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Diffusion-Wave Type Solutions to the Second-Order Evolutionary Equation with Power Nonlinearities

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Abstract: The paper deals with a nonlinear second-order one-dimensional evolutionary equation related to applications and describes various diffusion, filtration, convection, and other processes. The particular cases of this equation are the well-known porous medium equation and its generalizations. We construct solutions that describe perturbations propagating over a zero background with a finite velocity. Such effects are known to be atypical for parabolic equations and appear as a consequence of the degeneration of the equation at the points where the desired function vanishes. Previously, we have constructed it, but here the case of power nonlinearity is considered. It allows for conducting a more detailed analysis. We prove a new theorem for the existence of solutions of this type in the class of piecewise analytical functions, which generalizes and specifies the earlier statements. We find and study exact solutions having the diffusion wave type, the construction of which is reduced to the second-order Cauchy problem for an ordinary differential equation (ODE) that inherits singularities from the original formulation. Statements that ensure the existence of global continuously differentiable solutions are proved for the Cauchy problems. The properties of the constructed solutions are studied by the methods of the qualitative theory of differential equations. Phase portraits are obtained, and quantitative estimates are determined by constructing and analyzing finite difference schemes. The most significant result is that we have shown that all the special cases for incomplete equations take place for the complete equation, and other configurations of diffusion waves do not arise.

Keywords: nonlinear partial differential equation; porous medium equation; diffusion wave; existence theorem; analytical solution; power series; majorant method; exact solution

MSC: 35K57

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1. Introduction

This article continues our study of one special class of solutions to a second-order nonlinear evolutionary equation [1]. We consider the equation having the following general form:

$$T_t = (\Phi_1(T))_{xx} + (\Phi_2(T))_x + \Phi_3(T). \tag{1}$$

Here t, x are independent variables: t is time, x is a spatial variable, $T(t,x)$ is an unknown function, and $\Phi_i,\ i = 1,2,3$ are the specified functions. From a physical point of view, the function Φ_1 describes diffusion processes (diffusion term), Φ_2 corresponds to convection processes (convection term), and Φ_3 is a source or a sink.

Equation (1) is parabolic if $\Phi_1'(T) \geq 0$. Solutions that hold the parabolic type of the equation are usually studied. However, for the completeness of the study, negative case can also be considered.

A detailed bibliography overview is given in our first article devoted to the problem considered [1]. Let us briefly recollect some essential points. First, we should mention classical monographs that significantly influenced developing the theory of nonlinear parabolic equations [2–4]. Second, we point out the articles in which, apparently, the authors presented diffusion wave-type solutions for the first time [5,6]. Let us especially note book [7], which presents the mathematical theory of the porous medium equation and thorough state-of-art.

Recall that the case where Φ_1 is a power function, and $\Phi_2 = \Phi_3 \equiv 0$ is the porous medium Equation [7]. It is rich in applications and describes the filtration of an ideal gas in porous formations [6], the radiant (nonlinear) thermal conductivity [5], as well as population dynamics processes [8].

If Φ_1 and Φ_3 are power functions, and $\Phi_2 \equiv 0$, then (1) becomes the generalized porous medium Equation [7] or the nonlinear heat equation with a source [9]. This equation describes the same processes as the porous medium equation, but allows us to consider the inflow or outflow of matter or heat.

Assuming $\Phi_3 \equiv 0$, and Φ_1 and Φ_2 are nonzero leads Equation (1) to the convection-diffusion Equation [10,11]. Several mathematical models of fluid mechanics, which simultaneously describe the diffusion and convective [12] mechanisms of energy and matter transfer, are reduced to such an equation. The phonon transport within silicon structures, which is subjected to internal heat generation, can also be explored [13,14]. In [15], the authors proposed the equation considered as a suitable governing equation for the gas flow through a Graphene Oxide membrane. A mathematical model describing the flow of a mixture of ideal gases in a highly porous electrode for fuel cell engineering is proposed in [16]. Its particular case is the well-known Burgers equation [7].

Finally, Equation (1), if $\Phi_2(c)$ is a linear function, which describes the non-stationary thermal conductivity in a medium moving at a constant speed, when the thermal conductivity coefficient and the reaction rate are arbitrary functions of the temperature [17].

Note that the problem is also being studied in the case of several spatial variables, and solutions of different types are constructed. In [18,19], the author considers the anisotropic case and construct weak solutions. In [20], the authors present weak supersolutions for different functional spaces. Analytical travelling waves for the nonlinear convection-diffusion equation are studied in [21], including the use of Lie symmetry [22]. Various models of a similar but more general form are used, for example, in the study of diffusion processes in metallurgy [23], as well as the thermal fields located in the permafrost area [24]. The list could be continued, so the study of Equation (1) is still relevant.

In this paper, we deal with the problem of constructing and studying diffusion-wave-type solutions in the case of power functions Φ_i. The existence and uniqueness theorem is proved. It, unlike the known ones, allows us to set the boundary condition at a moving point. In addition, exact solutions are found and investigated in detail in one particular case. Their construction is reduced to the integration of the Cauchy problem for an ordinary differential equation.

In contrast to similar solutions that we dealt with in [1], this study is more systematic. Firstly, here these Cauchy problems are investigated in a general formulation. Secondly, we do not limit ourselves to considering cases when equations can be integrated explicitly but perform their qualitative analysis and constructed phase portraits, which allowed us to investigate the behavior of solutions. We also construct finite difference schemes and prove their convergence, which, in particular, makes it possible to construct accurate estimates for the solutions obtained.

2. Problem Formulation

If the functions $\Phi_1(T), \Phi_2(T)$ are differentiable, Equation (1) can be written as:

$$T_t = (\Phi_1'(T)T_x)_x + \Phi_2'(T)T_x + \Phi_3(T). \tag{2}$$

We assume that $\Phi_i(T),\ i = 1, 2, 3$ are power functions:

$$\Phi_1'(T) = \lambda_1 T^{\sigma_1},\ \Phi_2'(T) = \lambda_2 T^{\sigma_2},\ \Phi_3(T) = \lambda_1 T^{\sigma_3},$$

where σ_i, $i = 1,2,3$ are positive constants, $\sigma_1 + \sigma_3 > 1$, λ_i, $i = 1,2,3$ are constants, and $\lambda_1 > 0$.

The substitution $u = \Phi_1'(T) = \lambda_1 T^{\sigma_1}$ and effortless transformations lead Equation (2) to the form:

$$u_t = uu_{xx} + \frac{1}{\sigma}u_x^2 + Au^\theta u_x + Bu^\beta. \tag{3}$$

Here $\sigma = \sigma_1 > 0$, $\theta = \sigma_1/\sigma_2 > 0$, $\beta = \sigma_3/\sigma_1 + 1 - 1/\sigma_1 > 0$, $A = \lambda_2\lambda_1^{-1/\sigma}$, $B = \lambda_3\lambda_1^{1/\sigma - 1/\sigma_3}$. Obviously, Equation (3) has the trivial solution $u \equiv 0$.

Let us set for Equation (3) the boundary conditions:

$$u(t,x)|_{x=a(t)} = f(t),\ f(0) = 0,\ f'(0) \geq 0. \tag{4}$$

Previously, the same conditions for the porous medium Equation [25] were considered. In this paper, we prove the solvability of problem (3) and (4) in the class of analytical functions. Moreover, we show that if there exists a sufficiently smooth solution to problem (3) and (4), then together with the trivial one, it forms a diffusion wave.

3. Main Theorem

Let us formulate and prove the existence and uniqueness theorem. Here and further, an analytical solution means a solution in the class of analytical functions, i.e., it coincides in a neighborhood with its Taylor expansion.

Recall that the diffusion wave-type solution means a piecewise smooth solution to Equation (1), consisting of a trivial $u \equiv 0$ part and a nontrivial $u = u(t,x) \geq 0$ one, continuously joined on some line in the plane of variables t, x. This line is called the wave front.

Theorem 1. *Let the functions $a(t)$ and $f(t)$ be analytical in some neighborhood of $t = 0$; $f'(0) \geq 0$; $[a'(0)]^2 + f'(0) > 0$; and let θ and β be natural (positive integer) numbers. Then problem (3) and (4) has a nonzero analytical solution of diffusion-wave type in some neighborhood of the point $(t = 0, x = 0)$, which is unique if the direction of the diffusion-wave front moving is chosen.*

Proof. Let us give a brief scheme of the further reasoning. First, we construct the solution in the form of a power series. Then we reduce problem (3) and (4), which is not a Cauchy–Kovalevskaya type, to the standard form by the consequence of non-degenerate substitutions. This standard form is subject to the Cauchy–Kovalevskaya theorem.

To simplify the boundary conditions, we make the substitution $t_1 = t, r = x - a(t)$. It is easy to show that the Jacobian of the substitution is nonzero. As a result, we get the problem:

$$u_t - a'(t)u_r = uu_{rr} + \frac{1}{\sigma}u_r^2 + Au^\theta u_r + Bu^\beta, \tag{5}$$

$$u(t,r)|_{r=0} = f(t). \tag{6}$$

Here and to further simplify the notation, the first independent variable retains t without index 1.

We construct the solution to problem (5) and (6) as the series:

$$u(t,r) = \sum_{k,m=0}^{\infty} u_{k,m}\frac{t^k r^m}{k!m!},\ u_{k,m} = \left.\frac{\partial^{k+m}u}{\partial t^k \partial r^m}\right|_{t=r=0}. \tag{7}$$

This method develops the method of special series, which was proposed and widely used in the scientific school of A.F. Sidorov [26,27].

Since the construction essentially coincides [28] (see also [25]), we try to avoid repetitions, focusing on new points in the proof and emerging difficulties.

Since the functions $a(t), f(t)$ are analytical, they allow the expansions:

$$f(t)=\sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \quad a(t)=\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}.$$

Boundary condition (6) implies the equalities $u_{n,0}=f_n$, and $f_0=0$. The remaining coefficients of (7) are determined by recursive induction on the total order of differentiation $n=k+m$.

First, we establish the induction base by considering the case $k+m=1$. As it has been shown, $u_{1,0}=f_1$. Assume, that $t=r=0$ in (5). Then it is possible to consider the equation obtained as quadratic with respect to $u_{0,1}$ and find its roots:

$$u^{\pm}_{0,1}=\frac{\sigma}{2}\left(-a_1 \pm \sqrt{a_1^2+4f_1}\right).$$

Since $f'(0)\geq 0$, $[a'(0)]^2+f'(0)>0$, both roots are real.

The direction of the diffusion wave moving depends on the choice of the sign of $u_{0,1}$. The value $u^{-}_{0,1}$ corresponds to a diffusion wave whose front lies to the right of the line $x=a(t)$ in the plane of variables t, x. A diffusion wave whose front is located to the left of the line $x=a(t)$ corresponds to $u^{+}_{0,1}$. These cases can be considered separately, or one can be chosen based on additional reasons. Looking ahead, we note that the procedure for constructing a solution is similar in both cases.

If the sign is chosen, then the series (7) is constructed uniquely.

We differentiate (5) k times with respect to r, $n-k$ times by t, and set $t=r=0$. After collecting terms, we arrive at the equality:

$$b_{n-k}u_{n-k-1,k+2}+c_k u_{n-k,k+1}+u_{n-k+1,k}=R_{n-k,k}, \tag{8}$$

where:

$$b_{n-k}=-(n-k)f_1,\ c_k=-\left(k+\frac{2}{\sigma}\right)u_{0,1}-a_1.$$

We do not show here the explicit form of $R_{n-k,k}$ since it is cumbersome. Their form for the particular case $A=B=0$ can be found in [25], where it is presented since the convergence proof technique used there requires direct estimates. Here, we use another technique for constructing the majorant problem based on the hodograph transformation. In this regard, it is enough to point out that $R_{n-k,k}$ depend on the derivatives of the unknown function of order at most n, which are known by the induction hypothesis. The condition $\theta, \beta \in \mathbb{N}$ ensures infinite differentiability of Equation (5).

Changing in (8) k from 0 to n and taking into account that $u_{n+1,0}=f_n$, and $b_0=0$, we obtain the following system of linear algebraic equations:

$$\begin{pmatrix} c_0 & b_n & 0 & \dots & 0 & 0 & 0 \\ 1 & c_1 & b_{n-1} & \dots & 0 & 0 & 0 \\ & \dots & & & & \dots & \\ 0 & 0 & 0 & \dots & 1 & c_{n-1} & b_1 \\ 0 & 0 & 0 & \dots & 0 & 1 & c_n \end{pmatrix} \times \begin{pmatrix} u_{n,1} \\ u_{n-1,2} \\ \dots \\ u_{1,n} \\ u_{0,n+1} \end{pmatrix} = \begin{pmatrix} R^*_{n,0} \\ R_{n-1,1} \\ \dots \\ R_{1,n-1} \\ R_{0,n} \end{pmatrix}. \tag{9}$$

Here $R^*_{n,0}=R_{n,0}-f_{n+1}$. You can see that the matrix A_n of system (9) is tridiagonal of order $n+1$, and the condition of diagonal dominance is not satisfied. Let us prove that its determinant is nonzero.

Indeed, it is necessary to consider three cases: (1) $f_1=0$; (2) $f_1>0$, $u_{0,1}=u^{+}_{0,1}$; (3) $f_1>0$, $u_{0,1}=u^{-}_{0,1}$.

1. Let $f_1=0$. Then $b_k=0$ for all k, i.e., the matrix A_n is triangular two-diagonal and its determinant is equal to the product of the elements of the main diagonal:

$$\det A_n=\prod_{i=0}^{n} c_i.$$

Two subcases are possible here: (a) $u_{0,1} = u_{0,1}^{+} = 0$, then $c_k = -a_1$ for all k; (b) $u_{0,1} = u_{0,1}^{-} = -a_1\sigma$, then $c_k = (k\sigma + 1)a_1$. For the both subcases $\det A_n \neq 0$ since $a_1 \neq 0$.

2. Let $f_1 > 0$, $u_{0,1} = u_{0,1}^{+}$. Then $b_k < 0$ for $k \geq 1$ and $c_k > 0$ for all k, i.e., all elements on the main diagonal and subdiagonal are positive, and all elements of the superdiagonal are negative. Hence, all the principal minors of the matrix A_n are positive, which means its non-degeneracy.

3. Let $f_1 > 0$, $u_{0,1} = u_{0,1}^{-}$. Then $b_k < 0$ for $k \geq 1$ and $c_k < 0$ for all k. Let us introduce an auxiliary numeric sequence $\Delta_{n,k}$ as follows:

$$\Delta_{n,0} = 1,\ \Delta_{n,1} = c_0 < 0,\ \Delta_{n,k} = c_{k-1}\Delta_{n,k-1} - b_{n-k+2}\Delta_{n,k-2},\ k = 2, 3, \ldots, n.$$

It can be shown by induction on k that $\Delta_{n,k}$ consists of two positive terms for even k and two negative ones for odd k. Hence we have that $\Delta_{n,k} \neq 0$ for all admissible n and k. On the other hand, it is easy to show that $\Delta_{n,n} = \det A_n$ by induction on n. Thus, $\det A_n \neq 0$, moreover, $\det A_n > 0$ for even n and $\det A_n < 0$ for odd n.

Thus, we have proved that system (9) is non-degenerate, and the coefficients of series (7) are uniquely determined if one of the two possible values of $u_{0,1}^{\pm}$ is chosen. This finishes the first step of the proof.

We refuse the direct estimates applied in [25] in the proof of convergence. Here we use an alternative methodology, which reduces the problem to a special form previously considered in [1,28].

Since $u_{0,0} = 0$, $u_{0,1}^2 + u_{1,0}^2 \neq 0$, then if series (7) converges, there exists a line $r = g(t)$ in the plane t, r, on which the unknown function vanishes:

$$u|_{r=g(t)} = 0,\ g(0) = 0.$$

In problem (5) and (6), which is equivalent to the original one, let us make the substitution $t_2 = t, s = r - g(t)$. We arrive at the problem that consists of one equation and two boundary conditions:

$$u_t - [a'(t) + g'(t)]u_s = uu_{ss} + \frac{1}{\sigma}u_s^2 + Au^\theta u_s + Bu^\beta, \tag{10}$$

$$u(t,s)|_{s=-g(t)} = f(t),\ u(t,s)|_{s=0} = 0. \tag{11}$$

To simplify the notation, the first independent variable retains t without index 2.

The function $g(t)$ is still unknown, and it will be determined simultaneously with the construction of the function u. Thus, we obtain one of the problems with a free boundary. The most famous of them for parabolic equations is the Stefan problem [29,30].

The following substitution changes the roles of the unknown function u and the independent variable s, i.e., it is a variant of the hodograph transformation. Equation (10) becomes:

$$us_{uu} = Bu^\beta s_u^3 + [s_t + a'(t) + g'(t) + Au^\theta]s_u^2 + \frac{1}{\sigma}s_u. \tag{12}$$

Conditions (11) take the form:

$$s(t,u)|_{u=f(t)} = -g(t),\ s(t,u)|_{u=0} = 0. \tag{13}$$

Let us differentiate the first condition of (13) and substitute the resulting expression $[s_t + s_u f'(t)]|_{u=f(t)} = -g'(t)$ into Equation (12). We obtain that:

$$us_{uu} = Bu^\beta s_u^3 + \Big\{s_t + a'(t) - [s_t + s_u f'(t)]|_{u=f(t)} + Au^\theta\Big\}s_u^2 + \frac{1}{\sigma}s_u. \tag{14}$$

The positive trait of Equation (14) is that it no longer contains the unknown function $g(t)$. The boundary condition for (14) takes the form:

$$s(t,u)|_{u=0} = 0. \tag{15}$$

Having constructed a solution to problem (14) and (15) which, recall, does not contain the function $g(t)$, we can find $g(t)$ from the first condition of (13). Thus, we have decomposed problem (10) and (11), which includes two unknown functions into two separate tasks. They contain one unknown function and can be solved sequentially.

As a result of the substitutions performed, we have obtained the problem with the known diffusion front, which was previously considered in [1]. As already noted, the detailed proof of the similar theorem for the porous medium equation with two spatial variables is given in [28]. In this regard, we will be brief so as to not overload the paper.

Completing the series of substitutions, let us introduce the variable $y = u - f$, which allows us to make the surface $u = f$ as a new coordinate plane. Next, the unknown function is represented as $s(u,y) = us_1(y) + u^2 Z(u,y)$, where s_1 is the known analytical function, and $Z = Z(u,y)$ is a new unknown function. Note that in this case, the second boundary condition of (13) is satisfied automatically, and the problem is reduced to one equation of the form:

$$\begin{aligned} &\Psi_0(y)Z|_{y=0} + \Psi_1(y)u(Z_u|_{y=0}) + \Psi_2(y)u^2(Z_{uu}|_{y=0}) \\ &+B_0 Z + B_1 u S_u + u^2 Z_{uu} = h_0 + uh_1 + u^2 h_2 + u^3 h_3. \end{aligned} \tag{16}$$

Here $B_0 = 2(1 + 1/\sigma)$, $B_1 = (4 + 1/\sigma)$ are constants; Ψ_i, $i = 0,1,2$ and h_j, $j = 0,1,2,3$ are analytical functions of their variables. Moreover, $h_0 = h_0(u,y)$, and the remaining h_j depend on independent variables and derivatives of the function Z with respect to u of order at most $j - 1$. The functions $\Psi_i(y)$ are positive for $y = 0$. Thus, Equation (16) obeys Lemma 2 from [28]. Therefore, it is solvable in the class of analytical functions. The construction of the majorant problem and the proof of the existence of its analytical solution are also carried out similarly. □

Remark 1. *We have constructed an analytical solution to problem* (3) *and* (4) *and simultaneously determined the line $x = a(t) + g(t)$, which is the diffusion wave front. The non-negative part of the specified solution $u = u^+$ and the trivial solution $u \equiv 0$, which are joined on the diffusion front, give the required diffusion wave.*

Remark 2. *A particular case of problem* (3) *and* (4)*, when $f(t) \equiv 0$, is a problem about the moving of a diffusion wave with a given diffusion front, which obeys the theorem proved in* [1].

4. Exact Solutions

Theorem 1 ensures the existence and uniqueness of the solution to the problem of diffusion wave initiating and, remarkably, gives an algorithm for its construction in the form of a double series. Unfortunately, it is local, and, as attempts to use such constructions for a numerical modeling show [31], the radius of convergence of the series is usually small. Thus, the theorem does not allow us to study the global properties of diffusion waves. Besides, the requirement that the parameters β and θ in Equation (3) are natural numbers significantly limits the generality. In general, these problems are far from being solved, as well as for most other nonlinear degenerate partial differential equations. Therefore, we investigate the properties of diffusion-wave type solutions to Equation (3) for an arbitrary $\beta > 0$ and $\theta > 0$ in the particular case. Exact solutions of parabolic equations are widely used in solving applied problems: From modeling the well clogging process [32] to the description of bubble dynamics [33].

4.1. Reduction to Ordinary Differential Equations (ODEs)

Consider for Equation (3) the boundary condition:

$$u(t,x)|_{x=a(t)} = 0, \tag{17}$$

which, as already noted, is a particular case of (4) when $f \equiv 0$. Problem (3) and (17) is the problem of the diffusion wave moving with a given front.

Note that problem (3) and (17) has the trivial solution $u \equiv 0$. However, in this case, the uniqueness of the solution is violated, and a nonzero solution can also exist. Its existence in the class of analytical functions follows from Theorem 1.

Current and further sections are devoted to finding and studying non-trivial exact solutions to problem (3) and (17), the constructing of which is reduced to the integration of Cauchy problems for ODEs. Previously, we studied this problem in detail for the nonlinear heat equation [34] and for the nonlinear heat equation with a source [35] and found new classes of diffusion-wave type solutions. Those problems corresponded to the case $A = 0$. Here let us consider the case when $A \neq 0$.

Following [1], we look for solutions to Equation (3) having the form:

$$u = \psi(t)v(x - a(t)). \tag{18}$$

Solution (18) is a generalized traveling wave, which becomes a simple traveling wave if $a(t)$ is a linear function. Substituting (18) into Equation (3), we obtain:

$$vv'' + \frac{1}{\sigma}(v')^2 + A\psi^{\theta-1}(t)v^\theta v' + B\psi^{\beta-2}(t)v^\beta + \frac{a'(t)}{\psi(t)}v' - \frac{\psi'(t)}{\psi^2(t)}v = 0. \tag{19}$$

In order for (19) to become an ODE with respect to $v(z)$, $z = x - a(t)$, it is necessary and sufficient to satisfy the conditions:

$$\frac{a'(t)}{\psi(t)} = \text{const}, \ \frac{\psi'(t)}{\psi^2(t)} = \text{const}, \ \psi^{\theta-1}(t) = \text{const}, \ \psi^{\beta-2}(t) = \text{const}. \tag{20}$$

Here the first two conditions form a first-order ODE system. The third and fourth conditions are additional compatibility conditions that can be satisfied, for example, by choosing θ and β.

Let us consider two possible cases.

1. Let $\psi(t) = \psi = \text{const}$. Without losing the generality of consideration, we can set $\psi = 1$. Then $a(t) = \mu t + \eta$, where μ, η are constants, and (19) takes the form of the following ODE:

$$vv'' + \frac{1}{\sigma}(v')^2 + (Av^\theta + \mu)v' + Bv^\beta = 0. \tag{21}$$

We assume that in this case $\eta = 0, \mu > 0$, which also does not reduce the generality of consideration.

2. Let now $\psi(t) \neq \text{const}$. Then from the first two equations of (20) we have that $\psi(t) = \omega/(\mu t + \eta)$, $a(t) = \omega \ln(\mu t + \eta)$, where μ, η, and ω are nonzero constants, $\eta > 0$. You can see that the necessary and sufficient conditions to satisfy the third and fourth equalities of (20) are $\theta = 1$, $\beta = 2$. Then (19) takes the form of the following ODE:

$$vv'' + \frac{1}{\sigma}(v')^2 + (Av + \mu)v' + Bv^2 + \frac{\mu}{\omega}v = 0. \tag{22}$$

We can bring (21) and (22) to the general form:

$$vv'' + \frac{1}{\sigma}(v')^2 + (Av^\theta + \mu)v' + Bv^\beta + Cv = 0. \tag{23}$$

4.2. Cauchy Conditions for ODEs

It is easy to see that condition (17) becomes $v(0) = 0$ for a solution having the form (18). Then, obviously, Equation (23) has the trivial solution $v \equiv 0$. Assuming $v = 0$ in (23), we obtain the quadratic equation with respect to $v_1 = v'(0)$:

$$\frac{1}{\sigma}v_1^2 + \mu v_1 = 0, \tag{24}$$

which has roots $v_1 = 0$ and $v_1 = -\mu\sigma$. Accordingly, we will consider Equation (23) with the Cauchy conditions of two types:

$$v(0) = 0,\ v'(0) = 0; \tag{25}$$

$$v(0) = 0,\ v'(0) = -\mu\sigma. \tag{26}$$

The trivial solution corresponds to conditions (25). However, as it is shown below, there may also exist a non-trivial solution that is not analytical, i.e., it cannot be represented as a Taylor series.

Theorem 1 implies that problem (23) and (26) for positive integer values of θ and β has the unique analytic solution in the form of a convergent series in powers of z. Unfortunately, the theorem does not hold for non-integer values of these constants.

Note that Equation (23), although it is an ODE, stays still complex to study. First, it is nonlinear. Second, the Cauchy problems inherit singularities from the original statements, which does not allow for using standard methods and theorems of ODE theory. Thus, the general case is quite complex and cannot be explored within the framework of a single article. Therefore, we consider here one of the particular cases. On the one hand, this case is significant and has interesting properties. On the other hand, it gives a clear idea of the difficulties encountered in studying the properties of the obtained classes of exact solutions and what techniques can be applied to overcome them.

5. Traveling Wave. Qualitative Analysis

In this section, we consider the exact solutions having the form of traveling waves, which, as shown above, are described by Equation (21) with Cauchy conditions (25) or (26). We study them using the methods of ODE theory, including qualitative analysis with the construction of a phase portrait and some quantitative estimates.

5.1. Transition to Phase Variables

Using the fact that the equation does not explicitly depend on z, we proceed to the phase plane. Let us introduce a new independent variable w and an unknown function p:

$$w = v^\theta, p = v'. \tag{27}$$

The substitution is non-degenerate if $\theta \geq 1$. Equation (21) takes the form:

$$\theta w p \frac{dp}{dw} + \frac{p^2}{\sigma} + Awp + \mu p + Bw^{\beta/\theta} = 0. \tag{28}$$

Let $\theta = \beta \geq 1$, i.e., in the third and fourth terms of Equation (21) v has the same degree. Due to the linear change of variables, we can reduce the number of constants. Let,

$$w = \tilde{w}\tilde{A}, \tilde{A} = \frac{\mu}{A};\ p = \tilde{p}\tilde{B}, \tilde{B} = \frac{\mu}{\beta}.$$

Then Equation (28) takes the form ($\sim$ is omitted for simplicity):

$$wp\frac{dp}{dw} + \frac{p^2}{\gamma} + wp + p + \alpha w = 0, \tag{29}$$

where $\gamma = \sigma\theta > 0$, $\alpha = B/\mu > 0$. Note that Equation (29) is similar to (39) from [1], however, the appearance of the term pw in (29) significantly complicates the study.

For (29), let us consider solutions corresponding to the initial condition given at $w = 0$. Since $\theta > 0$, nontrivial solutions of this kind generate solutions to the original problem having the diffusion-wave type. Looking ahead, we note that some of them may not have physical meaning.

If we substitute $w = 0$ into Equation (29), we obtain the algebraic relation $p^2(0)/\gamma + p(0) = 0$, which is an analogue of equality (24). You can make sure that it has roots (1) $p(0) = 0$ and (2) $p(0) = -\gamma$, which correspond to conditions (26) and (25), respectively. Now let us consider Equation (29) with Cauchy conditions (1) and (2) in more detail.

5.2. Singular Points

First, we study the singular points of Equation (29). Since it is autonomous, let us turn to the phase plane $(v, v' = w)$. We use the classic technique proposed in [36] (see, also [34]). The following dynamic system corresponds to Equation (29):

$$\frac{dw}{d\zeta} = wp, \quad \frac{dp}{d\zeta} = -\frac{p^2}{\gamma} - p - pw - \alpha w, \tag{30}$$

where $dz = w\,d\zeta$.

Consider now the equilibrium states (singular points) of system (30). There are two equilibrium states $(0, -\gamma)$ and $(0, 0)$.

Let us introduce the following notation:

$$R(w,p) = wp, \quad Q(w,p) = -p^2/\gamma - p - pw - \alpha w,$$

$$M(v,w) = \begin{pmatrix} R_w & R_p \\ Q_w & Q_p \end{pmatrix} = \begin{pmatrix} p & w \\ -p-\alpha & -2p/\gamma - w - 1 \end{pmatrix},$$

$$\Delta(w,p) = \det M(w,p) = -\frac{2p^2}{\gamma} - p + \alpha w,$$

$$\delta(w,p) = \operatorname{Tr} M(w,p) = \frac{(\gamma-2)}{\gamma}p - w - 1.$$

Let us define the type of each singular point.

1. Consider the point $(0, -\gamma)$. Since $\Delta(0, -\gamma) = -\gamma \neq 0$, it is a simple equilibrium point. From $\det(M - \lambda E)|_{w=0, p=-\gamma} = (\lambda + \gamma)(\lambda - 1)$, it follows that $\lambda_1 = -\gamma$ and $\lambda_2 = 1$ are the roots of the characteristic equation. Therefore, the point $(0, -\gamma)$ is the topological saddle since $\Delta < 0$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1\lambda_2 < 0$.

2. Consider the point $(0, 0)$. Since $\Delta(0, 0) = 0$, this is a compound equilibrium point. Here $\delta(0, 0) = -1 \neq 0$, and the equation that is obtained from system (30) by the elimination of the independent variable ζ can be written as:

$$wpdw - [lp - p^2/\gamma - wp - \alpha w]\,dw = 0,$$

where $l = -1$. We represent the solution to the equation:

$$-lp + p^2/\gamma + pw + \alpha w = 0$$

as a series in powers of w, which we substitute into pw. As a result, we have:

$$p = \phi(w) = -\alpha w + \ldots, \quad \xi(w) = (wp)|_{p=\phi(w)} = -\alpha w^2 + \ldots$$

Since the lowest power of w in the expansion $\xi(w)$ equals two, the point $(0, 0)$ is a saddle-node with one nodal and two saddle sectors. The nodal sector is stable because $l < 0$. Moreover, if $\alpha < 0$, then the trajectories of the nodal sector tend to $(0, 0)$ when $\zeta \to -\infty$ on the left of the Op axis. If $\alpha > 0$, as in the considered case, the trajectories of the nodal sector tend to $(0, 0)$ when $\zeta \to +\infty$ on the right of the Op axis.

5.3. Phase Portrait

Let us construct and explore the phase portrait of system (30) for $\gamma, \alpha > 0$. Note that in all the considered cases:

1. The phase trajectories change the direction of motion when passing through the Ow axis, as well as when crossing the quadric $p^2/\gamma + p + pw + \alpha w = 0$, which, in particular, singular points belong;
2. Both singular points have vertical semi-separatrices, since they are located on the Op axis.

Let us first determine the properties of the second-order curve:

$$p^2/\gamma + p + pw + \alpha w = 0.$$

Bringing it to its canonical form, we obtain:

$$\left(p + \frac{\gamma w}{2} + \frac{\gamma}{2}\right)^2 - \frac{\gamma^2}{4}\left(w + 1 - \frac{2\alpha}{\gamma}\right)^2 = \alpha(\gamma - \alpha). \quad (31)$$

It is easy to see that for $\alpha = \gamma$, we have a pair of intersecting straight lines $p_1(w) = -\gamma w - \gamma + \alpha$, $p_2(w) \equiv -\alpha$. If $\alpha \neq \gamma$, then we obtain hyperbolas with the same asymptotes $p = p_1(w)$ and $p = p_2(w)$ and different positions of the branches depending on the sign of the difference $\gamma - \alpha$.

Let us consider all possible cases. Note here that in all cases, there are three semi-separatrices. The first is a monotonically decreasing curve coming to the singular point $(0, 0)$ and located in the second quadrant (bold curve S_1 in Figure 1–3). The second and third are vertical semi-separatrices lying on the Op axis.

Case $\gamma = \alpha$. Figure 1 shows the phase portrait of system (30) for this case. As already noted, the quadric (31) degenerates into two intersecting lines (dashed and green lines). Besides the separatrices mentioned above, there is also a separatrix that coincides with the line $p = -\gamma$ (green line). The nodal sector is bounded by the Op axis and the straight line $p = -\gamma$.

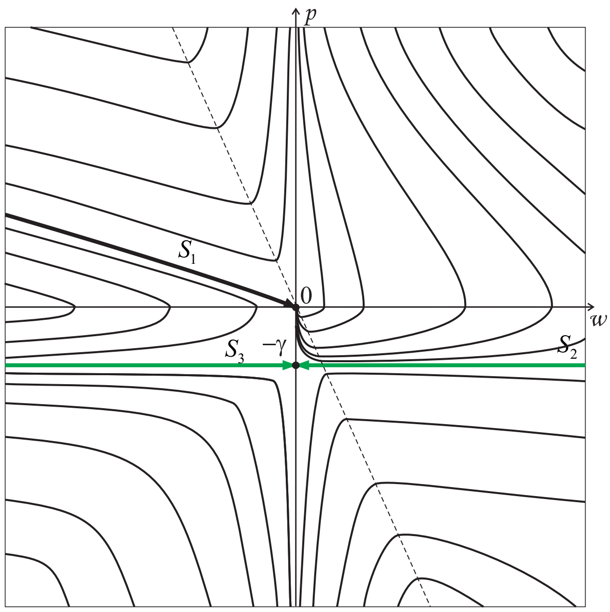

Figure 1. Phase portrait for $\gamma = \alpha$.

Case $\gamma > \alpha$. Figure 2 shows the phase portrait of system (30). You can see that half-hyperbolas (31) are located in the right upper and left lower quarters, into which the lines $p_1(w) = -\gamma w - \gamma + \alpha$, $p_2(w) \equiv -\alpha$ divide the coordinate plane (dashed curves). Here we have two additional separatrices S_2 and S_3 coming into the point $(0, -\gamma)$ (purple curves).

Both S_2 and S_3 are monotonically decreasing functions; S_2 tends to $-\infty$ when $\zeta \to +\infty$; S_3 tends to $p = -\alpha$ when $\zeta \to -\infty$. The nodal sector is bounded by the Op axis and the semi-separatrix S_2 located in the fourth quadrant.

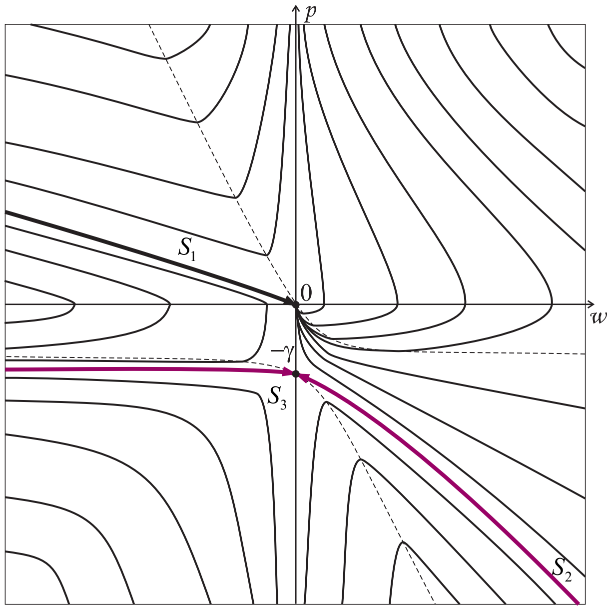

Figure 2. Phase portrait for $\gamma > \alpha$.

Case $\gamma < \alpha$. Here half-hyperbolas (31) are located in the left upper and right lower quarters, into which the lines $p_1(w) = -\gamma w - \gamma + \alpha$, $p_2(w) \equiv -\alpha$ divide the coordinate plane (see Figure 3). Again, in addition to the same separatrices for all cases, we have two semi-separatrices going out the point $(0, -\gamma)$ (blue curves). The separatrix S_2 first increases to the intersection with the Ow axis, then decreases and asymptotically tends to the Op axis when $\zeta \to +\infty$, bounding the nodal sector. The separatrix S_3 is a monotonically increasing function and tends to the line $p = -\alpha$ when $\zeta \to -\infty$.

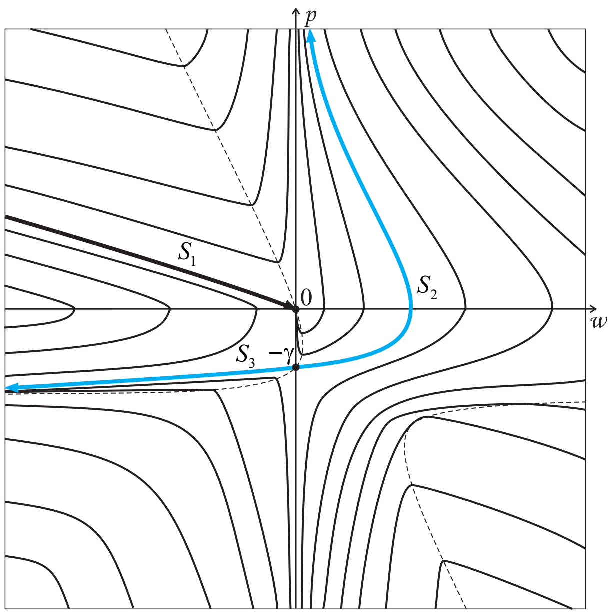

Figure 3. Phase portrait for $\gamma < \alpha$.

The properties of separatrices that do not coincide with the Op axis and the interpretation of the results from the original problem point of view will be discussed below.

6. Zero Initial Condition

Let us first consider the case when the initial condition for Equation (29) has the form:

$$p(0) = 0. \tag{32}$$

Previously, this case has not been considered. The only exception is paper [34], where we showed the existence of a semi-separatrix lying in the second quadrant and passing through the origin for the porous medium equation. However, the properties of the corresponding solution were not studied.

Obviously, in this case, the classical existence theorems are inapplicable due to degeneracy. Therefore, we attempt to eliminate the singularity.

6.1. Solution in the Form of a Series

Following [35], we try to construct an analytical solution to problem (29) and (32) as the series:

$$p(w) = \sum_{k=0}^{\infty} \frac{p_k}{k!} z^k, \; p_k = p^{(k)}(0). \tag{33}$$

Let us construct the coefficients for (33) using the following recurrent procedure. From (32) we have $p_0 = p(0) = 0$. To find p_1, we differentiate Equation (29) with respect to w, set $w = 0, p(0) = 0$, and obtain that $p_1 = p'(0) = -\alpha < 0$. Similarly, we get $p_2 = 2\alpha(1 - \alpha - \alpha/\gamma)$. Thus, the induction base is found.

Assume that $p_0, p_1, \ldots, p_{k-1}, k \geq 3$ are determined. To find p_k, we differentiate Equation (29) k times with respect to w and set $w = 0$. Then we arrive at the equality:

$$k\sum_{i=0}^{k-1} C_{k-1}^{i} p_i p_{k-i} + \frac{1}{\gamma}\sum_{i=0}^{k} C_k^i p_i p_{k-i} + kp_{k-1} + p_k = 0, \tag{34}$$

where $C_k^i = k!/[i!(k-i)!], k \geq i$. Resolving (34) with respect to p_k and taking into account $p_0 = 0, p_1 = -\alpha$, we have that:

$$p_k = k\left(\alpha k + \frac{2\alpha}{\gamma} - 1\right)p_{k-1} - k\sum_{i=2}^{k-2} C_{k-1}^{i} p_i p_{k-i} - \frac{1}{\gamma}\sum_{i=2}^{k-2} C_k^i p_i p_{k-i}. \tag{35}$$

You can see that all terms on the right-hand side of (35) are known by the induction hypothesis. Thus, all the coefficients of series (33) are uniquely determined by the formul obtained.

Now we study the properties of the constructed series. To do this, consider:

$$p_2 = 2\alpha\left[1 - \frac{\alpha(1+\gamma)}{\gamma}\right].$$

If $\alpha = \gamma/(1+\gamma)$ we have $p_2 = 0$. Then, it is easy to show by induction on k that $p_3 = p_4 = \ldots = p_k = \ldots = 0$. This means that the series breaks off, and the solution has the form $p = -\alpha w$.

If $\alpha > \gamma/(1+\gamma)$, then $p_2 < 0$, and we can make sure that $p_k < 0, k = 3, \ldots$ Therefore, from (35) we have that:

$$p_k < k\left(\alpha k + \frac{2\alpha}{\gamma} - 1\right)p_{k-1} < 0.$$

Hence we get that:

$$\lim_{k\to\infty} \frac{|p_k|(k-1)!}{|p_{k-1}|k!} > \lim_{k\to\infty} \frac{k(k-1)!}{k!}\left(\alpha k + \frac{2\alpha}{\gamma} - 1\right) = +\infty.$$

Thus, we have proved the divergence of series (33) and the validity of the following proposition.

Proposition 1. *Problem* (29) *and* (32) *has:*

1. *The analytical solution $p = -\alpha w$, if $\alpha = \gamma/(1+\gamma)$;*
2. *The solution having the form of a formal power series that converges only at the point $w = 0$, if $\alpha > \gamma/(1+\gamma)$.*

Note that for $0 < \alpha < \gamma/(1+\gamma)$ the terms in formula (35) have, generally speaking, different signs, and the question of the convergence or divergence of series (33) is much more challenging. Nevertheless, the results of numerical calculations allow us to make a reasonable assumption that the series diverges.

Remark 3. *There are also simpler examples based on a similar idea. Indeed, consider the Cauchy problem:*

$$xyy' - y + x + 1 = 0, y(0) = 1.$$

We can easily make sure that in this case $y'(0) = 1, y''(0) = 2$, and $y^{(k)}(0) \geq k^2 y^{(k-1)}(0) > 0$, $k \geq 2$, which means the divergence of the Maclaurin series for the function $y(x)$ at $x \neq 0$.

6.2. Euler Polygonal Approximations

As you know, the absence of an analytical solution to the Cauchy problem does not mean that it is impossible to construct a smooth (classical) solution. The simplest example is the problem $y' = \sqrt{x}, y(0) = 0$, which has a unique continuously differentiable solution for $x \geq 0$. In this section, we show that problem (29) and (32) has a similar property for $w \leq 0$, especially since the results of qualitative analysis evidence the existence of such a solution.

We use the classical Euler method. Therefore, it is necessary to construct a finite difference approximation of Equation (29). Calculations have shown that explicit difference schemes, in this case, turn out to be unstable. Therefore, we consider an implicit one, which at an arbitrary point $w_k, k \geq 1$ has the form:

$$w_k p_k \frac{p_k - p_{k-1}}{w_k - w_{k-1}} + \frac{1}{\gamma} p_k^2 + p_k + w_k p_k + \alpha w_k = 0. \tag{36}$$

From Cauchy condition (32) we have that $p_0 = p(0) = 0$. For convenience, we use a finite difference approximation with a constant step h, i.e., $w_k = kh$. Then (36) takes the form

$$\left(k + \frac{1}{\gamma}\right) p_k^2 + (1 + kh - kp_{k-1}) p_k + \alpha kh = 0. \tag{37}$$

The roots of Equation (37) are:

$$p_k = \frac{-1 - kh + kp_{k-1}}{2(k + 1/\gamma)} \pm \sqrt{\frac{(-1 - kh + kp_{k-1})^2}{4(k + 1/\gamma)^2} - \frac{\alpha kh}{k + 1/\gamma}}.$$

We choose the root that corresponds to + sign, otherwise $p_1 \to -\gamma/(1+\gamma)$ if $h \to 0$, i.e., there is a discontinuity of the first kind at zero. So, we have the recurrent sequence:

$$p_0 = 0,\ p_k = \frac{-1 - kh + kp_{k-1}}{2(k + 1/\gamma)} + \sqrt{\frac{(-1 - kh + kp_{k-1})^2}{4(k + 1/\gamma)^2} - \frac{\alpha kh}{k + 1/\gamma}}, k = 1, 2, \ldots \tag{38}$$

For $h > 0$, the radical expression in (38) rapidly becomes negative as k increases, i.e., the scheme is not applicable in this case. This fact goes with the results of the qualitative analysis, which showed that if $\alpha \neq \gamma/(\gamma + 1)$ problem (29) and (32) for $w > 0$ does not have a solution.

We investigate the properties of the sequence p_k for $w < 0$, i.e., when $h < 0$. To do this, we formulate the following auxiliary lemma.

Lemma 1. *Let $x > y$, $A \geq B > 0$. Then $x + \sqrt{x^2 + A} - y - \sqrt{y^2 + B} > 0$.*

Proof. If $y \geq 0$, then the lemma is obvious. Let $-y > 0$, $y^2 > x^2$. Since $A \geq B$, the inequality holds:

$$x + \sqrt{x^2 + A} - y - \sqrt{y^2 + B} \geq x + \sqrt{x^2 + A} - y - \sqrt{y^2 + A}.$$

To prove the Lemma, it is enough to show that the right-hand side is greater than zero. We use the rule of contraries. Let $x + \sqrt{x^2 + A} - y - \sqrt{y^2 + A} < 0$, then $x - y < \sqrt{y^2 + A} - \sqrt{x^2 + A}$. Since $x > y$, we can square this inequality and collect the terms:

$$xy + A > \sqrt{(y^2 + A)(x^2 + A)}.$$

If we square this inequality one more time and collect the terms, we obtain the inequality:

$$2xy > y^2 + x^2,$$

which is wrong. The contradiction proves the Lemma. □

Now we formulate and prove the lemma about the properties of the sequence p_k.

Lemma 2. *Let $h < 0$, $\alpha > 0$, $\gamma > 0$. Then the sequence p_k is monotonically increasing with respect to k, and the estimate holds:*

$$p_k \geq -kh \min\left\{\alpha, \frac{\gamma}{\gamma + 1}\right\}, \ k = 0, 1, 2 \ldots \tag{39}$$

Proof. We carry out the proof by induction on k. Assume for certainty that $\alpha \leq \gamma/(\gamma + 1) = 1/(1 + 1/\gamma)$. Then,

$$p_1 - p_0 = p_1 \geq -\frac{(h+1)\alpha}{2} + \sqrt{\frac{(h+1)^2\alpha^2}{4} - h\alpha^2} = -\frac{(h+1)\alpha}{2} + \sqrt{\frac{(1-h)^2\alpha^2}{4}} = -\alpha h,$$

and the induction base is determined.

Let $0 = p_0 < p_1 < \ldots < p_k$. Consider the difference $p_{k+1} - p_k$. Using (38), we can rewrite it as:

$$p_{k+1} - p_k = \frac{-1 - (k+1)h + (k+1)p_k}{2(k+1+1/\gamma)} - \frac{-1 - kh + kp_{k-1}}{2(k+1/\gamma)} +$$

$$\sqrt{\frac{(-1-(k+1)h+(k+1)p_k)^2}{4(k+1+1/\gamma)^2} - \frac{\alpha(k+1)h}{k+1+1/\gamma}} - \sqrt{\frac{(-1-kh+kp_{k-1})^2}{4(k+1/\gamma)^2} - \frac{\alpha kh}{k+1/\gamma}}.$$

Consider the first difference on the right side:

$$\frac{-1 - (k+1)h + (k+1)p_k}{2(k+1+1/\gamma)} - \frac{-1 - kh + kp_{k-1}}{2(k+1/\gamma)} =$$

$$\frac{(k+1)(k+1/\gamma)(p_k - p_{k-1}) + p_{k-1}/\gamma + 1 - h/\gamma}{2(k+1+1/\gamma)(k+1/\gamma)} > 0.$$

It is valid since all terms and factors are positive both in the numerator and in the denominator by the Lemma condition and the assumption of induction. It is easy to make sure that the inequality holds:

$$-\frac{\alpha(k+1)h}{k+1+1/\gamma} > -\frac{\alpha kh}{k+1/\gamma} > 0.$$

Thus, we can apply Lemma 1 to the difference of the roots, which ensures that it is positive. Therefore, we obtain that $p_{k+1} - p_k > 0$. The monotonic increase of the sequence p_k is proved.

Let us turn to justify estimate (39). We carry out the proof again by induction on k. The induction base was determined earlier. Let $p_i \geq -i\alpha h,\ i = 1, 2\ldots, k-1$. Then,

$$p_k \geq -\frac{1+kh+k(k-1)\alpha h}{2(k-1+1/\alpha)} + \sqrt{\frac{(1+kh+k(k-1)\alpha h)^2}{4(k-1+1/\alpha)^2} - \frac{k\alpha h}{k-1+1/\alpha}}$$

$$= -\frac{kh\alpha}{2} - \frac{1}{2(k-1+1/\alpha)} + \sqrt{\left[\frac{1}{2(k-1+1/\alpha)} - \frac{kh\alpha}{2}\right]^2} = -kh\alpha.$$

The case $\alpha \geq \gamma/(\gamma+1)$ is treated similarly. □

Remark 4. *In the case $\alpha = \gamma/(\gamma+1)$, the double unstrict inequality (39) becomes an equality, and we get the previously found linear solution $p = -\alpha w$.*

With the help of the lemmas, we now prove the main theorem of this section. Let us introduce the notation:

$$\alpha_m = \min\left\{\alpha, \frac{\gamma}{\gamma+1}\right\}, \alpha_M = \max\left\{\alpha, \frac{\gamma}{\gamma+1}\right\}.$$

Theorem 2. *Problem (29) and (32) for $w \leq 0$ has a decreasing continuously differentiable solution $p = p(w)$ satisfying the inequality:*

$$\alpha_m w \leq p(w) \leq \alpha_M w \leq 0. \tag{40}$$

Proof. To prove the existence of the solution with the desired properties, we consider and estimate the difference:

$$\Delta p_k = p_k - p_{k-1} = \frac{-1-kh+kp_{k-1}}{2(k+1/\gamma)} + \sqrt{\frac{(-1-kh+kp_{k-1})^2}{4(k+1/\gamma)^2} - \frac{\alpha kh}{k+1/\gamma}} - p_{k-1} =$$

$$-\frac{1}{2(k+1/\gamma)} + \frac{k(p_{k-1}-h)}{2(k+1/\gamma)} + \sqrt{\left[-\frac{1}{2(k+1/\gamma)} + \frac{k(p_{k-1}-h)}{2(k+1/\gamma)}\right]^2 + \frac{-\alpha kh}{k+1/\gamma}} - p_{k-1}.$$

It follows from Lemma 2 that $\Delta p_k > 0$.

Let first, as in the proof of Lemma 2, $\alpha \leq \gamma/(\gamma+1)$. Then by Lemma 2, the following chain of inequalities holds:

$$\frac{k(p_{k-1}-h)}{k+1/\gamma} \geq \frac{-\alpha kh(k-1+1/\alpha)}{k+1/\gamma} \geq \frac{-\alpha kh(k-1+1+1/\gamma)}{k+1/\gamma} = -\alpha kh > 0.$$

Hence we get that:

$$\Delta p_k \leq -\frac{1}{2(k+1/\gamma)} + \frac{k(p_{k-1}-h)}{2(k+1/\gamma)} + \sqrt{\left[-\frac{1}{2(k+1/\gamma)} + \frac{k(p_{k-1}-h)}{2(k+1/\gamma)}\right]^2 + \frac{k(p_{k-1}-h)}{(k+1/\gamma)^2}}$$

$$-p_{k-1} = -\frac{1}{2(k+1/\gamma)} + \frac{k(p_{k-1}-h)}{2(k+1/\gamma)} + \sqrt{\left[\frac{1}{2(k+1/\gamma)} + \frac{k(p_{k-1}-h)}{2(k+1/\gamma)}\right]^2} - p_{k-1}$$

$$= \frac{k(p_{k-1}-h)}{k+1/\gamma} - p_{k-1} = -\frac{p_{k-1}}{k\gamma+1} - h \leq -\frac{\gamma h}{\gamma+1}.$$

The case $\alpha \geq \gamma/(\gamma+1)$ is treated similarly and gives the estimate $0 < \Delta p_k \leq -\alpha h$. Thus, it has been shown that:

$$0 < \Delta p_k \leq -\max\left\{\alpha, \frac{\gamma}{\gamma+1}\right\}h,\ k = 1, 2, \ldots \tag{41}$$

It follows from (41) that the constructed difference scheme is stable. According to the Lax equivalence theorem, since it also has the approximation property (by construction), it is convergent. This means that the sequence of Euler polygonal lines with vertices at the points (kh, p_k) converges to a continuously differentiable solution to problem (29) and (32) if $k \to \infty, h \to 0$. Moreover, the estimates show that the solution exists for all $w \leq 0$ and decreases.

Inequality (41) also gives the upper estimate from (40). The lower estimate (40) follows from Lemma 2. □

Remark 5. *As the results of the qualitative analysis show, if $w > 0$, Problem (29) and (32) is solvable only for $\alpha = \gamma/(\gamma+1)$. This explains the divergence of series (33) for $\alpha \neq \gamma/(\gamma+1)$.*

7. Nonzero Initial Condition

Let us now consider the second case when the initial condition for Equation (29) is $p(0) = -\gamma$, i.e., the problem:

$$wp\frac{dp}{dw} + \frac{p^2}{\gamma} + wp + p + \alpha w = 0,\ p(0) = -\gamma. \tag{42}$$

Looking ahead, we note that this case leads to results that can be clearly interpreted from the point of view of Problem (3) and (17).

7.1. Solution in the Form of a Series

Let us show that the solution to problem (42) can be found as a power series that converges in a small neighborhood of zero. We construct the series having form (33).

From the boundary condition, we have $p_0 = p(0) = -\gamma$. To find p_1, we differentiate Equation (42) with respect to w, set $w = 0, p(0) = -\gamma$, and obtain that $p_1 = (\alpha-\gamma)/(\gamma+1)$. Similarly, we get:

$$p_2 = \frac{2\alpha(\alpha-\gamma)}{\gamma(\gamma+1)(2\gamma+1)}.$$

Thus, the induction base is found.

Assume that $p_0, p_1, \ldots, p_{k-1}, k \geq 3$ are determined. To find p_k, we differentiate Equation (42) k times with respect to w and set $w = 0$. Then we arrive at the equality (34). Resolving with respect to p_k gives:

$$p_k = \frac{1}{\gamma k+1}\left[\sum_{i=1}^{k-1} C_k^i\left(k-i+\frac{1}{\gamma}\right)p_i p_{k-i} + kp_{k-1}\right]. \tag{43}$$

You can see that all terms on the right-hand side of (43) are known by the induction hypothesis. Thus, all the coefficients of series (33) are uniquely determined by formula (43). If $\gamma = \alpha$, then $p_i = 0,\ i = 1, 2, \ldots$, i.e., the series breaks off and $p \equiv -\gamma = -\alpha$.

The local convergence of series (43) follows from Theorem 1 (see also Theorem 1 in [1]). We have not yet estimated the radius of convergence, but the results of previous studies allow us to make a reasonable assumption that it is small [37]. Thus, we have justified the following proposition.

Proposition 2. *Problem (42) has an analytical solution having the form of the locally convergent series (33), whose coefficients are determined by formula (43). The series breaks off if $\alpha = \gamma$, and the solution is $p = -\gamma$.*

7.2. Euler Polygonal Approximations

The constructed series locally converges in some neighborhood of the point $(0, -\gamma)$. To find the global properties of the solution to problem (42), as above, we use the Euler method. Consider the following finite difference approximation of (42):

$$w_k p_{k-1}\frac{p_k - p_{k-1}}{h} + \frac{1}{\gamma}p_k p_{k-1} + p_{k-1}(1+w_k) + \alpha w_k = 0,$$

where $w_k = kh$. Then we yield the recurrent formula:

$$p_k = \frac{1}{1+1/(k\gamma)}\left(p_{k-1} - h - \frac{1}{k} - \frac{\alpha h}{p_{k-1}}\right). \tag{44}$$

From the Cauchy condition, we have $p_0 = -\gamma$.

Lemma 3. *The following formula is valid:*

$$p_{k+1} = -\gamma - \frac{(k+1)\gamma h}{\gamma+1} - \alpha h \sum_{j=1}^{k+1} \frac{1}{p_{j-1}\prod\limits_{i=j}^{k+1}[1+1/(i\gamma)]}. \tag{45}$$

The lemma is proved by induction on k. The proof is simple and based on direct substitutions.

Lemma 3 is the basis for proving the properties of the difference scheme, which are given below.

Proposition 3. *Sequence* (45) *for* $h > 0, \gamma > \alpha$ *is decreasing, and* $0 < (\gamma-\alpha)h/(\gamma+1) \le p_{k-1} - p_k < (\alpha/\gamma+\gamma)h/(\gamma+1)$, $\lim\limits_{k\to\infty}(p_{k-1}-p_k) = \gamma h/(\gamma+1)$.

Proof. We carry out the proof by induction on k. Indeed,

$$p_0 - p_1 = -\gamma + \gamma + \frac{(\gamma-\alpha)h}{\gamma+1} = \frac{(\gamma-\alpha)h}{\gamma+1} > 0.$$

Thus, the induction base is found. Let $-\gamma = p_0 > p_1 > \ldots > p_{k-1}$, then

$$p_{k-1} - p_k = \frac{\gamma h}{\gamma+1} + \alpha h \sum_{j=1}^{k} \frac{1}{p_{j-1}\prod\limits_{i=j}^{k}[1+1/(i\gamma)]} - \alpha h \sum_{j=1}^{k-1} \frac{1}{p_{j-1}\prod\limits_{i=j}^{k-1}[1+1/(i\gamma)]} =$$

$$= \frac{\gamma h}{\gamma+1} - \frac{\alpha h}{k\gamma}\sum_{j=1}^{k-1} \frac{1}{p_{j-1}\prod\limits_{i=j}^{k}[1+1/(i\gamma)]} + \frac{\alpha h}{p_{k-1}[1+1/(\gamma k)]} \ge$$

$$\ge \frac{\gamma h}{\gamma+1} - \frac{\alpha h}{k\gamma p_{k-1}}\frac{k\gamma}{\gamma+1} = \frac{\gamma h}{\gamma+1} + \frac{\alpha\gamma h}{(\gamma+1)p_{k-1}} \ge \frac{\gamma h}{\gamma+1} - \frac{\alpha\gamma h}{(\gamma+1)\gamma} = \frac{(\gamma-\alpha)h}{\gamma+1} > 0.$$

On the other hand, the last estimates show that:

$$p_{k-1} - p_k < \frac{\gamma h}{\gamma+1} - \frac{\alpha h}{k\gamma}\sum_{j=1}^{k-1} \frac{1}{p_{j-1}\prod\limits_{i=j}^{k}[1+1/(i\gamma)]}$$

if we cast out the negative term. Hence, we have that:

$$p_{k-1} - p_k < \frac{\gamma h}{\gamma+1} - \frac{\alpha h}{k\gamma p_0}\sum_{j=1}^{k-1} \frac{1}{\prod\limits_{i=j}^{k}[1+1/(i\gamma)]} =$$

$$\frac{\gamma h}{\gamma+1} + \frac{\alpha h}{k\gamma^2}\frac{k}{1+1/\gamma} = \frac{\gamma+\alpha/\gamma}{\gamma+1}h < h.$$

□

Proposition 4. *Sequence* (45) *for* $h > 0, \gamma < \alpha$ *is increasing, and* $0 < (\alpha - \gamma)h/(\gamma + 1) \leq p_k - p_{k-1}$. *Moreover, there exists* $k_* < \gamma(\gamma + 1)/[(\alpha - \gamma)h]$ *such that* $p_{k_*} \geq 0$.

Proof. The inequality $p_k - p_{k-1} \geq (\alpha - \gamma)h/(\gamma + 1) > 0$ is proved similarly to Proposition 3. The difference is that due to the change of the sign of $\gamma - \alpha$, the sign of the difference estimate $\Delta p_k = p_k - p_{k-1}$ changes, starting from Δp_1. Hence, in particular, it follows that $p_k \geq -\gamma + (\alpha - \gamma)kh/(\gamma + 1)$. The right side of the last inequality equals to zero for $k_* = \gamma(\gamma + 1)/[(\alpha - \gamma)h]$. If the resulting value is not an integer, then it is necessary to round it with excess, and then we obtain $p_{k_*} > 0$. □

Proposition 5. *Sequence* (45) *for* $h < 0, \gamma > \alpha$ *is increasing, and for* $h < 0, \gamma < \alpha$ *is decreasing, and in both cases* $\lim\limits_{k \to \infty} p_k = -\alpha$.

Proof. The increase and decrease of the sequence p_n for $h < 0$ are proved similarly to Propositions 3 and 4, respectively. On the other hand, since the signs in front of p_{k-1} and $1/p_{k-1}$ on the right side of (44) in this case are the same, the limit is not equal to infinity. Obviously, the limiting value p_∞ satisfies the following equation, which is obtained if we tend $k \to \infty$ in (44):

$$p_\infty = p_\infty - h - \frac{\alpha h}{p_\infty}.$$

It is easy to see that the solution is $p_\infty = -\alpha$. □

Theorem 3. *Problem* (44) *has a continuously differentiable solution, which monotonically tends to* $-\alpha$ *when* $w \to -\infty$. *For* $w > 0$, *three cases are possible:*

1. *If* $\gamma > \alpha$, *then the solution is monotonically decreasing, and the estimate holds:*

$$-\gamma - \frac{\alpha/\gamma + \gamma}{\gamma + 1} w < p \leq -\gamma - \frac{\gamma - \alpha}{\gamma + 1} w;$$

2. *If* $\gamma > \alpha$, *then the solution monotonically increases and at some point* $w = w^* < \gamma(\gamma + 1)/(\alpha - \gamma)$ *vanishes;*
3. *If* $\gamma = \alpha$, *then the solution is the constant* $p \equiv -\gamma = -\alpha$.

The theorem statement follows from Propositions 3–5 by the reasoning similar to those we carried out in the proof of Theorem 2.

8. Discussion

This section is devoted to interpreting the results obtained in the previous sections from the point of view of the corresponding diffusion waves properties. Recall that Equation (29) has been obtained from Equation (21) by changing variables. Equation (21), in turn, follows from Equation (3) if the diffusion-wave front $x = a(t)$ is a linear function.

The results for problem (29) and (32) appear to be non-physical. At any rate, we cannot interpret the negative values of w (for which the solution was constructed) from the point of view of applications. The same situation occurs to the «left branches» of the solution to problem (42).

However, the «right branches» of solutions (42), along which $w \geq 0$, allow a clear physical interpretation.

We have the function $p = p^*(w)$, which is the solution to problem 42 for $w \geq 0$. Returning to the space of variables z, w, we obtain that:

$$z = \int\limits_0^w \frac{d\zeta}{p^*(\zeta)}. \tag{46}$$

As shown above, there are three different cases in which the function $p(w)$ behaves differently. Let us consider them separately.

Case $\gamma = \alpha$. Here $p^*(w) \equiv -\gamma$, whence we have that $v = -\gamma z$, i.e., $u = -\sigma\mu(x - \mu t)$ for $\mu\sigma = B/A$ (see Figure 4). Previously, we constructed a similar solution for the porous medium Equation [1]. In this case, the diffusion wave has the form of a plane in the space of variables t, x, and u.

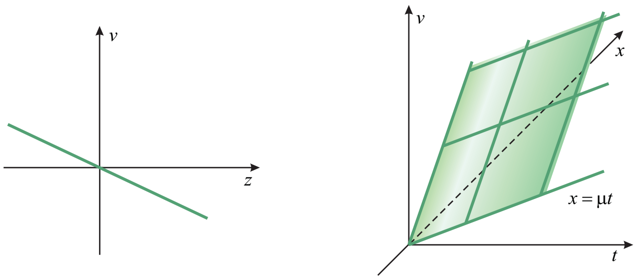

Figure 4. Solution $v(z) = -\gamma z$ and diffusion wave $u = -\sigma\mu(x - \mu t)$.

Case $\gamma > \alpha$. Then $p = p^*(w)$ is infinitely decreasing, and it is bounded upper and below by two straight lines. Hence we have that the function $w(z) = v^\theta(z)$ located between two exponents with negative powers when $z \to \infty$. Returning to the plane of variables v, z, we obtain a monotone infinitely decreasing function, which is defined for all $z \in [0, -\infty)$. The diffusion wave is a cylindrical surface in the space of variables t, x, u, and the unknown function increases with exponential velocity along the generatrix of the cylinder with distance from the wave front (Figure 5).

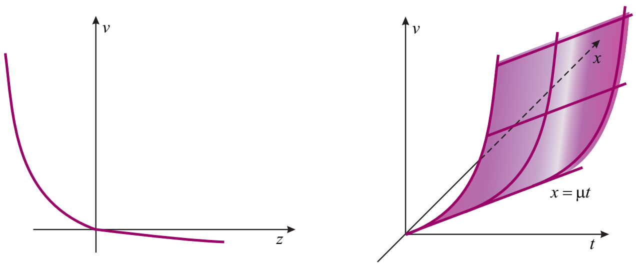

Figure 5. Solution $v(z)$ and the diffusion wave.

Case $\gamma < \alpha$. The study has shown that the function $p(w)$ first increases, and there is a point $w^* > 0$ such that $p(w^*) = 0$, $\lim\limits_{w \to w^* - 0} p'(w) = +\infty$. The point can be determined numerically, since the problem does not have singularities on the interval $[0, w^*)$. Consider the problem:

$$\frac{dw}{dp} = -\frac{wp}{\frac{p^2}{\gamma} + wp + p + \alpha w} = 0, \; w(0) = w^*, \tag{47}$$

where w is an unknown function, and p is an independent variable. It follows from the results of the qualitative analysis that the solution to problem (47) is decreasing on the ray $[0, +\infty)$, and $\lim\limits_{p \to +\infty} w(p) = +0$. Returning to the plane of variables v, z, we get the solution $v = v_*(x)$. From the original problem point of view, there exists a solution $u = v_*(x - \mu t)$, which is a solitary wave (soliton) (see Figure 6).

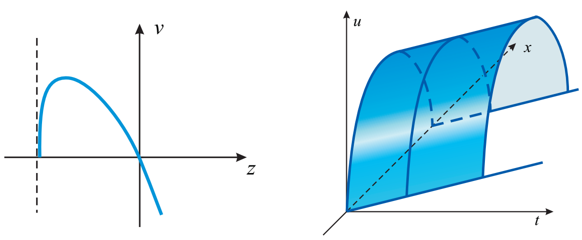

Figure 6. Solution $v(z)$ and the soliton.

Summing up, we note that for different values of the coefficients $\gamma = \sigma\theta > 0$, $\alpha = B/\mu > 0$, we obtained the same basic configurations of diffusion waves that we described earlier (see [1]) for incomplete variants of Equations (3):

- A linear heat wave for the porous medium equation;
- An infinitely increasing wave with a nonzero second derivative with distance from the wave front for the convection–diffusion equation;
- A diffusion wave in the form of a soliton for the generalized porous medium equation (the heat equation with source).

Parameter γ characterizes the diffusion and convection terms, and parameter α characterizes the source and velocity of the wave front. It seems pretty natural that if $\gamma > \alpha$, then the diffusion wave for the complete Equation (3) behaves similarly to the case without a source but with a convection term (the convection-diffusion equation). The case $\gamma < \alpha$ corresponds to the case without convection term but with a source (the generalized porous medium equation). Finally, if the parameters are equal, the diffusion wave behaves similarly to the case when there is neither a source nor a convection term (the porous medium equation).

9. Conclusions

For a second-order one-dimensional singular evolutionary equation with power nonlinearities, we studied diffusion-wave-type solutions propagating along a zero background with a finite velocity. Such properties of solutions usually appear for hyperbolic equations and are atypical for parabolic ones. Apparently, their occurrence is associated with the degeneracy mentioned above, which, in turn, is caused by vanishing the term multiplying the highest (second) derivative. Besides being a fascinating mathematical object, such solutions are also valuable for applications. They allow us to describe nonlinear filtration and diffusion processes with a finite velocity of perturbation propagation by parabolic models. Such models are considered better described physical processes at a distance from the degeneracy line.

This paper is the second step in a large research cycle started in [1]. We have considered an equation with power nonlinearities, a specification of the general equation discussed earlier. The choice of the type of functions was related to the fact that such nonlinearities usually arise in applications. Due to this, we have been able to get more profound results. Thus, in the existence and uniqueness theorem, we have chosen a more complex type of boundary conditions, raising a diffusion wave. As a result, both the technique of constructing the solution and the procedure for proving the convergence of series have become significantly more complicated. Besides, we have studied in detail one of the particular but quite natural cases, where the degree of the convection term coincides with the degree of the source. We have performed both a qualitative analysis of ODEs with the construction of phase portraits and obtained quantitative estimates for the solutions.

The most significant result is that we have shown that all the special cases for incomplete equations take place for the complete equation, and other configurations of diffusion waves do not arise. In addition, a nontrivial solution to the Cauchy problem with zero initial conditions has been found. Although this solution has no physical interpretation since it is negative, its presence is an interesting and non-obvious mathematical fact.

Further research in this direction in the short term, in our opinion, should be associated with the development of a practical computational technique for diffusion waves construction. In this context, the boundary element approach, which we have been developing in recent years in collaboration with colleagues, looks promising. It is also advisable to consider other special cases, for example, to construct and study generalized self-similar solutions to the considered problem.

In the long term, it would be helpful to increase the dimensionality and consider cases where an unknown function depends on two or three spatial variables, as well as consider systems of partial differential equations. In the end, the final stage of the research cycle should be the application of the developed model-algorithmic apparatus for solving applied problems related to modeling diffusion processes occurring in Lake Baikal.

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