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Stieltjes Property of Quasi-Stable Matrix Polynomials

Xuzhou Zhan [1], Bohui Ban [2] and Yongjian Hu [1,2,*]

[1] Department of Mathematics, Beijing Normal University at Zhuhai, Zhuhai 519087, China
[2] School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
* Correspondence: yongjian@bnu.edu.cn

Abstract: In this paper, basing on the theory of matricial Hamburger moment problems, we establish the intrinsic connections between the quasi-stability of a monic or comonic matrix polynomial and the Stieltjes property of a rational matrix-valued function built from the even–odd split of the original matrix polynomial. As applications of these connections, we obtain some new criteria for quasi-stable matrix polynomials and Hurwitz stable matrix polynomials, respectively.

Keywords: matrix polynomial; quasi-stability; Hurwitz stability; Hamburger moment problem; Nevanlinna function; Stieltjes function; stability index; degeneracy index

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1. Introduction

To begin with, we introduce some notations and conventions that are used throughout this paper. Let $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{N}_0$ stand for the sets of all complex, positive integer and non-negative integer numbers, respectively. Given a pair of $p, q \in \mathbb{N}$, we denote by $\mathbb{C}^{p\times q}$ the set of all complex $p \times q$ matrices, and by $\mathbb{C}[z]^{p\times p}$ the set of $p \times p$ matrix polynomials; that is, the ring of polynomials in z with matrix coefficients from $\mathbb{C}^{p\times p}$. In particular, $\mathbb{C}[z] = \mathbb{C}[z]^{1\times 1}$. For convenience, the zero and the identity $p \times p$ matrices are, respectively, written as 0_p and I_p for short. Given a matrix $A \in \mathbb{C}^{p\times p}$, we denote its transpose by A^{T}, its conjugate transpose by A^* and its Moore–Penrose inverse by $A^\dagger$, i.e., $A^\dagger$ is a unique solution of the matrix equations:

$$AXA = A, \quad XAX = X, \quad AX = (AX)^*, \quad XA = (XA)^*.$$

Let A be a Hermitian matrix, i.e., $A = A^*$. We write $A \succ 0$ if A is an Hermitian positive definite matrix, and $A \succeq 0$ if A is an Hermitian non-negative definite matrix.

Given a non-zero matrix polynomial $F(z) \in \mathbb{C}[z]^{p\times p}$, $F(z)$ can be represented in the form

$$F(z) = \sum_{k=0}^{n} A_k z^{n-k}, \quad \text{with } A_0, \ldots, A_n \in \mathbb{C}^{p\times p} \text{ and } A_0 \text{ is a non-zero matrix,} \tag{1}$$

where A_n is called the constant term of $F(z)$, A_0 is called the leading coefficient of $F(z)$ and n is called the degree of $F(z)$, denoted by $\deg F(z)$. $F(z)$ is called monic if A_0 is equal to the identity matrix I_p and it is called comonic if $A_n = I_p$. A matrix polynomial $F(z)$ is said to be regular if $\det F(z)$ is not identically zero. For a regular matrix polynomial $F(z)$, we say that $\lambda \in \mathbb{C}$ is a zero (also called a latent root) of $F(z)$ if the determinant $\det F(\lambda) = 0$. Its multiplicity is the multiplicity of λ as a zero of $\det F(z)$. The spectrum $\sigma(F)$ of $F(z)$ is the set of all zeros of $F(z)$. The study of the zero localization of a regular matrix polynomial can be converted to the comonic or monic situation via the translation and reversal of the original polynomial (see, e.g., [1]).

The structured feature of dynamical systems can be intimately related to the zero localization of the characteristic matrix polynomials or matrix-valued functions. For example, certain differential algebraic systems are asymptotically stable if and only if the zeros of the characteristic matrix polynomials $F(z)$ are located in the open left half-plane $\mathbb{C}_l$ (see, e.g., [2–8]). In this case, $F(z)$ is called Hurwitz stable. More system features involving bifurcation and marginal stability are connected with the study of the location of the characteristic zeros in the closed left half-plane (see, e.g., [9–12]). In general, a regular matrix polynomial $F(z)$ is called quasi-stable if $\sigma(F)$ is contained in the closed left half-plane.

Recently, the stability analysis for matrix polynomials in [13] connects with the theory of holomorphic matrix-valued functions. Denote by $\mathbb{C}_+$ the open upper half of the complex plane. Recall that a function $R : \mathbb{C}_+ \to \mathbb{C}^{p\times p}$ is said to be a matrix-valued Herglotz–Nevanlinna (In the scalar case $p = 1$, other popular titles in the literature for the same function are "Nevanlinna", "Pick", "Nevanlinna-Pick", "Herglotz", etc.) function if it is holomorphic on $\mathbb{C}_+$ and its imaginary part satisfies that

$$\operatorname{Im} R(z) = \frac{1}{2\mathrm{i}}(R(z) - R(z)^*) \succeq 0, \quad z \in \mathbb{C}_+.$$

Each matrix-valued Herglotz–Nevanlinna function, $R(z)$ can be continued into the open lower half-plane $\mathbb{C}_-$ by reflection (see, e.g., [14]):

$$R(z) = R(\bar{z})^*, \quad z \in \mathbb{C}_-.$$

A function $R : \mathbb{C} \setminus [0, +\infty) \to \mathbb{C}^{p\times p}$ is said to be a matrix-valued Stieltjes function if it satisfies the following three conditions:

(i) $R(z)$ is a matrix-valued Herglotz–Nevanlinna function;
(ii) $R(z)$ is holomorphic in $\mathbb{C} \setminus [0, +\infty)$;
(iii) for each $z \in (-\infty, 0)$, $R(z) \succeq 0$.

It is clear that $R(z)$ is a matrix-valued Stieltjes function if and only if both $R(z)$ and $zR(z)$ are matrix-valued Herglotz–Nevanlinna functions (see, e.g., [15]).

For a matrix polynomial $F(z)$ written as in (1), it can be split into the even part $F_e(z)$ and the odd part $F_o(z)$ as

$$F_e(z) = \sum_{k=0}^{m} A_{2k} z^{m-k} \quad \text{and} \quad F_o(z) = \sum_{k=1}^{m} A_{2k-1} z^{m-k}$$

when $n = 2m$, and

$$F_e(z) = \sum_{k=0}^{m} A_{2k+1} z^{m-k} \quad \text{and} \quad F_o(z) = \sum_{k=0}^{m} A_{2k} z^{m-k}$$

when $n = 2m + 1$, so that $F(z) = F_e(z^2) + zF_o(z^2)$. This leads to the construction of two $p \times p$ rational matrix-valued functions (i.e., matrices whose entries are rational functions)

$$R_F(z) = F_o(-z)(F_e(-z))^{-1} \tag{2}$$

if $F_e(z)$ is regular, and

$$\widetilde{R}_F(z) = -F_e(-z)(zF_o(-z))^{-1} \tag{3}$$

if $F_o(z)$ is regular.

It has been shown in [13] (Theorems 1.1 and 1.2) that, for a monic matrix polynomial $F(z)$, its Hurwitz stability can be checked via its Stieltjes property. Here, we say $F(z)$ has Stieltjes property if $R_F(z)$ or $\widetilde{R}_F(z)$ is a matrix-valued Stieltjes function. These results give some matrix generalizations of a classical stability criterion by Gantmacher, Chebotarev theorem, Grommer theorem and some aspects of the modified Hermite–Biehler theorem (see [13] (Section 3)).

This paper continues our investigations in [13] on the relation between the stability analysis and Stieltjes property of matrix polynomials. It turns out that, for a matrix polynomial $F(z)$ under some natural assumptions, its quasi-stability can also be checked via its Stieltjes property. The basic strategy for the quasi-stability of a monic matrix polynomial $F(z)$ is based on the theory of matricial Hamburger moment problem (see, e.g., [16–18]). As for the comonic case, it can be converted into the monic case via the reversal of $F(z)$. We remark that when $F(z)$ is comonic and $\deg F$ is odd, the Stieltjes property of $F(z)$ is characterized by $R_F(z)$, which is different from the corresponding monic case. Furthermore, these relations between the quasi-stability and the Stieltjes property of matrix polynomials lead to a Hurwitz stability criterion for comonic matrix polynomials. Note that the comonic situation is a natural assumption for Hurwitz stable matrix polynomials. Indeed, when the constant term of $F(z)$ is singular, $0 \in \sigma(F)$ and then $F(z)$ is not Hurwitz stable. Therefore, for a Hurwitz stable matrix polynomial $F(z)$, its constant term is necessarily non-singular. In this case, without loss of generality, we always assume that the tested matrix polynomial $F(z)$ is comonic. Our results in this paper generalize some results in [13,19].

We conclude the introduction with the outline of this paper. Sections 2 and 3 build relations between the Stieltjes property and the quasi-stability of matrix polynomials, respectively, in the monic case and in the comonic case. Section 4 is devoted to the Hurwitz stability criterion for comonic matrix polynomials.

2. Stieltjes Property of Quasi-Stable Matrix Polynomials: The Monic Case

Let $D(z), P(z), Q(z) \in \mathbb{C}[z]^{p\times p}$. We say that $D(z)$ is a right divisor of $P(z)$ if there exists a $C(z) \in \mathbb{C}[z]^{p\times p}$ such that

$$P(z) = C(z)D(z).$$

In this case, if $D(z)$ is also a right divisor of $Q(z)$, then $D(z)$ is called a right common divisor of $P(z)$ and $Q(z)$. For a right common divisor $D(z)$ of $P(z)$ and $Q(z)$, we call $D(z)$ a GRCD of $P(z)$, and $Q(z)$ if any other right common divisor of $P(z)$ and $Q(z)$ is a right divisor of $D(z)$. Furthermore, $P(z)$ and $Q(z)$ are said to be right coprime if any right common divisor of $P(z)$ and $Q(z)$ is unimodular; that is, its determinant is a non-zero constant. For the calculation for GRCDs, we refer the reader to the methods based on the use of the Hermite or Popov form (see, e.g., [20] (Section 6.3), [21]) and a fast algorithm via elimination for the generalized Sylvester matrices (see [22]).

Let $R(z)$ be a rational matrix-valued function. If $\lambda \in \mathbb{C}$ is a zero of the monic least common multiple of the denominators of the entries of $R(z)$, then λ is called a pole of $R(z)$ (see, e.g., [20]). Moreover, $R(z)$ is called symmetric with respect to the real line if it obeys that $R(z) = R(\bar{z})^*$ for all $z \in \mathbb{C}$ except the poles of the entries of $R(z)$.

To test the Hurwitz stability of a monic matrix polynomial $F(z) \in \mathbb{C}[z]^{p\times p}$, it is necessary to assume that $F_e(z)$ and $F_o(z)$ are right coprime and the constant term of $F(z)$ is non-singular. In fact, if the constant term of $F(z)$ is singular or $F_e(z)$ and $F_o(z)$ are not right coprime, $F(z)$ cannot be Hurwitz stable. Another precondition is that the rational matrix-valued function $R_F(z)$ or $\widetilde{R}_F(z)$ is symmetric with respect to the real line.

Theorem 1 ([13] (Theorem 1.1))**.** *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be a monic matrix polynomial with the non-singular constant term in which $F_e(z)$ and $F_o(z)$ are right coprime, and let $R_F(z)$ defined by (2) be a symmetric rational matrix-valued function with respect to the real line. Then, $F(z)$ is Hurwitz stable if and only if $R_F(z)$ is a matrix-valued Stieltjes function.*

Theorem 2 ([13] (Theorem 1.2))**.** *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be a monic matrix polynomial with the non-singular constant term in which $F_e(z)$ and $F_o(z)$ are right coprime, $F_o(z)$ be regular when* $\deg F$ *is even, and let $\widetilde{R}_F(z)$ defined by (3) be a symmetric rational matrix-valued function with respect to the real line. Then, $F(z)$ is Hurwitz stable if and only if $\widetilde{R}_F(z)$ is a matrix-valued Stieltjes function.*

With regard to Theorems 1 and 2, we are naturally to consider the relations between the quasi-stability of a monic matrix polynomial and its Stieltjes property.

For a monic matrix polynomial $F(z)$ which is quasi-stable, each GRCD $\widetilde{F}(z)$ of $F_e(z)$ and $F_o(z)$ necessarily satisfies that $\sigma(\widetilde{F}) \subseteq (-\infty, 0]$. In fact, note that $\widetilde{F}(z^2)$ is a right divisor of $F(z)$. If $\widetilde{F}(z)$ has a zero $re^{\mathrm{i}\theta}$ $(r > 0, -\pi < \theta < \pi)$ located outside the interval $(-\infty, 0]$, then $\sqrt{r}e^{\mathrm{i}\theta/2}$ is a zero of $F(z)$, which is located in the open right half-plane. In this case, $F(z)$ is not quasi-stable. So, to test the quasi-stability of $F(z)$, we make the following assumption:

Assumption 1. *The spectrum of a/each GRCD of $F_e(z)$ and $F_o(z)$ is contained in the interval $(-\infty, 0]$.*

Recently, Zhan et al. [23] have presented several criteria for the quasi-stability of $F(z)$ under Assumption 1. Here, we establish the relationships between the quasi-stability of a monic matrix polynomial and its Stieltjes property. For this goal, we invoke some basic results on the matricial Hamburger moment problem. For a more comprehensive study, we refer the reader to some references, e.g., [16–18,24,25].

Given an infinite sequence of Hermitian matrices $\mathscr{S} = (\mathbf{s}_k)_{k=0}^{\infty}$, the full matricial Hamburger moment problem (FHM($\mathscr{S}$) for short) is to find all the non-negative Hermitian $p \times p$ Borel measures τ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} u^k \mathrm{d}\tau(u) = \mathbf{s}_k, \quad k \in \mathbb{N}_0.$$

In view of [17], if there exists a solution τ of Problem FHM($\mathscr{S}$), then the Stieltjes transform $\int_{\mathbb{R}} \frac{\mathrm{d}\tau(u)}{u-z}$ of τ admits the following asymptotic expansion

$$\int_{\mathbb{R}} \frac{1}{u-z} \mathrm{d}\tau(u) = -\sum_{j=0}^{\infty} \frac{\mathbf{s}_j}{z^{j+1}} \tag{4}$$

when $z \to \infty$ in the sector $\{z \mid \varepsilon \le \arg z \le \pi - \varepsilon\}$, $0 < \varepsilon < \frac{\pi}{2}$. Conversely, if there exists a non-negative Hermitian $p \times p$ Borel measures τ on $\mathbb{R}$, such that its Stieltjes transform admits the asymptotic expansion (4), then τ is a solution of Problem FHM($\mathscr{S}$).

The solvability of Problem FHM($\mathscr{S}$) is intimately related to the Hermitian non-negative definiteness of block Hankel matrices built from the moment sequence $\mathscr{S}$. Denote the block Hankel matrices associated with $\mathscr{S}$ by

$$H_{j,k}^{\mathscr{S}} := \begin{bmatrix} \mathbf{s}_j & \mathbf{s}_{j+1} & \cdots & \mathbf{s}_{j+k} \\ \mathbf{s}_{j+1} & \mathbf{s}_{j+2} & \cdots & \mathbf{s}_{j+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{s}_{j+k} & \mathbf{s}_{j+k+1} & \cdots & \mathbf{s}_{j+2k} \end{bmatrix}, \quad j,k \in \mathbb{N}_0.$$

For simplicity, $H_{0,k}^{\mathscr{S}}$ is written as $H_k^{\mathscr{S}}$. Moreover, we denote by $\mathscr{S}_{[k]}$ the generalized Schur complement of $H_{k-1}^{\mathscr{S}}$ in $H_k^{\mathscr{S}}$, i.e.,

$$\mathscr{S}_{[k]} = \mathbf{s}_{2k} - \begin{bmatrix} \mathbf{s}_k & \cdots & \mathbf{s}_{2k-1} \end{bmatrix} (H_{k-1}^{\mathscr{S}})^{\dagger} \begin{bmatrix} \mathbf{s}_k \\ \vdots \\ \mathbf{s}_{2k-1} \end{bmatrix}, \quad k \in \mathbb{N}.$$

Lemma 1 ([17] (Theorem 2.2))**.** *Let $\mathscr{S}$ be an infinite sequence of $p \times p$ Hermitian matrices. Problem FHM($\mathscr{S}$) is solvable if and only if $H_k^{\mathscr{S}} \succeq 0$ for $k \in \mathbb{N}_0$.*

Lemma 2 ([18] (Proposition 4.9))**.** *Let $\mathscr{S}$ be an infinite sequence of $p \times p$ Hermitian matrices, such that $H_k^{\mathscr{S}} \succeq 0$ for $k \in \mathbb{N}_0$. If $\mathscr{S}_{[k]} = 0_p$ for some $k \in \mathbb{N}$, then Problem FHM($\mathscr{S}$) has a unique solution.*

Now, we present the relations between the quasi-stability and the Stieltjes property of a monic matrix polynomial.

Theorem 3. *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be a monic matrix polynomial under Assumption 1.*

(i) *When $\deg F(z)$ is even, let $R_F(z)$ defined by (2) be a symmetric rational matrix-valued function with respect to the real line. Then, $F(z)$ is quasi-stable if and only if $R_F(z)$ is a matrix-valued Stieltjes function.*

(ii) *When $\deg F(z)$ is odd, let $\widetilde{R}_F(z)$ defined by (3) be a symmetric rational matrix-valued function with respect to the real line. Then, $F(z)$ is quasi-stable if and only if $\widetilde{R}_F(z)$ is a matrix-valued Stieltjes function.*

Proof. We only give a proof of Part (i). As for Part (ii), it can be proved in an analogous way. Let $\deg F(z) = 2m$ for some integer m. Since $\deg F_e(z) - \deg F_o(z) \geq 1$, we suppose that the rational matrix-valued function $R_F(z)$ has the following asymptotic expansion at $z = \infty$:

$$R_F(z) = -\sum_{j=0}^{\infty} \frac{\mathbf{s}_j}{z^{j+1}}, \tag{5}$$

where $\mathbf{s}_j = \mathbf{s}_j^*$.

We first prove the "if" part. Let $R_F(z)$ defined by (2) be a matrix-valued Stieltjes function and $\{\lambda_j\}_{j=1}^r$ be the set of all different poles of $R_F(z)$. Then, $R_F(z)$ admits an integral representation (see, e.g., [15])

$$R_F(z) = A + \int_0^{+\infty} \frac{1}{u-z} \mathrm{d}\tau(u), \quad z \in \mathbb{C}_+, \tag{6}$$

where $A \succeq 0$ and τ is a non-negative Hermitian $p \times p$ matrix-valued Borel measure on $[0, +\infty)$, such that

$$\int_0^{+\infty} \frac{1}{1+u} \mathrm{trace}(\mathrm{d}\tau(u)) < +\infty.$$

Noting that $R_F(z)$ is a rational matrix-valued function, such that $\lim_{z\to\infty} R_F(z) = 0$, we can rewrite (6) into the following discrete form

$$R_F(z) = \sum_{j=1}^{r} \frac{E_j}{\lambda_j - z}, \tag{7}$$

where $E_j \succeq 0$, $E_j \neq 0_p$, and $\lambda_j \geq 0$ for $j = 1, \dots, r$. It follows from (7) that

$$H_{m-1}^{\mathscr{S}} = \sum_{j=1}^{r} \left(\begin{bmatrix} 1 \\ \vdots \\ \lambda_j^{m-1} \end{bmatrix} \left[1, \cdots, \lambda_j^{m-1}\right] \right) \otimes E_j \succeq 0,$$

in which $\mathscr{S} = (\mathbf{s}_j)_{j=0}^{\infty}$ and $\otimes$ stands for the Kronecker product of two matrices. On the other hand, by (7) we have

$$zR_F(z) = -\sum_{j=1}^{r} E_j + \sum_{j=1}^{r} \frac{\lambda_j E_j}{\lambda_j - z}. \tag{8}$$

Similarly, from (8), one can derive that

$$H_{1,m-1}^{\mathscr{S}} = \sum_{j=1}^{r} \left(\begin{bmatrix} 1 \\ \vdots \\ \lambda_j^{m-1} \end{bmatrix} \left[1, \cdots, \lambda_j^{m-1}\right] \right) \otimes (\lambda_j E_j) \succeq 0.$$

Therefore, by [23] (Theorem 3.1) $F(z)$ is a quasi-stable matrix polynomial.

Now, we prove the "only if" part. Suppose that $F(z)$ is quasi-stable. By [23] (Theorem 3.1), $H^{\mathscr{S}}_{m-1} \succeq 0$. Assume that

$$(-1)^m F_e(-z) = \sum_{k=0}^{m} P_k z^{m-k}.$$

Due to (5) and the fact that $P_0 = I_p$, we have

$$H^{\mathscr{S}}_{k+m-1}\begin{bmatrix} 0_p \\ \vdots \\ 0_p \\ P_m \\ \vdots \\ P_1 \end{bmatrix} = -\begin{bmatrix} \mathbf{s}_{k+m} \\ \vdots \\ \mathbf{s}_{2k+2m-1} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{s}_{k+m} & \cdots & \mathbf{s}_{2k+2m-1} \end{bmatrix}\begin{bmatrix} 0_p \\ \vdots \\ 0_p \\ P_m \\ \vdots \\ P_1 \end{bmatrix} = -\mathbf{s}_{2k+2m}, \quad k \in \mathbb{N}_0.$$

It follows from the last equations that

$$\begin{aligned} \mathbf{s}_{2k+2m} &= \begin{bmatrix} 0_p & \cdots & 0_p & P_m^* & \cdots & P_1^* \end{bmatrix} H^{\mathscr{S}}_{k+m-1}\begin{bmatrix} 0_p \\ \vdots \\ 0_p \\ P_m \\ \vdots \\ P_1 \end{bmatrix} \\ &= \begin{bmatrix} 0_p & \cdots & 0_p & P_m^* & \cdots & P_1^* \end{bmatrix} H^{\mathscr{S}}_{k+m-1}\left(H^{\mathscr{S}}_{k+m-1}\right)^{\dagger} H^{\mathscr{S}}_{k+m-1}\begin{bmatrix} 0_p \\ \vdots \\ 0_p \\ P_m \\ \vdots \\ P_1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{s}_{k+m} & \cdots & \mathbf{s}_{2k+2m-1} \end{bmatrix}\left(H^{\mathscr{S}}_{k+m-1}\right)^{\dagger}\begin{bmatrix} \mathbf{s}_{k+m} \\ \vdots \\ \mathbf{s}_{2k+2m-1} \end{bmatrix}, \quad k \in \mathbb{N}_0. \end{aligned}$$

Then, $\mathscr{S}_{[k+m]} = 0_p$, $k \in \mathbb{N}_0$. Together with the Hermitian non-negative definiteness of $H^{\mathscr{S}}_{m-1}$, we have $H^{\mathscr{S}}_{k} \succeq 0$ for $k \in \mathbb{N}_0$. In view of Lemma 2, there exists a unique non-negative Hermitian $p \times p$ matrix-valued Borel measure τ on $\mathbb{R}$, such that

$$\int_{\mathbb{R}} u^k \mathrm{d}\tau(u) = \mathbf{s}_k, \quad k \in \mathbb{N}_0,$$

or equivalently,

$$\int_{\mathbb{R}} \frac{1}{u-z}\mathrm{d}\tau(u) = -\sum_{j=0}^{\infty} \frac{\mathbf{s}_j}{z^{j+1}}, \quad z \to \infty. \tag{9}$$

Combining (5) and (9), we have that

$$R_F(z) = \int_{\mathbb{R}} \frac{1}{u-z}\mathrm{d}\tau(u).$$

Therefore, $R_F(z)$ is a matrix-valued Herglotz–Nevanlinna function.

To prove $R_F(z)$ is a matrix-valued Stieltjes function, we must prove that $zR_F(z)$ is also a matrix-valued Herglotz–Nevanlinna function. To this end, we invoke the Anderson–Jury Bezoutian matrix of a pair of matrix polynomials (see, e.g., [1,26–28]). Let $P(z), Q(z) \in \mathbb{C}[z]^{p\times p}$ satisfy

$$P^{\vee}(z)Q(z) = Q^{\vee}(z)P(z),$$

where $P^{\vee}(z) = P(\bar{z})^*$, $Q^{\vee}(z) = Q(\bar{z})^*$. The Anderson–Jury Bezoutian matrix $\mathscr{B}(P,Q)$ of $P(z)$ and $Q(z)$ is defined via the formula

$$[I_p, zI_p, \cdots, z^{m-1}I_p]\mathscr{B}(P,Q)\begin{bmatrix} I_p \\ uI_p \\ \vdots \\ u^{m-1}I_p \end{bmatrix} = \frac{1}{z-u}\big(P^{\vee}(z)Q(u) - Q^{\vee}(z)P(u)\big),$$

where $m = \max\{\deg P(z), \deg Q(z)\}$.

Note that $zR_F(z)$ is holomorphic in $\mathbb{C}_+$. If we choose $P(z) = F_e(-z)$, $Q(z) = -zF_o(-z)$, then $zR_F(z) = -Q(z)P(z)^{-1}$. An application of [23] (Theorem 3.1) and [13] (Lemma A1) yields that the Anderson–Jury Bezoutian matrix $\mathscr{B}(P,Q)$ is Hermitian non-negative definite, and subsequently,

$$P(z)^*\frac{zR_F(z) - (zR_F(z))^*}{z-\bar{z}}P(z) = [I_p, \cdots, \bar{z}^{m-1}I_p]\mathscr{B}(P,Q)\begin{bmatrix} I_p \\ \vdots \\ z^{m-1}I_p \end{bmatrix} \succeq 0, \quad z \in \mathbb{C}_+.$$

This implies that $\operatorname{Im} zR_F(z) \succeq 0$ for all $z \in \mathbb{C}_+$. Then, $zR_F(z)$ is also a matrix-valued Herglotz–Nevanlinna function. The proof of the "only if" part is complete. □

Now, we provide an example to illustrate Theorem 3.

Example 1. *Let $F(z) \in \mathbb{C}[z]^{2\times 2}$ be a monic matrix polynomial of degree 5, given as*

$$F(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}z^5 + \begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 2 \end{bmatrix}z^4 + \begin{bmatrix} 8 & -\mathrm{i} \\ \mathrm{i} & 8 \end{bmatrix}z^3 + \begin{bmatrix} 4 & 3\mathrm{i} \\ -2\mathrm{i} & 9 \end{bmatrix}z^2 + \begin{bmatrix} 19 & -\mathrm{i} \\ 7\mathrm{i} & 13 \end{bmatrix}z + \begin{bmatrix} 5 & \mathrm{i} \\ 3\mathrm{i} & 9 \end{bmatrix}.$$

Then, the even and odd parts of $F(z)$ are

$$F_e(z) = \begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 2 \end{bmatrix}z^2 + \begin{bmatrix} 4 & 3\mathrm{i} \\ -2\mathrm{i} & 9 \end{bmatrix}z + \begin{bmatrix} 5 & \mathrm{i} \\ 3\mathrm{i} & 9 \end{bmatrix},$$
$$F_o(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}z^2 + \begin{bmatrix} 8 & -\mathrm{i} \\ \mathrm{i} & 8 \end{bmatrix}z + \begin{bmatrix} 19 & -\mathrm{i} \\ 7\mathrm{i} & 13 \end{bmatrix},$$

respectively. By a direct computation, we have that

$$\widetilde{F}(z) = \begin{bmatrix} z+5 & \mathrm{i} \\ 2\mathrm{i} & z+2 \end{bmatrix}$$

is a GRCD of $F_e(z)$ and $F_o(z)$, and $\sigma(\widetilde{F}) = \{-3, -4\} \subseteq (-\infty, 0]$. Moreover,

$$\widetilde{R}_F(z) = -F_e(-z)(zF_o(-z))^{-1} = \frac{E_1}{-z} + \frac{E_2}{4-z} + \frac{E_3}{5-z}$$

is a symmetric rational matrix-valued function with respect to the real line, where

$$E_1 = \frac{1}{10}\begin{bmatrix} 3 & \mathrm{i} \\ -\mathrm{i} & 7 \end{bmatrix} \succeq 0, \quad E_2 = \frac{1}{2}\begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix} \succeq 0, \quad E_3 = \frac{1}{5}\begin{bmatrix} 1 & 2\mathrm{i} \\ -2\mathrm{i} & 4 \end{bmatrix} \succeq 0.$$

Hence, $R_F(z)$ is a matrix-valued Stieltjes function. In view of Theorem 3, $F(z)$ is quasi-stable.

If the rational matrix-valued function $R_F(z)$ defined by (2) in Theorem 3 is replaced by $\widetilde{R}_F(z)$ defined by (3), then we obtain the following criterion for the quasi-stability of $F(z)$.

Theorem 4. *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be a monic matrix polynomial under Assumption 1.*

(i) *When $\deg F(z)$ is even, let $F_o(z)$ be regular and $\widetilde{R}_F(z)$ defined by (3) be a symmetric rational matrix-valued function with respect to the real line. Then, $F(z)$ is quasi-stable if and only if $\widetilde{R}_F(z)$ is a matrix-valued Stieltjes function.*

(ii) *When $\deg F(z)$ is odd, let $F_e(z)$ be regular and $R_F(z)$ defined by (2) be a symmetric rational matrix-valued function with respect to the real line. Then, $F(z)$ is quasi-stable if and only if $R_F(z)$ is a matrix-valued Stieltjes function.*

Proof. We only give a proof of the first part of Theorem 4. The second part can be proved in a similar way. Under the assumptions of Theorem 4, the rational matrix-valued function $R_F(z)$ defined by (2) is also symmetric with respect to the real line, and satisfies

$$R_F(z) = -(z\widetilde{R}_F(z))^{-1}, \quad zR_F(z) = -(\widetilde{R}_F(z))^{-1}.$$

The last two equations imply that $R_F(z)$ is a matrix-valued Stieltjes function if and only if $\widetilde{R}_F(z)$ is a matrix-valued Stieltjes function. Hence, the first part of Theorem 4 follows directly from Part (i) of Theorem 3. □

For a quasi-stable matrix polynomial $F(z)$, the stability index of $F(z)$, denoted by $\nu(F)$, is the number of zeros of $F(z)$ with negative real parts, and the degeneracy index of $F(z)$ is denoted by $\delta(F)$, which stands for the number of zeros of $F(z)$ lying on the imaginary axis, counting their multiplicities. Note that a monic matrix polynomial $F(z)$ is Hurwitz stable if and only if $F(z)$ is quasi-stable and $\nu(F) = \deg\det F(z)$. Thus, a combination of Theorem 3, Corollary 2 below and [13] (Lemma A.2) leads to Theorems 1 and 2 for the Hurwitz stability of matrix polynomials.

A $p \times p$ rational matrix-valued function $R(z)$ is called proper if $R(z)$ converges to a constant matrix as z tends to ∞. Recall that each $p \times p$ proper rational matrix-valued function $R(z)$ can be reduced to the following Smith–McMillan form via two $p \times p$ unimodular matrix polynomials $U_L(z)$ and $U_R(z)$ as follows:

$$U_L(z)R(z)U_R(z) = \operatorname{diag}\left[\frac{n_1(z)}{d_1(z)}, \frac{n_2(z)}{d_2(z)}, \cdots, \frac{n_r(z)}{d_r(z)}, 0, \cdots, 0\right],$$

in which

(i) For $k = 1\ldots, r$, $n_k(z)$ and $d_k(z)$ are coprime;
(ii) For $k = 1, \ldots, r-1$, $n_{k+1}(z)$ is divisible by $n_k(z)$;
(iii) For $k = 1, \ldots, r-1$, $d_k(z)$ is divisible by $d_{k+1}(z)$.

The sum $\sum_{k=1}^{r} \deg d_k(z)$ is called the McMillan degree of $R(z)$ and denoted by $\mu(R)$ (see, e.g., [20] (Section 6.5.2)).

In what follows, we represent the stability index $\nu(F)$ of a quasi-stable matrix polynomial $F(z)$ in terms of the McMillan degrees of $R_F(z)$ and $zR_F(z)$, or the McMillan degrees of $\widetilde{R}_F(z)$ and $z\widetilde{R}_F(z)$.

Lemma 3. *Let $P(z), Q(z) \in \mathbb{C}[z]^{p\times p}$, $s = \deg P(z) \geq 1$, and let the leading coefficient of $P(z)$ be non-singular. If $R(z) = Q(z)(P(z))^{-1}$ is a symmetric rational matrix-valued function with respect to the real line and admits the following Laurent series*

$$R(z) = \sum_{j=0}^{\infty} z^{-(j+1)}\mathbf{s}_j,$$

Then, $\operatorname{rank} H_{s-1}^{\mathscr{S}} = \mu(R)$*, in which* $\mathscr{S} = (\mathbf{s}_j)_{j=0}^{\infty}$.

Proof. By [13] (Lemma A.2), we have

$$\operatorname{dimKer} H_{s-1}^{\mathscr{S}} = \deg\det D(z),$$

where $D(z)$ is a GRCD of $P(z)$ and $Q(z)$. On the other hand, it follows from [20] (P. 445, (13)) that

$$\mu(R) = \deg\det P(z) - \deg\det D(z) = ps - \deg\det D(z).$$

Then, we obtain

$$\operatorname{rank} H_{s-1}^{\mathscr{S}} = ps - \operatorname{dimKer} H_{s-1}^{\mathscr{S}} = ps - \deg\det D(z) = \mu(R),$$

as required. □

A combination of [13] (Corollary 3.2) and Lemma 3 yields that

Corollary 1. *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be a monic quasi-stable matrix polynomial.*

(i) *When $\deg F(z)$ is even, let $R_F(z)$ defined by (2) be a symmetric rational matrix-valued function with respect to the real line. Then,*

$$\nu(F) = \mu(R_F) + \mu(zR_F).$$

(ii) *When $\deg F(z)$ is odd, let $\widetilde{R}_F(z)$ defined by (3) be a symmetric rational matrix-valued function with respect to the real line. Then,*

$$\nu(F) = \mu(\widetilde{R}_F) + \mu(z\widetilde{R}_F).$$

At the end of this section, we consider the quasi-stability of scalar polynomials. Let $F(z)$ be a monic scalar polynomial under Assumption 1. In this case, one of $R_F(z)$ and $\widetilde{R}_F(z)$ is a well-defined and symmetric rational function with respect to the real line if and only if $F(z)$ is a polynomial with real coefficients. For simplicity, and without loss of generality, we assume further that $F(z)$ is a monic real polynomial and the constant term of $F(z)$ is non-zero.

When $\deg F(z)$ is even, by Theorem 3, $F(z)$ is quasi-stable if and only if $R_F(z)$ is a rational Stieltjes function. In this case, the degeneracy index $\delta(F)$ is even, and thus the stability index $\nu(F) = \deg F(z) - \delta(F)$ is even as well. Since

$$\mu(zR_F) \leq \mu(R_F) \leq \mu(zR_F) + 1,$$

by Corollary 1, we have

$$\frac{\nu(F)}{2} \leq \mu(R_F) \leq \frac{\nu(F)+1}{2}.$$

This implies that

$$\mu(R_F) = \frac{\nu(F)}{2}.$$

In view of the fact that $\deg F_o(-z) < \deg F_e(-z)$, the Stieltjes function $R_F(z)$ can be rewritten as the following discrete form

$$R_F(z) = \sum_{i=1}^{r} \frac{e_i}{\lambda_i - z},$$

in which $r = \frac{\nu(F)}{2}$, $e_i > 0$, $i = 1, \cdots, r$ and $\lambda_1, \cdots, \lambda_r$ are distinct positive real numbers.

When $\deg F(z)$ is odd, both $R_F(z)$ and $\widetilde{R}_F(z)$ are well defined and, by Theorem 3, $F(z)$ is quasi-stable if and only if $\widetilde{R}_F(z)$ is a rational Stieltjes function, or equivalently, $R_F(z)$ is a rational Stieltjes function. In this case, $\deg F_e(-z) \leq \deg F_o(-z)$. In view of the fact that the limit $A = \lim_{z\to\infty} R_F(z)$ exists and is non-negative, we obtain $\deg F_e(-z) \geq \deg F_o(-z)$. Then $\deg F_e(-z) = \deg F_o(-z)$, and thus $A > 0$. On the other hand,

$$\mu(z\widetilde{R}_F) \leq \mu(\widetilde{R}_F) \leq \mu(z\widetilde{R}_F) + 1.$$

By Corollary 1, we have

$$\frac{\nu(F) - 1}{2} \leq \mu(z\widetilde{R}_F) \leq \frac{\nu(F)}{2}.$$

Since $F(z)$ is a real polynomial and $F(0) \neq 0$, the degeneracy index $\delta(F)$ is even, and thus the stability index $\nu(F)$ is odd. Then,

$$\mu(z\widetilde{R}_F) = \frac{\nu(F) - 1}{2}.$$

Note that

$$\begin{aligned}
\mu(R_F) &= \deg F_e(-z) - \deg\gcd(F_e(-z), F_o(-z)) \\
&= \deg F_o(-z) - \deg\gcd(F_e(-z), F_o(-z)) \\
&= \mu(z\widetilde{R}_F).
\end{aligned}$$

Then, $\mu(R_F) = \frac{\nu(F)-1}{2}$. Therefore, the Stieltjes function $R_F(z)$ admits the following discrete form

$$R_F(z) = A + \sum_{i=1}^{r} \frac{e_i}{\lambda_i - z},$$

in which $A > 0$, $r = \frac{\nu(F)-1}{2}$, $e_i > 0$, $i = 1, \cdots, r$ and $\lambda_1, \cdots, \lambda_r$ are distinct positive real numbers.

Summarizing the analysis above, we obtain a criterion for the quasi-stability of scalar polynomials.

Corollary 2. *Let $F(z)$ be a monic real polynomial under Assumption 1 and the constant term of $F(z)$ is non-zero. Then, $F(z)$ is quasi-stable if and only if*

$$R_F(z) = A + \sum_{i=1}^{r} \frac{e_i}{\lambda_i - z},$$

in which $r \in \mathbb{N}_0$, $A = 0$ when $\deg F(z)$ is even and $A > 0$ when $\deg F(z)$ is odd, $e_i > 0$, $\lambda_i > 0$ for $i = 1, \cdots, r$, and $\lambda_1, \cdots, \lambda_r$ are distinct. In this case,

$$r = \left\lfloor \frac{\nu(F)}{2} \right\rfloor.$$

where the symbol $\lfloor x \rfloor$ stands for the largest integer not exceeding x.

We remark that the assertion of Theorem 4.8 in [19] is not valid if the real polynomial $p(z)$ does not satisfy Assumption 1. For example, $p(z) = (z+1)^3(z-1)$ is a real polynomial with a non-zero constant term. We check easily that the associated function $\Phi(z) = \frac{2}{z+1}$ defined by (4.2)–(4.6) in [19] is a R-function and each pole of $\Phi(z)$ is negative. However, $f(z)$ is not quasi-stable since 1 is a zero of $f(z)$. In fact, under Assumption 1 Theorem 4.8 in [19] is equivalent to Corollary 2 above.

3. Stieltjes Property of Quasi-Stable Matrix Polynomials: The Comonic Case

This section continues our investigations on the Stieltjes property of quasi-stable matrix polynomials. Different from Section 2, we focus on the quasi-stability of comonic matrix polynomials.

Let $F(z)$ be a non-zero matrix polynomial of degree n. For each $d \in \mathbb{N}$, $d \geq n$, the d-reversal matrix polynomial of $F(z)$ is defined as follows:

$$Rev_d[F](z) = z^d F(z^{-1}).$$

For simplicity, we denote $Rev_n[F](z)$ by $Rev[F](z)$. A matrix polynomial $F(z) \in \mathbb{C}[z]^{p\times p}$ is comonic if and only if $Rev[F](z)$ is a monic matrix polynomial. In this case, we check easily that the quasi-stability of both matrix polynomials $F(z)$ and $Rev[F](z)$ are equivalent.

Lemma 4. *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be comonic. Then, $F(z)$ is quasi-stable if and only if $Rev[F](z)$ is quasi-stable.*

Proof. Since $F(0) = I_p$, we have

$$Rev[Rev[F]](z) = F(z).$$

So we only need to prove the "only if" part of this lemma. We use proof by contradiction. Suppose that $Rev[F](z)$ is quasi-stable. If $F(z)$ is not quasi-stable, then $F(z)$ has a zero λ located in the open right half-plane. Note that

$$0 = \det F(\lambda) = \det(\lambda^n Rev[F](\lambda^{-1})) = \lambda^{np} \det Rev[F](\lambda^{-1}).$$

The last equation implies that $\lambda^{-1} \in \sigma(Rev[F])$. This contradicts the quasi-stability of $Rev[F](z)$. Then, $F(z)$ is also quasi-stable. □

Owing to Lemma 4, the quasi-stability study of a comonic matrix polynomial can be reduced to that of a monic matrix polynomial via reversal. Now, we present two lemmas to deduce the quasi-stability criterion of comonic matrix polynomials.

Lemma 5. *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be of degree $n = 2m + j$ for $j = 0$ or $j = 1$ under Assumption 1. Then,*

$$\sigma(\widetilde{F}) \subseteq (-\infty, 0],$$

where $\widetilde{F}(z)$ is a GRCD of $Rev_m[F_e](z)$ and $Rev_{m+j-1}[F_o](z)$.

Proof. In view of [29] (Proposition A.3), there exist two unimodular matrix polynomials $U_1(z)$ and $U_2(z)$ (i.e., the determinants of $U_1(z)$ and $U_2(z)$ are non-zero constants), such that

$$U_1(z)\begin{bmatrix} Rev_m[F_e](z) \\ Rev_{m+j-1}[F_o](z) \end{bmatrix} = \begin{bmatrix} \widetilde{F}(z) \\ 0_p \end{bmatrix}, \quad U_2(z)\begin{bmatrix} F_e(z) \\ F_o(z) \end{bmatrix} = \begin{bmatrix} \widehat{F}(z) \\ 0_p \end{bmatrix}, \tag{10}$$

where $\widehat{F}(z)$ is a GRCD of $F_e(z)$ and $F_o(z)$. Now, we use proof by contradiction to deduce that $\sigma(\widetilde{F}) \subseteq (-\infty, 0]$. Suppose that there exists a zero λ ($\lambda \notin (-\infty, 0]$) of $\widetilde{F}(z)$. Due to (10), we have

$$\begin{aligned}\operatorname{rank}\widehat{F}(\lambda^{-1}) =& \operatorname{rank}\begin{bmatrix}\widehat{F}(\lambda^{-1})\\ 0_p\end{bmatrix} = \operatorname{rank}\begin{bmatrix}F_e(\lambda^{-1})\\ F_o(\lambda^{-1})\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\lambda^m F_e(\lambda^{-1})\\ \lambda^{m+j-1}F_o(\lambda^{-1})\end{bmatrix}\\ =& \operatorname{rank}\begin{bmatrix}Rev_m[F_e](\lambda)\\ Rev_{m+j-1}[F_o](\lambda)\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\widetilde{F}(\lambda)\\ 0_p\end{bmatrix} = \operatorname{rank}\widetilde{F}(\lambda) < p.\end{aligned}$$

Therefore, $\lambda^{-1} \in \sigma(\widehat{F})$ and $\lambda^{-1} \notin (-\infty, 0]$, which contradicts Assumption 1. □

Lemma 6. *Let $R(z)$ be a $p \times p$ matrix-valued function which is holomorphic in $\mathbb{C} \setminus [0, +\infty)$ and symmetric with respect to the real line. Then, $R(z)$ is a matrix-valued Stieltjes function if and only if $\widetilde{R}(z) = -z^{-1}R(z^{-1})$ is a matrix-valued Stieltjes function.*

Proof. Since $R(z) = -z^{-1}\widetilde{R}(z^{-1})$ for all $z \in \mathbb{C} \setminus [0, +\infty)$, we only need to prove the "only if" part of this lemma. We suppose that $R(z)$ is a matrix-valued Stieltjes function, or equivalently, $R(z)$ and $zR(z)$ are matrix-valued Herglotz–Nevanlinna functions. Obviously, $\widetilde{R}(z)$ and $z\widetilde{R}(z)$ are holomorphic in $\mathbb{C} \setminus \mathbb{R}$. Moreover, for every $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\begin{aligned}\frac{\widetilde{R}(z) - \widetilde{R}(z)^*}{z - \bar{z}} =& \frac{-z^{-1}R(z^{-1}) + \bar{z}^{-1}R(z^{-1})^*}{z - \bar{z}}\\ =& |z|^{-2} \cdot \frac{z^{-1}R(z^{-1}) - (z^{-1}R(z^{-1}))^*}{z^{-1} - \overline{z^{-1}}} \succeq 0,\end{aligned}$$

and

$$\begin{aligned}\frac{z\widetilde{R}(z) - (z\widetilde{R}(z))^*}{z - \bar{z}} =& \frac{-R(z^{-1}) + R(z^{-1})^*}{z - \bar{z}}\\ =& |z|^{-2} \cdot \frac{R(z^{-1}) - R(z^{-1})^*}{z^{-1} - \overline{z^{-1}}} \succeq 0.\end{aligned}$$

Then, both $\widetilde{R}(z)$ and $z\widetilde{R}(z)$ are matrix-valued Herglotz–Nevanlinna functions, and thus, $\widetilde{R}(z)$ is a matrix-valued Stieltjes function. □

For a comonic matrix polynomial $F(z) \in \mathbb{C}[z]^{p\times p}$, whether its degree is even or not, the rational matrix-valued function $R_F(z)$ is always well defined. This enables us to describe the Stieltjes property of a quasi-stable matrix polynomial $F(z)$ in terms of the rational matrix-valued function $R_F(z)$, which is different from Theorem 3 for the odd case.

Theorem 5. *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be a comonic matrix polynomial under Assumption 1, and let $R_F(z)$ defined by (2) be a symmetric rational matrix-valued function with respect to the real line. Then, $F(z)$ is quasi-stable if and only if $R_F(z)$ is a matrix-valued Stieltjes function.*

Proof. Case 1. $\deg F = 2m$. First, we prove the "if" part of this theorem. Note that

$$Rev[F](z) = Rev_m[F_e](z^2) + zRev_{m-1}[F_o](z^2).$$

This implies that $Rev_m[F_e](z)$ and $Rev_{m-1}[F_o](z)$ are the even part and odd part of $Rev[F](z)$, respectively. Since $Rev_m[F_e](z)$ is monic, the rational matrix-valued function $R_{Rev[F]}(z)$ is well defined and

$$R_{Rev[F]}(z) = \frac{Rev_{m-1}[F_o](-z)}{Rev_m[F_e](-z)} = \frac{(-1)^{m-1}z^{m-1}F_o(-z^{-1})}{(-1)^m z^m F_e(-z^{-1})} = -z^{-1}R_F(z^{-1})$$

is symmetric with respect to the real line. Suppose that $\widetilde{F}(z)$ is a GRCD of $Rev_m[F_e](z)$ and $Rev_{m-1}[F_o](z)$. Due to Lemma 5, we have $\sigma(\widetilde{F}) \subseteq (-\infty, 0]$, which means that the monic matrix polynomial $Rev[F](z)$ satisfies Assumption 1. Then, by Lemmas 4, 6 and Theorem 3, we have

$$\begin{aligned}
&R_F(z) \text{ is a matrix-valued Stieltjes function} \\
&\iff R_{Rev[F]}(z) \text{ is a matrix-valued Stieltjes function} \\
&\iff Rev[F](z) \text{ is quasi-stable} \\
&\iff F(z) \text{ is quasi-stable.}
\end{aligned}$$

Case 2. $\deg F = 2m+1$. In this case,

$$Rev[F](z) = Rev_m[F_o](z^2) + zRev_m[F_e](z^2).$$

This implies that $Rev_m[F_o](z)$ and $Rev_m[F_e](z)$ are the even part and odd part of $Rev[F](z)$, respectively. Since $Rev_m[F_e](z)$ is monic, the rational matrix-valued function $\widetilde{R}_{Rev[F]}(z)$ is well defined and

$$\widetilde{R}_{Rev[F]}(z) = -\frac{Rev_m[F_o](-z)}{zRev_m[F_e](-z)} = -\frac{F_o(-z^{-1})}{zF_e(-z^{-1})} = -z^{-1}R_F(z^{-1})$$

is symmetric with respect to the real line. Let $\widetilde{F}(z)$ be a GRCD of $Rev_m[F_e](z)$ and $Rev_m[F_o](z)$. Using Lemma 5, we have that $\sigma(\widetilde{F}) \subseteq (-\infty, 0]$. Then, the monic matrix polynomial $Rev[F](z)$ satisfies Assumption 1. Due to Lemmas 4, 6 and Theorem 3, we have

$$\begin{aligned}
&R_F(z) \text{ is a matrix-valued Stieltjes function} \\
&\iff \widetilde{R}_{Rev[F]}(z) \text{ is a matrix-valued Stieltjes function} \\
&\iff Rev[F](z) \text{ is quasi-stable} \\
&\iff F(z) \text{ is quasi-stable.}
\end{aligned}$$

Then, the proof is complete. □

Under some conditions, the Stieltjes property of quasi-stable comonic matrix polynomials can also be described in terms of $\widetilde{R}_F(z)$ defined by (3).

Theorem 6. *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be a comonic matrix polynomial under Assumption 1, $F_o(z)$ be regular, and let $\widetilde{R}_F(z)$ defined by (3) be a symmetric rational matrix-valued function with respect to the real line. Then, $F(z)$ is quasi-stable if and only if $\widetilde{R}_F(z)$ is a matrix-valued Stieltjes function.*

The following example shows how to use Theorem 5 to test the quasi-stability of a comonic matrix polynomial.

Example 2. *Let $F(z) \in \mathbb{C}[z]^{2\times 2}$ be a comonic matrix polynomial of degree 3, given as*

$$F(z) = \begin{bmatrix} 2 & 7\mathrm{i} \\ 0 & 10 \end{bmatrix} z^3 + \begin{bmatrix} 4 & -\mathrm{i} \\ 3\mathrm{i} & 6 \end{bmatrix} z^2 + \begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 2 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with the even and odd parts

$$\begin{aligned}
F_e(z) &= \begin{bmatrix} 4 & -\mathrm{i} \\ 3\mathrm{i} & 6 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
F_o(z) &= \begin{bmatrix} 2 & 7\mathrm{i} \\ 0 & 10 \end{bmatrix} z + \begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 2 \end{bmatrix},
\end{aligned}$$

, respectively. By a direct computation, we have that $F_e(z)$ and $F_o(z)$ are right coprime and

$$R_F(z) = F_o(-z)(F_e(-z))^{-1} = A + \frac{E_1}{\frac{1}{3} - z} + \frac{E_2}{\frac{1}{7} - z}$$

is a symmetric rational matrix-valued function with respect to the real line, where

$$A = \frac{1}{21}\begin{bmatrix} 33 & 30\mathrm{i} \\ -30\mathrm{i} & 40 \end{bmatrix} \succeq 0, \quad E_1 = -\frac{1}{36}\begin{bmatrix} 9 & 3\mathrm{i} \\ -3\mathrm{i} & 1 \end{bmatrix} \preceq 0, \quad E_2 = \frac{5}{196}\begin{bmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix} \succeq 0.$$

Obviously, $R_F(z)$ is not a matrix-valued Stieltjes function. Then, by Theorem 5, $F(z)$ is not quasi-stable.

4. Stieltjes Property of Hurwitz Stable Matrix Polynomials: The Comonic Case

In this section, we extend Theorems 1 and 2 to the comonic case, in which the leading coefficient of the tested matrix polynomial is unnecessarily non-singular. For this reason, the following lemmas are needed.

Lemma 7. *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be a comonic matrix polynomial. Then, $F(z)$ is Hurwitz stable if and only if $Rev[F](z)$ is quasi-stable and $\sigma(Rev[F]) \subseteq \mathbb{C}_l \cup \{0\}$.*

Proof. Since $F(z)$ is comonic of degree n, we have that $0 \notin \sigma(F)$. For any non-zero $\lambda \in \mathbb{C}$,

$$\det F(\lambda) = \det(\lambda^n Rev[F](\lambda^{-1})) = \lambda^{np} \det Rev[F](\lambda^{-1}),$$

which implies that $\lambda \in \sigma(F)$ if and only if $\lambda^{-1} \in \sigma(Rev[F])$. Obviously, $\lambda \in \mathbb{C}_l$ if and only if $\lambda^{-1} \in \mathbb{C}_l$. Then, we have

$$\begin{aligned} F(z) \text{ is Hurwitz stable} &\iff \sigma(F) \subseteq \mathbb{C}_l \\ &\iff \sigma(Rev[F]) \setminus \{0\} \subseteq \mathbb{C}_l \\ &\iff \sigma(Rev[F]) \subseteq \mathbb{C}_l \cup \{0\}. \end{aligned}$$

In this case, $Rev[F](z)$ is apparently quasi-stable. Then, we complete the proof. □

Lemma 8. *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be a comonic matrix polynomial of degree $n = 2m + j$ for $j = 0$ or $j = 1$, and $\widetilde{F}(z)$ be a GRCD of $Rev_m[F_e](z)$ and $Rev_{m+j-1}[F_o](z)$. If $F_e(z)$ and $F_o(z)$ are right coprime, then $\sigma(\widetilde{F}) \subseteq \{0\}$.*

Proof. Under the assumption of the lemma, there exist two unimodular matrix polynomials $U_1(z)$ and $U_2(z)$, such that

$$U_1(z)\begin{bmatrix} F_e(z) \\ F_o(z) \end{bmatrix} = \begin{bmatrix} I_p \\ 0 \end{bmatrix}, \quad U_2(z)\begin{bmatrix} Rev_m[F_e](z) \\ Rev_{m+j-1}[F_o](z) \end{bmatrix} = \begin{bmatrix} \widetilde{F}(z) \\ 0 \end{bmatrix}.$$

Note that, for any non-zero $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} \operatorname{rank}\widetilde{F}(\lambda) &= \operatorname{rank}\begin{bmatrix} \widetilde{F}(\lambda) \\ 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} Rev_m[F_e](\lambda) \\ Rev_{m+j-1}[F_o](\lambda) \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} \lambda^m F_e(\lambda^{-1}) \\ \lambda^{m+j-1} F_o(\lambda^{-1}) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} F_e(\lambda^{-1}) \\ F_o(\lambda^{-1}) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} I_p \\ 0 \end{bmatrix} = p, \end{aligned}$$

which means that $\widetilde{F}(\lambda)$ is non-singular. Then, $\sigma(\widetilde{F}) \subseteq \{0\}$. □

Lemma 9. *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be a monic matrix polynomial and $\widehat{F}(z)$ be a GRCD of $F_e(z^2)$ and $zF_o(z^2)$. If $R_F(z)$ defined by (2) (resp. $\widetilde{R}_F(z)$ defined by (3)) is a symmetric rational matrix-valued function with respect to the real line when* $\deg F(z)$ *is even (resp. odd), then*

$$\delta(F) = \delta(\widehat{F}).$$

Proof. The proof can be divided into two cases.

Case I: $\deg F(z) = 2m$. Let $R_F(z)$ admit the Laurent series expansion

$$R_F(z) = -\sum_{j=0}^{\infty} z^{-(j+1)}\mathbf{s}_j, \quad z \to \infty.$$

Then, $\mathbf{s}_j$ $(j = 0, 1, \cdots)$ are Hermitian matrices, and the rational matrix-valued function $zF_o(-z^2)(F_e(-z^2))^{-1}$ admits the following Laurent series expansion

$$zF_o(-z^2)(F_e(-z^2))^{-1} = zR_F(z^2) = -\sum_{j=0}^{\infty} z^{-(2j+1)}\mathbf{s}_j, \quad z \to \infty.$$

Let H be a Hermitian block Hankel matrix defined by

$$H = \begin{bmatrix} \mathbf{s}_0 & 0_p & \mathbf{s}_1 & \cdots & \mathbf{s}_{m-1} & 0_p \\ 0_p & \mathbf{s}_1 & 0_p & \cdots & 0_p & \mathbf{s}_m \\ \mathbf{s}_1 & 0_p & \mathbf{s}_2 & \cdots & \mathbf{s}_m & 0_p \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{s}_{m-1} & 0_p & \mathbf{s}_m & \cdots & \mathbf{s}_{2m-2} & 0_p \\ 0_p & \mathbf{s}_m & 0_p & \cdots & 0_p & \mathbf{s}_{2m-1} \end{bmatrix}.$$

We can easily check that H is congruent to the following block diagonal matrix

$$\begin{bmatrix} H_{m-1}^{\mathscr{S}} & \\ & H_{1,m-1}^{\mathscr{S}} \end{bmatrix},$$

in which $\mathscr{S} = (\mathbf{s}_j)_{j=0}^{\infty}$. Since $\widehat{F}(\mathrm{i}z)$ is a GRCD of $zF_o(-z^2)$ and $F_e(-z^2)$, by [13] (Lemma A.2) we have

$$\dim\mathrm{Ker}\, H = \deg\det \widehat{F}(\mathrm{i}z) = \deg\det \widehat{F}(z),$$

or equivalently,

$$\delta(H_{m-1}^{\mathscr{S}}) + \delta(H_{1,m-1}^{\mathscr{S}}) = \deg\det \widehat{F}(z). \tag{11}$$

Due to [23] (Lemma 2.1), we have

$$\delta(F) = \delta(H_{m-1}^{\mathscr{S}}) + \delta(H_{1,m-1}^{\mathscr{S}}) - \pi(\widehat{F}) - \nu(\widehat{F}). \tag{12}$$

A combination of (11) and (12) yields that $\delta(F) = \delta(\widehat{F})$.

Case I: $\deg F(z) = 2m+1$. Let $\widetilde{R}_F(z)$ admit the Laurent series expansion

$$\widetilde{R}_F(z) = \sum_{j=0}^{\infty} z^{-j}\mathbf{s}_j, \quad z \to \infty.$$

Then, $\mathbf{s}_j$ $(j = 0, 1, \cdots)$ are Hermitian matrices and the Laurent series expansion of the rational matrix-valued function $F_e(-z^2)(zF_o(-z^2))^{-1}$ is of the form:

$$F_e(-z^2)(zF_o(-z^2))^{-1} = -z\widetilde{R}_F(z^2) = -\sum_{j=0}^{\infty} z^{-(2j+1)}\mathbf{s}_j, \quad z \to \infty.$$

We define a Hermitian block Hankel matrix $\widetilde{H}$ by

$$\widetilde{H} = \begin{bmatrix} \mathbf{s}_0 & 0_p & \mathbf{s}_1 & \cdots & \mathbf{s}_m \\ 0_p & \mathbf{s}_1 & 0_p & \cdots & 0_p \\ \mathbf{s}_1 & 0_p & \mathbf{s}_2 & \cdots & \mathbf{s}_{m+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{s}_m & 0_p & \mathbf{s}_{m+1} & \cdots & \mathbf{s}_{2m} \end{bmatrix}.$$

It is not difficult to check that $\widetilde{H}$ is congruent to the following block diagonal matrix

$$\begin{bmatrix} H_m^{\mathscr{S}} & \\ & H_{1,m-1}^{\mathscr{S}} \end{bmatrix},$$

in which $\mathscr{S} = (\mathbf{s}_j)_{j=0}^{\infty}$. Note that $\widehat{F}(\mathrm{i}z)$ is a GRCD of $zF_o(-z^2)$ and $F_e(-z^2)$. Then, by [13] (Lemma A.2), we have

$$\dim\mathrm{Ker}\, \widetilde{H} = \deg\det \widehat{F}(\mathrm{i}z) = \deg\det \widehat{F}(z),$$

or equivalently,

$$\delta(H_m^{\mathscr{S}}) + \delta(H_{1,m-1}^{\mathscr{S}}) = \deg\det \widehat{F}(z). \tag{13}$$

Due to [23] (Lemma 2.1), we have

$$\delta(F) = \delta(H_m^{\mathscr{S}}) + \delta(H_{1,m-1}^{\mathscr{S}}) - \pi(\widehat{F}) - \nu(\widehat{F}). \tag{14}$$

A combination of (13) and (14) leads to $\delta(F) = \delta(\widehat{F})$. Then, the proof is complete. □

Based on the above lemmas, we obtain the following relationship between the Hurwitz stability and the Stieltjes property of comonic matrix polynomials.

Theorem 7. *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be comonic, in which $F_e(z)$ and $F_o(z)$ are right coprime, and let $R_F(z)$ defined by (2) be symmetric with respect to the real line. Then, $F(z)$ is Hurwitz stable if and only if $R_F(z)$ is a matrix-valued Stieltjes function.*

Proof. Since $F_e(z)$ and $F_o(z)$ is right coprime, $F(z)$ satisfies Assumption 1. Then, the "only if" part is a direct consequence of Theorem 5. Now, we prove the "if" part. Suppose that $R_F(z)$ is a matrix-valued Stieltjes function. Due to Theorem 5, $F(z)$ is quasi-stable, and thus by Lemma 4, $Rev[F](z)$ is quasi-stable as well.

Let $\deg F = 2m + j$ for $j = 0$ or $j = 1$. Note that $Rev[F](z)$ is monic and $Rev_m[F_o](z)$ and $Rev_{m+j-1}[F_e](z)$ are the even part and the odd part of $Rev[F](z)$, respectively. Let $\widehat{F}(z)$ be a GRCD of $Rev_m[F_e](z^2)$ and $zRev_{m+j-1}[F_o](z^2)$. In view of Lemma 9, we have

$$\delta(Rev[F]) = \delta(\widehat{F}).$$

Observe that $\widehat{F}(z)$ is a right common divisor of $Rev[F](z)$. The last equation implies that $Rev[F](z)$ and $\widehat{F}(z)$ have the same zeros (if they exist) on the imaginary axis. Due to the quasi-stability of $Rev[F](z)$, we have

$$\sigma(Rev[F]) \subseteq \mathbb{C}_l \cup \sigma(\widehat{F}). \tag{15}$$

Let $\widetilde{F}(z)$ be a GRCD of $Rev_m[F_e](z)$ and $Rev_{m+j-1}[F_o](z)$. It follows from [23] (Lemma 2.3) that there exists a $L(z) \in \mathbb{C}[z]^{p\times p}$, such that $\sigma(L) \subseteq \{0\}$ and

$$\widehat{F}(z) = L(z)\widetilde{F}(z^2). \tag{16}$$

On the basis of Lemma 8, $\sigma(\widetilde{F}) \subseteq \{0\}$, and thus by (15) and (16), $\sigma(Rev[F]) \subseteq \mathbb{C}_l \cup \{0\}$. Hence, by Lemma 7, $F(z)$ is Hurwitz stable. $\square$

Now, we provide an example to test the Hurwitz stability of a comonic matrix polynomial by Theorem 7.

Example 3. *Let $F(z) \in \mathbb{C}[z]^{2\times 2}$ be a comonic matrix polynomial of degree 4, given as*

$$F(z) = \begin{bmatrix} 3-2\mathrm{i} & -4-6\mathrm{i} \\ -5-2\mathrm{i} & -4+10\mathrm{i} \end{bmatrix} z^4 + \begin{bmatrix} -2-2\mathrm{i} & -4+4\mathrm{i} \\ -5-4\mathrm{i} & -8+10\mathrm{i} \end{bmatrix} z^3 + \begin{bmatrix} 1-2\mathrm{i} & -7 \\ 0 & 5+2\mathrm{i} \end{bmatrix} z^2 + \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The even and odd parts of $F(z)$ are

$$F_e(z) = \begin{bmatrix} 3-2\mathrm{i} & -4-6\mathrm{i} \\ -5-2\mathrm{i} & -4+10\mathrm{i} \end{bmatrix} z^2 + \begin{bmatrix} 1-2\mathrm{i} & -7 \\ 0 & 5+2\mathrm{i} \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$F_o(z) = \begin{bmatrix} -2-2\mathrm{i} & -4+4\mathrm{i} \\ -5-4\mathrm{i} & -8+10\mathrm{i} \end{bmatrix} z + \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix},$$

respectively. A direct calculation shows that $F_e(z)$ and $F_o(z)$ are right coprime and

$$R_F(z) = F_o(-z)(F_e(-z))^{-1} = \frac{E_1}{\frac{1}{2} - z} + \frac{E_2}{\frac{1}{4} - z}$$

is a symmetric rational matrix-valued function with respect to the real line, where

$$E_1 = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \succeq 0, \quad E_2 = \frac{1}{8}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \succeq 0.$$

Hence, $R_F(z)$ is a matrix-valued Stieltjes function. In view of Theorem 7, $F(z)$ is Hurwitz stable.

In a similar way, we can prove that the rational matrix-valued function $R_F(z)$ in Theorem 7 can be replaced by $\widetilde{R}_F(z)$ if it is well defined.

Theorem 8. *Let $F(z) \in \mathbb{C}[z]^{p\times p}$ be comonic, in which $F_e(z)$ and $F_o(z)$ are right coprime and $F_o(z)$ is regular, and let $\widetilde{R}_F(z)$ defined by (3) be symmetric with respect to the real line. Then, $F(z)$ is Hurwitz stable if and only if $\widetilde{R}_F(z)$ is a matrix-valued Stieltjes function.*

We remark that Theorems 7 and 8 are direct generalizations of Theorems 1 and 2, respectively.

5. Conclusions

In this paper, we have revealed some intrinsic connections between the quasi-stability of a monic or comonic matrix polynomial and the Stieltjes property of a rational matrix-valued function constructed by the even–odd split of the original matrix polynomial. These connections provide us with new ways to test the quasi-stability of matrix polynomials under some natural assumptions. Moreover, applying these results, we have obtained two criteria for the Hurwitz stability of comonic matrix polynomials. We remark that the constant term of a Hurwitz stable matrix polynomial is always non-singular, and the Hurwitz stability of a matrix polynomial with a non-singular constant term is equivalent to that of a certain comonic matrix polynomial. Then, to investigate the Hurwitz stability of a matrix polynomial, we assume that it is comonic without loss of generality. Hence, these two Hurwitz stability criteria presented here are direct generalizations of Theorems 1.1 and

1.2 in [13], where the tested matrix polynomial is assumed to be monic, or equivalently, the leading coefficient matrix is non-singular.

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