{2})$$

$$+J^{p,\alpha}_{\left(\left(\frac{q}{2}(b^p-a^p)+\frac{a^p}{2}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}_*(\frac{q}{2}(b^p-a^p)+a^p)\bigg).$$

From which, we obtain the following:

$$\frac{2}{\alpha p}\mathscr{F}\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right)\leqslant_p$$

$$\frac{2^\alpha\Gamma(\alpha)}{p^{1-\alpha}(b^p-a^p)^\alpha}\bigg(J^{p,\alpha}_{\left(\left(\frac{q}{2}(a^p-b^p)+\frac{b^p}{2}+\frac{a^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}((\frac{q}{2}(a^p-b^p)+b^p))+\frac{b^p}{2}+\frac{a^p}{2})$$

$$+J^{p,\alpha}_{\left(\left(\frac{q}{2}(b^p-a^p)+\frac{a^p}{2}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}(\frac{q}{2}(b^p-a^p)+a^p)\bigg).$$

Now, to obtain the right hand side inequality, we apply the definition of the LR-p convex IVF on the expression

$$\mathscr{F}\left([\frac{q+t^p}{2}a^p+(1-\frac{q+t^p}{2})b^p]^{\frac{1}{p}}\right)+\mathscr{F}\left([\frac{q+t^p}{2}b^p+(1-\frac{q+t^p}{2})a^p]^{\frac{1}{p}}\right).$$

Multiplying everything by $t^{\alpha p-1}$ and integrating with respect to t from 0 to 1, we get the original inequality

$$\mathscr{F}\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \frac{2^{\alpha-1}p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{\left(\left(\frac{q}{2}(a^p-b^p)+\frac{b^p}{2}+\frac{a^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}\left(\frac{q}{2}(a^p-b^p)+b^p\right)\right.$$

$$\left.+J^{p,\alpha}_{\left(\left(\frac{q}{2}(b^p-a^p)+\frac{a^p}{2}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}\left(\frac{q}{2}(b^p-a^p)+a^p\right)\right)$$

$$\leqslant_p \frac{p\alpha}{2}(\mathscr{F}(a)+\mathscr{F}(b))\int_0^1\frac{q+t^p}{2}t^{\alpha p-1}dt+\frac{p\alpha}{2}(\mathscr{F}(a)+\mathscr{F}(b))\int_0^1(1-\frac{q+t^p}{2})t^{\alpha p-1}dt.$$

□

Setting $q=\frac{1}{3}$, we have the following new inequality:

Corollary 3.

$$\mathscr{F}\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \frac{2^{\alpha-1}p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{\left(\left(\frac{1}{6}(a^p-b^p+\frac{b^p}{2}+\frac{a^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}\left(\frac{1}{6}(a^p-b^p)+b^p\right)\right.$$

$$\left.+J^{p,\alpha}_{\left(\left(\frac{1}{6}(b^p-a^p)+\frac{a^p}{2}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}\left(\frac{1}{6}(b^p-a^p)+a^p\right)\right)$$

$$\leqslant_p \frac{p\alpha}{2}(\mathscr{F}(a)+\mathscr{F}(b))\int_0^1\frac{\frac{1}{3}+t^p}{2}t^{\alpha p-1}dt$$

$$+\frac{p\alpha}{2}(\mathscr{F}(a)+\mathscr{F}(b))\int_0^1\left(1-\frac{\frac{1}{3}+t^p}{2}\right)t^{\alpha p-1}dt.$$

Example 2. *Let p be an odd number and the I-V-F $\mathscr{F}:[a,b]=[-1,1]\to\mathbb{R}_I^+$ defined by $\mathscr{F}(\omega)=[\omega^p,e^{w^p}]$. Since end point functions $\mathscr{F}_*(\omega)=\omega^p$ and $\mathscr{F}^*(\omega)=e^{\omega^p}$ are both p-convex functions, $\mathscr{F}$ is LR-p-convex-I-V-F. Setting $p=1,\alpha=2,q=\frac{1}{3}$ in the above inequality, we have:*

$$\mathscr{F}\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \frac{2^{\alpha-1}p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{\left(\left(\frac{q}{2}(a^p-b^p)+\frac{b^p}{2}+\frac{a^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}\left(\frac{q}{2}(a^p-b^p)+b^p\right)\right.$$

$$\left.+J^{p,\alpha}_{\left(\left(\frac{q}{2}(b^p-a^p)+\frac{a^p}{2}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}\left(\frac{q}{2}(b^p-a^p)+a^p\right)\right)$$

$$\leqslant_p \frac{p\alpha}{2}(\mathscr{F}(a)+\mathscr{F}(b))\int_0^1\frac{q+t^p}{2}t^{\alpha p-1}dt+\frac{p\alpha}{2}(\mathscr{F}(a)+\mathscr{F}(b))\int_0^1(1-\frac{q+t^p}{2})t^{\alpha p-1}dt.$$

$$\mathscr{F}_*\left((\frac{a^p+b^p}{2})^{\frac{1}{p}}\right)=\mathscr{F}_*(0)=0,$$

$$\frac{2^{\alpha-1}p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{\left(\left(\frac{q}{2}(a^p-b^p)+\frac{b^p}{2}+\frac{a^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}_*\left(\frac{q}{2}(a^p-b^p)+b^p\right)\right.$$

$$\left.+J^{p,\alpha}_{\left(\left(\frac{q}{2}(b^p-a^p)+\frac{a^p}{2}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}_*\left(\frac{q}{2}(b^p-a^p)+a^p\right)\right)=0,$$

and

$$\frac{p\alpha}{2}(\mathscr{F}_*(a)+\mathscr{F}_*(b))\int_0^1 \frac{q+t^p}{2}t^{\alpha p-1}dt+\frac{p\alpha}{2}(\mathscr{F}_*(a)+\mathscr{F}_*(b))\int_0^1 (1-\frac{q+t^p}{2})t^{\alpha p-1}dt=0.$$

This implies

$$0\leqslant 0\leqslant 0.$$

Computing the upper end point function, we get

$$\mathscr{F}^*\left((\frac{a^p+b^p}{2})^{\frac{1}{p}}\right)=\mathscr{F}^*(0)=1,$$

$$\frac{2^{\alpha-1}p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\bigg(J^{p,\alpha}_{\left(\left(\frac{q}{2}(a^p-b^p)+\frac{b^p}{2}+\frac{a^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}^*\Big(\frac{q}{2}(a^p-b^p)+b^p\Big)$$

$$+J^{p,\alpha}_{\left(\left(\frac{q}{2}(b^p-a^p)+\frac{a^p}{2}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}^*\Big(\frac{q}{2}(b^p-a^p)+a^p\Big)\bigg)\sim 1.00076,$$

$$\frac{p\alpha}{4}(\mathscr{F}^*(a)+\mathscr{F}^*(b))\int_0^1 t^p t^{\alpha p-1}dt+\frac{p\alpha}{2}(\mathscr{F}^*(a)+\mathscr{F}^*(b))\int_0^1 (1-\frac{t^p}{2})t^{\alpha p-1}dt$$

$$=\frac{3}{8}\left(\frac{1}{e}+e\right)\sim 1.15731$$

Hence,

$$1\leqslant 1.00076\leqslant 1.15731.$$

From which we get

$$[0,1]\leqslant_p [0,1.00076]\leqslant_p [0,1.15731].$$

Theorem 4. *Let $p,\alpha>0, u,v\in\mathbb{I}$ with $v>u$ and $\mathscr{F},\mathscr{G}\in L([u,v])$. If $\mathscr{F},\mathscr{G}\in LRSX([u,v],\mathbb{K}^+,p)$, then we have*

$$\frac{2^{\alpha}p^\alpha\Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+}\mathscr{F}(b)\mathscr{G}(b)+J^{p,\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-}\mathscr{F}(a)\mathscr{G}(a)\right)$$

$$\leqslant_p \frac{\alpha}{4(\alpha+2)}(\mathscr{F}(a)\mathscr{G}(a)+\mathscr{F}(b)\mathscr{G}(b))+\frac{\alpha(\alpha+3)}{4(\alpha^2+3\alpha+2)}(\mathscr{F}(a)\mathscr{G}(b)+\mathscr{F}(b)\mathscr{G}(a))$$

$$+\frac{\alpha^2+5\alpha+8}{4(\alpha^2+3\alpha+2)}(\mathscr{F}(b)\mathscr{G}(b)+\mathscr{F}(a)\mathscr{G}(a)).$$

Proof. Since $\mathscr{F},\mathscr{G}\in LRSX([u,v],K_c^+,p)$, then for $\rho\in[0,1]$, we have

$$\mathscr{F}\left([\frac{t^pa^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)\leqslant_p \frac{t^p}{2}\mathscr{F}(a)+(1-\frac{t^p}{2})\mathscr{F}(b)$$

and

$$\mathscr{G}\left([\frac{t^pa^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)\leqslant_p \frac{t^p}{2}\mathscr{G}(a)+(1-\frac{t^p}{2})\mathscr{G}(b).$$

From the definition of the IVF, we have

$$\mathscr{F}_*\left([\frac{t^pa^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)\leqslant \frac{t^p}{2}\mathscr{F}_*(a)+(1-\frac{t^p}{2})\mathscr{F}_*(b),$$

$$\mathscr{F}^*\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2} b^p]^{\frac{1}{p}}\right) \leqslant \frac{t^p}{2}\mathscr{F}^*(a) + (1 - \frac{t^p}{2})\mathscr{F}^*(b),$$

$$\mathscr{G}_*\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2} b^p]^{\frac{1}{p}}\right) \leqslant \frac{t^p}{2}\mathscr{G}_*(a) + (1 - \frac{t^p}{2})\mathscr{G}_*(b)$$

and

$$\mathscr{G}^*\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2} b^p]^{\frac{1}{p}}\right) \leqslant \frac{t^p}{2}\mathscr{G}^*(a) + (1 - \frac{t^p}{2})\mathscr{G}^*(b).$$

Now multiplying the corresponding functions of $\mathscr{F}, \mathscr{G}$, we get

$$\mathscr{F}_*\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2} b^p]^{\frac{1}{p}}\right)\mathscr{G}_*\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2} b^p]^{\frac{1}{p}}\right)$$

$$\leqslant \left(\frac{t^p}{2}\mathscr{F}_*(a) + \frac{2-t^p}{2}\mathscr{F}_*(b)\right)\left(\frac{t^p}{2}\mathscr{G}_*(a) + \frac{2-t^p}{2}\mathscr{G}_*(b)\right)$$

$$= \frac{t^{2p}}{4}\mathscr{F}_*(a)\mathscr{G}_*(a) + \frac{(2-t^p)^2}{4}\mathscr{F}_*(b)\mathscr{G}_*(b) + \frac{t^p}{2}\frac{2-t^p}{2}\mathscr{G}_*(b)\mathscr{F}_*(a) + \frac{2-t^p}{2}\frac{t^p}{2}\mathscr{F}_*(b)\mathscr{G}_*(a)$$

$$= \frac{t^{2p}}{4}\mathscr{F}_*(a)\mathscr{G}_*(a) + \frac{(2-t^p)^2}{4}\mathscr{F}_*(b)\mathscr{G}_*(b) + \frac{t^p(2-t^p)}{4}(\mathscr{F}_*(a)\mathscr{G}_*(b) + \mathscr{F}_*(b)\mathscr{G}_*(a))$$

and

$$\mathscr{F}^*\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2} b^p]^{\frac{1}{p}}\right)\mathscr{G}^*\left([\frac{t^p a^p}{2} + \frac{2-t^p}{2} b^p]^{\frac{1}{p}}\right)$$

$$\leqslant \left(\frac{t^p}{2}\mathscr{F}^*(a) + \frac{2-t^p}{2}\mathscr{F}^*(b)\right)\left(\frac{t^p}{2}\mathscr{G}^*(a) + \frac{2-t^p}{2}\mathscr{G}^*(b)\right)$$

$$= \frac{t^{2p}}{4}\mathscr{F}^*(a)\mathscr{G}^*(a) + \frac{(2-t^p)^2}{4}\mathscr{F}^*(b)\mathscr{G}^*(b) + \frac{t^p}{2}\frac{2-t^p}{2}\mathscr{G}^*(b)\mathscr{F}^*(a) + \frac{2-t^p}{2}\frac{t^p}{2}\mathscr{F}^*(b)\mathscr{G}^*(a)$$

$$= \frac{t^{2p}}{4}\mathscr{F}^*(a)\mathscr{G}^*(a) + \frac{(2-t^p)^2}{4}\mathscr{F}^*(b)\mathscr{G}^*(b) + \frac{t^p(2-t^p)}{4}(\mathscr{F}^*(a)\mathscr{G}^*(b) + \mathscr{F}^*(b)\mathscr{G}^*(a)).$$

Analogously, we have

$$\mathscr{F}_*\left([\frac{t^p b^p}{2} + \frac{2-t^p}{2} a^p]^{\frac{1}{p}}\right)\mathscr{G}_*\left([\frac{t^p b^p}{2} + \frac{2-t^p}{2} a^p]^{\frac{1}{p}}\right)$$

$$\leqslant \left(\frac{t^p}{2}\mathscr{F}_*(b) + \frac{2-t^p}{2}\mathscr{F}_*(a)\right)\left(\frac{t^p}{2}\mathscr{G}_*(b) + \frac{2-t^p}{2}\mathscr{G}_*(a)\right)$$

$$= \frac{t^{2p}}{4}\mathscr{F}_*(b)\mathscr{G}_*(b) + \frac{(2-t^p)^2}{4}\mathscr{F}_*(a)\mathscr{G}_*(a) + \frac{t^p}{2}\frac{2-t^p}{2}\mathscr{G}_*(a)\mathscr{F}_*(b) + \frac{2-t^p}{2}\frac{t^p}{2}\mathscr{F}_*(a)\mathscr{G}_*(b)$$

$$= \frac{t^{2p}}{4}\mathscr{F}_*(b)\mathscr{G}_*(b) + \frac{(2-t^p)^2}{4}\mathscr{F}_*(a)\mathscr{G}_*(a) + \frac{t^p(2-t^p)}{4}(\mathscr{F}_*(b)\mathscr{G}_*(a) + \mathscr{F}_*(a)\mathscr{G}_*(b))$$

and

$$\mathscr{F}^*\left([\frac{t^p b^p}{2} + \frac{2-t^p}{2} a^p]^{\frac{1}{p}}\right)\mathscr{G}^*\left([\frac{t^p b^p}{2} + \frac{2-t^p}{2} a^p]^{\frac{1}{p}}\right)$$

$$\leqslant \left(\frac{t^p}{2}\mathscr{F}^*(b) + \frac{2-t^p}{2}\mathscr{F}^*(a)\right)\left(\frac{t^p}{2}\mathscr{G}^*(b) + \frac{2-t^p}{2}\mathscr{G}^*(a)\right)$$

$$= \frac{t^{2p}}{4}\mathscr{F}^*(b)\mathscr{G}^*(b) + \frac{(2-t^p)^2}{4}\mathscr{F}^*(a)\mathscr{G}^*(a) + \frac{t^p}{2}\frac{2-t^p}{2}\mathscr{G}^*(a)\mathscr{F}^*(b) + \frac{2-t^p}{2}\frac{t^p}{2}\mathscr{F}^*(a)\mathscr{G}^*(b)$$

$$= \frac{t^{2p}}{4}\mathscr{F}^*(b)\mathscr{G}^*(b) + \frac{(2-t^p)^2}{4}\mathscr{F}^*(a)\mathscr{G}^*(a) + \frac{t^p(2-t^p)}{4}(\mathscr{F}^*(b)\mathscr{G}^*(a) + \mathscr{F}^*(a)\mathscr{G}^*(b)).$$

Adding the corresponding parts of the inequalities, we get

$$\begin{aligned}
&\mathscr{F}_*\left([\frac{t^p b^p}{2}+\frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right)\mathscr{G}_*\left([\frac{t^p b^p}{2}+\frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right)\\
&+\mathscr{F}_*\left([\frac{t^p a^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)\mathscr{G}_*\left([\frac{t^p a^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)\\
&\leqslant \frac{t^{2p}}{4}(\mathscr{F}_*(a)\mathscr{G}_*(a)+\mathscr{F}_*(b)\mathscr{G}_*(b))+\frac{t^p(2-t^p)}{4}(\mathscr{F}_*(a)\mathscr{G}_*(b)+\mathscr{F}_*(b)\mathscr{G}_*(a))\\
&+\frac{(2-t^p)^2}{4}(\mathscr{F}_*(b)\mathscr{G}_*(b)+\mathscr{F}_*(a)\mathscr{G}_*(a))
\end{aligned}$$

and

$$\begin{aligned}
&\mathscr{F}^*\left([\frac{t^p b^p}{2}+\frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right)\mathscr{G}^*\left([\frac{t^p b^p}{2}+\frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right)\\
&+\mathscr{F}^*\left([\frac{t^p a^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)\mathscr{G}^*\left([\frac{t^p a^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)\\
&\leqslant \frac{t^{2p}}{4}(\mathscr{F}^*(a)\mathscr{G}^*(a)+\mathscr{F}^*(b)\mathscr{G}^*(b))+\frac{t^p(2-t^p)}{4}(\mathscr{F}^*(a)\mathscr{G}^*(b)+\mathscr{F}^*(b)\mathscr{G}^*(a))\\
&+\frac{(2-t^p)^2}{4}(\mathscr{F}^*(b)\mathscr{G}^*(b)+\mathscr{F}^*(a)\mathscr{G}^*(a)).
\end{aligned}$$

Multiplying both inequalities above with $t^{\alpha p-1}$ and integrating with respect to t from 0 to 1, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha p-1}\mathscr{F}_*\left([\frac{t^p b^p}{2}+\frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right)\mathscr{G}_*\left([\frac{t^p b^p}{2}+\frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right)dt\\
&+\int_0^1 t^{\alpha p-1}\mathscr{F}_*\left([\frac{t^p a^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)\mathscr{G}_*\left([\frac{t^p a^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)dt\\
&\leqslant \frac{\alpha}{4(\alpha+2)}(\mathscr{F}_*(a)\mathscr{G}_*(a)+\mathscr{F}_*(b)\mathscr{G}_*(b))+\frac{\alpha(\alpha+3)}{4(\alpha^2+3\alpha+2)}(\mathscr{F}_*(a)\mathscr{G}_*(b)+\mathscr{F}_*(b)\mathscr{G}_*(a))\\
&+\frac{\alpha^2+5\alpha+8}{4(\alpha^2+3\alpha+2)}(\mathscr{F}_*(b)\mathscr{G}_*(b)+\mathscr{F}_*(a)\mathscr{G}_*(a))
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 t^{\alpha p-1}\mathscr{F}^*\left([\frac{t^p b^p}{2}+\frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right)\mathscr{G}^*\left([\frac{t^p b^p}{2}+\frac{2-t^p}{2}a^p]^{\frac{1}{p}}\right)dt\\
&+\int_0^1 t^{\alpha p-1}\mathscr{F}^*\left([\frac{t^p a^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)\mathscr{G}^*\left([\frac{t^p a^p}{2}+\frac{2-t^p}{2}b^p]^{\frac{1}{p}}\right)dt\\
&\leqslant \frac{\alpha}{4(\alpha+2)}(\mathscr{F}^*(a)\mathscr{G}^*(a)+\mathscr{F}^*(b)\mathscr{G}^*(b))+\frac{\alpha(\alpha+3)}{4(\alpha^2+3\alpha+2)}(\mathscr{F}^*(a)\mathscr{G}^*(b)+\mathscr{F}^*(b)\mathscr{G}^*(a))\\
&+\frac{\alpha^2+5\alpha+8}{4(\alpha^2+3\alpha+2)}(\mathscr{F}^*(b)\mathscr{G}^*(b)+\mathscr{F}^*(a)\mathscr{G}^*(a)).
\end{aligned}$$

Identifying the product of the functions under the integral to be Katagumpola type integral, we obtain

$$\begin{aligned}
&\frac{2^\alpha p^\alpha \Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+}\mathscr{F}_*(b)\mathscr{G}_*(b)+J^{p,\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-}\mathscr{F}_*(a)\mathscr{G}_*(a)\right)\\
&\leqslant \frac{\alpha}{4(\alpha+2)}(\mathscr{F}_*(a)\mathscr{G}_*(a)+\mathscr{F}_*(b)\mathscr{G}_*(b))+\frac{\alpha(\alpha+3)}{4(\alpha^2+3\alpha+2)}(\mathscr{F}_*(a)\mathscr{G}_*(b)+\mathscr{F}_*(b)\mathscr{G}_*(a))
\end{aligned}$$

$$+\frac{\alpha^2+5\alpha+8}{4(\alpha^2+3\alpha+2)}(\mathscr{F}_*(b)\mathscr{G}_*(b)+\mathscr{F}_*(a)\mathscr{G}_*(a))$$

and

$$\frac{2^\alpha p^\alpha \Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+}\mathscr{F}^*(b)\mathscr{G}^*(b)+J^{p,\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-}\mathscr{F}^*(a)\mathscr{G}^*(a)\right)$$

$$\leqslant \frac{\alpha}{4(\alpha+2)}(\mathscr{F}^*(a)\mathscr{G}^*(a)+\mathscr{F}^*(b)\mathscr{G}^*(b))+\frac{\alpha(\alpha+3)}{4(\alpha^2+3\alpha+2)}(\mathscr{F}^*(a)\mathscr{G}^*(b)+\mathscr{F}^*(b)\mathscr{G}^*(a))$$

$$+\frac{\alpha^2+5\alpha+8}{4(\alpha^2+3\alpha+2)}(\mathscr{F}^*(b)\mathscr{G}^*(b)+\mathscr{F}^*(a)\mathscr{G}^*(a)).$$

As a consequence, we obtain

$$\frac{2^\alpha p^\alpha \Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\Bigg(J^{p,\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+}\mathscr{F}^*(b)\mathscr{G}^*(b)+J^{p,\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-}\mathscr{F}^*(a)\mathscr{G}^*(a),$$

$$J^{p,\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+}\mathscr{F}_*(b)\mathscr{G}_*(b)+J^{p,\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-}\mathscr{F}_*(a)\mathscr{G}_*(a)\Bigg)$$

$$\leqslant_p \frac{\alpha}{4(\alpha+2)}(\mathscr{F}_*(a)\mathscr{G}_*(a)+\mathscr{F}_*(b)\mathscr{G}_*(b),\mathscr{F}^*(a)\mathscr{G}^*(a)+\mathscr{F}^*(b)\mathscr{G}^*(b))$$

$$+\frac{\alpha(\alpha+3)}{4(\alpha^2+3\alpha+2)}(\mathscr{F}_*(a)\mathscr{G}_*(b)+\mathscr{F}_*(b)\mathscr{G}_*(a),\mathscr{F}^*(a)\mathscr{G}^*(b)+\mathscr{F}^*(b)\mathscr{G}^*(a))$$

$$+\frac{\alpha^2+5\alpha+8}{4(\alpha^2+3\alpha+2)}(\mathscr{F}_*(b)\mathscr{G}_*(b)+\mathscr{F}_*(a)\mathscr{G}_*(a),\mathscr{F}^*(b)\mathscr{G}^*(b)+\mathscr{F}^*(a)\mathscr{G}^*(a)).$$

From which we obtain the original inequality, namely

$$\frac{2^\alpha p^\alpha \Gamma(\alpha+1)}{(b^p-a^p)^\alpha}\left(J^{p,\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^+}\mathscr{F}(b)\mathscr{G}(b)+J^{p,\alpha}_{((\frac{a^p+b^p}{2})^{\frac{1}{p}})^-}\mathscr{F}(a)\mathscr{G}(a)\right)$$

$$\leqslant_p \frac{\alpha}{4(\alpha+2)}(\mathscr{F}(a)\mathscr{G}(a)+\mathscr{F}(b)\mathscr{G}(b))+\frac{\alpha(\alpha+3)}{4(\alpha^2+3\alpha+2)}(\mathscr{F}(a)\mathscr{G}(b)+\mathscr{F}(b)\mathscr{G}(a))$$

$$+\frac{\alpha^2+5\alpha+8}{4(\alpha^2+3\alpha+2)}(\mathscr{F}(b)\mathscr{G}(b)+\mathscr{F}(a)\mathscr{G}(a)).$$

□

Setting $p=1$ in the above theorem, we obtain Theorem 7 of Srivastava et al. [36].

Corollary 4.

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J^{1,\alpha}_{((\frac{a+b}{2}))^+}\mathscr{F}(b)\mathscr{G}(b)+J^{1,\alpha}_{((\frac{a+b}{2}))^-}\mathscr{F}(a)\mathscr{G}(a)\right)$$

$$\leqslant_p \frac{\alpha}{4}\left(\frac{1}{\alpha+2}-\frac{2}{\alpha+1}+\frac{2}{\alpha}\right)\Psi(a,b)+\frac{\alpha}{4}\left(\frac{2}{\alpha+1}-\frac{1}{\alpha+2}\right)\Omega(a,b).$$

where

$$\Psi(a,b)=\mathscr{F}(a)\mathscr{G}(a)+\mathscr{F}(b)\mathscr{G}(b),$$
$$\Omega(a,b)=\mathscr{F}(a)\mathscr{G}(b)+\mathscr{F}(b)\mathscr{G}(a)$$

Corollary 5. *Setting $p=7$ in the previously derived theorem, we obtain the following new inequality:*

$$
\frac{2^{\alpha}7^{\alpha}\Gamma(\alpha+1)}{(b^7-a^7)^{\alpha}}\left(J^{7,\alpha}_{((\frac{a^7+b^7}{2})^{\frac{1}{7}})^+}\mathscr{F}(b)\mathscr{G}(b)+J^{7,\alpha}_{((\frac{a^7+b^7}{2})^{\frac{1}{7}})^-}\mathscr{F}(a)\mathscr{G}(a)\right)
$$
$$
\leqslant_p \frac{\alpha}{4(\alpha+2)}(\mathscr{F}(a)\mathscr{G}(a)+\mathscr{F}(b)\mathscr{G}(b))+\frac{\alpha(\alpha+3)}{4(\alpha^2+3\alpha+2)}(\mathscr{F}(a)\mathscr{G}(b)+\mathscr{F}(b)\mathscr{G}(a))
$$
$$
+\frac{\alpha^2+5\alpha+8}{4(\alpha^2+3\alpha+2)}(\mathscr{F}(b)\mathscr{G}(b)+\mathscr{F}(a)\mathscr{G}(a)).
$$

4. Conclusions

In this paper, various new inequalities have been obtained in the IVF setting. As a consequence of the IVF setting, we recover the previously obtained inequalities by setting the lower and upper bound to be the same. Our theorems generalize the results reported in the recent past, which have been supplemented with suitable numerical examples. It is an open problem as to whether our inequalities can be generalized further using various different fractional operators and inequality settings using suitable techniques.

Author Contributions: Conceptualization, V.S. and S.R.; methodology, V.S., S.R., R.R. and O.A.A.A.; formal analysis, V.S., S.R. and O.A.A.A.; writing—original draft preparation, V.S. and S.R.; supervision, R.R., O.A.A.A. and S.R. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deputyship for Research & Innovation, Ministry of Education in Saudi Arabia for funding this research work through the project number (IF-PSAU-2021/01/18689).

Acknowledgments: The research is supported by the Deanship of Scientific Research, Prince Sattam Bin Abulaziz University, Alkharj, Saudi Arabia. The authors are thankful to the anonymous reviewers for their valuable comments, which helped to bring the manuscript to its present form.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Siricharuanun, P.; Erden, S.; Ali, M.A.; Budak, H.; Chasreechai, S.; Sitthiwirattham, T. Some New Simpson's and Newton's Formulas Type Inequalities for Convex Functions in Quantum Calculus. *Mathematics* **2021**, *9*, 1992. [CrossRef]
2. You, X.; Ali, M.A.; Budak, H.; Reunsumrit, J.; Sitthiwirattham, T. Hermite–Hadamard–Mercer-Type Inequalities for Harmonically Convex Mappings. *Mathematics* **2021**, *9*, 2556. [CrossRef]
3. Sarikaya, M.Z.; Yildirim, H. On Hermite–Hadamard type inequalities for Riemann-Liouville fractional integrals. *Miskolc Math. Notes* **2016**, *17*, 1049–1059. [CrossRef]
4. Chen, H.; Katugampola, U.N. Hermite–Hadamard and Hermite–Hadamard-Fejr type inequalities for generalized fractional integrals. *J. Math. Anal. Appl.* **2017**, *446*, 1274–1291. [CrossRef]
5. Han, J.; Mohammed, P.O.; Zeng, H. Generalized fractional integral inequalities of Hermite–Hadamard-type for a convex function. *Open Math.* **2020**, *18*, 794–806. [CrossRef]
6. Awan, M.U.; Talib, S.; Chu, Y.M.; Noor, M.A.; Noor, K.I. Some new refinements of Hermite–Hadamard-type inequalities involving-Riemann-Liouville fractional integrals and applications. *Math. Probl. Eng.* **2020**, *2020*, 3051920. [CrossRef]
7. Aljaaidi, T.A.; Pachpatte, D.B. The Minkowski's inequalities via f-Riemann-Liouville fractional integral operators. *Rend. Circ. Mat. Palermo Ser. 2* **2021**, *70*, 893–906. [CrossRef]
8. Mohammed, P.O.; Aydi, H.; Kashuri, A.; Hamed, Y.S.; Abualnaja, K.M. Midpoint inequalities in fractional calculus defined using positive weighted symmetry function kernels. *Symmetry* **2021**, *13*, 550. [CrossRef]
9. Mohammed, P.O.; Abdeljawad, T.; Jarad, F.; Chu, Y.M. Existence and uniqueness of uncertain fractional backward difference equations of Riemann-Liouville type. *Math. Probl. Eng.* **2020**, *2020*, 6598682. [CrossRef]
10. Mitrinović, D.S. *Analytic Inequalities*; Springer: Berlin, Germany, 1970.
11. Pečarić, J.; Proschan, F.; Tong, Y. *Convex Functions, Partial Orderings, and Statistical Applications*; Academic Press, Inc.: Cambridge, MA, USA, 1992.
12. Hadamard, J. Étude sur les propriétés des fonctions entières en particulier d'une fonction considéréé par Riemann. *J. Math. Pures Appl.* **1893**, *58*, 171–215.
13. Cristescu, G. Hadamard type inequalities for convolution of h-convex functions. *Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity* **2010**, *8*, 3–11.
14. Dragomir, S.S. An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **2002**, *3*, 31.

15. Dragomir, S.S.; Fitzpatrick, S. The Hadamard inequalities for sconvex functions in the second sense. *Demonstratio Math.* **1999**, *32*, 687–696.
16. El Farissi, A. Simple proof and refeinment of Hermite-Hadamard inequality. *J. Math. Ineq.* **2010**, *4*, 365–369. [CrossRef]
17. Kikianty, E.; Dragomir, S.S. Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space. *Math. Inequal. Appl.* **2010**, *in press*. [CrossRef]
18. Mitrinović, D.S.; Lacković, I.B. Hermite and convexity. *Aequationes Math.* **1985**, *28*, 229–232. [CrossRef]
19. Hermann, R. *Fractional Calculus an Introduction for Physicists*; World Scientific Publishing Co. Pte. Ltd.: Singapore, 2011; p. 596224.
20. Oldham, K.B.; Spanier, J. *The Fractional Calculus Theory and Applications of Differentation and Integration to Arbitrary Order*; Academic Press, Inc.: London, UK, 1974.
21. Yang, X.J. *General Fractional Derivatives Theory, Methods and Applications*; Taylor and Francis Group: London, UK, 2019.
22. Stojiljković, V.; Ramaswamy, R.; Alshammari, F.; Ashour, O.A.; Alghazwani, M.L.H.; Radenović, S. Hermite–Hadamard Type Inequalities Involving (k-p) Fractional Operator for Various Types of Convex Functions. *Fractal Fract.* **2022**, *6*, 376. [CrossRef]
23. Khan, M.B.; Cătas, A.; Saeed, T. Generalized Fractional Integral Inequalities for p-Convex Fuzzy Interval-Valued Mappings. *Fractal Fract.* **2022**, *6*, 324. [CrossRef]
24. Khan, M.B.; Mohammed, P.O.; Noor, M.A.; Baleanu, D.; Guirao, J.L.G. Some New Fractional Estimates of Inequalities for LR-p-Convex Interval-Valued Functions by Means of Pseudo Order Relation. *Axioms* **2021**, *10*, 175. [CrossRef]
25. Zhang, D.; Guo, C.; Chen, D.; Wang, G. Jensen's inequalities for set-valued and fuzzy set-valued functions. *Fuzzy Sets Syst.* **2021** , *404*, 178–204. [CrossRef]
26. Costa, T.M. Jensen's inequality type integral for fuzzy-interval-valued functions. *Fuzzy Sets Syst.* **2017**, *327*, 31–47. [CrossRef]
27. Srivastava, H.M.; Sahoo, S.K.; Mohammed, P.O.; Kodamasingh, B.; Nonlaopon, K.; Abualnaja, K.M. Interval valued Hadamard-Fejér and Pachpatte Type inequalities pertaining to a new fractional integral operator with exponential kernel. *AIMS Math.* **2022**, *7*, 15041–15063. [CrossRef]
28. Srivastava, H.M.; Sahoo, S.K.; Mohammed, P.O.; Baleanu, D.; Kodamasingh, B. Hermite–Hadamard Type Inequalities for Interval-Valued Preinvex Functions via Fractional Integral Operators. *Int. J. Comput. Intell. Syst.* **2022**, *15*, 8. [CrossRef]
29. Varošanec, S. On h-convexity. *J. Math. Anal. Appl.* **2007**, *326*, 303–311. [CrossRef]
30. Moore, R.E. *Interval Analysis*; Prentice Hall: Englewood Cliffs, NJ, USA, 1966.
31. Roman-Flores, H.; Chalco-Cano, Y.; Silva, G.N. A note on Gronwall type inequality for interval-valued functions. In Proceedings of the IEEE IFSA World Congress and NAFIPS Annual Meeting, Edmonton, AB, Canada, 24–28 June 2013; Volume 35, pp. 1455–1458.
32. Katugampola, U.N. A new approach to generalized fractional derivatives. *Bull. Math. Anal. Appl.* **2014**, *6*, 1–15.
33. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*; Dover Publications: New York, NY, USA, 1992.
34. Fang, Z.; Shi, R. On the (p,h)-convex function and some integral inequalities. *J. Inequalities Appl.* **2014**, *45*, 1–16 . [CrossRef]
35. Khan, M.B.; Srivastava, H.M.; Mohammed, P.O.; Baleanu, D.; Jawa, T.M. Fuzzy-interval inequalities for generalized convex fuzzy-interval-valued functions via fuzzy Riemann integrals. *AIMS Math.* **2022**, *7*, 1507–1535. [CrossRef]
36. Srivastava, H.M.; Sahoo, S.K.; Mohammed, P.O.; Kodamasingh, B.; Hamed, Y.S. New Riemann–Liouville Fractional-Order Inclusions for Convex Functions via Interval-Valued Settings Associated with Pseudo-Order Relations. *Fractal Fract.* **2022**, *6*, 212. [CrossRef]