mathematics

Article

Two Linearized Schemes for One-Dimensional Time and Space Fractional Differential Equations

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Abstract: This paper investigates the solution to one-dimensional fractional differential equations with two types of fractional derivative operators of orders in the range of $(1, 2)$. Two linearized schemes of the numerical method are constructed. The considered FDEs are equivalently transformed by the Riemann–Liouville integral into their integro-partial differential problems to reduce the requirement for smoothness in time. The analysis of stability and convergence is rigorously discussed. Finally, numerical experiments are described, which confirm the obtained theoretical analysis.

Keywords: time and space fractional differential equations; linearized schemes; integro-differential equation; stability; convergence; weighted and shifted Grünwald difference operator

MSC: 35R11; 65M06

Citation: Orlov, V.N.; Elsayed, A.M.; Mahmoud, E.I. Two Linearized Schemes for One-Dimensional Time and Space Fractional Differential Equations. *Mathematics* **2022**, *10*, 3651. https://doi.org/10.3390/math10193651

Academic Editor: Luigi Rodino

Received: 17 August 2022
Accepted: 1 October 2022
Published: 5 October 2022

1. Introduction

Many different branches of science and engineering use fractional partial differential equations (FPDEs), such as hydrodynamics, electroanalytical chemistry, quantum science, viscoelastic mechanics, signal image processing, chain-breaking of polymer materials, molecular spectrum, and anomalous ion diffusion in nerve cells [1,2]. Moreover, PDEs with a fractional order were used to simulate the flow and filtration of a fluid in a porous fractal medium. The use of fractional derivatives (FDs)for modeling real physical processes or environments leads to the appearance of equations containing derivatives and integrals of fractional order in addition to the classical ones. Researchers have focused their efforts on fractional-order physical models [3] because of the material's dynamic behavior and viscoelastic behavior [4]. As a result, the model of fractional order is widely employed to model the frequency apportionment of structural damping mechanisms [5], the electrical and physical characteristics of a process in relation to the order of fractional operator. An intrinsic multiscale existence of these operators is an interesting feature. As a consequence, memory effects (i.e., a system's response is a function of its previous history) are enabled by time-fractional operators, while non-local and scale effects are enabled by space fractional operators. Fractional analysis is used in many areas of science, including nonlinear biological processes, solid-state mechanics, field theory, control theory, friction, fluid dynamics, and so on [6]. For the study of fractionally damped viscoelastic material, Josefson and Enelund [7] employed the finite element scheme. The surface of a concrete structure is susceptible to major damaging consequences. Therefore, a composite with enhanced operating characteristics is currently being developed based on a concrete blend, polymer concrete, which is characterized by greater tolerance to moisture, chemical compounds, low temperatures, and toughness relative to concrete. It is possible to depict polymer concrete as a collection of solid filler granules contained in a viscoelastic medium in simulation [8]. The fractional oscillator equation describes the transverse movement under the control of the force of gravity or the exterior force of a filler granule. Thus, the substitution of concrete for polymer concrete refers to the substitution of the differential equations of the second order with the FDEs. Special attention is given to the use of fractional calculus to establish better mathematical models of many real-world issues. Many scholars have presented the theoretical evolution and implementation of fractional calculus in earlier works [9–11]. Special attention is required to be given to the following vibrating string equation:

$$\frac{\partial^2 w(x,t)}{\partial t^2} + b^C D_t^{\beta} w(x,t) = \frac{\partial^2 w(x,t)}{\partial x^2} + c^R D_x^{\alpha} w(x,t) + \mathtt{f}(x,t),$$

where $w(x,t)$ represents the displacement of a granule through the x- axis at a time t, b, c are arbitrary constants, and $\mathtt{f}(x,t)$ is an external forcing function. In the description of the vibration models, fractional differentiation operators are commonly utilized. It is well known that FD equations accurately describe the motion of vibrations with elastic and viscoelastic components [12]. The findings of [13] demonstrate that the outcome of solving problems can be used to simulate alteration in the deformation-strength properties of polymer concrete under the effect of gravity force. Researchers examined samples of polyester resin-based polymer concrete (chloride-1, Diane,1-dichloro-2, diacyl, and 2-diethylene). FDs are frequently employed to characterize the viscoelastic characteristics of sophisticated materials, as well as the dissipative forces in structural dynamics [14].

The aim of this article is to analyze the following 1-D time-space FDEs in the range $D = \{0 < x < \mathfrak{L}, 0 < t < \mathfrak{T}\}$

$$^C D_t^{\beta} w(x,t) =^R D_x^{\alpha} w(x,t) + \frac{\partial^2 w(x,t)}{\partial x^2} + \mathtt{f}(x,t), \tag{1}$$

with initial conditions

$$\begin{aligned} w(x,0) &= \varphi(x), \\ w_t(x,0) &= \psi(x), \end{aligned} \tag{2}$$

and boundary conditions

$$w(0,t) = w(\mathfrak{L},t) = 0, \tag{3}$$

where $^C D_t^{\beta}$ denotes the Caputo derivative with respect to the variable t of order $\beta (1 < \beta < 2)$, which is defined as

$$^C D_t^{\beta} w(x,t) = \frac{1}{\Gamma(2-\beta)} \int_0^t \frac{\partial^2 w(x,v)}{\partial v^2} (t-v)^{1-\beta} dv,$$

and $^R D_x^{\alpha}$ is the Liouville derivative with respect to the variable x of order $\alpha (1 < \alpha < 2)$, i.e.,

$$^R D_x^{\alpha} w(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_0^x (x-\xi)^{1-\alpha} w(\xi,t) d\xi.$$

Our fundamental objective in this work is to form a numerical strategy for Equation (1) and perform the comparing numerical examination for the suggested method. Analytical methods have the advantage of explaining the fundamentals of mechanical engineering problems and physical connotation making it possible to analyze a variety of physical and mechanical engineering problems and taking less time than the numerical method. However, scholars found that obtaining exact solutions to PDEs is extremely complicated. To obtain numerical solutions to FPDEs, a variety of numerical methods have been studied and developed, including the homotopy perturbation methods, Adomian decomposition method, spectral method, finite difference scheme, Galerkin method, and finite element method [15–19].

By combining the compact difference method for spatial discretization and L_1 approximation for temporal discretization, a finite difference scheme was derived in [20–23]. For the diffusion problem in the time derivative term, Li and Xu [24] established a time–space spectral system.

In 2019, Huang et al. [25] considered a two-dimensional nonlinear super-diffusion problem in the time derivative term and proposed conservative linearized two ADI schemes to obtain the approximate solution of the model. For the space and time fractional telegraph equation, Li and Zhao [26] proposed a linearized fractional difference/finite element approximation, and the unconditional stability of the suggested scheme was proven by the mathematical induction and energy method. In [27], Sun and Wu introduced a finite difference method by adding two additional parameters to turn the original equation into a low order equation system enabling the analysis of the error. To construct a compact difference method for diffusion-wave equations of fractional order, the equivalent integro-differential equations and product trapezoidal law were used by Chen and Li [28]. In 2016, Wang et al. [29] studied finite difference methods for both temporal and spatial fractional derivatives for differential equations. They also presented a precondition in order to improve the effectiveness of the schemes' implementation in this case.

Many of the published papers on this topic concerned both one specific type of fractional derivatives and a specific range of parameters for of these derivatives. Therefore, the novelty of the presented research lies in the difference between the considered combination of different fractional derivatives and the parameters of the fractional derivatives of Equation (1) and assembles high-order numerical schemes by constructing the equivalent partial integro-differential equation form. We also perform the corresponding numerical evaluation for the proposed schemes. It is common knowledge that the numerical methods for integral equations have higher numerical stability than those created for equivalent differential equations. To reduce the requirement for smoothness in time, the considered FDEs (1) are equivalently transformed by the Riemann–Liouville integral into their integro-partial differential problems. We discretize the Riemann–Liouville derivative using the Crank–Nicholson scheme combined with the weighted and shifted Grünwald-difference scheme. The first order derivative uses the midpoint scheme, and the second order derivative is approximated using the classical central difference scheme, which refers to the implicit difference scheme. Furthermore, we discuss their unconditional stability and convergence. The convergence rates of these two schemes are the second-order accuracy in time and space.

The manuscript proceeds as follows. In Section 2, the equivalence between the space–time (FDEs) and a partial integro-differential equation is proved. Then, for this integro-differential equation, we discuss two difference methods, and we derive some preparations and essential lemmas. In Section 3, the first scheme for the space–time FDEs is derived and studied; in addition, it is rigorously proven that the proposed method is convergent and unconditionally stable. In Section 4, the second scheme is constructed and analyzed. To validate our theoretical results, numerical experiments are performed in Section 5. Finally, a brief conclusion of the manuscript is presented in the last section.

2. Two Difference Schemes

Considering Equation (1) with conditions (2) and (3), we find that if we assume for Equation (1) an equivalent form, the precision of the discrete approximations could be improved. We indicate readers to [25] for the details of this analogous form. For the completeness of our analysis, we will only detail the main elements here.

From the Caputo derivative definition, clearly, ${}^{C}D_t^{\beta}$ is the composition of ${}^{C}D_t^{\beta-1}$ and D_t, such that

$$\begin{aligned} {}^{C}D_t^{\beta}w(x,t) &= \frac{1}{\Gamma(1-(\beta-1))}\int_0^t \frac{\partial}{\partial v}\frac{\partial w(x,v)}{\partial v}(t-v)^{-(\beta-1)}dv \\ &= {}^{C}D_t^{(\beta-1)}{}^{C}D_t w(x,t) = {}^{C}D_t^{\theta}{}^{C}D_t w(x,t), \end{aligned}$$

where $0 < \theta = \beta - 1 < 1$. Let us integrate both sides of Equation (1) under $\theta-$fractional integral Riemann–Liouville ${}_0J_t^{\theta}$ to obtain

$$\begin{aligned} w_t(x,t) = \psi(x) + \frac{1}{\Gamma(\theta)}\int_0^t {}^R D_x^{\alpha} u(x,v)(t-v)^{\theta-1}dv \\ + \frac{1}{\Gamma(\theta)}\int_0^t \frac{\partial^2 w(x,v)}{\partial x^2}(t-v)^{\theta-1}dv + \mathsf{F}(x,t), \end{aligned} \tag{4}$$

where $\mathsf{F}(x,t) ={}_0 J_t^{\theta}\mathsf{f}(x,t)$, so for a function $\mathsf{f}(x,t)$ we can define $\theta-$ fractional integral Riemann–Liouville as

$${}_0J_t^{\theta}\mathsf{f}(x,t) = \frac{1}{\Gamma(\theta)}\int_0^t (t-v)^{\theta-1}\mathsf{f}(x,v)dv.$$

The weighted and shifted Grünwald difference (WSGD) formula for the $\theta-$fractional integral Riemann–Liouville was used to developing our scheme described by Equation (4). In order to discretize Equation (4), we introduced the temporal step size $\tau = \frac{\mathfrak{T}}{\mathbb{N}}$ with a positive integer $\mathbb{N}$, and $t_n = n\tau; n = 0,1,...,\mathbb{N}$; we also defined a grid function time $\Omega_\tau = \{t_n | n \geq 0\}$. For a spatial discretization, let $h = \frac{\mathfrak{L}}{\mathbb{M}}$ and $x_i = ih; 0 \leq i \leq \mathbb{M}$, where $\mathbb{M}$ is a non-zero integer number, and we also defined a grid function space $\Omega_h = \{x_i | \quad 0 \leq i \leq \mathbb{M}\}$. Suppose on $\Omega_h \times \Omega_\tau$, their exist grid functions $\mathcal{W} = \{w_i^n | \quad 0 \leq i \leq \mathbb{M}, n \geq 0\}$, such that for any $w, g \in \mathcal{W}$, we define the following norms, semi-norm $\|.\|_{\breve{H}}$, and the inner product, as follows

$$w_i^{n+\frac{1}{2}} = \frac{1}{2}\Big[w_i^{n+1} + w_i^n\Big], \qquad \delta_t w_i^{n+\frac{1}{2}} = \frac{1}{\tau}\Big[w_i^{n+1} - w_i^n\Big],$$

$$\langle w^n, g^n\rangle = h\sum_{i=1}^{\mathbb{M}-1} w_i g_i, \qquad \|w^n\|^2 = \langle w, w\rangle,$$

$$\|w^n\|_\infty = \max_{0\leq i\leq \mathbb{M}} |w_i^n|, \qquad \langle \delta_x^2 w, g\rangle = -\langle \delta_x w, \delta_x g\rangle,$$

$$\langle \delta_x w, \delta_x g\rangle_{\breve{H}} = \langle \delta_x w, \delta_x g\rangle - \frac{h^2}{12}\langle \delta_x^2 w, \delta_x^2 g\rangle, \qquad \|\delta_x w\|_{\breve{H}} = \sqrt{\langle \delta_x w, \delta_x w\rangle_{\breve{H}}}.$$

Moreover, we utilized the discretization [30] for the spatial derivatives supplied by

$${}^R D_x^{\alpha}\mathsf{f}(x_i) = \frac{1}{\Gamma(4-\alpha)h^{\alpha}}\sum_{s=0}^{i+1} q_s^{\alpha}\mathsf{f}(x_{i-s+1}) + \mathcal{O}(h^2), \qquad \delta_x^2 w_i^n = \frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{h^2},$$

where $\mathsf{f} \in \mathcal{C}^4(R)$ and is defined by

$$q_s^{\alpha} = \begin{cases} 1, & s = 0 \\ 2^{3-\alpha} - 4, & s = 1 \\ 3^{3-\alpha} - 4\times 2^{3-\alpha} + 6, & s = 2 \\ (s+1)^{3-\alpha} - 4s^{3-\alpha} + 6(s-1)^{3-\alpha} - 4(s-2)^{3-\alpha} + (s-3)^{3-\alpha}, & s \geq 3, \end{cases} \tag{5}$$

and

$$\mathcal{H}w_i = \begin{cases} (1 + \frac{h^2}{12}\delta_x^2)w_i = \frac{1}{12}(w_{i-1} + 10w_i + w_{i+1}), & 1 \leq i \leq \mathbb{M}-1 \\ w_i, & i = 0, \mathbb{M}. \end{cases}$$

To raise the accuracy in $\frac{\partial^2 w}{\partial x^2}$, we used the following lemma.

Lemma 1 ([31]). *Suppose* $w(x) \in \mathcal{C}^6[x_{i-1}, x_{i+1}], 1 \leq i \leq \mathbb{M}-1$, *let* $\xi(s) = (1-s)^3[5 - 3(1-s)^2]$; *then,*

$$\frac{1}{12}[w''(x_{i-1}) + 10w''(x_i) + w''(x_{i+1})] = \frac{1}{h^2}[w(x_{i-1}) - 2w(x_i) + w(x_{i+1})] + \frac{h^4}{360}\int_0^1 [w^{(6)}(x_i - sh) + w^{(6)}(x_i + sh)]\xi(s)ds.$$

For time discretization, the high-order accuracy of our suggested method is presented on the second-order approximation of the θ—$R\check{}L$ fractional integral; this approximation established by the shifted operator for the θ—fractional integral Riemann–Liouville was defined as:

$$J^{\theta}_{\tau,\zeta}\mathsf{f}(t) = \tau^{\theta}\sum_{j=0}^{\infty}\varpi_j \mathsf{f}(t-(j-\zeta)\tau),$$

where ζ is an integer, and $\varpi_j = (-1)^j\binom{-\theta}{j}$ for $j \geq 0$. The second-order estimate for $\theta-$fractional integral Riemann–Liouville was set out in [32–34].

Lemma 2 ([34])**.** *Assume $\theta > 0, \mathsf{f}(t) \in L^p(R), p \geq 1$. The Fourier transform belonging to $L^p(R)$ of the $\theta-$ fractional integral Riemann–Liouville holds that*

$$\aleph[{}_{-\infty}I^{\theta}_t \mathsf{f}(t)] = (i\omega)^{-\theta}\widehat{\mathsf{f}}(\omega),$$

such that $\widehat{\mathsf{f}}(\omega) = \int_R e^{-i\omega t}\mathsf{f}(t)dt$ is the Fourier transform of the function $\mathsf{f}(t)$.

Lemma 3 ([33])**.** *Consider $\mathsf{f}(t),{}_{-\infty} I^{\theta}_t\mathsf{f}(t)$ and its Fourier transform in $L^1(R)$, and let us state the WSGD operator by*

$$\Im^{\theta}_{v,p,q}\mathsf{f}(t) = \frac{2q+\theta}{2(q-p)}J^{\theta}_{v,p}\mathsf{f}(t) + \frac{2p+\theta}{2(p-q)}J^{\theta}_{v,q}\mathsf{f}(t);$$

so, we have

$$\Im^{\theta}_{\tau,p,q}\mathsf{f}(t) ={}_{-\infty} I^{\theta}_t\mathsf{f}(t) + \mathcal{O}(\tau^2),$$

for integers $p \neq q; t \in R$.

Without loss of generality, for $t < 0$, $w(x,t)$ could be continuously expanded to be equal to zero, via selecting $(p,q) = (0,-1)$ in Lemma 2, which produces $\frac{2q+\theta}{2(q-p)} = 1-\frac{\theta}{2}$, $\frac{2p+\theta}{2(p-q)} = \frac{\theta}{2}$, by indicating

$$^{R}D^{\alpha}_x w^k_i = \frac{1}{\Gamma(4-\alpha)h^{\alpha}}\sum_{s=0}^{i+1} q^{\alpha}_s w^k_{i-s+1} + \mathcal{O}(h^2).$$

For the time discretization at point (x_i, t_{n+1})

$$\begin{aligned} {}_0J^{\theta}_t\delta^{\alpha}_x w(x_i,t_{n+1}) &= \tau^{\theta}\left[(1-\frac{\theta}{2})\sum_{k=0}^{n+1}\varpi_k\delta^{\alpha}_x w^{n+1-k}_i + \frac{\theta}{2}\sum_{k=0}^{n}\varpi_k\delta^{\alpha}_x w^{n-k}_i\right] + \mathcal{O}(\tau+h^2) \\ &= \tau^{\theta}\sum_{k=0}^{n+1}\lambda_k\delta^{\alpha}_x w^{n+1-k}_i + \mathcal{O}(\tau+h^2), \end{aligned} \tag{6}$$

where

$$\lambda_0 = (1-\frac{\theta}{2})\varpi_0, \quad \lambda_k = (1-\frac{\theta}{2})\varpi_k + \frac{\theta}{2}\varpi_{k-1}; \quad k \geq 1.$$

Lemma 4 ([16])**.** *If $f(t) \in C^2([0,\mathfrak{T}])$, then it holds that at $t = t_{n+\frac{1}{2}}$,*

$${}_0J^{\theta}_t f(t_{n+\frac{1}{2}}) = \frac{1}{2}\left({}_0J^{\theta}_t f(t_{n+1}) +{}_0 J^{\theta}_t f(t_n)\right) + \mathcal{O}(\tau^2).$$

Lemma 5 ([30])**.** *For any $\alpha \in (1,2)$, the sequence $q_s^{(\alpha)}$, which is defined in Equation (5), fulfills the next characteristics:*

$$q_1^{(\alpha)} < 0, \quad q_0^{(\alpha)} \geq q_3^{(\alpha)} \ldots \geq 0, \quad q_0^{(\alpha)} + q_2^{(\alpha)} \geq 0, \quad q_2^{(\alpha)} \begin{cases} > 0, & \alpha \in (\alpha_0, 2) \\ \leq 0, & \alpha \in (1, \alpha_0), \end{cases} \sum_{s=0}^{\infty} q_s^{(\alpha)} = 0,,$$

and $\alpha_0 \approx 1.5545$ is the root of the Equation $3^{3-\alpha} - 4 \times 2^{3-\alpha} + 6; \alpha \in (1,2)$.

Lemma 6 (Grownall's inequality [35])**.** *Suppose that ν_n and ϑ_n are nonnegative sequences, and $\{\phi_n\}$ is a sequence that satisfies*

$$\phi_0 \leq \hbar_0, \quad \phi_n \leq \hbar_0 + \sum_{s=0}^{n-1} \vartheta_s + \sum_{s=0}^{n-1} \nu_s \phi_s, \quad \hbar_0 \geq 0; n \geq 1;$$

so, a sequence $\{\phi_n\}$ fulfills

$$\phi_n \leq \left(\hbar_0 + \sum_{s=0}^{n-1} \vartheta_s\right) \exp\left(\sum_{s=0}^{n-1} \nu_s\right); \quad n \geq 1.$$

Lemma 7 ([33])**.** *Let $\left\{\lambda_j^{\gamma}\right\}_{j=0}^{\infty}$, defined as in Equation (6). Then, for any real vector $(w_1, w_2, \ldots, w_k)^T \in R^k$, k integer, the following inequality holds*

$$\sum_{m=0}^{k-1} \left(\sum_{j=0}^{m} \lambda_j^{\gamma} w_{m+1-j}\right) w_{m+1} \geq 0.$$

Then, using a weighted Crank–Nicolson method for Equation (4) at the point $(x_i, t_{n+\frac{1}{2}})$ and using Lemma 4, it can be written as

$$\begin{aligned} \frac{w_i^{n+1} - w_i^n}{\tau} = \psi(x) &+ \frac{\tau^{\theta}}{2}\left[\sum_{k=0}^{n+1} \lambda_k \delta_x^{\alpha} w_i^{n+1-k} + \sum_{k=0}^{n} \lambda_k \delta_x^{\alpha} w_i^{n-k}\right] \\ &+ \frac{\tau^{\theta}}{2}\left[\sum_{k=0}^{n+1} \lambda_k \delta_x^{2} w_i^{n+1-k} + \sum_{k=0}^{n} \lambda_k \delta_x^{2} w_i^{n-k}\right] \\ &+ \frac{1}{2}(\mathsf{F}_i^{n+1} + \mathsf{F}_i^n) + \mathcal{O}(\tau^2 + h^2), \end{aligned} \tag{7}$$

where $1 \leq i \leq \mathbb{M} - 1, 0 \leq n \leq \mathbb{N} - 1$, w_i^n is a numerical value of $w(x_i, t_n)$, $\psi_i = \psi(x_i)$, and $\mathsf{F}_i^n = \mathsf{F}(x_i, t_n)$.

3. Construction and Analysis of Scheme 1

3.1. Construction of Scheme 1

Rearranging Equation (7) yields

$$\begin{aligned} w_i^{n+1} - w_i^n = \tau\psi(x) &+ \frac{\tau^{\theta+1}}{2}\left[\sum_{k=0}^{n+1} \lambda_k \delta_x^{\alpha} w_i^{n+1-k} + \sum_{k=0}^{n} \lambda_k \delta_x^{\alpha} w_i^{n-k}\right] \\ &+ \frac{\tau^{\theta+1}}{2}\left[\sum_{k=0}^{n+1} \lambda_k \delta_x^{2} w_i^{n+1-k} + \sum_{k=0}^{n} \lambda_k \delta_x^{2} w_i^{n-k}\right] \\ &+ \frac{\tau}{2}(\mathsf{F}_i^{n+1} + \mathsf{F}_i^n) + \tau\mathcal{O}(\tau^2 + h^2). \end{aligned} \tag{8}$$

Denoting $\rho = \tau^{\theta+1}/2$, we suggest the following compact scheme for Equation (8), which is based on Lemma 1.

$$\begin{aligned}\mathcal{H}(w_i^{n+1} - w_i^n) = \tau\mathcal{H}\psi(x) + \rho\left[\sum_{k=0}^{n+1}\lambda_k\delta_x^\alpha w_i^{n+1-k} + \sum_{k=0}^{n}\lambda_k\delta_x^\alpha w_i^{n-k}\right] \\ + \rho\left[\sum_{k=0}^{n+1}\lambda_k\delta_x^2 w_i^{n+1-k} + \sum_{k=0}^{n}\lambda_k\delta_x^2 w_i^{n-k}\right] \\ + \frac{\tau}{2}\mathcal{H}(\mathsf{F}_i^{n+1} + \mathsf{F}_i^n) + \tau\varrho_i^{n+1},\end{aligned} \tag{9}$$

where $\varrho_i^{n+1} \leq \mathcal{O}(\tau^2 + h^4)$. Ignoring the truncation error term in Equation (9) and replacing w_i^n with its numerical solution $\mathcal{W}_i^n$, we obtain the following scheme for Equation (9)

$$\begin{aligned}\mathcal{H}(\mathcal{W}_i^{n+1} - \mathcal{W}_i^n) = \tau\mathcal{H}\psi(x) + \rho\left[\sum_{k=0}^{n+1}\lambda_k\delta_x^\alpha \mathcal{W}_i^{n+1-k} + \sum_{k=0}^{n}\lambda_k\delta_x^\alpha \mathcal{W}_i^{n-k}\right] \\ + \rho\left[\sum_{k=0}^{n+1}\lambda_k\delta_x^2 \mathcal{W}_i^{n+1-k} + \sum_{k=0}^{n}\lambda_k\delta_x^2 \mathcal{W}_i^{n-k}\right] \\ + \frac{\tau}{2}\mathcal{H}(\mathsf{F}_i^{n+1} + \mathsf{F}_i^n), \qquad 1 \leq i \leq \mathbb{M}-1, \quad 0 \leq n \leq \mathbb{N}-1,\end{aligned} \tag{10}$$

$$\begin{aligned}&\mathcal{W}_0^n = \mathcal{W}_M^n = 0, && 1 \leq n \leq \mathbb{N}, \\ &\mathcal{W}_i^0 = \varphi_i, \quad \mathcal{W}_i^n = 0, && 0 \leq i \leq \mathbb{M}.\end{aligned} \tag{11}$$

3.2. Analysis of Scheme 1

Theorem 1. *Let $w(x,t) \in \mathcal{C}_{x,t}^{6,3}([0,\mathfrak{L}] \times [0,\mathfrak{T}])$ be the exact solution of Equations (1)–(3) and $\mathcal{W}(x,t)$ be a numerical solution of scheme (10)–(11), which is defined as $\{W_i^n | 0 \leq i \leq \mathbb{M}, 0 \leq n \leq \mathbb{N}\}$. Then, for $n\tau \leq T$, it holds that*

$$\|\mathcal{W}^n - w^n\| \leq \tilde{c}(\tau^2 + h^4), \qquad 0 \leq n \leq \mathbb{N}$$

Proof. Subtracting Equation (10) from Equation (9) and denoting the error $\mathcal{E}_i^n = w_i^n - \mathcal{W}_i^n$, then we have

$$\mathcal{H}(\mathcal{E}_i^{n+1} - \mathcal{E}_i^n) = \rho\sum_{s=0}^{n}\lambda_s\delta_x^\alpha(\mathcal{E}_i^{n+1-s} + \mathcal{E}_i^{n-s}) + \rho\sum_{s=0}^{n}\lambda_s\delta_x^2(\mathcal{E}_i^{n+1-s} + \mathcal{E}_i^{n-s}) + \tau\varrho_i^{n+1}, \tag{12}$$

where $\mathcal{E}_i^0 = 0, \quad 0 \leq i \leq \mathbb{M}$.

We can readily rewrite Equation (12) in a matrix form and by multiplying with the identity matrix I of size N, we obtain

$$\mathsf{C}(\mathcal{E}^{n+1} - \mathcal{E}^n) = \rho_1\sum_{s=0}^{n}\lambda_s A(\mathcal{E}^{n+1-s} + \mathcal{E}^{n-s}) + \rho_2\sum_{s=0}^{n}\lambda_s B(\mathcal{E}^{n+1-s} + \mathcal{E}^{n-s}) + \tau\varrho^{n+1}, \tag{13}$$

where $\rho_1 = \rho/\Gamma(4-\alpha)h^{\alpha}, \rho_2 = \rho/h^2, \|\varrho^{n+1}\| \leq c(\tau^2 + h^4)$, and

$$A = \begin{pmatrix} q_1^{(\alpha)} & q_0^{(\alpha)} & 0 & \cdots & 0 \\ q_2^{(\alpha)} & q_1^{(\alpha)} & q_0^{(\alpha)} & \cdots & \\ \vdots & q_2^{(\alpha)} & q_1^{(\alpha)} & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & q_0^{(\alpha)} \\ q_N^{(\alpha)} & q_{N-1}^{(\alpha)} & \cdots & q_2^{(\alpha)} & q_1^{(\alpha)} \end{pmatrix}, \qquad C = \frac{1}{12}\begin{pmatrix} 10 & 2 & & \cdots & 0 \\ 1 & 10 & 1 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & 1 & 10 & 1 \\ & & \cdots & 2 & 10 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & -2 & & \cdots & 0 \\ -1 & 2 & -1 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & -1 & 2 & -1 \\ & & \cdots & -2 & 2 \end{pmatrix}. \tag{14}$$

Multiplying Equation (13) by $h(\mathcal{E}^{n+1} + \mathcal{E}^n)^T$, we obtain

$$\begin{aligned} h(\mathcal{E}^{n+1} + \mathcal{E}^n)^T C(\mathcal{E}^{n+1} - \mathcal{E}^n) =& \rho_1 \sum_{s=0}^{n} \lambda_s (\mathcal{E}^{n+1} + \mathcal{E}^n)^T A(\mathcal{E}^{n+1-s} + \mathcal{E}^{n-s}) \\ &+ \rho_2 \sum_{s=0}^{n} \lambda_s (\mathcal{E}^{n+1} + \mathcal{E}^n)^T B(\mathcal{E}^{n+1-s} + \mathcal{E}^{n-s}) \\ &+ \tau h(\mathcal{E}^{n+1} + \mathcal{E}^n)^T \varrho^{n+1}. \end{aligned} \tag{15}$$

By the Gershgorin theorem, Lemma 5, and Lemma 7, we could investigate whether A and B are negative definite matrices, following

$$(\mathcal{E}^{n+1} + \mathcal{E}^n)^T A(\mathcal{E}^{n+1-s} + \mathcal{E}^{n-s}) < 0, \quad (\mathcal{E}^{n+1} + \mathcal{E}^n)^T B(\mathcal{E}^{n+1-s} + \mathcal{E}^{n-s}) < 0;$$

then, summing over n from 0 to $J-1$, we deduce

$$h(\mathcal{E}^{n+1} + \mathcal{E}^n)^T C(\mathcal{E}^{n+1} - \mathcal{E}^n) = h\Big[(\mathcal{E}^{n+1})^T C\mathcal{E}^{n+1} - (\mathcal{E}^n)^T C\mathcal{E}^n\Big], \quad h(\mathcal{E}^J)^T C\mathcal{E}^J \geq \frac{2}{3}\|\mathcal{E}^J\|^2,$$

$$\begin{aligned} \frac{2}{3}\|\mathcal{E}^J\|^2 \leq& \tau \sum_{n=0}^{J-1} \langle \varrho^{n+1}, (\mathcal{E}^{n+1} + \mathcal{E}^n) \rangle \leq \frac{1}{3}\|\mathcal{E}^J\|^2 + \frac{\tau}{3}\|\mathcal{E}^{J-1}\|^2 + \frac{3\tau^2}{4}\|\varrho^J\|^2 + \frac{3\tau}{4}\|\varrho^J\|^2 \\ &+ \frac{\tau}{3}\sum_{n=1}^{J-1}\|\mathcal{E}^n\|^2 + \frac{\tau}{3}\sum_{n=1}^{J-2}\|\mathcal{E}^n\|^2 + \frac{3\tau}{2}\sum_{n=0}^{J-2}\|\varrho^{n+1}\|^2 \leq \frac{1}{3}\|\mathcal{E}^J\|^2 + \frac{3\tau^2}{4}\|\varrho^J\|^2 \\ &+ \frac{2\tau}{3}\sum_{n=1}^{J-1}\|\mathcal{E}^n\|^2 + \frac{3\tau}{2}\sum_{n=0}^{J-1}\|\varrho^{n+1}\|^2, \end{aligned} \tag{16}$$

which gives

$$\|\mathcal{E}^J\|^2 \leq 2\tau \sum_{n=1}^{J-1}\|\mathcal{E}^n\|^2 + \frac{9\tau^2}{4}\|\varrho^J\|^2 + \frac{9\tau}{2}\sum_{n=0}^{J-1}\|\varrho^{n+1}\|^2 \leq 2\tau\sum_{n=1}^{J-1}\|\mathcal{E}^n\|^2 + C(\tau^2 + h^4)^2;$$

then, the required results follow by Lemma 5. □

Theorem 2. *The numerical solution of scheme* (10)–(11) $\mathcal{W}_i^n$ *is stable, and for* $1 < K < \mathbb{N}$, *it holds that*

$$\|\mathcal{W}^K\|_\infty^2 \leq 2\|\mathcal{W}^0\|_{\check{H}}^2 + 2\tau\sum_{n=0}^{K-1}\|\mathcal{W}^n\|^2 + \tau\sum_{n=0}^{K-1}\|\tau^\theta \lambda_{n+1}(\delta_x^\alpha + \delta_x^2)\mathcal{W}^0 + 2\mathcal{H}\psi + 2\mathcal{H}\mathrm{f}^{\frac{1}{2}}\|^2.$$

Proof. Multiplying Equation (10) by $h\mathcal{H}(\mathcal{W}_i^{n+1}+\mathcal{W}_i^n)$ and obtaining the sum over $1 \leq i \leq \mathbb{M}-1$,

$$\begin{aligned}\langle \mathcal{H}(\mathcal{W}_i^{n+1}-\mathcal{W}_i^n),\mathcal{H}(\mathcal{W}_i^{n+1}+\mathcal{W}_i^n)\rangle =&\tau\langle \mathcal{H}\psi,\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\rho\sum_{k=0}^{n}\lambda_k\langle \delta_x^{\alpha}(\mathcal{W}^{n+1-k}+\mathcal{W}^{n-k}),\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\rho\sum_{k=0}^{n}\lambda_k\langle \delta_x^{2}(\mathcal{W}^{n+1-k}+\mathcal{W}^{n-k}),\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\rho\lambda_{n+1}\langle \delta_x^{\alpha}\mathcal{W}^0,\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\rho\lambda_{n+1}\langle \delta_x^{2}\mathcal{W}^0,\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\tau\langle \mathcal{H}\mathsf{F}^{n+1/2},\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n\rangle.\end{aligned}\tag{17}$$

Further calculations provide

$$\begin{aligned}\|\mathcal{H}\mathcal{W}^{n+1}\|^2-\|\mathcal{H}\mathcal{W}^{n}\|^2 =&\tau\langle \mathcal{H}\psi,\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle+\rho\lambda_{n+1}\langle \delta_x^{\alpha}\mathcal{W}^0,\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\rho\sum_{k=0}^{n}\lambda_k\langle \delta_x^{\alpha}(\mathcal{W}^{n+1-k}+\mathcal{W}^{n-k}),\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\rho\lambda_{n+1}\langle \delta_x^{2}\mathcal{W}^0,\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\rho\sum_{k=0}^{n}\lambda_k\langle \delta_x^{2}(\mathcal{W}^{n+1-k}+\mathcal{W}^{n-k}),\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\tau\langle \mathcal{H}\mathsf{F}^{n+1/2},\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle.\end{aligned}\tag{18}$$

After applying the Cauchy–Schwarz inequality and obtaining the sum of Equation (18) over n from 0 to $K-1$,

$$\begin{aligned}\|\mathcal{H}\mathcal{W}^{K}\|^2-\|\mathcal{H}\mathcal{W}^{0}\|^2 \leq&\tau\sum_{n=0}^{K-1}\langle \mathcal{H}\psi,\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\rho\sum_{n=0}^{K-1}\sum_{k=0}^{n}\lambda_k\langle \delta_x^{\alpha}(\mathcal{W}^{n+1-k}+\mathcal{W}^{n-k}),\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\rho\sum_{n=0}^{K-1}\sum_{k=0}^{n}\lambda_k\langle \delta_x^{2}(\mathcal{W}^{n+1-k}+\mathcal{W}^{n-k}),\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\rho\sum_{n=0}^{K-1}\lambda_{n+1}\langle \delta_x^{\alpha}\mathcal{W}^0,\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\rho\sum_{n=0}^{K-1}\lambda_{n+1}\langle \delta_x^{2}\mathcal{W}^0,\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle\\&+\tau\sum_{n=0}^{K-1}\langle \mathcal{H}\mathsf{F}^{n+1/2},\mathcal{H}(\mathcal{W}^{n+1}+\mathcal{W}^n)\rangle.\end{aligned}\tag{19}$$

According to Lemmas 5 and 7, we infer that the first two terms on the whole right side of Equation (19) are negative; then,

$$
\begin{aligned}
\|\mathcal{H}\mathcal{W}^K\|^2 - \|\mathcal{H}\mathcal{W}^0\|^2 \leq & \tau \sum_{n=0}^{K-1} \|\mathcal{H}\psi\| \|\mathcal{H}(\mathcal{W}^{n+1} + \mathcal{W}_i^n)\| \\
& + \rho \sum_{n=0}^{K-1} \lambda_{n+1} \|\delta_x^\alpha \mathcal{W}^0\| \|\mathcal{H}(\mathcal{W}^{n+1} + \mathcal{W}^n)\| \\
& + \rho \sum_{n=0}^{K-1} \lambda_{n+1} \|\delta_x^2 \mathcal{W}^0\| \|\mathcal{H}(\mathcal{W}^{n+1} + \mathcal{W}^n)\| \\
& + \tau \sum_{n=0}^{K-1} \|\mathcal{H}\mathsf{F}^{n+1/2}\| \|\mathcal{H}(\mathcal{W}^{n+1} + \mathcal{W}^n)\|. \qquad (20)
\end{aligned}
$$

By applying Young's inequality,

$$
\begin{aligned}
\|\mathcal{H}\mathcal{W}^K\|^2 \leq & \|\mathcal{H}\mathcal{W}^0\|^2 + \frac{\tau}{2} \sum_{n=0}^{K-1} \|\mathcal{H}\psi\|^2 + \frac{\tau}{2} \sum_{n=0}^{K-1} \|\mathcal{H}(\mathcal{W}^{n+1} + \mathcal{W}^n)\|^2 + \frac{\rho}{2} \sum_{n=0}^{K-1} \lambda_{n+1} \|\delta_x^\alpha \mathcal{W}^0\|^2 \\
& + \frac{\rho}{2} \sum_{n=0}^{K-1} \lambda_{n+1} \|\delta_x^2 \mathcal{W}^0\|^2 + \frac{\rho}{2} \sum_{n=0}^{K-1} \lambda_{n+1} \|\mathcal{H}(\mathcal{W}^{n+1} + \mathcal{W}^n)\|^2 \\
& + \frac{\rho}{2} \sum_{n=0}^{K-1} \lambda_{n+1} \|\mathcal{H}(\mathcal{W}^{n+1} + \mathcal{W}^n)\|^2 + \frac{\tau}{2} \sum_{n=0}^{K-1} \|\mathcal{H}\mathsf{F}^{n+1/2}\|^2 + \frac{\tau}{2} \sum_{n=0}^{K-1} \|\mathcal{H}(\mathcal{W}^{n+1} + \mathcal{W}^n)\|^2. \qquad (21)
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
\|\mathcal{W}^K\|_\infty^2 \leq & 2\|\mathcal{W}^0\|_{\check{H}}^2 + 2\tau \sum_{n=0}^{K-1} \|\mathcal{H}\psi\|^2 + 2\tau \sum_{n=0}^{K-1} \|\mathcal{W}^n\|^2 + \tau^{\theta+1} \sum_{n=0}^{K-1} \lambda_{n+1} \|\delta_x^\alpha \mathcal{W}^0\|^2 \\
& + \tau^{\theta+1} \sum_{n=0}^{K-1} \lambda_{n+1} \|\delta_x^2 \mathcal{W}^0\|^2 + 2\tau \sum_{n=0}^{K-1} \|\mathcal{H}\mathsf{F}^{n+1/2}\|^2.
\end{aligned}
$$

This implies,

$$
\|\mathcal{W}^K\|_\infty^2 \leq e^{2T} \left(2\|\mathcal{W}^0\|_{\check{H}}^2 + \tau \sum_{n=0}^{K-1} \|2\mathcal{H}\psi + \tau^\theta \lambda_{n+1} (\delta_x^\alpha + \delta_x^2) \mathcal{W}^0 + 2\mathcal{H}\mathsf{F}^{n+1/2}\|^2 \right).
$$

□

4. Construction and Analysis of Scheme 2

4.1. Construction of Scheme 2

Similarly, if the Crank–Nicolson method is used to discretize Equation (4) at the point (x_i, t_{n+1}), for the time derivative, with the help of Equation (6), the second scheme presented in Section 4 is applied.

$$
w_i^{n+1} - w_i^n = \tau\psi(x) + \tau^{\theta+1} \sum_{k=0}^{n+1} \lambda_k \delta_x^\alpha w_i^{n+1-k} + \tau^{\theta+1} \sum_{k=0}^{n+1} \lambda_k \delta_x^2 w_i^{n+1-k} + \tau \mathsf{F}_i^{n+1} + \mathcal{O}(\tau^2 + \tau h^4) \qquad (22)
$$

Denoting $\rho = \tau^{\theta+1}/2$, we suggest the following compact Crank–Nicolson scheme at the point (x_i, t_{n+1}), which is based on Lemma 1.

$$
\mathcal{H}(w_i^{n+1} - w_i^n) = \tau\mathcal{H}\psi(x) + \rho \sum_{k=0}^{n+1} \lambda_k \delta_x^\alpha w_i^{n+1-k} + \rho \sum_{k=0}^{n+1} \lambda_k \delta_x^2 w_i^{n+1-k} + \tau\mathcal{H}\mathsf{F}_i^{n+1} + \tau\mathcal{O}(\tau + h^4), \qquad (23)
$$

ignoring the truncation error term $\mathcal{O}(\tau^2 + \tau h^4)$ from Equation (23) and replacing w_i^n with its numerical solution $\mathcal{W}_i^n$, we obtain the following scheme for Equation (23)

$$\mathcal{H}(\mathcal{W}_i^{n+1}-\mathcal{W}_i^n)=\tau\mathcal{H}\psi(x)+\rho\sum_{k=0}^{n+1}\lambda_k\delta_x^\alpha\mathcal{W}_i^{n+1-k}+\rho\sum_{k=0}^{n+1}\lambda_k\delta_x^2\mathcal{W}_i^{n+1-k}+\tau\mathcal{H}\mathsf{F}_i^{n+1},$$
$$1\le i\le\mathbb{M}-1, 0\le n\le\mathbb{N}-1. \tag{24}$$

4.2. Analysis of Scheme 2

Similar to Theorems 1 and 2, and after utilizing the matrix form of Equation (24), we can easily obtain the following theorems

Theorem 3. *The numerical solution of scheme* (24) $\mathcal{W}_i^n$ *is stable, and for* $1<K<\mathbb{N}$*, it holds that*

$$\|\mathcal{W}^K\|_\infty^2\le\|\mathcal{W}^0\|_{\check{H}}^2+\tau\sum_{n=0}^{K-1}\|\rho\lambda_{n+1}\mathcal{H}(\delta_x^\alpha+\delta_x^2)\mathcal{W}^0+\mathcal{H}\psi+\mathcal{H}\mathsf{f}^{n+1}\|^2.$$

Proof. Similar to proving Theorem 2, multiplying Equation (24) by $h\mathcal{H}\mathcal{W}_i^{n+1}$ and obtaining the sum over $1\le i\le\mathbb{M}-1$, we obtain

$$\begin{aligned}\langle\mathcal{H}(\mathcal{W}_i^{n+1}-\mathcal{W}_i^n),\mathcal{H}\mathcal{W}_i^{n+1}\rangle=&\tau\langle\mathcal{H}\psi,\mathcal{H}\mathcal{W}^{n+1}\rangle+\rho\sum_{k=0}^{n}\lambda_k\langle\delta_x^\alpha\mathcal{W}^{n+1-k},\mathcal{H}\mathcal{W}^{n+1}\rangle\\&+\rho\sum_{k=0}^{n}\lambda_k\langle\delta_x^2\mathcal{W}^{n+1-k},\mathcal{H}\mathcal{W}^{n+1}\rangle+\rho\lambda_{n+1}\langle\delta_x^\alpha\mathcal{W}^0,\mathcal{H}\mathcal{W}^{n+1}\rangle\\&+\rho\lambda_{n+1}\langle\delta_x^2\mathcal{W}^0,\mathcal{H}\mathcal{W}^{n+1}\rangle+\tau\langle\mathcal{H}\mathsf{F}^{n+1},\mathcal{H}\mathcal{W}^{n+1}\rangle.\end{aligned}\tag{25}$$

Applying the Cauchy–Schwarz inequality, further calculations give

$$\begin{aligned}\frac{\|\mathcal{H}\mathcal{W}^{n+1}\|^2-\|\mathcal{H}\mathcal{W}^n\|^2}{2}=&\tau\langle\mathcal{H}\psi,\mathcal{H}\mathcal{W}^{n+1}\rangle+\rho\sum_{k=0}^{n}\lambda_k\langle\delta_x^\alpha\mathcal{W}^{n+1-k},\mathcal{H}\mathcal{W}^{n+1}\rangle\\&+\rho\sum_{k=0}^{n}\lambda_k\langle\delta_x^2\mathcal{W}^{n+1-k},\mathcal{H}\mathcal{W}^{n+1}\rangle+\rho\lambda_{n+1}\langle\delta_x^\alpha\mathcal{W}^0,\mathcal{H}\mathcal{W}^{n+1}\rangle\\&+\rho\lambda_{n+1}\langle\delta_x^2\mathcal{W}^0,\mathcal{H}\mathcal{W}^{n+1}\rangle+\tau\langle\mathcal{H}\mathsf{F}^{n+1},\mathcal{H}\mathcal{W}^{n+1}\rangle.\end{aligned}\tag{26}$$

Summing Equation (26) over n, from 0 to $K-1$, yields

$$\begin{aligned}\frac{\|\mathcal{H}\mathcal{W}^K\|^2-\|\mathcal{H}\mathcal{W}^0\|^2}{2}\le&\tau\sum_{n=0}^{K-1}\langle\mathcal{H}\psi,\mathcal{H}\mathcal{W}^{n+1}\rangle+\rho\sum_{n=0}^{K-1}\sum_{k=0}^{n}\lambda_k\langle\delta_x^\alpha\mathcal{W}^{n+1-k},\mathcal{H}\mathcal{W}^{n+1}\rangle\\&+\rho\sum_{n=0}^{K-1}\sum_{k=0}^{n}\lambda_k\langle\delta_x^2\mathcal{W}^{n+1-k},\mathcal{H}\mathcal{W}^{n+1}\rangle+\rho\sum_{n=0}^{K-1}\lambda_{n+1}\langle\delta_x^\alpha\mathcal{W}^0,\mathcal{H}\mathcal{W}^{n+1}\rangle\\&+\rho\sum_{n=0}^{K-1}\lambda_{n+1}\langle\delta_x^2\mathcal{W}^0,\mathcal{H}\mathcal{W}^{n+1}\rangle+\tau\sum_{n=0}^{K-1}\langle\mathcal{H}\mathsf{F}^{n+1},\mathcal{H}\mathcal{W}^{n+1}\rangle.\end{aligned}\tag{27}$$

According to Lemmas 5 and 7, we infer that the first two terms on the whole right side of Equation (27) are negative; therefore,

$$\begin{aligned}\|\mathcal{H}\mathcal{W}^K\|^2\le&2\|\mathcal{H}\mathcal{W}^0\|^2+2\tau\sum_{n=0}^{K-1}\|\mathcal{H}\psi\|\|\mathcal{H}\mathcal{W}^{n+1}\|+2\rho\sum_{n=0}^{K-1}\lambda_{n+1}\|\delta_x^\alpha\mathcal{W}^0\|\|\mathcal{H}\mathcal{W}^{n+1}\|\\&+2\rho\sum_{n=0}^{K-1}\lambda_{n+1}\|\delta_x^2\mathcal{W}^0\|\|\mathcal{H}\mathcal{W}^{n+1}\|+2\tau\sum_{n=0}^{K-1}\|\mathcal{H}\mathsf{F}^{n+1}\|\|\mathcal{H}\mathcal{W}^{n+1}\|,\end{aligned}\tag{28}$$

and by applying Young's inequality, then,

$$
\begin{aligned}
\|\mathcal{H}\mathcal{W}^K\|^2 \leq & 2\|\mathcal{H}\mathcal{W}^0\|^2 + \tau \sum_{n=0}^{K-1} \|\mathcal{H}\psi\|^2 + \tau \sum_{n=0}^{K-1} \|\mathcal{H}\mathcal{W}^{n+1}\|^2 + \rho \sum_{n=0}^{K-1} \lambda_{n+1} \|\delta_x^\alpha \mathcal{W}^0\|^2 \\
& + \rho \sum_{n=0}^{K-1} \lambda_{n+1} \|\delta_x^2 \mathcal{W}^0\|^2 + \rho \sum_{n=0}^{K-1} \lambda_{n+1} \|\mathcal{H}\mathcal{W}^{n+1}\|^2 + \rho \sum_{n=0}^{K-1} \lambda_{n+1} \|\mathcal{H}\mathcal{W}^{n+1}\|^2 \\
& + \tau \sum_{n=0}^{K-1} \|\mathcal{H}\mathsf{F}^{n+1}\|^2 + \tau \sum_{n=0}^{K-1} \|\mathcal{H}\mathcal{W}^{n+1}\|^2,
\end{aligned} \tag{29}
$$

which implies that

$$
\|\mathcal{W}^K\|_\infty^2 \leq 2\|\mathcal{W}^0\|_{\breve{H}}^2 + \tau \sum_{n=0}^{K-1} \|\mathcal{H}\psi + \tau^\theta \lambda_{n+1} (\delta_x^\alpha + \delta_x^2) \mathcal{W}^0 + \mathcal{H}\mathsf{F}^{n+1}\|^2. \tag{30}
$$

This completes the proof. □

Theorem 4. *Let $w(x,t) \in C_{x,t}^{6,3}([0,\mathfrak{L}] \times [0,\mathfrak{T}])$ be the exact solution of Equations (1)–(3) and the numerical solution of scheme (24), which is defined as W_i^n. Then, for $n\tau \leq T$, it holds that*

$$
\|\mathcal{W}^n - w^n\| \leq \tilde{c}(\tau + h^4).
$$

Proof. Subtracting Equation (24) from Equation (23) and denoting the error $\mathcal{E}_i^n = w_i^n - \mathcal{W}_i^n$, we have

$$
\mathcal{H}(\mathcal{E}_i^{n+1} - \mathcal{E}_i^n) = \rho \sum_{k=0}^{n+1} \lambda_k \delta_x^\alpha (\mathcal{E}_i^{n+1-k} + \rho \sum_{k=0}^{n} \lambda_k \delta_x^2 (\mathcal{E}_i^{n+1-k} + \mathcal{E}_i^{n-s}) + \tau \varrho_i^{n+1}, \tag{31}
$$

where $\mathcal{E}_i^0 = 0, \quad 0 \leq i \leq \mathbb{M}, \quad \varrho_i^{n+1} < c(\tau + h^4), c > 0.$

Similar to proving Theorem 1, it is easy to check the convergence of scheme 2. □

5. Numerical Example

In this section, we introduce numerical examples to demonstrate the computational performance and theoretical findings of our proposed methods.

Example 1. *Consider the following 1-D time–space FDEs*

$$
\begin{aligned}
& {}^C D_t^\beta w(x,t) = {}^R D_x^\alpha w(x,t) + \frac{\partial^2 w(x,t)}{\partial x^2} + \mathsf{f}(x,t), \quad 1 < \beta, \alpha < 2 \\
& w(0,t) = w(1,t) = 0, \quad 0 < t < 1, \\
& w(x,0) = 0, \quad w_t(x,0) = 0, \quad 0 < x < 1,
\end{aligned} \tag{32}
$$

where

$$
\begin{aligned}
\mathsf{f}(x,t) = & \frac{\Gamma(2+\beta)}{\Gamma(2)} x^2 (1-x)^2 t - (12x^2 - 12x + 2) t^{1+\beta} \\
& - t^{1+\beta} \left(\frac{\Gamma(5)}{\Gamma(5-\alpha)} x^{4-\alpha} - 2 \frac{\Gamma(4)}{\Gamma(4-\alpha)} x^{3-\alpha} + \frac{\Gamma(3)}{\Gamma(3-\alpha)} x^{2-\alpha} \right)
\end{aligned}
$$

is the exact solution of (32), given as $w(x,t) = x^2(1-x)^2 t^{1+\beta}$.

First, we note that the exact solution $w(x,t)$ of Equation (32) fulfills all the smoothness conditions needed by the schemes (10) and (24). In Figure 1, the approximate and the exact solution of scheme (10) are shown for $\beta = 1.5$ and $\alpha = 1.5$, respectively. Then, in Figures 2 and 3, we take step size $\tau = h = 0.025$ to plot the curves of the exact and numerical solutions of the two schemes at $t = \mathfrak{T} = 1$ for $\alpha = 1.5$ and $\beta = 1.5$. This assures us that the exact solutions accord well with the *numerical results of the two schemes. To analyze the error in the numerical solution, we consider the* L_2*-norm*

$$\mathcal{E}(\tau,h)=\sqrt{h\sum_{i=1}^{\mathbb{M}-1}|w_i^{\mathbb{N}}-W_i^{\mathbb{N}}|^2},$$

where we can approximately calculate the order of the convergence rate of R_x *and* R_t *from*

$$R_x \simeq log_2[\varepsilon(\tau,2h)/\varepsilon(\tau,h)], \quad \tau \longrightarrow 0$$
$$R_t \simeq log_2[\varepsilon(\tau,h)/\varepsilon(2\tau,h)], \quad h \longrightarrow 0.$$

Furthermore, in Table 1, we fix the time at $\tau = 0.02$*, and analyze how the error* $\varepsilon(\tau,h)$ *and the convergence orders of scheme 1 with non-compact form change with* $\mathbb{M}$ *for different values of* α, β*. Moreover, in Table 2, we compute the errors and the convergence orders of scheme 1 with compact form, for different step sizes and in Table 3, we refer to the errors and the convergence orders of scheme 2 for different values of* α, β*. with change in time steps. The errors of the two methods reduced as the step size* τ *and h decreased; we used high-level technical computing language (Wolfram Mathematica) to calculate the numerical results in Tables 1 and 2.*

Table 1. The errors and the convergence orders of (32) with finite difference scheme (8), for different α and β, when $\tau = 0.02$.

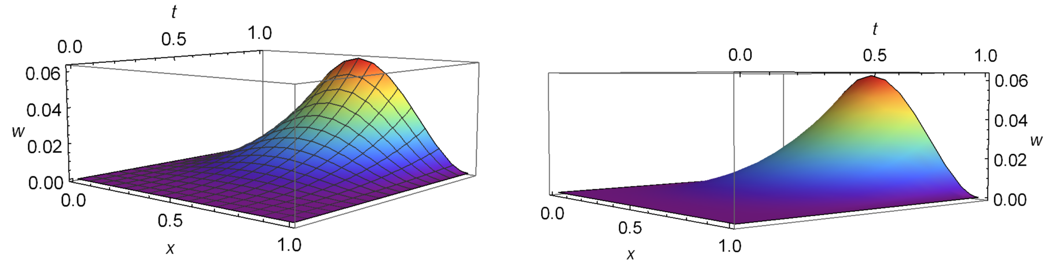
	$\beta = 1.3, \alpha = 1.8$		$\beta = 1.5, \alpha = 1.5$		$\beta = 1.7, \alpha = 1.2$	
h	$\varepsilon(\tau,h)$	R_x	$\varepsilon(\tau,h)$	R_x	$\varepsilon(\tau,h)$	R_x
1/4	9.770×10^{-3}		8.711×10^{-3}		6.931×10^{-3}	
1/8	2.485×10^{-3}	1.975	2.237×10^{-3}	1.961	1.790×10^{-3}	1.9528
1/16	6.274×10^{-4}	1.985	5.648×10^{-4}	1.986	4.511×10^{-4}	1.988
1/32	1.581×10^{-4}	1.988	1.418×10^{-4}	1.994	1.127×10^{-4}	1.999
1/64	3.983×10^{-5}	1.988	3.550×10^{-5}	1.998	2.798×10^{-5}	2.0107

Table 2. The errors and the convergence orders of (32) with finite difference schemes (10) and (11), for different α and β, when $\tau = 0.001$.

	$\beta = 1.7, \alpha = 1.3$		$\beta = 1.5, \alpha = 1.5$		$\beta = 1.2, \alpha = 1.9$	
h	$\varepsilon(\tau,h)$	Order	$\varepsilon(\tau,h)$	Order	$\varepsilon(\tau,h)$	Order
1/4	9.727×10^{-4}		1.323×10^{-3}		1.999×10^{-3}	
1/8	6.632×10^{-5}	3.874	8.361×10^{-5}	3.9843	1.369×10^{-4}	3.868
1/16	4.099×10^{-6}	4.016	4.998×10^{-6}	4.064	8.695×10^{-6}	3.976
1/32	2.690×10^{-7}	3.929	3.163×10^{-7}	3.981	5.312×10^{-7}	4.032
1/64	1.443×10^{-8}	4.220	1.909×10^{-8}	4.050	3.083×10^{-8}	4.106

Table 3. The errors and the convergence orders of (32) with finite difference scheme (24), for different α and β, when $h = 0.02$.

	$\beta = 1.7, \alpha = 1.3$		$\beta = 1.5, \alpha = 1.5$		$\beta = 1.2, \alpha = 1.9$	
τ	$\varepsilon(\tau,h)$	Order	$\varepsilon(\tau,h)$	Order	$\varepsilon(\tau,h)$	Order
1/10	9.102×10^{-4}		5.453×10^{-4}		2.905×10^{-4}	
1/20	4.363×10^{-4}	1.060	2.620×10^{-4}	1.057	1.482×10^{-4}	0.971
1/40	2.169×10^{-4}	1.008	1.321×10^{-4}	0.987	8.044×10^{-5}	0.881
1/80	1.098×10^{-4}	0.981	6.947×10^{-5}	0.927	4.208×10^{-5}	0.934
1/160	5.686×10^{-5}	0.950	3.881×10^{-5}	0.839	2.052×10^{-5}	1.035



(a) Exact solution. (b) Numerical solution.

Figure 1. The exact and numerical solution (32) at $\alpha = 1.5$ and $\beta = 1.5$.

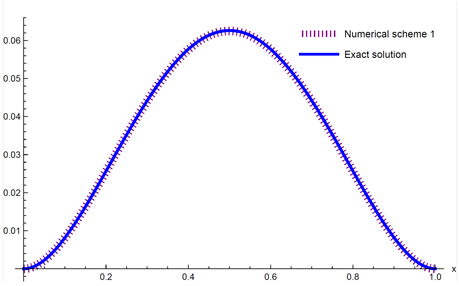


Figure 2. A comparison between the exact solution and the numerical solution of Equation (32) according to numerical scheme (10) for $\alpha = 1.5, \beta = 1.5$, and $t = \mathfrak{T} = 1$.

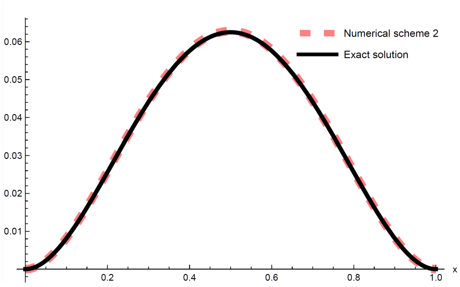


Figure 3. A comparison between the exact solution and the numerical solution of Equation (32) according to numerical scheme (24) for $\alpha = 1.5, \beta = 1.5$, and $t = \mathfrak{T} = 1$.

6. Conclusions

In this manuscript, we considered a special form of time–space FDEs with viscoelastic damping, associated with using two types of fractional derivatives operators Caputo and Riemann–Liouville in the temporal and spatial directions, respectively, by fractional order derivatives in interval $(1, 2)$. We used two linearized Crank–Nicolson finite difference schemes to deduce the numerical solution of the problem (1). The suggested Crank–Nicolson scheme was demonstrated to be unconditionally stable and with a convergence with a second order of accuracy in space for a noncompact weighted and shifted Grunwald difference approximation scheme and a fourth order for a compact weighted and shifted Grunwald difference approximation scheme. Moreover, there was a convergence in time with a second order of accuracy at a point $t = t_{n+\frac{1}{2}}$ and a first order of accuracy at a point $t = t_{n+1}$, which was in perfect agreement with the exact solution of (1). Both the numerical schemes and the theoretical analysis showed that the suggested methods were efficient for solving one-dimensional time and space FDEs.

Author Contributions: Conceptualization, E.I.M., A.M.E. and V.N.O.; formal analysis, A.M.E. and E.I.M.; funding acquisition, V.N.O.; methodology, A.M.E., E.I.M. and V.N.O.; data curation, A.M.E.; investigation, A.M.E., E.I.M. and V.N.O.; project administration, E.I.M. and V.N.O.; resources, A.M.E. and E.I.M.; software, A.M.E. and E.I.M.; supervision, A.M.E., E.I.M. and V.N.O.; writing—original

draft preparation, A.M.E. and E.I.M.; writing—review and editing, A.M.E., E.I.M. and V.N.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors express their gratitude to the Editorial Board and reviewers for their attention to our work and comments that allowed us to improve the quality of our work.

Conflicts of Interest: The authors declare no conflict of interest.

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