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Solutions for Multitime Reaction–Diffusion PDE

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Abstract: A previous paper by our research group introduced the nonlinear multitime reaction–diffusion PDE (with oblique derivative) as a generalized version of the single-time model. This paper states and uses some hypotheses that allow the finding of some important explicit families of the exact solutions for multitime reaction–diffusion PDEs of any dimension that have a multitemporal directional derivative term. Some direct methods for determining the exact solutions of nonlinear PDEs from mathematical physics are presented. In the single-time case, our methods present many advantages in comparison with other known approaches. Particularly, we obtained classes of ODEs and classes of PDEs whose solutions generate solutions of the multitime reaction–diffusion PDE.

Keywords: multitime reaction–diffusion PDE; oblique derivative; exact solutions; first integrals

MSC: 35B40; 35C05; 35K55; 35K57

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1. Introduction

Let $\mu \in \mathbb{R}$, $\mu \neq 0$ and let $m, n \in \mathbb{N}$, $m \geq 2$, $n \geq 2$. We consider the functions

- $f\colon \Omega_1 \to \mathbb{R}$, f of class $\mathcal{C}^1$, with $\Omega_1 \subseteq \mathbb{R}^n$, Ω_1 open;
- for $\alpha \in \{1, 2, \dots, m\}$, $h^\alpha\colon \Omega_2 \to \mathbb{R}$, h^α of class $\mathcal{C}^1$, with $\Omega_2 \subseteq \mathbb{R}^m$, Ω_2 open; $h(t) = \big(h^1(t), h^2(t), \dots, h^m(t)\big)$, for all $t = (t^1, t^2, \dots, t^m) \in \Omega_2$.

These functions define the *multitime reaction–diffusion* PDE

$$h^\alpha(t)\,\frac{\partial u}{\partial t^\alpha} = \mu \frac{\partial^n u}{\partial x^n} + f\left(u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^{n-1} u}{\partial x^{n-1}}\right). \tag{1}$$

The present paper deals with new exact solutions admitted by multitime nonlinear reaction–diffusion PDEs (1) of a fairly general form that depends on one or more arbitrary functions. Particularly, solutions are also being built for simplified multitime reaction–diffusion PDEs of type (11). It is important to underline that exact solutions of mathematical physics PDEs, which contain arbitrary functions and, therefore, have a significant generality, is of great practical interest for evaluating the accuracy of various numerical and approximate analytical methods in order to solve corresponding initial boundary value problems.

The additional information we need is distributed as follows: (i) exact solutions of a model nonlinear single-time reaction–diffusion equation [1–4], single-time soliton solutions and fractional effects [5], single-time wave solutions [6], single-time Schrödinger solitons [7]; (ii) methods for constructing complex solutions of nonlinear PDEs using simpler solutions [8]; (iii) techniques to create multitime extensions of single-time ODEs and PDEs [9]; (iv) multitime Floquet theory [10]; (v) multitime wave functions [11,12]; (vi) multitime Boussinesq solitons [13], multitime Rayleigh solitons [14], multitime reaction–diffusion solitons [15]; (vii) discrete diagonal recurrences [16], discrete multiple recurrences [17], linear discrete multitime multiple recurrences [18];

The ingredients used to find the solutions of multitime reaction–diffusion PDEs (1) are: (i) the solutions of PDE (3) (i.e., first integrals of the ODE system (5)); (ii) the solutions of ODE (6); and a particular solution of PDE (4).

Here is the plan for the rest of the article. In Section 2, we describe how we obtain the solutions via the first integrals of an associated ODE system. In Section 3, we try to argue that, despite the obvious impossibility of developing a useful general theory that encompasses all, there nevertheless exists an ODE that generates solutions for PDE. Section 4 further develops this idea for simplified multitime reaction–diffusion PDEs. In Section 5, we discuss one of the most fundamental problems in PDE, namely, simplified multitime reaction–diffusion PDEs in a Riemannian setting. Finally, Section 6 tries to identify some of the main goals and open problems for exact solutions to PDEs.

2. Solutions via First Integrals

The first integrals of an associated symmetric ODE system are generators of solutions of the multitime reaction–diffusion PDE.

Definition 1. *Let $\Omega_0 \subseteq \mathbb{R} \times \Omega_2$ be an open subset. A function $u\colon \Omega_0 \to \mathbb{R}$ is called the solution of PDE (1) if*

- *u is of class $\mathcal{C}^1$;*
- *for all $(x,t) \in \Omega_0$, there exist $\dfrac{\partial^2 u}{\partial x^2}(x,t), \dots, \dfrac{\partial^n u}{\partial x^n}(x,t)$;*
- *the functions $\dfrac{\partial^2 u}{\partial x^2}, \dots, \dfrac{\partial^{n-1} u}{\partial x^{n-1}}\colon \Omega_0 \to \mathbb{R}$ are continuous (if $n = 2$, this condition does not appear);*
- *for all $(x,t) \in \Omega_0$, we have $\left(u(x,t), \dfrac{\partial u}{\partial x}(x,t), \dots, \dfrac{\partial^{n-1} u}{\partial x^{n-1}}(x,t)\right) \in \Omega_1$;*
- *function u verifies the relation (1) on Ω_0.*

From relation (1), it follows that function $\dfrac{\partial^n u}{\partial x^n}$ is continuous on the set Ω_0.

Let $k \in \mathbb{R}$ be a fixed constant. We attached another PDE to multitime reaction–diffusion PDE (1)

$$h^\alpha(t)\,\frac{\partial \varphi}{\partial t^\alpha} = \mu \frac{\partial^n \varphi}{\partial x^n} - k\frac{\partial \varphi}{\partial x} + f\left(u, \frac{\partial \varphi}{\partial x}, \dots, \frac{\partial^{n-1} \varphi}{\partial x^{n-1}}\right). \tag{2}$$

Let us consider that $\varphi(x,t;c)$, with $c = (c_1, \dots, c_r)$, is a solution of PDE (2), which depends on r parameters. Suppose that φ is a $\mathcal{C}^1$ with respect to all arguments. We accept that $\dfrac{\partial^2 \varphi}{\partial x^2}, \dots, \dfrac{\partial^{n-1} \varphi}{\partial x^{n-1}}$ are continuous with respect to all arguments (if $n = 2$, this condition does not appear).

Let us consider the $\mathcal{C}^1$ functions $P_1(\cdot), P_2(\cdot), \dots, P_r(\cdot), P_{r+1}(\cdot)$, as $r+1$ solutions of the PDE

$$h^\alpha(t)\,\frac{\partial w}{\partial t^\alpha}(t) = 0. \tag{3}$$

We note $P(t) = \big(P_1(t), P_2(t), \dots, P_r(t)\big)$.

Suppose that $v_0(\cdot)$ is a $\mathcal{C}^1$ solution of the linear PDE

$$h^\alpha(t)\,\frac{\partial v}{\partial t^\alpha}(t) = 1. \tag{4}$$

Let us show that $u(x,t) := \varphi\big(x + kv_0(t) + P_{r+1}(t), t; P(t)\big)$ is a solution of PDE (1) (we assume that functions can be composed); in the case $k = 0$, $v_0(\cdot)$ is no longer considered. We denote $\sigma(x,t) := x + kv_0(t) + P_{r+1}(t)$. Then

$$
\begin{aligned}
h^\alpha(t)\frac{\partial u}{\partial t^\alpha}(x,t) &= h^\alpha(t)\frac{\partial \varphi}{\partial x}(\sigma(x,t),t;P(t))\left(k\frac{\partial v_0}{\partial t^\alpha}(t)+\frac{\partial P_{r+1}}{\partial t^\alpha}(t)\right)\\
&+h^\alpha(t)\frac{\partial \varphi}{\partial t^\alpha}(\sigma(x,t),t;P(t))+h^\alpha(t)\sum_{j=1}^{r}\frac{\partial \varphi}{\partial c_j}(\sigma(x,t),t;P(t))\frac{\partial P_j}{\partial t^\alpha}(t)\\
&=\left(kh^\alpha(t)\frac{\partial v_0}{\partial t^\alpha}(t)+h^\alpha(t)\frac{\partial P_{r+1}}{\partial t^\alpha}(t)\right)\frac{\partial \varphi}{\partial x}(\sigma(x,t),t;P(t))\\
&+h^\alpha(t)\frac{\partial \varphi}{\partial t^\alpha}(\sigma(x,t),t;P(t))+\sum_{j=1}^{r}\frac{\partial \varphi}{\partial c_j}(\sigma(x,t),t;P(t))h^\alpha(t)\frac{\partial P_j}{\partial t^\alpha}(t).
\end{aligned}
$$

Since $h^\alpha(t)\frac{\partial v_0}{\partial t^\alpha}(t)=1$ and $h^\alpha(t)\frac{\partial P_j}{\partial t^\alpha}(t)=0$ (if $1\le j\le r+1$), we obtain

$$h^\alpha(t)\frac{\partial u}{\partial t^\alpha}(x,t)=h^\alpha(t)\frac{\partial \varphi}{\partial t^\alpha}(\sigma(x,t),t;P(t))+k\frac{\partial \varphi}{\partial x}(\sigma(x,t),t;P(t)).$$

$$
\begin{aligned}
&\mu\frac{\partial^n u}{\partial x^n}(x,t)+f\left(u(x,t),\frac{\partial u}{\partial x}(x,t),\dots,\frac{\partial^{n-1}u}{\partial x^{n-1}}(x,t)\right)=\mu\frac{\partial^n \varphi}{\partial x^n}(\sigma(x,t),t;P(t))\\
&+f\left(u(\sigma(x,t),t;P(t)),\frac{\partial \varphi}{\partial x}(\sigma(x,t),t;P(t)),\dots,\frac{\partial^{n-1}\varphi}{\partial x^{n-1}}(\sigma(x,t),t;P(t))\right)\\
&=h^\alpha(t)\frac{\partial \varphi}{\partial t^\alpha}(\sigma(x,t),t;P(t))+k\frac{\partial \varphi}{\partial x}(\sigma(x,t),t;P(t));
\end{aligned}
$$

the last equality is true for the functions $\varphi(\cdot,\cdot;c)$, which are solutions of PDE (2).

It follows that u is a solution of PDE (1).

Analogously, if $P_1(t)$ is a solution of PDE (3) and $v_0(t)$ is a solution of PDE (4), and if $\varphi(x,t)$ is a solution of the PDE (2), then $u(x,t):=\varphi(x+kv_0(t)+P_1(t),t)$ is a solution of PDE (1) (assuming that the composition of functions can be performed).

Therefore, we proved the following four propositions.

Proposition 1. *Let $k\in\mathbb{R}$ be a fixed constant. Let $\varphi\colon\Omega\to\mathbb{R}$ be a solution of PDE (2), with $\Omega\subseteq\mathbb{R}\times\Omega_2$, Ω open; let $P_1\colon D_1\to\mathbb{R}$ be a $\mathcal{C}^1$ solution of PDE (3), with $D_1\subseteq\Omega_2$, D_1 open; let $v_0\colon D_0\to\mathbb{R}$ be a $\mathcal{C}^1$ solution of PDE (4), with $D_0\subseteq\Omega_2$, D_0 open.*

Let $\Omega_0:=\left\{(x,t)\in\mathbb{R}\times(D_0\cap D_1)\,\middle|\,(x+kv_0(t)+P_1(t),t)\in\Omega\right\}$.

Suppose that set Ω_0 is nonvoid (it is obvious that Ω_0 is an open set).

Then, function $u\colon\Omega_0\to\mathbb{R}$,

$$u(x,t)=\varphi(x+kv_0(t)+P_1(t),t),\quad \textit{for all } (x,t)\in\Omega_0,$$

is a solution of PDE (1).

Proposition 2. *Let $k\in\mathbb{R}$ be a fixed constant. Let $\Omega\subseteq\mathbb{R}\times\Omega_2\times\mathbb{R}^r$, Ω open, and let $\varphi\colon\Omega\to\mathbb{R}$ be a $\mathcal{C}^1$ function with respect to all arguments. Suppose there exist $\frac{\partial^2\varphi}{\partial x^2},\dots,\frac{\partial^{n-1}\varphi}{\partial x^{n-1}}$ continuous functions with respect to all arguments (if $n=2$, this condition does not appear).*

Denote $pr_3(\Omega):=\left\{c\in\mathbb{R}^r\,\middle|\,\text{there exists }(x,t)\in\mathbb{R}\times\Omega_2,\text{ such that }(x,t,c)\in\Omega\right\}$. Suppose that, for all $c\in pr_3(\Omega)$, function $\varphi(\cdot,\cdot;c)\colon\left\{(x,t)\in\mathbb{R}\times\Omega_2\,\middle|\,(x,t,c)\in\Omega\right\}\to\mathbb{R}$ is a solution of PDE (2).

Let $P_1,\dots,P_r,P_{r+1}\colon D_1\to\mathbb{R}$ be $\mathcal{C}^1$ solutions of PDE (3), with $D_1\subseteq\Omega_2$, D_1 open; let $v_0\colon D_0\to\mathbb{R}$ be a $\mathcal{C}^1$ solution of PDE (4), with $D_0\subseteq\Omega_2$, D_0 open.

Let $\Omega_0:=\left\{(x,t)\in\mathbb{R}\times(D_0\cap D_1)\,\middle|\,(x+kv_0(t)+P_{r+1}(t),t;P_1(t),\dots,P_r(t))\in\Omega\right\}$.

Suppose the Ω_0 is nonvoid (it is obvious that Ω_0 is an open set).

Then, function $u\colon\Omega_0\to\mathbb{R}$,

$$u(x,t)=\varphi(x+kv_0(t)+P_{r+1}(t),t;P_1(t),\dots,P_r(t)),\quad \textit{for all } (x,t)\in\Omega_0,$$

is a solution of PDE (1).

Proposition 3. *Let $\varphi\colon \Omega \to \mathbb{R}$ be a solution of the PDE (1), with $\Omega \subseteq \mathbb{R} \times \Omega_2$, Ω open; let $P_1\colon D_1 \to \mathbb{R}$ be a $\mathcal{C}^1$ solution of PDE (3), with $D_1 \subseteq \Omega_2$, D_1 open.*

Denote $\Omega_0 := \left\{(x,t) \in \mathbb{R} \times D_1 \,\middle|\, \big(x + P_1(t), t\big) \in \Omega\right\}$.

Suppose the set Ω_0 is nonvoid (it is obvious that Ω_0 is an open set).

Then, function

$$u\colon \Omega_0 \to \mathbb{R},\ u(x,t) = \varphi\big(x + P_1(t), t\big), \quad \text{for all } (x,t) \in \Omega_0,$$

is a solution of PDE (1).

Proposition 4. *Let $\Omega \subseteq \mathbb{R} \times \Omega_2 \times \mathbb{R}^r$, Ω open, and let $\varphi\colon \Omega \to \mathbb{R}$ be a $\mathcal{C}^1$ function with respect to all arguments. Suppose there exist the functions $\dfrac{\partial^2\varphi}{\partial x^2}, \ldots, \dfrac{\partial^{n-1}\varphi}{\partial x^{n-1}}$, which are continuous with respect to all arguments (if $n = 2$, this condition does not appear).*

Denote $pr_3(\Omega) := \left\{c \in \mathbb{R}^r \,\middle|\, \text{there exists } (x,t) \in \mathbb{R} \times \Omega_2, \text{ such that } (x,t,c) \in \Omega\right\}$. Suppose that, for all $c \in pr_3(\Omega)$, the function $\varphi(\cdot,\cdot;c)\colon \left\{(x,t) \in \mathbb{R} \times \Omega_2 \,\middle|\, (x,t,c) \in \Omega\right\} \to \mathbb{R}$ is a solution of PDE (1).

Let $P_1, \ldots, P_r, P_{r+1}\colon D_1 \to \mathbb{R}$ be $\mathcal{C}^1$ solutions of PDE (3), with $D_1 \subseteq \Omega_2$, D_1 open.

Denote $\Omega_0 := \left\{(x,t) \in \mathbb{R} \times D_1 \,\middle|\, \big(x + P_{r+1}(t), t; P_1(t), \ldots, P_r(t)\big) \in \Omega\right\}$.

Suppose that set Ω_0 is nonvoid (it is obvious that Ω_0 is open set).

Then, function $u\colon \Omega_0 \to \mathbb{R}$,

$$u(x,t) = \varphi\big(x + P_{r+1}(t), t; P_1(t), \ldots, P_r(t)\big), \quad \text{for all } (x,t) \in \Omega_0,$$

is a solution of PDE (1).

The $\mathcal{C}^1$ solutions of PDE (3) are the first integrals of the ODE system

$$\begin{cases} \dfrac{\mathrm{d}t^1}{\mathrm{d}\tau}(\tau) = h^1\big(t(\tau)\big) \\ \dfrac{\mathrm{d}t^2}{\mathrm{d}\tau}(\tau) = h^2\big(t(\tau)\big) \\ \qquad\vdots \\ \dfrac{\mathrm{d}t^m}{\mathrm{d}\tau}(\tau) = h^m\big(t(\tau)\big), \end{cases} \tag{5}$$

written in symmetric form $\dfrac{\mathrm{d}t^1}{h^1(t)} = \dfrac{\mathrm{d}t^2}{h^2(t)} = \ldots = \dfrac{\mathrm{d}t^m}{h^m(t)}$.

We use the following remark (similar to the one in paper [19]), in which it is shown how PDE (3) solutions are obtained, i.e., first integrals of ODEs system (5).

Remark 1. *Let $t_0 \in \Omega_2$, such that $h(t_0) \neq (0,0,\ldots,0)$; hence there exists $\alpha_0 \in \{1,2,\ldots,m\}$, that satisfies $h^{\alpha_0}(t_0) \neq 0$. Let F_1, F_2, …, $F_{m-1}\colon V \to \mathbb{R}$, be $\mathcal{C}^1$ functional independent first integrals of the ODE system (5) (in the sense specified in [19]), with $V \subseteq \Omega_2$, $t_0 \in V$, V open and connected such that, for all $t \in V$ we have $h^{\alpha_0}(t) \neq 0$.*

Denote $F(t) := \big(F_1(t), F_2(t), \ldots, F_{m-1}(t)\big)$, $t \in V$.

Any solution of PDE (3), defined in a neighborhood of t_0 has the form $w(t) = E\big(F(t)\big)$, for all $t \in W_0$, where $E\colon U \to \mathbb{R}$ is a $\mathcal{C}^1$ function, $U \subseteq \mathbb{R}^{m-1}$, U open; $t_0 \in W_0 \subseteq V$, W_0 open; $F(t) \in U$, for all $t \in W_0$ (see the results from Section 4.1, of the paper [19]).

Remark 2. *It follows that functions $P_j(\cdot)$, which appeared above in this section, have (in a neighborhood of t_0) the form specified in Remark 1; i.e., $P_j(t) = E_j\big(F(t)\big)$, for all $t \in W_j$, where*

$E_j \colon U_j \to \mathbb{R}$ is a $\mathcal{C}^1$ function, $U_j \subseteq \mathbb{R}^{m-1}$, U_j open; $t_0 \in W_j \subseteq V$, W_j open; $F(t) \in U_j$, for all $t \in W_j$ (for each $j \in \{1, 2, \ldots, r+1\}$).

We can consider that functions $P_1, P_2, \ldots, P_{r+1}$ have the same domain of definition D, choosing $D = W_1 \cap W_2 \cap \ldots \cap W_{r+1}$. We notice that D is an open set, $t_0 \in D \subseteq V$, and, for all $t \in D$, we have $F(t) \in U_1 \cap U_2 \cap \ldots \cap U_{r+1}$.

3. Solutions via an Adapted ODE

We associate to the PDE (1) an n-th order ODE

$$\mu y^{(n)} - ky' + f\left(y, y', y'', \ldots, y^{(n-1)}\right) = 0 \tag{6}$$

(which depends on the parameter $k \in \mathbb{R}$).

For $k = 0$, Equation (6) becomes

$$\mu y^{(n)} + f\left(y, y', y'', \ldots, y^{(n-1)}\right) = 0. \tag{7}$$

It can be seen that if $y(\cdot)$ is a solution of Equation (6) (respectively of Equation (7)), then the function $\varphi(x,t) := y(x)$ is a solution of PDE (2) (respectively of PDE (1)). From Propositions 1, 2, 3, and 4, four other statements are immediately obtained.

Proposition 5. *Let $k \in \mathbb{R}$ be a fixed constant. Let $\psi \colon I \to \mathbb{R}$ be a solution of ODE (6), with $I \subseteq \mathbb{R}$, I open interval; let $P_1 \colon D_1 \to \mathbb{R}$ be a $\mathcal{C}^1$ solution of PDE (3), with $D_1 \subseteq \Omega_2$, D_1 open; let $v_0 \colon D_0 \to \mathbb{R}$ be a $\mathcal{C}^1$ solution of PDE (4), with $D_0 \subseteq \Omega_2$, D_0 open.*

Denote $\Omega_0 := \left\{(x,t) \in \mathbb{R} \times (D_0 \cap D_1) \,\middle|\, x + kv_0(t) + P_1(t) \in I\right\}$.

Suppose that set Ω_0 is nonvoid (obviously, Ω_0 is an open set).

Then, function $u \colon \Omega_0 \to \mathbb{R}$,

$$u(x,t) = \psi\big(x + kv_0(t) + P_1(t)\big), \quad \text{for all } (x,t) \in \Omega_0,$$

is a solution of PDE (1).

Proposition 6. *Let $k \in \mathbb{R}$ be a fixed constant, and let $I \subseteq \mathbb{R}$, I be an open interval. Suppose $G \subseteq \mathbb{R}^r$, G is open.*

Let $\psi \colon I \times G \to \mathbb{R}$ be a $\mathcal{C}^1$ function with respect to all arguments. Suppose there exist $\dfrac{\partial^2 \psi}{\partial x^2}, \ldots, \dfrac{\partial^{n-1} \psi}{\partial x^{n-1}}$ continuous functions (with respect to all arguments) (if $n = 2$, this condition does not appear). Suppose that, for all $c \in G$, function $\psi(\,\cdot\,; c) \colon I \to \mathbb{R}$ is a solution of ODE (6).

Suppose $P_1, \ldots, P_r, P_{r+1} \colon D_1 \to \mathbb{R}$ are $\mathcal{C}^1$ solutions of PDE (3), with $D_1 \subseteq \Omega_2$, D_1 open; let $v_0 \colon D_0 \to \mathbb{R}$ be a $\mathcal{C}^1$ solution of PDE (4), with $D_0 \subseteq \Omega_2$, D_0 open.

We define $\Omega_0 := \left\{(x,t) \in \mathbb{R} \times (D_0 \cap D_1) \,\middle|\, \big(x + kv_0(t) + P_{r+1}(t); P_1(t), \ldots, P_r(t)\big) \in I \times G\right\}$.

Suppose that set Ω_0 is nonvoid (obviously, set Ω_0 is open).

Then, function $u \colon \Omega_0 \to \mathbb{R}$,

$$u(x,t) = \psi\big(x + kv_0(t) + P_{r+1}(t); P_1(t), \ldots, P_r(t)\big), \quad \text{for all } (x,t) \in \Omega_0,$$

is a solution of PDE (1).

Proposition 7. *Let $\psi \colon I \to \mathbb{R}$ be a solution of ODE (7), with $I \subseteq \mathbb{R}$, I open interval; let $P_1 \colon D_1 \to \mathbb{R}$ be a $\mathcal{C}^1$ solution of PDE (3), with $D_1 \subseteq \Omega_2$, D_1 open.*

Denote $\Omega_0 := \left\{(x,t) \in \mathbb{R} \times D_1 \,\middle|\, x + P_1(t) \in I\right\}$.

Suppose that set Ω_0 is nonvoid (obviously, Ω_0 is an open set).

Then, function $u \colon \Omega_0 \to \mathbb{R}$,

$$u(x,t) = \psi\big(x + P_1(t)\big), \quad \text{for all } (x,t) \in \Omega_0,$$

is a solution of PDE (1).

Proposition 8. *Let $I \subseteq \mathbb{R}$, I be an open interval, and $G \subseteq \mathbb{R}^r$, G open.*

Let $\psi\colon I \times G \to \mathbb{R}$ be a $\mathcal{C}^1$ function with respect to all arguments. Suppose there exist functions $\dfrac{\partial^2\psi}{\partial x^2}, \dots, \dfrac{\partial^{n-1}\psi}{\partial x^{n-1}}$, which are continuous with respect to all arguments (if $n = 2$, this condition does not appear). Suppose that, for all $c \in G$, function $\psi(\,\cdot\,;c)\colon I \to \mathbb{R}$ is a solution of ODE (7).

We introduce $P_1, \dots, P_{r+1}\colon D_1 \to \mathbb{R}$ as $\mathcal{C}^1$ solutions of PDE (3), with $D_1 \subseteq \Omega_2$, D_1 open.

Denote $\Omega_0 := \Big\{(x,t) \in \mathbb{R} \times D_1 \,\Big|\, \big(x + P_{r+1}(t); P_1(t), \dots, P_r(t)\big) \in I \times G\Big\}$.

Suppose that set Ω_0 is nonvoid (obviously, Ω_0 is an open set).

Then, function $u\colon \Omega_0 \to \mathbb{R}$,

$$u(x,t) = \psi\big(x + P_{r+1}(t); P_1(t), \dots, P_r(t)\big), \quad \textit{for all } (x,t) \in \Omega_0,$$

is a solution of PDE (1).

For $\lambda \in \mathbb{R}$, $c = (c_1, \dots, c_n) \in \Omega_1$, $k \in \mathbb{R}$, let $\eta(\,\cdot\,;\lambda;c;k)$ be maximal solution $y(\,\cdot\,)$ of Equation (6), which verifies $y(\lambda) = c_1$, $y'(\lambda) = c_2$, $y''(\lambda) = c_3$, $\dots$, $y^{(n-1)}(\lambda) = c_n$. The domain of definition of this solution is an open interval $I(\lambda;c;k)$, with $\lambda \in I(\lambda;c;k)$.

Let $A := \big\{(x;\lambda;c;k) \in \mathbb{R} \times \mathbb{R} \times \Omega_1 \times \mathbb{R} \,\big|\, x \in I(\lambda;c;k)\big\}$. Set A is open. The functions $\eta, \dfrac{\partial\eta}{\partial x}, \dfrac{\partial^2\eta}{\partial x^2}, \dots, \dfrac{\partial^{n-1}\eta}{\partial x^{n-1}}\colon A \to \mathbb{R}$ are of class $\mathcal{C}^1$. For any $(x;\lambda;c;k) \in A$, we have $\left(\eta(x;\lambda;c;k), \dfrac{\partial\eta}{\partial x}(x;\lambda;c;k), \dfrac{\partial^2\eta}{\partial x^2}(x;\lambda;c;k), \dots, \dfrac{\partial^{n-1}\eta}{\partial x^{n-1}}(x;\lambda;c;k)\right) \in \Omega_1$.

For any $\lambda \in \mathbb{R}$, $c = (c_1, \dots, c_n) \in \Omega_1$, $k \in \mathbb{R}$, we have $(\lambda;\lambda;c;k) \in A$, and
$\eta(\lambda;\lambda;c;k) = c_1$, $\dfrac{\partial\eta}{\partial x}(\lambda;\lambda;c;k) = c_2$, $\dfrac{\partial^2\eta}{\partial x^2}(\lambda;\lambda;c;k) = c_3$, $\dots$, $\dfrac{\partial^{n-1}\eta}{\partial x^{n-1}}(\lambda;\lambda;c;k) = c_n$.

For any $\lambda \in \mathbb{R}$, $c \in \Omega_1$, $k \in \mathbb{R}$, we have $I(\lambda;c;k) = \lambda + I(0;c;k)$, and
$\eta(x;\lambda;c;k) = \eta(x-\lambda;0;c;k)$, for all $x \in I(\lambda;c;k)$,
or, more generally, $I(\lambda+\xi;c;k) = \xi + I(\lambda;c;k)$, and
$\eta(x+\xi;\lambda+\xi;c;k) = \eta(x;\lambda;c;k)$, for all $x \in I(\lambda;c;k)$, for all $\xi \in \mathbb{R}$.
(For the interval (α,β), the set $\xi + (\alpha,\beta)$ is the interval $(\xi+\alpha, \xi+\beta)$.)

Let $x_0 \in \mathbb{R}$, $c_0 \in \Omega_1$, $k_0 \in \mathbb{R}$. Since $(x_0;x_0;c_0;k_0) \in A$, and A is an open set, it follows that there exists $\rho > 0$, and there exist the open and bounded intervals Λ, L, and there exists the open set $G \subseteq \Omega_1$, such that: $x_0 \in \Lambda$, $c_0 \in G$, $k_0 \in L$, and $(x_0-\rho, x_0+\rho) \times \Lambda \times G \times L \subseteq A$.

Proposition 9. *Let $x_0 \in \mathbb{R}$, $t_0 \in \Omega_2$, with $h(t_0) \neq (0,0,\dots,0)$.*

Denote $\rho \in \mathbb{R}$, $\rho > 0$. Let $L \subseteq \mathbb{R}$, L be an open and bounded interval, and let $G \subseteq \Omega_1$, G open, such that: $(x_0-\rho, x_0+\rho) \times \{x_0\} \times G \times L \subseteq A$.

Let F_1, F_2, $\dots$, $F_{m-1}\colon V \to \mathbb{R}$, be $\mathcal{C}^1$, functional independent first integrals for ODEs system (5), with $V \subseteq \Omega_2$, $t_0 \in V$, V open. Let $F(t) := \big(F_1(t), F_2(t), \dots, F_{m-1}(t)\big)$, $t \in V$.

We consider the $\mathcal{C}^1$ functions $E_1, E_2, \dots, E_{n+1}\colon U \to \mathbb{R}$, $E_{n+2}\colon U \to L$, with $U \subseteq \mathbb{R}^{m-1}$, U open. Suppose that, for all $s \in U$, we have $\big(E_1(s), E_2(s), \dots, E_n(s)\big) \in G$.

Suppose that there exists $D \subseteq V$, D open, with $t_0 \in D$, such that $F(D) \subseteq U$.

Let $v_0\colon D_0 \to \mathbb{R}$ be a $\mathcal{C}^1$ solution of PDE (4), with $D_0 \subseteq \Omega_2$, D_0 open, $t_0 \in D_0$.

For all $j \in \{1,2,\dots,n-1,n,n+2\}$, and for all $t \in D$, we denote $P_j(t) := E_j\big(F(t)\big)$.

For all $t \in D$, we denote $P_{n+1}(t) := E_{n+1}\big(F(t)\big) - E_{n+1}\big(F(t_0)\big)$.

For all $t \in D_0$, we take $\widetilde{v}_0(t) := v_0(t) - v_0(t_0)$.

Then, there exists $D_1 \subseteq D \cap D_0$, D_1 open, with $t_0 \in D_1$, such that:

(i) for all $x \in \left(x_0 - \frac{\rho}{2}, x_0 + \frac{\rho}{2}\right)$*, for all* $t \in D_1$*, we have*

$$x + P_{n+2}(t) \cdot \widetilde{v}_0(t) + P_{n+1}(t) \in (x_0 - \rho, x_0 + \rho); \tag{8}$$

(ii) the function $u\colon \left(x_0 - \frac{\rho}{2}, x_0 + \frac{\rho}{2}\right) \times D_1 \to \mathbb{R}$,

$$u(x,t) = \eta\big(x + P_{n+2}(t) \cdot \widetilde{v}_0(t) + P_{n+1}(t); x_0; P_1(t), \ldots, P_n(t); P_{n+2}(t)\big),$$
$$\text{for all } (x,t) \in \left(x_0 - \frac{\rho}{2}, x_0 + \frac{\rho}{2}\right) \times D_1,$$

is a solution of PDE (1).

Proof. (i) Let $I := (x_0 - \rho, x_0 + \rho)$, $I_0 := \left(x_0 - \frac{\rho}{2}, x_0 + \frac{\rho}{2}\right)$.
Because the interval L is bounded, there exists $M > 0$, such that, for any $k \in L$, we have $|k| \le M$.

Since the functions $\widetilde{v}_0$ and P_{n+1} are continuous in t_0, and $\widetilde{v}_0(t_0) = P_{n+1}(t_0) = 0$, it follows that there exists $R > 0$, such that, for any $t \in D \cap D_0 \cap B(t_0, R)$, the following relations hold

$$\big|\widetilde{v}_0(t)\big| < \frac{\rho}{2(M+1)} \quad \text{and} \quad \big|P_{n+1}(t)\big| < \frac{\rho}{2(M+1)} \tag{9}$$

(set $B(t_0, R)$ is an open ball of center t_0 and radius R, included in $\mathbb{R}^m$).

Denote $D_1 := D \cap D_0 \cap B(t_0, R)$; D_1 is open, $t_0 \in D_1$, and $D_1 \subseteq D \cap D_0$.

Let us show that

$$x + k\widetilde{v}_0(t) + P_{n+1}(t) \in I, \quad \text{for all } x \in I_0, \text{ for all } t \in D_1, \text{ for all } k \in L. \tag{10}$$

We note $x \in I_0$, $t \in D_1$, $k \in L$. We use the obvious inequality
$\big|x + k\widetilde{v}_0(t) + P_{n+1}(t) - x_0\big| \le |x - x_0| + |k| \cdot \big|\widetilde{v}_0(t)\big| + \big|P_{n+1}(t)\big|$.

From $x \in I_0$, we deduce $|x - x_0| < \frac{\rho}{2}$. Since $k \in L$, we have $|k| \le M$. Further, using inequalities (9), we obtain

$|x - x_0| + |k| \cdot \big|\widetilde{v}_0(t)\big| + \big|P_{n+1}(t)\big| < \frac{\rho}{2} + M \cdot \frac{\rho}{2(M+1)} + \frac{\rho}{2(M+1)} = \rho.$

It follows that $\big|x + k\widetilde{v}_0(t) + P_{n+1}(t) - x_0\big| < \rho$, i.e., $x + k\widetilde{v}_0(t) + P_{n+1}(t) \in I$.

We proved statement (10); from this statement (i) follows, since for all $t \in D_1$, we have $P_{n+2}(t) = E_{n+2}\big(F(t)\big) \in L$, and if in (10) we choose $k = P_{n+2}(t)$, we find (8).

(ii) For each $k \in L$, we consider the function $\psi_k\colon I \times G \to \mathbb{R}$,
$\psi_k(x; c) = \eta(x; x_0; c; k)$, for all $(x; c) \in I \times G$.

For all $c \in G$, function $\psi_k(\,\cdot\,; c)\colon I \to \mathbb{R}$ is a solution of Equation (6).

The functions $P_1(\cdot), P_2(\cdot), \ldots, P_n(\cdot), P_{n+1}(\cdot)$, defined on D_1, are $\mathcal{C}^1$ solutions of PDE (3). Function $\widetilde{v}_0\colon D_0 \to \mathbb{R}$ is a $\mathcal{C}^1$ solution of PDE (4).

Denote $\Omega_0(k) = \Big\{(x,t) \in \mathbb{R} \times D_1 \,\Big|\, \big(x + k\widetilde{v}_0(t) + P_{n+1}(t); P_1(t), \ldots, P_n(t)\big) \in I \times G\Big\}$.

Since for all $t \in D_1$, we have $\big(P_1(t), P_2(t), \ldots, P_n(t)\big) \in G$, we deduce that
$\Omega_0(k) = \Big\{(x,t) \in \mathbb{R} \times D_1 \,\Big|\, x + k\widetilde{v}_0(t) + P_{n+1}(t) \in I\Big\}$.

According to Proposition 6, function $\theta_k(\cdot,\cdot)\colon \Omega_0(k) \to \mathbb{R}$,

$$\theta_k(x,t) = \psi_k\big(x + k\widetilde{v}_0(t) + P_{n+1}(t); P_1(t), \ldots, P_n(t)\big), \quad \text{for all } (x,t) \in \Omega_0(k),$$

is a solution of PDE (1) (for all $k \in L$).

From statement (10) we find that, for all $k \in L$, we have $I_0 \times D_1 \subseteq \Omega_0(k)$; it follows that we can define the function
$\varphi\colon I_0 \times D_1 \times L \to \mathbb{R}$, $\varphi(x,t;k) = \theta_k(x,t)$, for all $(x,t;k) \in I_0 \times D_1 \times L$, i.e.
$\varphi(x,t;k) = \psi_k\big(x + k\widetilde{v}_0(t) + P_{n+1}(t); P_1(t), \ldots, P_n(t)\big)$, or

$\varphi(x,t;k) = \eta\big(x + k\widetilde{v}_0(t) + P_{n+1}(t); x_0; P_1(t), \ldots, P_n(t); k\big)$, for all $(x,t;k) \in I_0 \times D_1 \times L$.

For any $k \in L$, function $\theta_k(\cdot,\cdot)$ is a solution of PDE (1). We deduce that for any $k \in L$, function $\varphi(\cdot,\cdot;k) \colon I_0 \times D_1 \to \mathbb{R}$ is a solution of PDE (1).

Function $P_{n+2}(\cdot)$, defined on D_1, is a $\mathcal{C}^1$ solution of PDE (3).

We apply Proposition 4 for function φ; we have $r = 1$ and $\Omega = I_0 \times D_1 \times L$. Instead of $P_1(\cdot)$ from Proposition 4, we set $P_{n+2}(\cdot)$, and $P_{r+1}(\cdot)$ will be the null function.

Let $\Omega_0 = \Big\{(x,t) \in \mathbb{R} \times D_1 \,\Big|\, \big(x,t;P_{n+2}(t)\big) \in I_0 \times D_1 \times L\Big\}$.

Since, for all $t \in D_1$, we have $P_{n+2}(t) \in L$, and hence, $\Omega_0 = I_0 \times D_1$.

According to Proposition 4, function $u\colon I_0 \times D_1 \to \mathbb{R}$,

$$u(x,t) = \varphi\big(x,t;P_{n+2}(t)\big), \quad \text{for all } (x,t) \in I_0 \times D_1,$$

is a solution of PDE (1). We obtain statement (ii). □

4. Simplified Multitime Reaction–Diffusion PDE

The simplified multitime reaction–diffusion PDE, in the direction $h = (h^1, h^2, \ldots, h^m)$, is a PDE of the form (see [15])

$$h^\alpha(t)\,\frac{\partial u}{\partial t^\alpha} = \mu\frac{\partial^2 u}{\partial x^2} + g(u), \tag{11}$$

with $g\colon J \to \mathbb{R}$ a $\mathcal{C}^1$ function, $J \subseteq \mathbb{R}$, J open interval.

PDE (11) is an equation of type (1), with $n = 2$; we have $\Omega_1 = J \times \mathbb{R}$, and $f(q_1,q_2) = g(q_1)$, for all $(q_1,q_2) \in J \times \mathbb{R}$.

Let us determine some solutions for a particular case of PDE (11), namely

$$t^1\frac{\partial u}{\partial t^1} + \ldots + t^{m-1}\frac{\partial u}{\partial t^{m-1}} + t^m\frac{\partial u}{\partial t^m} = \mu\frac{\partial^2 u}{\partial x^2} - au^3 + bu^2, \quad t^m \neq 0, \tag{12}$$

where μ, a and b are real constants, $\mu > 0$, $a > 0$, $b \neq 0$ (compare with soliton solutions, [15]).

In this case, the ODEs system (5) becomes

$$\begin{cases} \dfrac{\mathrm{d}t^1}{\mathrm{d}\tau}(\tau) = t^1(\tau) \\ \dfrac{\mathrm{d}t^2}{\mathrm{d}\tau}(\tau) = t^2(\tau) \\ \quad\vdots \\ \dfrac{\mathrm{d}t^{m-1}}{\mathrm{d}\tau}(\tau) = t^{m-1}(\tau) \\ \dfrac{\mathrm{d}t^m}{\mathrm{d}\tau}(\tau) = t^m(\tau). \end{cases} \tag{13}$$

Functions $F_1, F_2, \ldots, F_{m-1}\colon \mathbb{R}^{m-1} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$,

$$F_\alpha(t) = \frac{t^\alpha}{t^m}, \quad \text{for all } t \in \mathbb{R}^{m-1} \times (\mathbb{R} \setminus \{0\}), \quad \text{for all } \alpha \in \{1,2,\ldots,m-1\},$$

are $m-1$ functional independent first integrals of the ODEs system (13).

We note $F(t) = \big(F_1(t), F_2(t), \ldots, F_{m-1}(t)\big) = \left(\dfrac{t^1}{t^m}, \dfrac{t^2}{t^m}, \ldots, \dfrac{t^{m-1}}{t^m}\right)$, $t \in \mathbb{R}^{m-1} \times (\mathbb{R} \setminus \{0\})$.

Let $E\colon U \to \mathbb{R}$ be any $\mathcal{C}^1$ function ($U \subseteq \mathbb{R}^{m-1}$, U open).

The PDE (4) is written

$$t^1\frac{\partial v}{\partial t^1} + \ldots + t^{m-1}\frac{\partial v}{\partial t^{m-1}} + t^m\frac{\partial v}{\partial t^m} = 1. \tag{14}$$

The function $v_0 \colon \mathbb{R}^{m-1} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$,

$$v_0(t) = \ln |t^m|, \quad \text{for all } t \in \mathbb{R}^{m-1} \times (\mathbb{R} \setminus \{0\}),$$

is a particular solution for PDE (14).

Equations (6) and (7), associated to PDE (12), are

$$\mu y'' - k y' - a y^3 + b y^2 = 0 \tag{15}$$

$$\mu y'' - a y^3 + b y^2 = 0. \tag{16}$$

For statements (i), (ii) and (iii), the below are true.

(i) For $k = -b\sqrt{\dfrac{2\mu}{a}}$, the function $y_1(x) = \dfrac{1}{x}\sqrt{\dfrac{2\mu}{a}}$ is a solution for ODE (15).

For $k = b\sqrt{\dfrac{2\mu}{a}}$, the function $y_2(x) = -\dfrac{1}{x}\sqrt{\dfrac{2\mu}{a}}$ is a solution for ODE (15).

(ii) For $k = -b\sqrt{\dfrac{\mu}{2a}}$, the function $y_3(x) = \dfrac{b}{a} \cdot \dfrac{1}{1 + c e^{\frac{bx}{\sqrt{2\mu a}}}}$ is a solution for ODE (15).

For $k = b\sqrt{\dfrac{\mu}{2a}}$, the function $y_4(x) = \dfrac{b}{a} \cdot \dfrac{1}{1 + c e^{\frac{-bx}{\sqrt{2\mu a}}}}$ is a solution for ODE (15).

Solutions y_3 and y_4 depend on the real constant c.

(iii) Functions $y_5(x) = \dfrac{b}{a} \cdot \dfrac{\sqrt{2} + \sinh \dfrac{bx}{\sqrt{\mu a}}}{-2\sqrt{2} + \sinh \dfrac{bx}{\sqrt{\mu a}}}$, $y_6(x) = \dfrac{b}{a} \cdot \dfrac{-\sqrt{2} + \sinh \dfrac{bx}{\sqrt{\mu a}}}{2\sqrt{2} + \sinh \dfrac{bx}{\sqrt{\mu a}}}$,

$y_7(x) = \dfrac{12 b\mu}{9a\mu - 2b^2 x^2}$ are three solutions for ODE (16).

Using statement (i) and Proposition 5, we deduce that the functions

$$u_1(x,t) = \frac{1}{x - b\sqrt{\frac{2\mu}{a}} \ln |t^m| + E\big(F(t)\big)} \sqrt{\frac{2\mu}{a}} = \frac{\sqrt{2\mu}}{\sqrt{a}x - b\sqrt{2\mu} \ln |t^m| + \sqrt{a} E\big(F(t)\big)},$$

$$u_2(x,t) = -\frac{1}{x + b\sqrt{\frac{2\mu}{a}} \ln |t^m| + E\big(F(t)\big)} \sqrt{\frac{2\mu}{a}} = -\frac{\sqrt{2\mu}}{\sqrt{a}x + b\sqrt{2\mu} \ln |t^m| + \sqrt{a} E\big(F(t)\big)}$$

are solutions of PDE (12).

Using statement (ii) and Proposition 6 (with $r = 1$ and choosing $P_{r+1}(\cdot)$ as the null function), we deduce that the functions

$$u_3(x,t) = \frac{b}{a} \cdot \frac{1}{1 + E\big(F(t)\big) e^{\frac{b}{\sqrt{2\mu a}}\left(x - b\sqrt{\frac{\mu}{2a}} \ln |t^m|\right)}} = \frac{b}{a} \cdot \frac{1}{1 + E\big(F(t)\big) e^{\frac{bx}{\sqrt{2\mu a}}} \cdot |t^m|^{-\frac{b^2}{2a}}},$$

$$u_4(x,t) = \frac{b}{a} \cdot \frac{1}{1 + E\big(F(t)\big) e^{\frac{-b}{\sqrt{2\mu a}}\left(x + b\sqrt{\frac{\mu}{2a}} \ln |t^m|\right)}} = \frac{b}{a} \cdot \frac{1}{1 + E\big(F(t)\big) e^{\frac{-bx}{\sqrt{2\mu a}}} \cdot |t^m|^{-\frac{b^2}{2a}}}$$

are solutions for PDE (12).

Using statement (iii) and Proposition 7, we deduce that functions

$$u_5(x,t) = \frac{b}{a} \cdot \frac{\sqrt{2} + \sinh \dfrac{bx + bE\big(F(t)\big)}{\sqrt{\mu a}}}{-2\sqrt{2} + \sinh \dfrac{bx + bE\big(F(t)\big)}{\sqrt{\mu a}}}, \quad u_6(x,t) = \frac{b}{a} \cdot \frac{-\sqrt{2} + \sinh \dfrac{bx + bE\big(F(t)\big)}{\sqrt{\mu a}}}{2\sqrt{2} + \sinh \dfrac{bx + bE\big(F(t)\big)}{\sqrt{\mu a}}},$$

$$u_7(x,t) = \frac{12b\mu}{9a\mu - 2b^2\big(x + E\big(F(t)\big)\big)^2} = \frac{12b\mu}{9a\mu - 2\big(bx + bE\big(F(t)\big)\big)^2}$$

are solutions for PDE (12).

Since function E is arbitrary, we can replace E, which appears in the expression of u_1 and u_2, with $\frac{E}{\sqrt{a}}$; we can replace E, which appears in the expression of u_5 and u_6, with $\frac{\sqrt{\mu a}}{b}E$; we can replace E, which appears in the expression of u_7, with $\frac{E}{b}$.

Solutions $u_1, u_2, u_3, u_4, u_5, u_6, u_7$ of PDE (12) become

$$u_1(x,t) = \frac{\sqrt{2\mu}}{\sqrt{a}x - b\sqrt{2\mu}\ln|t^m| + E\big(F(t)\big)}$$

$$u_2(x,t) = -\frac{\sqrt{2\mu}}{\sqrt{a}x + b\sqrt{2\mu}\ln|t^m| + E\big(F(t)\big)}$$

$$u_3(x,t) = \frac{b}{a}\cdot\frac{|t^m|^{\frac{b^2}{2a}}}{|t^m|^{\frac{b^2}{2a}} + E\big(F(t)\big)e^{\frac{bx}{\sqrt{2\mu a}}}}$$

$$u_4(x,t) = \frac{b}{a}\cdot\frac{|t^m|^{\frac{b^2}{2a}}}{|t^m|^{\frac{b^2}{2a}} + E\big(F(t)\big)e^{\frac{-bx}{\sqrt{2\mu a}}}}$$

$$u_5(x,t) = \frac{b}{a}\cdot\frac{\sqrt{2} + \sinh\left(\frac{bx}{\sqrt{\mu a}} + E\big(F(t)\big)\right)}{-2\sqrt{2} + \sinh\left(\frac{bx}{\sqrt{\mu a}} + E\big(F(t)\big)\right)}$$

$$u_6(x,t) = \frac{b}{a}\cdot\frac{-\sqrt{2} + \sinh\left(\frac{bx}{\sqrt{\mu a}} + E\big(F(t)\big)\right)}{2\sqrt{2} + \sinh\left(\frac{bx}{\sqrt{\mu a}} + E\big(F(t)\big)\right)}$$

$$u_7(x,t) = \frac{12b\mu}{9a\mu - 2\big(bx + E\big(F(t)\big)\big)^2}.$$

For each u_j, the domain of definition is chosen as an open set in $\mathbb{R}\times\left(\mathbb{R}^{m-1}\times(\mathbb{R}\setminus\{0\})\right)$, such that $F(t)\in U$ and the respective denominator do not cancel. For example, for u_1, the domain of definition is

$$\left\{(x,t)\in\mathbb{R}\times\left(\mathbb{R}^{m-1}\times(\mathbb{R}\setminus\{0\})\right)\,\middle|\, F(t)\in U,\ \sqrt{a}x - b\sqrt{2\mu}\ln|t^m| + E\big(F(t)\big)\neq 0\right\}.$$

5. The Simplified Multitime Reaction–Diffusion PDE in Riemannian Setting

From the physical point of view, it would be more important to further study a simplified multitime reaction–diffusion PDE in a Riemannian setting, which is still an open problem. For that, let (M,g) be a smooth, compact Riemannian manifold of dimension n (particularly, $n = 2,3$) without a boundary. In the Riemannian setting, the simplified multitime reaction–diffusion PDE is

$$ih^{\alpha}(x,t)\frac{\partial u}{\partial t^{\alpha}} + \mu\Delta_g u + f(u) = 0, \quad (x,t)\in M\times\mathbb{R}^m. \tag{17}$$

Let dx_g denote the volume element of the compact Riemannian manifold (M,g) and

$$|u|^2 = u\bar{u},\ |\nabla_g u|_g^2 = \langle\nabla_g u, \nabla_g\bar{u}\rangle_g.$$

This simplified multitime reaction–diffusion PDE is mainly concerned with the interface between Riemannian geometry and heat theory, but it leads, in a natural way, to questions of functional analysis related to the theory of operators on Hilbert spaces. In some respects, these problems are similar to those studied in the standard Euclidean case, but depending on the Riemannian metric g these might go beyond and provide new aspects to the problem.

Open problems (i) What happens if we consider Riemannian metrics that vary periodically in multitime? (ii) Is a diffusion process determined by its intrinsic Riemannian metric?

6. Conclusions

The problem of exact solutions for PDEs was resuscitated by the publication of the book [20]. Although this book refers to geometrical PDEs (Exact Solutions of Einstein's Field Equations), there are many other PDEs with a physical sense (Broer–Kaup PDE, Burgers PDE, Fokker–Planck PDE, Hamilton–Jacobi equation PDE, Hamilton–Jacobi–Bellman equation PDE, Diffusion PDE, Klein–Gordon PDE, Korteweg–de Vries PDE, Maxwell PDEs, Navier–Stokes PDEs, Schrödinger PDE, Wave equation, etc.) for which we need exact solutions. Particularly, solitons are exact solutions. In order to show the advantages of the work on exact solutions, one can point out that certain solutions have played very important roles in the solving of physical problems and hence finding exact solutions that have a practical interpretation is of fundamental importance.

Our techniques are similar to the one in paper [19]. There are practical as well as theoretical reasons for studying exact solutions of the multitime reaction–diffusion PDEs (see also single-time case [3,5–8,21–23]). In some cases, the reader can be surprised by the enormous number of exact solutions. In other cases, there are problems with a unique exact solution, for instance, the Kerr and Schwarzschild solutions for the final collapsed state of massive bodies. The single-time reaction–diffusion PDE or the multitime reaction–diffusion PDE can be used as mathematical models in a wide series of domains. One example is represented by many processes during embryonic development, which involves the transport and reaction of molecules, or transport and proliferation of cells, within growing tissues.

Due to the possibilities offered by symbolic computation software such as Maple or Mathematica [24], the computation of the exact solutions of PDEs has become easier and more stimulating for mathematicians and other scientists.

In Maple, the basic Calling Sequences are:

pdsolve(PDE, f, HINT = hint, INTEGRATE, build);

pdsolve(PDEsystem, funcs, HINT, otheroptions).

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