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Dynamics of Fractional Stochastic Ginzburg–Landau Equation Driven by Nonlinear Noise

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Abstract: In this work, we focus on the long-time behavior of the solutions of the stochastic fractional complex Ginzburg–Landau equation defined on $\mathbb{R}^n$ with polynomial drift terms of arbitrary order. The well-posedness of the equation based on pathwise uniform estimates and uniform estimates on average are proved. Following this, the existence and uniqueness of weak pullback random attractors are establsihed.

Keywords: fractional complex Ginzburg–Landau equation; mean random attractor; nonlinear noise; unbounded domain; locally Lipschitz continuous

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1. Introduction

In this paper, we investigate the random dynamics of the stochastic fractional Ginzburg–Landau equation defined on $\mathbb{R}^n$ with polynomial drift terms of arbitrary order. To be specific, we consider the following stochastic fractional complex Ginzburg–Landau equation on $\mathbb{R}^n$, for $t > 0$ and given $\alpha \in (0,1)$,

$$\begin{aligned} du(t) + (1+\mathrm{i}\nu)(-\Delta)^\alpha u(t)dt + (1+\mathrm{i}\mu)|u(t)|^{2\beta}u(t)dt =& \rho u(t)dt + g(t,x)dt \\ &+ \sigma(t,\omega,u(t))dW(t), \end{aligned} \tag{1}$$

with initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where $u(x,t)$ is a complex-valued function on $\mathbb{R}^n \times [0,+\infty)$. In (1), i is the imaginary unit, α, β, μ, ν and ρ are real constants with $\rho > 0$ and $\beta > 0$, $(-\Delta)^\alpha$ is fractional Laplace operator, $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$ is given, σ is a local Lipschitz nonlinear diffusion coefficient, and W is a two-sided cylindrical Wiener process in a Hilbert space defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, P)$, $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub-σ-algebras of $\mathcal{F}$ that contains all P-null sets. For simplicity in our discussion, we write $p = 2\beta + 2$ and $q = \frac{2\beta+2}{2\beta+1}$.

The Ginzburg–Landau equation [1,2] is one of the most studied nonlinear equations in physics. It describes a vast variety of phenomena from nonlinear waves to second-order phase transitions, from superconductivity, superfluidity, and Bose–Einstein condensation to liquid crystals and strings in field theory. The Ginzburg–Landau equation with fractional derivatives [3] is used to describe processes in media with fractal dispersion or long-range interaction. In [4], the authors analyzed a one-dimensional fractional complex Ginzburg–Landau equation. In [5], the dynamics of a two-dimensional fractional complex Ginzburg–Landau equations is studied. In [6], the authors studied the dynamics of 3-D fractional complex Ginzburg–Landau equation. During the derivation of these ideal models, small perturbations (such as molecular collisions in gases and liquids and electric fluctuations in resistors) may be neglected. Therefore, one may represent the micro-effects by random perturbations in the dynamics of the macro observable through additive or multiplicative noise in the governing equation.

In the past two decades, a great deal with mathematical efforts has been devoted to the fractional Ginzburg–Landau equation which is driven by an additive noise or a linear multiplicative noise. Respectively, a fractional Ginzburg–Landau equation on the line with special nonlinearity and multiplicative noise was analyzed in [7]. A stochastic fractional complex Ginzburg–Landau equation with multiplicative noise in three spatial dimensions was studied in [8]. In [9], the author established fractional stochastic Ginzburg–Landau equation driven by colored noise with a nonlinear diffusion term to the case where $\alpha \in (\frac{1}{2}, 1)$. Time-space fractional stochastic Ginzburg–Landau equations are also studies in [10,11]. Considering the complexity of the environment, many disturbances can not be described by multiplicative noise or additive noise, and nonlinear noise can better fit the phenomenon, at this point it is very necessary to study nonlinear noise. However, in spite of quite contributions about these literature, there are no result taking into account of the existence of pathwise pullback random attractors for the stochastic equation (1) with a nonlinear diffusion term σ.

The purpose of this paper is to establish the well-posedness of (1) and (2) in $L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^n))$ and study the mean random dynamical system generated by the solution operators. The counterpart of the concept of mean random dynamical system is the pathwise random dynamical system. The global attractors for pathwise random dynamical system have been extensively studied, see, e.g., [12–23] and [24–35] for autonomous and non-autonomous stochastic equations, respectively. There are few results about mean random dynamical system ([36,37]), but these results are about real-valued functions. This paper is about complex-valued function.

In Equation (1), we assume that the diffusion coefficient $\sigma(t, \omega, u(t)) : \mathbb{R} \times \Omega \times H \to \mathcal{L}_2(U, H)$ is locally Lipschitz continuous in its third argument uniformly for $(t, \omega) \in \mathbb{R} \times \Omega$; namely, for every $r > 0$, there exists a positive number M_r depending on r such that for all $t \in \mathbb{R}, \omega \in \Omega$ and $u_1, u_2 \in H$ with $\|u_1\| \le r$ and $\|u_2\| \le r$,

$$\|\sigma(t, \omega, u_1) - \sigma(t, \omega, u_2)\|_{\mathcal{L}_2(U,H)} \le M_r \|u_1 - u_2\|. \tag{3}$$

In addition, $\sigma(t, \omega, u)$ grows linearly in $u \in H$ uniformly for $(t, \omega) \in \mathbb{R} \times \Omega$; that is, there exists a positive number L such that for all $t, \omega, u \in \mathbb{R} \times \Omega \times H$,

$$\|\sigma(t, \omega, u)\|_{\mathcal{L}_2(U,H)} \le L(1 + \|u\|). \tag{4}$$

We further assume that $\sigma(t, \omega, u) : \mathbb{R} \times \Omega \to \mathcal{L}_2(U, H)$ is progressively measurable for every fixed $u \in H$.

The arrangement of the article is as follows. In Section 2, we introduce some related concepts and preliminaries. In Section 3, we prove the well-posedness of (1) and (2) driven by regular additive noise. In Section 4, we study the existence and uniqueness of solutions with general additive noise. In Section 5 and Section 6, we respectively investage the well-posedness of (1) and (2) with globally and locally Lipschitz continuous diffusion coefficients. In the last Section, we focus on the existence and uniqueness of weak pullback random attractor for (1) and (2).

2. Preliminaries and Notations

In this section, we first recall the concept of the fractional Laplace operator on $\mathbb{R}^n$ as well as the definition of some spaces, norm and inner product. Then, we introduce the concept of weak pullback mean random attractors for mean random dynamical systems Φ over filtered probability spaces and the definition of solutions for the stochastic equations under investigation. At the last of this section, we list some inequalities and theorems which will be used in this paper.

Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions on $\mathbb{R}^n$. Then by [38], we have for $0 < \alpha < 1$ the fractional Laplace operator $(-\Delta)^\alpha$ is defined by

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}C(n,\alpha)\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2\alpha}}dy, \ \ x \in \mathbb{R}^n, for\ u \in \mathcal{S}, \quad (5)$$

where $C(n,\alpha)$ is a positive constant given by

$$C(n,\alpha) = \frac{\alpha 4^\alpha \Gamma(\frac{n+2\alpha}{2})}{\pi^{\frac{n}{2}}\Gamma(1-\alpha)}. \quad (6)$$

For $0 < \alpha < 1$, the fractional Sobolev space $H^\alpha(\mathbb{R}^n)$ is defined by

$$H^\alpha(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}dxdy < \infty\},$$

endowed with the norm

$$\|u\|_{H^\alpha(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n}|u(x)|^2dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}dxdy\right)^{\frac{1}{2}}.$$

By [39], The norm $\|u\|_{H^\alpha(\mathbb{R}^n)}$ is equivalent to the norm $(\|u\|^2_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{\frac{\alpha}{2}}u\|^2_{L^2(\mathbb{R}^n)})^{\frac{1}{2}}$ for $u \in H^\alpha(\mathbb{R}^n)$; more precisely, we have

$$\|u\|^2_{H^\alpha(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \frac{2}{C(n,\alpha)} \parallel (-\Delta)^{\frac{\alpha}{2}}u\|^2_{L^2(\mathbb{R}^n)}, for\ all\ u \in H^\alpha(\mathbb{R}^n). \quad (7)$$

The inner product of $H^\alpha(\mathbb{R}^n)$ in complex field is defined by

$$(u,v)_{H^\alpha(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)\bar{v}(x)dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(\bar{v}(x)-\bar{v}(y))}{|x-y|^{n+2\alpha}}dxdy,\ u,\ v\ \in\ H^\alpha(\mathbb{R}^n).$$

For convenience, we write $H = L^2(\mathbb{R}^n)$ and $V = H^\alpha(\mathbb{R}^n)$. Then, we have $V \hookrightarrow H = H^* \hookrightarrow V^*$, where H^* and V^* are the dual spaces of H and V, respectively, H^* is identified with H by Riesz's representation theorem. We respectively denote the norm and the inner product of $L^2(\mathbb{R}^n)$ by $\|\cdot\|$ and $(\cdot,\cdot)$. $\mathcal{L}_{2(U,H)}$ is used for the space of Hilbert-Schmidt operators from a separable Hilbert space U to H with norm $\|\cdot\|_{\mathcal{L}_2(U,H)}$.

Let $\mathcal{D}$ be a collection of some families of nonempty bounded subsets of $L^2(\Omega,\mathcal{F}_\tau;H)$ parametrized by $\tau \in \mathbb{R}$, that is

$$\mathcal{D} = \Big\{D = \{D(\tau) \subseteq L^2(\Omega,\mathcal{F}_\tau;H) : D(\tau) \neq \varnothing\ bounded,\ \tau \in \mathbb{R}\}$$

$$: D\ satisfies\ \lim_{\tau\to-\infty} e^{\rho\tau}\|D(\tau)\|^2_{L^2(\Omega,\mathcal{F}_\tau;H)=0}\Big\}, \quad (8)$$

where $\|D\|_{L^2(\Omega,\mathcal{F}_\tau;H)} = \sup_{u\in D}\|u\|_{L^2(\Omega,\mathcal{F}_\tau;H)}$ for a subset D in $L^2(\Omega,\mathcal{F}_\tau;H)$.

Definition 1 ([40]). *$\mathcal{D}$ is called inclusion-closed if $D \in \mathcal{D}$ and if $\tilde{D} = \{\tilde{D}(\tau)\}_{\tau\in\mathbb{R}}$ is a random subset of H with $\tilde{D}(\tau) \subseteq D(\tau)$ for all $\tau \in \mathbb{R}$ then $\tilde{D} \in \mathcal{D}$.*

Definition 2 ([36]). *A family $\Phi = \{\Phi(t,\tau) : t \in \mathbb{R}^+, \tau \in \mathbb{R}\}$ of mapping is called a mean random dynamical system on $L^2(\Omega,\mathcal{F};H)$ over $L^2(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},P)$ if for all $\tau \in \mathbb{R}$ and $t,s \in \mathbb{R}^+$,*

(i) $\Phi(t,\tau)$ *maps* $L^2(\Omega,\mathcal{F}_\tau,H)$ *to* $L^2(\Omega,\mathcal{F}_{t+\tau},H)$,

(ii) $\Phi(0,\tau)$ *is the identity operator on* $L^2(\Omega,\mathcal{F}_\tau,H)$*,*
(iii) $\Phi(t+s,\tau)=\Phi(t,\tau+s)\circ\Phi(s,\tau)$*.*

Definition 3 ([36])**.** *A family* $K=\{K(\tau):\tau\in\mathbb{R}\}\in\mathcal{D}$ *is called a* $\mathcal{D}$*-pullback weakly attracting set of mean random dynamical system* Φ *on* $L^2(\Omega,\mathcal{F};H)$ *over* $L^2(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},P)$*, if for every* $\tau\in\mathbb{R}$*,* $D\in\mathcal{D}$ *and every weak neighborhood* $\mathcal{N}^\omega(K(\tau))$ *of* $K(\tau)$ *in* $L^2(\Omega,\mathcal{F};H)$*, there exists* $T=T(\tau,D,\mathcal{N}^\omega(K(\tau)))>0$ *such that for all* $t\geq T$*,*

$$\Phi(t,\tau-t)(D(\tau-t))\subseteq\mathcal{N}^\omega(K(\tau)),$$

where $\mathcal{N}^\omega(K(\tau))$ *is the weak neighborhood of* $K(\tau)$*. For a subset* $K(\tau)\in L^2(\Omega,\mathcal{F},H)$*, every weakly open set containing* $K(\tau)$ *is called a weak neighborhood of* $K(\tau)$ *in* $L^2(\Omega,\mathcal{F},H)$*. In addition, if* $K(\tau)$ *is a weakly compact subset of* $L^2(\Omega,\mathcal{F};H)$ *for every* $\tau\in\mathbb{R}$*, then* $K=\{K(\tau):\tau\in\mathbb{R}\}$ *is called a* $\mathcal{D}$*-pullback weakly compact weakly attracting set for* Φ*.*

Definition 4 ([36])**.** *A family* $\mathcal{A}=\{\mathcal{A}(\tau):\tau\in\mathbb{R}\}\in\mathcal{D}$ *is called a weak* $\mathcal{D}$*-pullback mean random attractor for* Φ *on* $L^2(\Omega,\mathcal{F};H)$ *over* $L^2(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},P)$ *if the following conditions are fulfilled,*

(i) $\mathcal{A}(\tau)$ *is a weakly compact subset of* $L^2(\Omega,\mathcal{F}_\tau;H)$ *for every* $\tau\in\mathbb{R}$*,*
(ii) $\mathcal{A}$ *is a* $\mathcal{D}$*-pullback weakly attracting set of* Φ*,*
(iii) $\mathcal{A}$ *is the minimal element of* $\mathcal{D}$ *with properties (i) and (ii), that is, if* $D=\{D(\tau):\tau\in\mathbb{R}\}\in\mathcal{D}$ *is a* $\mathcal{D}$*-pullback weakly compact weakly attracting set of* Φ*, then* $\mathcal{A}(\tau)\subseteq D(\tau)$ *for all* $\tau\in\mathbb{R}$*.*

Theorem 1 ([36])**.** *Let* $\mathcal{D}$ *be an inclusion-closed collection of some families of nonempty bounded subsets of* $L^p(\Omega,\mathcal{F};H)$ *as given by* (8)*. If* Φ *has a weakly compact* $\mathcal{D}$*-pullback absorbing set* $B\in\mathcal{D}$ *on* $L^2(\Omega,\mathcal{F};H)$ *over* $L^2(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},P)$*, then* Φ *has a unique weak* $\mathcal{D}$*-pullback mean attractor* $\mathcal{A}\in\mathcal{D}$ *on* $L^2(\Omega,\mathcal{F};H)$ *over* $L^2(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},P)$*, which is given by, for each* $\tau\in\mathbb{R}$*,*

$$\mathcal{A}(\tau)=\Omega^\omega(B,\tau)=\bigcap_{r\geq 0}\overline{\bigcup_{t\geq r}\Phi(t,\tau-t)(B(\tau-t))}^{\omega},$$

where the closure is taken with respect to the weak topology of $L^2(\Omega,\mathcal{F};H)$*.*

Definition 5. *Let* $u_0\in L^2(\Omega,H)$ *be* $\mathcal{F}_0$*-measurable. Then, a continuous* H*-valued* $\mathcal{F}_t$*-adapted stochastic process* u *is called a solution of* (1) *and* (2) *if*

$$u\in L^2(\Omega,C([0,T],H))\bigcap L^2(\Omega,L^2(0,T;V))\bigcap L^p(\Omega,L^p(0,T;L^p(\mathbb{R}^n))),\ \forall T>0, \tag{9}$$

such that for all $t>0$ *and* $\xi\in V\bigcap L^p(\mathbb{R}^n)$*,*

$$(u(t),\xi)+(1+\mathrm{i}\nu)\int_0^t((-\Delta)^{\frac{\alpha}{2}}u(s),(-\Delta)^{\frac{\alpha}{2}}\xi)ds+\int_0^t\int_{\mathbb{R}^n}(1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s)\xi(x)dxds$$

$$=(u_0,\xi)+\rho\int_0^t(u(s),\xi)ds+\int_0^t(g(s),\xi)ds+\int_0^t\xi\sigma(s,u(s))dW, \tag{10}$$

P—almost surely, where ξ *in the stochastic term is identified with the element in* $H^*=H$ *by Riesz's representation theorem.*

Note that if u is a solution of (1) and (2) in the sense of Definition 5, then by (9) we have

$$(-\Delta)^\alpha u\in L^2(\Omega,L^2(0,T;V^*))\,.$$

Consequently, a continuous H-valued $\mathcal{F}_t$-adapted stochastic process u is a solution of (1) and (2) in the sense of Definition 5 if and only if u satisfies (9), and for all $t \geq 0$,

$$u(t) + (1 + \mathrm{i}\,\nu) \int_0^t (-\Delta)^\alpha u(s) ds + \int_0^t (1 + \mathrm{i}\mu)|u(s)|^{2\beta} u(s) ds = u_0$$

$$+\rho \int_0^t u(s) ds + \int_0^t g(s) ds + \int_0^t \sigma(s, u(s)) dW \; in \; (V \bigcap L^p(\mathbb{R}^n))^*, \tag{11}$$

P–almost surely. In other words, (10) and (11) are equivalent.

3. Existence of Solutions: Regular and Additive Noise

In this section, we study the well-posedness of solution to problem (1) and (2) with a diffusion term σ taking values in a regular space.

Let V_0 be a separable Hilbert space satisfies $V_0 \hookrightarrow V$ and $V_0 \hookrightarrow L^p(\mathbb{R}^n)$. In this section, we assume that $\sigma : \mathbb{R} \times \Omega \to \mathcal{L}_2(U, V_0)$ is a progressively measurable process such that

$$\sigma \in L^2(\Omega, L^2(0, T; \mathcal{L}_2(U, V_0))) \;\; for\ every\ T > 0. \tag{12}$$

Considering the following stochastic equation with additive noise:

$$\begin{aligned} du(t) + (1 + \mathrm{i}\nu)(-\Delta)^\alpha u(t) + (1 + \mathrm{i}\mu)|u(t, x)|^{2\beta} u(t, x) dt \\ = \rho u(t) dt + g(t, x) dt + \sigma(t, \omega) dW, \;\; x \in \mathbb{R}^n, \; t > 0, \end{aligned} \tag{13}$$

with the initial condition

$$u(0, x) = u_0(x), \;\; x \in \mathbb{R}^n. \tag{14}$$

We need to approximate the locally Lipschitz nonlinearity $(1 + \mathrm{i}\mu)|u|^{2\beta} u$ by a globally Lipschitz function to prove the existence and uniqueness of solutions to (13) and (14). Therefore, for every $n \in N$, we define a function $\xi_n : \mathbb{C} \to \mathbb{C}$ by

$$\xi_n(s) = \begin{cases} s & if\ |s| \leq n, \\ \frac{ns}{|s|} & if\ |s| > n. \end{cases}$$

Then, $\xi_n : \mathbb{C} \to \mathbb{C}$ is globally Lipschitz continuous. In fact, we have $\xi_n(0) = 0$,

$$|\xi_n(s_1) - \xi_n(s_2)| \leq |s_1 - s_2|, for\ all\ s_1, s_2 \in \mathbb{C}, \tag{15}$$

and

$$|\xi_n(s)| \leq n, \; |\xi_n(s)| \leq |s| \;\; for\ all\ s \in \mathbb{C}. \tag{16}$$

Given $n \in \mathbb{N}$, for almost all $(t, x) \in [0, T] \times \mathbb{R}^n$, we choose a globally Lipschitz continuous function $(1 + \mathrm{i}\mu)|\xi_n(u)|^{2\beta} \xi_n(u)$; exactly, for every $n \in \mathbb{N}$, there exists $c_n > 0$ such that

$$\left| (1 + \mathrm{i}\mu)|\xi_n(u_1)|^{2\beta} \xi_n(u_1) - (1 + \mathrm{i}\mu)|\xi_n(u_2)|^{2\beta} \xi_n(u_2) \right| \leq c_n |u_1 - u_2|, \tag{17}$$

for all $u_1, u_2 \in \mathbb{C}$ and for almost all $(t, x) \in [0, T] \times \mathbb{R}^n$. By (17) we obtain, for almost all $t \in [0, T]$,

$$\|(1 + \mathrm{i}\mu)|\xi_n(u)|^{2\beta} \xi_n(u) - (1 + \mathrm{i}\mu)|\xi_n(v)|^{2\beta} \xi_n(v)\| \leq c_n \|u - v\|, \;\; for\ all\ u, v \in H. \tag{18}$$

Since $(1+\mathrm{i}\mu)|\xi_n(0)|^{2\beta}\xi_n(0)=0$, by (18) we obtain, for almost all $t\in[0,T]$,

$$\|(1+\mathrm{i}\mu)|\xi_n(u)|^{2\beta}\xi_n(u)\|\leq c_n\|u\|,\quad for\ all\ u\in H. \tag{19}$$

In addition, for all $t\in\mathbb{R}$, we infer that

$$\mathrm{Re}\Big((1+\mathrm{i}\mu)|\xi_n(u)|^{2\beta}\xi_n(u)-(1+\mathrm{i}\mu)|\xi_n(v)|^{2\beta}\xi_n(v),u-v\Big)\geq 0,\quad for\ all\ u,v\in H \tag{20}$$

and by the definition of ξ_n, we deduce

$$\bar{\xi}_n(u_n)u_n\geq|\xi_n(u_n)|^2. \tag{21}$$

Given $n\in\mathbb{N}$, consider the following approximate stochastic equation for (13) and (14) in V^* for $t>0$:

$$du_n(t)+(1+\mathrm{i}\nu)(-\Delta)^{\alpha}u_n(t)dt+(1+\mathrm{i}\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n)dt=\rho u_n(t)dt+g(t)dt+\sigma(t,\omega)dW, \tag{22}$$

with initial condition

$$u_n(0)=u_0. \tag{23}$$

By (18)–(20), it follows from [41] that for every $\mathcal{F}_0$-measurable $u_0\in L^2(\Omega,H)$, problem (22) and (23) has a unique solution u_n in the sense that u_n is an H-valued $\mathcal{F}_t$-adapted continuous process such that

$$u_n\in L^2(\Omega,C([0,T],H))\bigcap L^2(\Omega,L^2(0,T;V)),\ \forall\, T>0,$$

and for all $t\geq 0$,

$$u_n(t)+(1+\mathrm{i}\nu)\int_0^t(-\Delta)^{\alpha}u_n(s)ds+\int_0^t(1+\mathrm{i}\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n)ds=u_0+\rho\int_0^t u_n(s)ds$$

$$+\int_0^t g(s)ds+\int_0^t\sigma(s)dW(s)\ in\ V^*, \tag{24}$$

P–almost surely.

Next, we will derive uniform estimates of the approximate solution u_n and prove the limit of this sequence is a solution of problem (13) and (14). The first uniform estimate of u_n is given below.

Lemma 1. *Suppose* (12) *holds, then there exists a subset Ω_0 of Ω with $P(\Omega_0)=1$ such that for all $\omega\in\Omega_0$, the solution u_n of* (22) *and* (23) *satisfies*

$$\|u_n(\omega)\|^2_{C([0,T],H)}+\|u_n(\omega)\|^2_{L^2(0,T;V)}+\|\xi_n(u_n(\omega)\|^p_{L^p(0,T;L^p(\mathbb{R}^n))}$$

$$+\|(1+i\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n)\|^q_{L^q(0,T;L^q(\mathbb{R}^n))}\leq L(T,\omega),$$

where $L(T,\omega)$ is a positive number depending only on T and ω, but independent of $n\in\mathbb{N}$.

Proof. Let $v_n(t)=u_n(t)-\int_0^t\sigma(s)dW(s)$, then we have $v_n(t)\in L^2(\Omega,L^2(0,T;V))$, which implies that there exists a subset Ω_1 of Ω with $P(\Omega_1)=1$ such that for all $\omega\in\Omega_1$,

$$v_n\in L^2(0,T;V). \tag{25}$$

On the other hand, by (24) we find that there exists a subset Ω_2 of Ω with $P(\Omega_2)=1$ such that for all $\omega\in\Omega_2$ and $t\geq 0$,

$$
\begin{aligned}
v_n(t) =& u_0 - (1+\mathrm{i}\nu)\int_0^t ((-\Delta)^\alpha u_n(s)ds - \int_0^t (1+\mathrm{i}\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n))ds \\
&+\rho\int_0^t u_n(s)ds + \int_0^t g(s)ds \ in \ V^*.
\end{aligned} \tag{26}
$$

By (26) we obtain that for all $\omega \in \Omega_2$,

$$
\frac{dv_n}{dt} = -(1+\mathrm{i}\nu)(-\Delta)^\alpha u_n(t) - (1+\mathrm{i}\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n) + \rho u_n(t) + g(t) \ \ in \ L^2(0,T;V^*). \tag{27}
$$

Let $\Omega_3 = \Omega_1 \bigcap \Omega_2$. Then, we have $P(\Omega_3) = 1$. Moreover, by (25) and (27) we obtain from [42] that, for all $\omega \in \Omega_3$,

$$
\frac{d\|v_n(t)\|^2}{dt} = 2\mathrm{Re}\langle \frac{dv_n}{dt}, v_n(t)\rangle_{(V^*,V)} \tag{28}
$$

on $(0,T)$ in the sense of scalar distribution. It follows from (27) and (28) that for all $\omega \in \Omega_3$,

$$
\frac{d\|v_n(t)\|^2}{dt} = -2\mathrm{Re}((1+\mathrm{i}\nu)(-\Delta)^\alpha u_n(t), v_n(t))
$$

$$
-2\mathrm{Re}\Big((1+\mathrm{i}\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n), v_n(t)\Big) + 2\mathrm{Re}(\rho u_n(t), v_n(t)) + 2\mathrm{Re}(g(t), v_n(t)), \tag{29}
$$

for almost all $t \in [0,T]$.

We now deal with each term on the right-hand side of (29). For the first term on the right-hand side of (29), by Young's inequality, we have

$$
\begin{aligned}
-2\mathrm{Re}((1+\mathrm{i}\nu)(-\Delta)^\alpha u_n(t), v_n(t)) &= -2\mathrm{Re}\Big((1+\mathrm{i}\nu)(-\Delta)^\alpha u_n(t), u_n(t) - \int_0^t \sigma(s)dW(s)\Big) \\
&\le -2\|(-\Delta)^{\frac{\alpha}{2}}u_n(t)\|^2 + \frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}}u_n(t)\|^2 + 2\|(-\Delta)^{\frac{\alpha}{2}}\int_0^t \sigma(s)dW(s)\|^2 \\
&\quad + \frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}}u_n(t)\|^2 + 2\nu^2\|(-\Delta)^{\frac{\alpha}{2}}\int_0^t \sigma(s)dW(s)\|^2 \\
&\le -\|(-\Delta)^{\frac{\alpha}{2}}u_n(t)\|^2 + 2(1+\nu^2)\|\int_0^t \sigma(s)dW(s)\|^2_{H^\alpha}.
\end{aligned} \tag{30}
$$

For the second term on the right-hand side of (29), By (21), we have

$$
\begin{aligned}
&-2\mathrm{Re}\Big((1+\mathrm{i}\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n), v_n(t)\Big) \\
&= -2\mathrm{Re}\Big((1+\mathrm{i}\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n), u_n(t) - \int_0^t \sigma(s)dW(s)\Big) \\
&= -2\int_{\mathbb{R}} |\xi_n(u_n)|^{2\beta}\xi_n(u_n)\bar{u}_n dx + 2\mathrm{Re}\Big((1+\mathrm{i}\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n), \int_0^t \sigma(s)dW(s)\Big) \\
&\le -2\int_{\mathbb{R}^n} |\xi_n(u_n)|^{2\beta+2}dx + 2\Big|\Big((1+\mathrm{i}\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n), \int_0^t \sigma(s)dW(s)\Big)\Big|.
\end{aligned}
$$

Then, we estimate the last term on the right-hand side of above inequality. By Young's inequality, we have

$$\begin{aligned}
&2\left|\left((1+\mathrm{i}\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n), \int_0^t \sigma(s)dW(s)\right)\right| \\
&\leq \int_{\mathbb{R}^n} 2\sqrt{1+\mu^2}|\xi_n(u_n)|^{2\beta+1}\cdot\left|\int_0^t \sigma(s)dW(s)\right|dx \\
&\leq \int_{\mathbb{R}^n} |\xi_n(u_n)|^{2\beta+2}dx + c_1\int_{\mathbb{R}^n}\left|\int_0^t \sigma(s)dW(s)\right|^{2\beta+2}dx,
\end{aligned} \tag{31}$$

where $c_1 = (2\sqrt{1+\mu^2})^{2\beta+2}\frac{(2\beta+1)^{2\beta+1}}{(2\beta+2)^{2\beta+2}}$, $p = 2\beta+2$ in (31). Then, we have,

$$-2\mathrm{Re}\Big((1+\mathrm{i}\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n), v_n(t)\Big) \leq c_1\|\int_0^t \sigma(s)dW(s)\|_{V_0}^p - \int_{\mathbb{R}^n}|\xi_n(u_n)|^p dx, \tag{32}$$

where $p = 2\beta+2$ in (32). For the third term on the right-hand side of (29), we have

$$2\rho\mathrm{Re}\left(v_n(t) + \int_0^t \sigma(s)dW(s), v_n(t)\right) \leq 2\rho\|v_n(t)\|^2 + 2\rho\|\int_0^t \sigma(s)dW(s)\|\cdot\|v_n(t)\|$$

$$\leq (2\rho+\rho^2)\|v_n(t)\|^2 + \|\int_0^t \sigma(s)dW(s)\|_{V_0}^2. \tag{33}$$

For the last term on the right-hand side of (29), we have

$$2\mathrm{Re}(g(t), v_n(t)) \leq \|v_n(t)\|^2 + \|g(t)\|^2, \tag{34}$$

for almost all $t \in [0,T]$, It follows from (29)–(34) that for all $\omega \in \Omega_3$,

$$\begin{aligned}
&\frac{d\|v_n(t)\|^2}{dt} + \|(-\Delta)^{\frac{\alpha}{2}}u_n(t)\|^2 + \int_{\mathbb{R}^n}|\xi_n(u_n(t))|^p dx \\
&\leq c_2\|v_n(t)\|^2 + c_3\|\int_0^t \sigma(s)dW(s)\|_{V_0}^p + \|g(t)\|^2,
\end{aligned} \tag{35}$$

for almost all $t \in [0,T]$, where $c_2 = (\rho+1)^2, c_3 = (c_1 + 2\nu^2 + 3)$. By (12) and Burkholder–Davis–Gundy Inequality, we obtain

$$\mathbb{E}(\sup_{0\leq t\leq T}\|\int_0^t \sigma(s)dW(s)\|_{V_0}^2) \leq c_4\mathbb{E}(\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2(U,V_0)}ds) < \infty,$$

which implies that there exists a subset Ω_4 of Ω with $P(\Omega_4) = 1$ such that for all $\omega \in \Omega_4$,

$$c_5(T,\omega) = \sup_{0\leq t\leq T}\|\int_0^t \sigma(s)dW(s)\|_{V_0} < \infty. \tag{36}$$

Let $\Omega_5 = \Omega_3\bigcap\Omega_4$. Then, $P(\Omega_5) = 1$ and for all $\omega \in \Omega_5$, by (35) and (36) we obtain,

$$\frac{d\|v_n(t)\|^2}{dt} + \|(-\Delta)^{\frac{\alpha}{2}}u_n(t)\|^2 + \int_{\mathbb{R}^n}|\xi_n(u_n(t))|^p dx \leq c_2\|v_n(t)\|^2 + \|g(t)\|^2 + c_3c_5, \tag{37}$$

for almost all $t \in [0,T]$. Multiplying (37) by e^{-c_2t} and then integrating on $(0,t)$, we obtain, for all $\omega \in \Omega_5$ and $0 \leq t \leq T$,

$$e^{-c_2t}\|v_n(t)\|^2 + \int_0^t\left(e^{-c_2s}\left(\|(-\Delta)^{\frac{\alpha}{2}}u_n(s)\|^2 + \int_{\mathbb{R}^n}|\xi_n(u_n(s))|^p dx\right)\right)ds \leq \|v_n(0)\|^2$$

$$+\int_0^t e^{-c_2 s}(\|g(s)\|^2 + c_3 c_5)ds.$$

Therefore,

$$\|v_n(t)\|^2 \le (\|u_0\|^2 + c_6)e^{c_2 t}. \tag{38}$$

By (38) and (36), we have that for all $\omega \in \Omega_5$,

$$\|u_n(t)\|_{C(0,T,H)} = \max_{0\le t\le T} \|u_n(t)\|_H \le c_7, \tag{39}$$

where c_7 is a positive number only depending on T and ω. Integrating (37) on $[0, T]$, by (38) we obtain, for all $\omega \in \Omega_5$,

$$\int_0^T \|(-\Delta)^{\frac{\alpha}{2}} u_n(t)\|^2 dt + \int_0^T \int_{\mathbb{R}^n} |\xi_n(u_n(t))|^p dxdt \le c_8, \tag{40}$$

where c_8 is a positive number depending only T and ω. By (39) and (40) we obtain, for all $\omega \in \Omega_5$,

$$\int_0^T \int_{\mathbb{R}_n} \left|(1 + i\mu|\xi_n(u_n)|^{2\beta}\xi_n(u_n))\right|^q dxdt = c_9 \int_0^T \int_{\mathbb{R}^n} |\xi_n(u_n(t))|^p dxdt, \tag{41}$$

which together with (39)–(41) completes the proof. □

Next, we establish uniform estimates on the expectation of the solution.

Lemma 2. *Suppose* (12) *holds, then the solution $u_n(t)$ of* (22) *and* (23) *satisfies*

$$\|u_n(t)\|^2_{L^2(\Omega,C([0,T],H))} + \|u_n(t)\|^2_{L^2(\Omega,L^2([0,T],V))} + \|\xi_n(u_n)\|^p_{L^p(\Omega,L^p(0,T;L^p(\mathbb{R}^n)))}$$
$$\le L_1(T)(\|u_0\|^2_{L^2(\Omega,H)} + \|g\|^2_{L^2(0,T;H)} + \|\sigma\|^2_{L^2(\Omega,L^2(0,T;\mathcal{L}_2(U,H)))}),$$

where $L_1(T)$ is a positive number only depending on T.

Proof. By (24) and integration by parts of Ito's formula, for all $0 \le t \le T$, we obtain

$$\begin{aligned}&\|u_n(t)\|^2 + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}} u_n(s)\|^2 ds + 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^n} (1 + i\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n)\bar{u}_n dxds \\ &= \|u_0\|^2 + 2\rho\int_0^t \|u_n(s)\|^2 ds + 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^n} g(s)u_n(s)dxds + 2\mathrm{Re}\int_0^t \sigma(s)u_n(s)dW(s) \\ &+ \int_0^t \|\sigma(s)\|^2_{\mathcal{L}_2(U,H)} ds\end{aligned} \tag{42}$$

P–almost surely, by Riesz's representation theorem, u_n in the stochastic term is identified with the element in $H^* = H$. For the third term on the left-hand side of (42), by (21) we have, for $0 \le t \le T$,

$$\begin{aligned}\mathrm{Re}\int_0^t \int_{\mathbb{R}^n} (1 + i\mu)|\xi_n(u_n)|^{2\beta}\xi_n(u_n)\bar{u}_n(s)dxds &= \int_0^t \int_{\mathbb{R}^n} |\xi_n(u_n)|^{2\beta}\xi_n(u_n)\bar{u}_n(s)dxds \\ &\ge \int_0^t \int_{\mathbb{R}^n} |\xi_n(u_n)|^{2\beta+2} dxds.\end{aligned} \tag{43}$$

By Young's inequality, we have

$$2\mathrm{Re}\int_0^t (g(s),u_n(s))ds \le 2\int_0^t |(g(s),u_n(s))|ds \\ \le \int_0^t \|u_n(s)\|^2 ds + \int_0^t \|g(s)\|^2 ds. \tag{44}$$

It follows from (42)–(44) that, for all $0 \le t \le T$,

$$\|u_n(t)\|^2 + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}} u_n(s)\|^2 ds + \int_0^t \int_{\mathbb{R}^n} |\xi_n(u_n(s))|^p dxds \\ \le \|u_0\|^2 + (2\rho+1)\int_0^t \|u_n(s)\|^2 ds + \int_0^t \|g(s)\|^2 ds + \int_0^t \|\sigma(s)\|^2_{\mathcal{L}_2(U,H)} ds \\ + 2\mathrm{Re}\int_0^t u_n(s)\sigma(s)dW(s). \tag{45}$$

By (45), we imply that for all $0 \le t \le T$,

$$\mathbb{E}(\sup_{0\le r\le t}(\|u_n(r)\|^2)) \le \mathbb{E}(\|u_0\|^2) + (2\rho+1)\int_0^t \mathbb{E}(\sup_{0\le r\le s}(\|u_n(r)\|^2)ds + \int_0^T \|g(s)\|^2 ds \\ + \mathbb{E}\left(\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2(U,H)} ds\right) + 2\mathbb{E}\left(\sup_{0\le r\le t}\left|\int_0^r u_n(s)\sigma(s)dW(s)\right|\right). \tag{46}$$

By the Burkholder–Davis–Gundy inequality, we have for all $0 \le t \le T$,

$$2\mathbb{E}\left(\sup_{0\le r\le t}\left|\int_0^r u_n(s)\sigma(s)dW(s)\right|\right) \le c_{10}\mathbb{E}\left(\left(\int_0^t \|u_n(s)\|^2\|\sigma(s)\|^2_{\mathcal{L}_2(U,H)} ds\right)^{\frac{1}{2}}\right) \\ \le c_{10}\mathbb{E}\left(\sup_{0\le s\le t}\|u_n(s)\|\left(\int_0^t \|\sigma(s)\|^2_{\mathcal{L}_2(U,H)} ds\right)^{\frac{1}{2}}\right) \\ \le \frac{1}{2}\mathbb{E}\left(\sup_{0\le r\le t}\|u_n(r)\|^2\right) + \frac{1}{2}c_{10}^2\mathbb{E}\left(\int_0^t \|\sigma(s)\|^2_{\mathcal{L}_2(U,H)} ds\right). \tag{47}$$

By (46) and (47) we obtain, for all $0 \le t \le T$,

$$\mathbb{E}\left(\sup_{0\le r\le t}\|u_n(r)\|^2\right) \le 2\mathbb{E}(\|u_0\|^2) + (4\rho+2)\int_0^t \mathbb{E}(\sup_{0\le r\le s}(\|u_n(r)\|^2)ds + 2\int_0^T \|g(s)\|^2 ds \\ + (c_{10}^2+2)\mathbb{E}\left(\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2(U,H)} ds\right). \tag{48}$$

By (48) and the Gronwall inequality, we find that for all $0 \le t \le T$,

$$\mathbb{E}(\sup_{0\le r\le t}\|u_n(r)\|^2) \le c_{11}e^{(4\rho+2)t}, \tag{49}$$

where $c_{11} = 2\mathbb{E}(\|u_0\|^2) + 2\int_0^T \|g(s)\|^2 ds + (c_{10}^2+2)\int_0^T \mathbb{E}\|\sigma(s)\|^2_{\mathcal{L}_2(U,H)} ds$.

On the other hand, by (45) with $t = T$, we obtain

$$2\mathbb{E}\left(\int_0^T \|(-\Delta)^{\frac{\alpha}{2}} u_n(s)\|^2 ds\right) + \mathbb{E}\left(\int_0^T \int_{\mathbb{R}^n} |\xi_n(u_n(s))|^p dxds\right)$$

$$\leq \mathbb{E}(\|u_0\|^2) + (2\rho+1)\mathbb{E}\bigg(\int_0^T \|u_n(s)\|^2 ds\bigg) + \int_0^T \|g(s)\|^2 ds$$
$$+(1+\frac{1}{2}c_{10}^2)\int_0^T \mathbb{E}(\|\sigma(s)\|^2_{\mathcal{L}_2(U,H)})ds,$$

which together with (49) implies that

$$\begin{aligned} &\mathbb{E}\bigg(\int_0^T \|(-\Delta)^{\frac{\alpha}{2}} u_n(s)\|^2 ds\bigg) + \mathbb{E}\bigg(\int_0^T \int_{\mathbb{R}^n} |\xi_n(u_n(s))|^p dxds\bigg) \\ &\leq c_{11}\bigg(\mathbb{E}(\|u_0\|^2) + \int_0^T \|g(s)\|^2 ds + \int_0^T \mathbb{E}(\|\sigma(s)\|^2_{\mathcal{L}_2(U,H)})ds\bigg), \end{aligned} \tag{50}$$

which together with (49) and (50) completes the proof. □

We will prove the existence and uniqueness of solutions to problem (13) and (14).

Lemma 3. *Suppose* (12) *holds and* $u_0 \in L^2(\Omega, H)$ *is* $\mathcal{F}_0$*-measurable, then problem* (13) *and* (14) *has a unique solution* u *in the sense of Definition* 5*. Moreover,* u *satisfies,*

$$\begin{aligned} &\|u(t)\|^2_{L^2(\Omega,C([0,T],H))} + \|u(t)\|^2_{L^2(\Omega,L^2([0,T],V))} + \|u(t)\|^p_{L^p(\Omega,L^p(0,T;L^p(\mathbb{R}^n)))} \\ &\leq L_2(T)(\|u_0\|^2_{L^2(\Omega,H)} + \|g\|^2_{L^2(0,T;H)} + \|\sigma\|^2_{L^2(\Omega,L^2(0,T;\mathcal{L}_2(U,H)))}), \end{aligned} \tag{51}$$

where $L_2(T)$ *is a positive number only depending on* T.

Proof. We first prove the existence, then the uniqueness, and finally the measurability of the solutions.

Step 1. Existence of solutions for almost every fixed $\omega \in \Omega$. Let Ω_0 be the subset of Ω in Lemma 12 with $P(\Omega_0) = 1$. Then, for every fixed $\omega \in \Omega_0$, there exist $\tilde{u}(\omega) \in H$ and $u(\omega) \in L^\infty(0,T;H)) \bigcap L^2(0,T;V)$, $\chi_1(\omega) \in L^p(0,T;L^p(\mathbb{R}^n))$, $\chi_2(\omega) \in L^q(0,T;L^q(\mathbb{R}^n))$ and a subsequence $\{n_m\}_{m=1}^\infty$ of $\{n\}_{n=1}^\infty$ such that

$$u_{n_m}(\omega,T) \to \tilde{u}(\omega) \ weakly \ in \ H, \tag{52}$$

$$u_{n_m}(\omega) \to u(\omega) \ weak-star \ in \ L^\infty(0,T;H), \tag{53}$$

$$u_{n_m}(\omega) \to u(\omega) \ weakly \ in \ L^2(0,T;V), \tag{54}$$

$$\xi_{n_m}(u_{n_m}(\omega)) \to \chi_1(\omega) \ weakly \ in \ L^p(0,T;L^p(\mathbb{R}^n)), \tag{55}$$

and

$$(1+\mathrm{i}\mu)|\xi_{n_m}(u_{n_m}(\omega))|^{2\beta}\xi_{n_m}(u_{n_m}(\omega)) \to \chi_2(\omega) \ weakly \ in \ L^q(0,T;L^q(\mathbb{R}^n)). \tag{56}$$

Let $v_{n_m}(\omega,t) = u_{n_m}(\omega,t) - \int_0^t \sigma(s)dW$ and $v(\omega,t) = u(\omega,t) - \int_0^t \sigma(s)dW$. Then, by (53) we have

$$v_{n_m}(\omega) \to v(\omega) \ weak-star \ in \ L^\infty(0,T;H). \tag{57}$$

By (54) we obtain

$$\{v_{n_m}(\omega)\}_{m=1}^\infty \ is \ bounded \ in \ L^2(0,T;V). \tag{58}$$

On the other hand, by (24), we see that there exists a subset Ω_1 of Ω with $P(\Omega_1) = 1$ such that for every $\omega \in \Omega_1$,

$$\begin{aligned} v_{n_m}(\omega, t) = &- \int_0^t (1 + \mathrm{i}\nu)(-\Delta)^\alpha u_{n_m}(\omega, s) ds + \rho \int_0^t u_{n_m}(\omega, s) ds \\ &- \int_0^t (1 + \mathrm{i}\mu)|\xi_{n_m}(u_{n_m}(\omega))|^{2\beta} \xi_{n_m}(u_{n_m}(\omega, s)) ds + u_0 + \int_0^t g(s) ds \ in \ V^*. \end{aligned} \tag{59}$$

Note that (59) is a deterministic equation parametrized by $\omega \in \Omega_1$, which implies that for every $\omega \in \Omega_1$,

$$\begin{aligned} \frac{dv_{n_m}(\omega, t)}{dt} = &- (1 + \mathrm{i}\nu)(-\Delta)^\alpha u_{n_m}(\omega, t) + \rho u_{n_m}(\omega, t) \\ &- (1 + \mathrm{i}\mu)|\xi_{n_m}(u_{n_m}(\omega))|^{2\beta} \xi_{n_m}(u_{n_m}(\omega, t)) + g(t) \ in \ V^*, \end{aligned} \tag{60}$$

for almost all $t \in [0, T]$. By (54), (56) and (60) we infer that

$$\frac{dv_{n_m}(\omega)}{dt} \ is \ bounded \ in \ L^q(0, T; (V \bigcap L^p(\mathbb{R}^n))^*). \tag{61}$$

Let $\hat{\rho} : \mathbb{R}^n \to [0, 1]$ be a smooth function satisfies $\hat{\rho}(x) = 1$ if $|x| \le 1$; and $\hat{\rho}(x) = 0$ if $|x| \ge 2$. Given $k \in \mathbb{N}$, denote by $V_k = \{u \in V : u = 0 \ for \ almost \ all \ |x| \ge 2k\}$, $H_k = \{u \in H : u = 0 \ for \ almost \ all \ |x| \ge 2k\}$ and $L_k^p = \{u \in L^p(\mathbb{R}^n) : u = 0 \ for \ almost \ all \ |x| \ge 2k\}$. For brevity, we also write $O_k = \{x \in \mathbb{R}^n : |x| < k\}$ and $\tilde{v}_{n_m}(\omega, t, x) = \hat{\rho}(\frac{x}{k}) v_{n_m}(\omega, t, x)$ for $\omega \in \Omega$, $t \in [0, T]$ and $x \in \mathbb{R}^n$. Then, by (58) we have

$$\{\tilde{v}_{n_m}(\omega)\}_{m=1}^\infty \ is \ bounded \ in \ L^2(0, T; V_k). \tag{62}$$

Similar to (61), by (60), we can verify that

$$\frac{d\tilde{v}_{n_m}(\omega)}{dt} \ is \ bounded \ in \ L^q(0, T; (V_k \bigcap L_k^p)^*). \tag{63}$$

Since the embedding $V_k \hookrightarrow H_k$ is compact and $H_k = (H_k)^* \hookrightarrow (V_k \bigcap L_k^p)^*$ is continuous, by (62) and (63) and the compactness theorem in [42] we infer from (57) that for every $\omega \in \Omega_2 = \Omega_0 \bigcap \Omega_1$ and $k \in \mathbb{N}$, there exists a further subsequence (not relabeled) such that

$$\{\tilde{v}_{n_m}(\omega)\} \to \hat{\rho}(\frac{x}{k}) v(\omega) \ strongly \ in \ L^2(0, T; H_k). \tag{64}$$

By (64), we have, up to a further subsequence,

$$\tilde{v}_{n_m}(\omega, t, x) \to \hat{\rho}(\frac{x}{k}) v(\omega, t, x) \ for \ almost \ all \ (t, x) \in (0, T) \times O_{2k},$$

and hence

$$v_{n_m}(\omega, t, x) \to v(\omega, t, x) \ for \ almost \ all \ (t, x) \in (0, T) \times O_k. \tag{65}$$

Based on (65), by a diagonal process, we find that, up to a subsequence,

$$v_{n_m}(\omega, t, x) \to v(\omega, t, x) \ for \ almost \ all \ (t, x) \in (0, T) \times \mathbb{R}^n. \tag{66}$$

By (66) we obtain, for $\omega \in \Omega_2$,

$$u_{n_m}(\omega, t, x) \to u(\omega, t, x) \ for \ almost \ all \ (t, x) \in (0, T) \times \mathbb{R}^n. \tag{67}$$

By (15) we have

$$|\xi_{n_m}(u_{n_m}(\omega)) - u(\omega)| \le |\xi_{n_m}(u_{n_m}(\omega) - \xi_{n_m}(u(\omega))| + |\xi_{n_m}(u(\omega) - u(\omega))|$$

$$\leq |u_{n_m}(\omega) - u(\omega)| + |\xi_{n_m}(u(\omega)) - u(\omega)|,$$

which together with (67) implies that, for $\omega \in \Omega_2$,

$$\xi_{n_m}(u_{n_m}(\omega, t, x)) \to u(\omega, t, x) \text{ for almost all } (t, x) \in (0, T) \times \mathbb{R}^n. \tag{68}$$

By (55), (68) and Mazur's theorem, we obtain $\chi_1(\omega) = u(\omega)$ and we have

$$\xi_{n_m}(u_{n_m}(\omega, t, x)) \to u(\omega, t, x) \text{ weakly in } L^p(0, T; L^p(\mathbb{R}^n)). \tag{69}$$

In addition, by (68), for almost all $(t, x) \in (0, T) \times \mathbb{R}^n$, we obtain

$$(1 + \mathrm{i}\mu)|\xi_{n_m}(u_{n_m}(\omega, t, x))|^{2\beta}\xi_{n_m}(u_{n_m}(\omega, t, x)) \to (1 + \mathrm{i}\mu)|u(\omega, t, x)|^{2\beta}u(\omega, t, x). \tag{70}$$

By (56), (70) and Mazur's theorem, we obtain $\chi_2(\omega) = (1 + \mathrm{i}\mu)|u(\omega)|^{2\beta}u(\omega)$ and

$$\begin{aligned}&(1 + \mathrm{i}\mu)|\xi_{n_m}(u_{n_m}(\omega))|^{2\beta}\xi_{n_m}(u_{n_m}(\omega, t, x)) \\ &\quad \to (1 + \mathrm{i}\mu)|u(\omega)|^{2\beta}u(\omega) \text{ weakly in } L^q(0, T; L^q(\mathbb{R}^n)).\end{aligned} \tag{71}$$

Next, we take the limits of (22) to prove that $u(\omega)$ is a solution of (13) and (14). By (60), we know that for every $\omega \in \Omega_2$, $\xi \in V \bigcap L^p(\mathbb{R}^n)$ and $\psi \in C_0^\infty(0, T)$,

$$\begin{aligned}&-\int_0^T (v_{n_m}(\omega, t), \xi)\psi'(t)dt + (1 + \mathrm{i}\nu)\int_0^T \psi(t)((-\Delta)^{\frac{\alpha}{2}}u_{n_m}(\omega, t), (-\Delta)^{\frac{\alpha}{2}}\xi)dt \\ &\quad + \int_0^T \psi(t)((1 + \mathrm{i}\mu)|\xi_{n_m}(u_{n_m})|^{2\beta}\xi_{n_m}(u_{n_m}), \xi)dt \\ &= \rho\int_0^T \psi(t)(u_{n_m}(\omega, t), \xi)dt + \int_0^T (g(t), \xi)\psi(t)dt.\end{aligned} \tag{72}$$

Letting $m \to \infty$ in (72), by (54), (57) and (71), we have

$$\begin{aligned}&-\int_0^T (v(\omega, t), \xi)\psi'(t)dt + (1 + \mathrm{i}\nu)\int_0^T \psi(t)\Big((-\Delta)^{\frac{\alpha}{2}}u(\omega, t), (-\Delta)^{\frac{\alpha}{2}}\xi\Big)dt \\ &+ \int_0^T \psi(t)\Big((1 + \mathrm{i}\mu)|u|^{2\beta}u, \xi\Big)dt = \rho\int_0^T \psi(t)(u(\omega, t), \xi)dt + \int_0^T (g(t), \xi)\psi(t)dt.\end{aligned} \tag{73}$$

By (73) for every $\omega \in \Omega_2$ and $\xi \in V \bigcap L^p(\mathbb{R}^n)$, we infer that

$$\begin{aligned}&\frac{d(v(\omega), \xi)}{dt} + (1 + \mathrm{i}\nu)((-\Delta)^{\frac{\alpha}{2}}u(\omega), (-\Delta)^{\frac{\alpha}{2}}\xi) + \int_{\mathbb{R}^n} (1 + \mathrm{i}\mu)|u(\omega)|^{2\beta}u(\omega)\xi(x)dx \\ &= \rho(u(\omega, t), \xi) + (g(t), \xi)\end{aligned} \tag{74}$$

on $(0, T)$ in the sense of scalar distribution.

We next prove $v(\omega) : [0, T] \to H$ is continuous. Firstly, by (54), (71) and (74) we have

$$\frac{dv(\omega)}{dt} \text{ is in } L^2(0, T; V^*) \bigcup L^q(0, T; L^q(\mathbb{R}^n)). \tag{75}$$

By (54) and (69) we see that

$$u(\omega) \text{ is in } L^2(0, T; V) \bigcap L^p(0, T; L^p(\mathbb{R}^n)),$$

and hence

$$v(\omega)\ is\ in\ L^2(0,T;V)\bigcap L^p(0,T;L^p(\mathbb{R}^n)). \tag{76}$$

By (75) and (76), it follows from [36] that $v(\omega)\in C([0,T],H)$ and

$$\frac{d\|v(\omega)\|^2}{dt}=2\langle\frac{dv(\omega)}{dt},v(\omega)\rangle_{(V^*\bigcup L^q(\mathbb{R}^n),V\bigcap L^p(\mathbb{R}^n))},$$

in the sense of scalar distribution on $(0,T)$. As a result, we find that $u(\omega)\in C([0,T],H)$. Next, we show that $u(\omega)$ has initial condition $u_0(\omega)$ when $t=0$. By (60), we infer that for every $\omega\in\Omega_2,\xi\in V\bigcap L^p(\mathbb{R}^n)$ and $\psi\in C^\infty([0,T])$,

$$\begin{aligned}&(v_{n_m}(\omega,T),\xi)\psi(T)-(v_{n_m}(\omega,0),\xi)\psi(0)-\int_0^T(v_{n_m}(\omega,t),\xi)\psi'(t)dt\\&+(1+\mathrm{i}\nu)\int_0^T((-\Delta)^{\frac{\alpha}{2}}u_{n_m}(\omega,t),(-\Delta)^{\frac{\alpha}{2}}\xi)\psi(t)dt\\&+\int_0^T\int_{\mathbb{R}^n}(1+\mathrm{i}\mu)|\xi_{n_m}(u_{n_m})|^{2\beta}\xi_{n_m}(u_{n_m})\xi(t)\psi(t)dxdt\\&=\rho\int_0^T\psi(t)(u_{n_m}(\omega,t),\xi)dt+\int_0^T(g(t),\xi)\psi(t)dt.\end{aligned} \tag{77}$$

Letting $m\to\infty$ in (77), it follows from (52), (54), (57), (69) and (71) that

$$\begin{aligned}&\left(\tilde{u}(\omega)-\int_0^T\sigma(s)dW,\xi\right)\psi(T)-(u_0(\omega),\xi)\psi(0)-\int_0^T(v(\omega),\xi)\psi'(t)dt\\&+(1+\mathrm{i}\nu)\int_0^T\left((-\Delta)^{\frac{\alpha}{2}}u(\omega,t),(-\Delta)^{\frac{\alpha}{2}}\xi\right)\psi(t)dt\\&+\int_0^T\int_{\mathbb{R}^n}(1+\mathrm{i}\mu)|\xi_{n_m}(u_{n_m})|^{2\beta}\xi_{n_m}(u_{n_m})\xi(t)\psi(t)dxdt\\&=\rho\int_0^T\psi(t)(u(\omega,t),\xi)dt+\int_0^T(g(t),\xi)\psi(t)dt.\end{aligned} \tag{78}$$

On the other hand, by (74), we obtain

$$\begin{aligned}&\left(u(\omega,T)-\int_0^T\sigma(s)dW,\xi\right)\psi(T)-(u(\omega,0),\xi)\psi(0)-\int_0^T(v(\omega),\xi)\psi'(t)dt\\&+(1+\mathrm{i}\nu)\int_0^T((-\Delta)^{\frac{\alpha}{2}}u(\omega,t),(-\Delta)^{\frac{\alpha}{2}}\xi)\psi(t)dt\\&+\int_0^T\int_{\mathbb{R}^n}(1+\mathrm{i}\mu)|u(\omega,t)|^{2\beta}u(\omega,t)\xi(t)\psi(t)dxdt\\&=\rho\int_0^T\psi(t)(u(\omega,t),\xi)dt+\int_0^T(g(t),\xi)\psi(t)dt.\end{aligned} \tag{79}$$

By (78) and (79), we obtain

$$(u(\omega,T)-\tilde{u}(\omega),\xi)\psi(T)=(u(\omega,0)-u_0(\omega),\xi)\psi(0). \tag{80}$$

Choosing $\psi\in C^\infty([0,T])$ with $\psi(0)=1$ and $\psi(T)=0$, we obtain from (80) that for every $\omega\in\Omega_2$ and $\xi\in V\bigcap L^p(\mathbb{R}^n)$,

$$(u(\omega,0)-u_0(\omega),\xi)=0,$$

which shows that

$$u(\omega,0) = u_0(\omega) \text{ in } H. \tag{81}$$

Similarly, choosing $\psi \in C^\infty([0,T])$ with $\psi(0) = 0$ and $\psi(T) = 1$, we can obtain from (80) that for every $\omega \in \Omega_2$,

$$u(\omega,T) = \tilde{u}(\omega) \text{ in } H. \tag{82}$$

By (52) and (82), we know that

$$u_{n_m}(\omega,T) \to u(\omega,T) \text{ weakly in } H. \tag{83}$$

By (83), we can also infer that for every $t \in [0,T]$,

$$u_{n_m}(\omega,t) \to u(\omega,t) \text{ weakly in } H. \tag{84}$$

By (74) we find that for every $\omega \in \Omega_2$, $\xi \in V \bigcap L^p(\mathbb{R}^n)$ and $t \in [0,T]$,

$$\begin{aligned}
&(v(\omega,t),\xi) + (1+\mathrm{i}\nu)\int_0^t ((-\Delta)^{\frac{\alpha}{2}} u(\omega,s),(-\Delta)^{\frac{\alpha}{2}}\xi)ds \\
&+ \int_0^t \int_{\mathbb{R}^n} (1+\mathrm{i}\mu)|u(\omega,s)|^{2\beta} u(\omega,s)\xi(x)dxds \\
&= (v(\omega,0),\xi) + \rho\int_0^t (u(\omega,s),\xi)ds + \int_0^t (g(s),\xi)ds.
\end{aligned} \tag{85}$$

By (81) and (85) we obtain, for every $\omega \in \Omega_2$, $\xi \in V \bigcap L^p(\mathbb{R}^n)$ and $t \in [0,T]$,

$$\begin{aligned}
&(u(\omega,t),\xi) + (1+\mathrm{i}\nu)\int_0^t ((-\Delta)^{\frac{\alpha}{2}} u(\omega,s),(-\Delta)^{\frac{\alpha}{2}}\xi)ds \\
&+ \int_0^t \int_{\mathbb{R}^n} (1+\mathrm{i}\mu)|u(\omega,s)|^{2\beta} u(\omega,s)\xi(x)dxds \\
&= (u_0(\omega),\xi) + \rho\int_0^t (u(\omega,s),\xi)ds + \int_0^t (g(s),\xi)ds + \int_0^t \xi\sigma(s)dW(s).
\end{aligned} \tag{86}$$

Note that for every fixed $\omega \in \Omega_2$,

$$u(\omega) \in C([0,T],H) \bigcap L^2(0,T;V) \bigcap L^p(0,T;L^p(\mathbb{R}^n)). \tag{87}$$

Next, we prove the uniqueness of solutions to (86) with property (87).

Step 2. Uniqueness of solutions for almost every fixed $\omega \in \Omega$. Given $\omega \in \Omega_2$, let $u_1(\omega)$ and $u_2(\omega)$ be the solutions of (86) satisfying (87). We want to show $u_1(\omega,t) = u_2(\omega,t)$ in H for all $t \in [0,T]$.

Let $u'(\omega,t) = u_1(\omega,t) - u_2(\omega,t)$. Then by (87) we have

$$u'(\omega) \in C([0,T],H) \bigcap L^2(0,T;V) \bigcap L^p(0,T;L^p(\mathbb{R}^n)). \tag{88}$$

On the other hand, by (86) we obtain, for all $\xi \in V \bigcap L^p(\mathbb{R}^n)$ and $t \in [0,T]$,

$$\begin{aligned}
&(u'(\omega,t),\xi) + (1+\mathrm{i}\nu)\int_0^t ((-\Delta)^{\frac{\alpha}{2}} u'(\omega,s),(-\Delta)^{\frac{\alpha}{2}}\xi)ds = \rho\int_0^t (u'(\omega,s),\xi)ds \\
&- \int_0^t \int_{\mathbb{R}^n} \Big((1+\mathrm{i}\mu)|u_1(\omega,s)|^{2\beta} u_1(\omega,s) - (1+\mathrm{i}\mu)|u_2(\omega,s)|^{2\beta} u_2(\omega,s)\Big)\xi(x)dxds,
\end{aligned}$$

which together with (88) implies that

$$\frac{du'(\omega)}{dt} = -(1+i\nu)(-\Delta)^{\alpha}u'(\omega) + \rho u'(\omega) - \Big((1+i\mu)|u_1(\omega)|^{2\beta}u_1(\omega) - (1+i\mu)|u_2(\omega)|^{2\beta}u_2(\omega)\Big), \tag{89}$$

in $L^2(0,T;V^*)\bigcup L^q(0,T;L^q(\mathbb{R}^n))$.

By (88) and (89) we obtain

$$\begin{aligned}\frac{d\|u'(\omega,t)\|^2}{dt} =& 2\mathrm{Re}\langle\frac{du'(\omega,t)}{dt}, u'(\omega,t)\rangle_{(V^*\bigcup L^q(\mathbb{R}^n), V\bigcap L^p(\mathbb{R}^n))} \\ =& -2\|(-\Delta)^{\frac{\alpha}{2}}u'(\omega)\|^2 + 2\rho\|u'(\omega)\|^2 \\ &- 2\mathrm{Re}\Big((1+i\mu|u_1|^{2\beta}u_1 - (1+i\mu|u_2|^{2\beta}u_2), u'(\omega)\Big).\end{aligned} \tag{90}$$

According to (20), we obtain

$$\frac{d\|u'(\omega,t)\|^2}{dt} \le 2\rho\|u'(\omega,t)\|^2,$$

which together with Gronwall's inequality, we obtain that for all $t\in[0,T]$,

$$\|u'(\omega,t)\|^2 \le e^{2\rho t}\|u'(\omega,0)\|^2, \tag{91}$$

and $u'(\omega,0) = u_0(\omega) - u_0(\omega) = 0$, therefore, $u_1(\omega,t) = u_2(\omega,t)$ for all $t\in[0,T]$.

Step 3. Measurability and regularity of solutions. By (84) we know that for every $\omega\in\Omega_2$, there exists a subsequence $\{u_{n_m}(\omega)\}_{m=1}^{\infty}$ of $\{u_n(\omega)\}_{n=1}^{\infty}$, which may depend on ω, such that

$$u_{n_m}(\omega,t) \to u(\omega,t) \textit{ weakly in } H. \tag{92}$$

Since $u(\omega)$ is the unique solution of (86) with property (87), we know from (92) that the entire sequence $u_n(\omega,t)$ (not just a subsequence) weakly converges in H; namely, for every $\omega\in\Omega_2$ and $t\in[0,T]$,

$$u_n(\omega,t) \to u(\omega,t) \textit{ weakly in } H. \tag{93}$$

Since for each $n\in\mathbb{N}$, the process u_n is $\mathcal{F}_t$-adapted, it follows from (93) that u is also $\mathcal{F}_t$-adapted.

Next, we show the measurability of $u:\Omega\to L^2(0,T;V)$. By Lemma 2, we see that u_n is bounded in $L^2(\Omega, L^2(0,T;V))$, hence there exists $u'\in L^2(\Omega, L^2(0,T;V))$ and a subsequence (not relabeled) such that

$$u_n \to u' \textit{ weakly in } L^2(\Omega, L^2(0,T;V)). \tag{94}$$

By (94), (67) and Mazur's theorem, we obtain that $u(\omega) = u'(\omega)$ in $L^2(0,T;V)$ for almost all $\omega\in\Omega$, and hence $u:\Omega\to L^2(0,T;V)$ is measurable and

$$\|u\|^2_{L^2(\Omega,L^2(0,T;V))} \le \liminf_{n\to\infty}\|u_n\|^2_{L^2(\Omega,L^2(0,T;V))}. \tag{95}$$

We now prove the measurability of $u:\Omega\to L^p(0,T;L^p(\mathbb{R}^n))$. As before, given $\omega\in\Omega_2$, since $u(\omega)$ is the unique solution of (86) with property (87), by (69) we obtain, for every $\omega\in\Omega_2$,

$$\xi_n(u_n(\omega)) \to u(\omega) \text{ weakly in } L^p(0,T;L^p(\mathbb{R}^n)). \tag{96}$$

In addition, by Lemma 2, the sequence $\{\xi_n(u_n)\}_{n=1}^{\infty}$ is bounded in $L^p(\Omega, L^p(0,T;L^p(\mathbb{R}^n)))$, and hence there exists $\xi \in L^p(\Omega, L^p(0,T;L^p(\mathbb{R}^n)))$ and a subsequence (not relabeled) such that

$$\xi_n(u_n) \to \xi \text{ weakly in } L^p(0,T;L^p(\mathbb{R}^n)). \tag{97}$$

By (96) and (97) and Mazur's theorem, we find that $u(\omega) = \xi(\omega)$ in $L^p(0,T;L^p(\mathbb{R}^n))$ for almost all $\omega \in \Omega$. This implies $u : \Omega \to L^p(0,T;L^p(\mathbb{R}^n))$ is measurable and

$$\|u\|^p_{L^p(\Omega, L^p(0,T;v))} \le \liminf_{n\to\infty} \|u_n\|^p_{L^p(\Omega,L^p(0,T;V))}. \tag{98}$$

Note that u is a continuous H-valued $\mathcal{F}_t$-adapted process. Therefore, $u : \Omega \to C([0,T],H)$ is measurable. By (53) and the uniqueness of solution of (86), for every $\omega \in \Omega_2$,

$$u_n(\omega) \to u(\omega) \text{ weak-star in } L^\infty(0,T;H),$$

which implies

$$\|u(\omega)\|_{L^\infty(0,T;H)} \le \liminf_{n\to\infty} \|u_n(\omega)\|_{L^\infty(0,T;H)}. \tag{99}$$

By (99) and Fatou's lemma we obtain

$$\begin{aligned}\int_\Omega \|u(\omega)\|^2_{L^\infty(0,T;H)} dP &\le \int_\Omega \liminf_{n\to\infty} \|u_n(\omega)\|^2_{L^\infty(0,T;H)} dP \\ &\le \liminf_{n\to\infty} \int_\Omega \|u_n(\omega)\|_{L^\infty(0,T;H)} dP.\end{aligned} \tag{100}$$

By (100) and Lemma 2, we obtain $\int_\Omega \|u(\omega)\|^2_{L^\infty(0,T;H)} dP < \infty$, which along with the path continuity of u implies $u \in L^2(\Omega, C([0,T],H))$. By (86) and the above measurability of u, we see that u is a solution of (13) and (14) in the sense of Definition 5. We obtain the uniqueness of the solutions follows from Step2, and the uniform estimates of (51) follows from (95), (98), (100) and Lemma 2. □

4. Existence of Solutions: General Additive Noise

In this section, we study the existence and uniqueness of solutions to problem (1) and (2) with a general additive noise,

$$\begin{aligned}du(t) + (1+\mathrm{i}\nu)(-\Delta)^\alpha u(t)dt + (1+\mathrm{i}\mu)|u(t)|^{2\beta}u(t)dt =& \rho u(t)dt + g(t,x)dt \\ &+ \sigma(t,\omega)dW,\end{aligned} \tag{101}$$

with initial condition

$$u(0) = u_0, \tag{102}$$

where $\sigma : \mathbb{R} \times \Omega \to \mathcal{L}_2(U,H)$ is a progressively measurable process such that

$$\sigma \in L^2(\Omega, L^2(0,T;\mathcal{L}_2(U,H))) \text{ for every } T > 0. \tag{103}$$

We investigate the existence and uniqueness of solutions to problem (101) and (102) under condition (103).

Lemma 4. *Suppose* (103) *holds and $u_0 \in L^2(\Omega,H)$ is $\mathcal{F}_0$-measurable, then problem* (101) *and* (102) *has a unique solution u in the sense of Definition* 5*. Moreover, u satisfies,*

$$\begin{aligned}&\|u(t)\|^2_{L^2(\Omega,C([0,T],H))} + \|u(t)\|^2_{L^2(\Omega,L^2([0,T],V))} + \|u(t)\|^p_{L^p(\Omega,L^p(0,T;L^p(\mathbb{R}^n)))} \\ &\le L_3(T)(\|u_0\|^2_{L^2(\Omega,H)} + \|g\|^2_{L^2(0,T;H)} + \|\sigma\|^2_{L^2(\Omega,L^2(0,T;\mathcal{L}_2(U,H)))}),\end{aligned} \tag{104}$$

where $L_3(T)$ is a positive number depending only on T.

Proof. We first approximate the drift coefficient σ with (103) by regular drift terms and construct a sequence of approximate solutions. We then derive uniform estimates, and prove that the limit of the approximate solution is a solution of (101) and (102). Finally, we show the uniqueness of the solutions.

Step 1. Approximate solutions. We first approximate σ with (103) by regular functions. Therefore, we choose a positive integer k_0 such that $k_0 > \frac{(p-2)n}{4p}$. Then, we obtain that $H^{2k_0}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$. Given $m \in \mathbb{N}$, denote by

$$\sigma_m = (I - \frac{1}{m}\Delta)^{-k_0}\sigma.$$

Then we have $\sigma_m \in L^2(\Omega, L^2(0,T;\mathcal{L}_2(U,V_0)))$. By Lemma 3 we find that, for every $m \in \mathbb{N}$, there exists a unique continuous H-valued $\mathcal{F}_t$-adapted stochastic process u_m with

$$u_m(\omega) \in L^2(\Omega, C([0,T],H)) \bigcap L^2(\Omega, L^2(0,T;V)) \bigcap L^p(\Omega, L^p(0,T;L^p(\mathbb{R}^n))), \quad \forall T > 0,$$

such that for all $t \geq 0$ and $\xi \in V \bigcap L^p(\mathbb{R}^n)$,

$$\begin{aligned}
&(u_m(\omega,t),\xi) + (1+\mathrm{i}\nu)\int_0^t ((-\Delta)^{\frac{\alpha}{2}} u_m(\omega,s), (-\Delta)^{\frac{\alpha}{2}}\xi)ds \\
&+ \int_0^t \int_{\mathbb{R}^n} (1+\mathrm{i}\mu)|u_m(\omega,s)|^{2\beta} u_m(\omega,s)\xi(x)dxds \\
&= (u_0(\omega),\xi) + \rho\int_0^t (u_m(\omega,s),\xi)ds + \int_0^t (g(s),\xi)ds + \int_0^t \xi\sigma_m(s)dW(s),
\end{aligned} \tag{105}$$

P–almost surely. Where ξ in the stochastic term is considered as an element of H^* by Riesz's representation theorem. Moreover, by (51), Lemma 3 and the contractility of the operator $(I - \frac{1}{m}\Delta)^{-k_0}$, we find that for all $m \in \mathbb{N}$, there exists a positive number $C_1 = C_1(T)$ independent of m such that

$$\begin{aligned}
&\|u_m(t)\|^2_{L^2(\Omega,C([0,T],H))} + \|u_m(t)\|^2_{L^2(\Omega,L^2([0,T],V))} + \|u_m\|^p_{L^p(\Omega,L^p(0,T;L^p(\mathbb{R}^n)))} \\
&\leq C_1(\|u_0\|^2_{L^2(\Omega,H)} + \|g\|^2_{L^2(0,T;H)} + \|\sigma\|^2_{L^2(\Omega,L^2(0,T;\mathcal{L}_2(U,H)))}).
\end{aligned} \tag{106}$$

By (41) and (106) we obtain that, for all $m \in \mathbb{N}$,

$$\begin{aligned}
&\|(1+\mathrm{i}\mu)|u_m|^{2\beta}u_m\|^q_{L^q(\Omega,L^q(0,T;L^q(\mathbb{R}^n)))} \\
&\leq c_9C_1(\|u_0\|^2_{L^2(\Omega,H)} + \|g\|^2_{L^2(0,T;H)} + \|\sigma\|^2_{L^2(\Omega,L^2(0,T;\mathcal{L}_2(U,H)))}).
\end{aligned} \tag{107}$$

Next, we derive further uniform estimates of the approximate solutions.

Step 2. Uniform estimates on $\{u_m\}_{m=1}^{\infty}$. Note that by the proof of Lemma 3, for every $m \in \mathbb{N}$, the solution u_m of (105) is given by the limit of the solution $u_{m,n}$ of the following equation in V^*,

$$\begin{aligned}
&u_{m,n}(t) + (1+\mathrm{i}\nu)\int_0^t ((-\Delta)^{\alpha} u_{m,n}(s) + (1+\mathrm{i}\mu)|\xi_n(u_{m,n}(s))|^{2\beta}\xi_n(u_{m,n}(s))ds \\
&= u_0 + \rho\int_0^t u_{m,n}(s)ds + \int_0^t g(s,x)ds + \int_0^t \sigma_m(s,\omega)dW.
\end{aligned} \tag{108}$$

By (108) and integration by parts of Ito's formula, we obtain, for all $m_1, m_2 \in \mathbb{N}$, with $\hat{u}_{m,n}(t) = u_{m_1,n}(t) - u_{m_2,n}(t)$,

$$\begin{aligned}
&\|\hat{u}_{m,n}(t)\|^2 + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}}\hat{u}_{m,n}(s)\|^2 ds \\
&+ 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^n} (1+\mathrm{i}\mu)\big(|\xi_n(u_{m_1,n}(s))|^{2\beta}\xi_n(u_{m_1,n}(s)) - |\xi_n(u_{m_2,n}(s))|^{2\beta}\xi_n(u_{m_2,n}(s))\big)dxds \\
&= 2\rho\int_0^t \|\hat{u}_{m,n}\|^2 ds + \int_0^t \|\sigma_{m_1}(\omega) - \sigma_{m_2}(\omega)\|^2_{\mathcal{L}_2(U,H)} ds \\
&+ 2\mathrm{Re}\int_0^t \hat{u}_{m,n}(s)(\sigma_{m_1}(\omega) - \sigma_{m_2}(\omega))dW.
\end{aligned} \tag{109}$$

Together with (20), we obtain

$$\|\hat{u}_{m,n}(t)\|^2 + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}}\hat{u}_{m,n}(s)\|^2 ds \le 2\rho\int_0^t \|\hat{u}_{m,n}\|^2 ds$$

$$+ \int_0^t \|\sigma_{m_1}(\omega) - \sigma_{m_2}(\omega)\|^2_{\mathcal{L}_2(U,H)} ds + 2\mathrm{Re}\int_0^t \hat{u}_{m,n}(s)(\sigma_{m_1}(\omega) - \sigma_{m_2}(\omega))dW,$$

from which we can deduce that, for each $T > 0$,

$$\begin{aligned}
&\mathbb{E}(\sup_{0\le r\le t} \|\hat{u}_{m,n}(r)\|^2) \\
&\le 2\rho\int_0^t \mathbb{E}(\sup_{0\le r\le s}\|\hat{u}_{m,n}(r)\|^2)ds + \mathbb{E}\bigg(\int_0^T \|\sigma_{m_1}(\omega) - \sigma_{m_2}(\omega)\|^2_{\mathcal{L}_2(U,H)} ds\bigg) \\
&+ 2\mathbb{E}\bigg(\sup_{0\le r\le t}\bigg|\int_0^r \hat{u}_{m,n}(s)(\sigma_{m_1}(\omega) - \sigma_{m_2}(\omega))dW(s)\bigg|\bigg).
\end{aligned}$$

By the Burkholder–Davis–Gundy inequality and Young's inequality, we infer

$$2\mathbb{E}\bigg(\sup_{0\le r\le t}\bigg|\int_0^r \hat{u}_{m,n}(s)(\sigma_{m_1}(\omega) - \sigma_{m_2}(\omega))dW(s)\bigg|\bigg)$$

$$\le \frac{1}{2}\mathbb{E}(\sup_{0\le r\le t}\|\hat{u}_{m,n}(r)\|^2) + c_{12}\mathbb{E}\bigg(\int_0^t \|\sigma_{m_1}(\omega) - \sigma_{m_2}(\omega)\|^2_{\mathcal{L}_2(U,H)} ds\bigg).$$

Therefore, we have

$$\begin{aligned}
\mathbb{E}(\sup_{0\le r\le t}\|\hat{u}_{m,n}(r)\|^2) \le& 4\rho\int_0^t \mathbb{E}(\sup_{0\le r\le s}\|\hat{u}_{m,n}(r)\|^2)ds \\
&+ 2(1+c_{12})\mathbb{E}\bigg(\int_0^T \|\sigma_{m_1}(\omega) - \sigma_{m_2}(\omega)\|^2_{\mathcal{L}_2(U,H)} ds\bigg).
\end{aligned}$$

Applying the Gronwall inequality, for all $t \in [0, T]$, we deduce,

$$\mathbb{E}(\sup_{0\le r\le t}\|\hat{u}_{m,n}(r)\|^2) \le 2(1+c_{12})e^{4\rho t}\mathbb{E}\bigg(\int_0^T \|\sigma_{m_1}(\omega) - \sigma_{m_2}(\omega)\|^2_{\mathcal{L}_2(U,H)} ds\bigg).$$

By (109), we obtain

$$\begin{aligned}
2\mathbb{E}\bigg(\int_0^T \|(-\Delta)^{\frac{\alpha}{2}}\hat{u}_{m,n}\|^2 ds\bigg) \le& 2\rho\mathbb{E}\bigg(\int_0^T \|\hat{u}_{m,n}\|^2 ds\bigg) \\
&+ (c_{12}+1)\mathbb{E}\bigg(\int_0^T \|\sigma_{m_1}(\omega) - \sigma_{m_2}(\omega)\|^2_{\mathcal{L}_2(U,H)} ds\bigg),
\end{aligned}$$

which together above we can deduce, there exists a positive number $c_{13} = c_{13}(T)$ independent of m_1, m_2 and n such that

$$\mathbb{E}\left(\sup_{0\le t\le T}\|u_{m_1,n}(t) - u_{m_2,n}(t)\|^2\right) + \mathbb{E}\left(\int_0^T \|u_{m_1,n}(t) - u_{m_2,n}(t)\|^2_{H^\alpha} dt\right)$$

$$\le c_{13}\mathbb{E}\left(\int_0^T \|\sigma_{m_1}(\omega) - \sigma_{m_2}(\omega)\|^2_{\mathcal{L}_2(U,H)} ds\right), \tag{110}$$

where $c_{13} = \left(2(1+c_{12})e^{4\rho T}[1+T+\frac{2\rho T}{C(n,\alpha)}] + \frac{c_{12}+1}{C(n,\alpha)}\right)$.

Note that the proof of Lemma 3, we know that there exists a subset Ω_1 of Ω with $P(\Omega_1) = 1$ such that for every $\omega \in \Omega_1$ and every fixed $m \in \mathbb{N}$, as $n \to \infty$,

$$u_{m,n}(\omega) \to u_m(\omega) \text{ weak-star in } L^\infty(0,T;H), \tag{111}$$

$$u_{m,n}(\omega) \to u_m(\omega) \text{ weakly in } L^2(0,T;V), \tag{112}$$

$$\xi_n(u_{m,n})(\omega) \to u_m(\omega) \text{ weakly in } L^p(0,T;L^p(\mathbb{R}^n)), \tag{113}$$

$$\begin{aligned}(1+\mathrm{i}\mu)|\xi_n(u_{m,n}(\omega))|^{2\beta}\xi_n(u_{m,n}(\omega)) \to &(1+\mathrm{i}\mu)|u_{m,n}(\omega)|^{2\beta}u_{m,n}(\omega) \\ &\text{weakly in } L^q(0,T;L^q(\mathbb{R}^n)).\end{aligned} \tag{114}$$

By (112) and Fatou's lemma, we obtain

$$\begin{aligned}\mathbb{E}\left(\|u_{m_1} - u_{m_2}\|^2_{L^2(0,T;V)}\right) \le &\liminf_{n\to\infty}\mathbb{E}\left(\|u_{m_1} - u_{m_2}\|^2_{L^2(0,T;V)}\right) \\ \le &c_{13}\mathbb{E}\left(\int_0^T \|\sigma_{m_1} - \sigma_{m_2}\|^2_{\mathcal{L}_2(U,H)} ds\right).\end{aligned} \tag{115}$$

Similarly, we obtain

$$\mathbb{E}\left(\|u_{m_1} - u_{m_2}\|^2_{C(0,T;H)}\right) \le c_{13}\mathbb{E}\left(\int_0^T \|\sigma_{m_1} - \sigma_{m_2}\|^2_{\mathcal{L}_2(U,H)} ds\right). \tag{116}$$

Note that $\sigma_m \to \sigma$ in $L^2(\Omega, L^2(0,T;\mathcal{L}_2(U,H)))$ as $m \to \infty$, and hence $\{\sigma_m\}_{m=1}^\infty$ is a Cauchy sequence in $L^2(\Omega, C([0,T],H)) \cap L^2(\Omega, L^2(0,T;V))$ such that

$$\lim_{m\to\infty} u_m = u \ in\ L^2(\Omega, C([0,T],H)) \bigcap L^2(\Omega, L^2(0,T;V)). \tag{117}$$

By (117) we see that u is a continuous H-valued $\mathcal{F}_t$-adapted process. On the other hand, by (117) we infer that, up to a subsequence (not relabeled) such that

$$u_m \to u \text{ almost everywhere in } \Omega\times[0,T]\times\mathbb{R}^n. \tag{118}$$

By (107), there exists $\chi \in L^q(\Omega, L^q(0,T;L^q(\mathbb{R}^n)))$ such that, up to a subsequence,

$$(1+\mathrm{i}\mu)|u_m|^{2\beta}u_m \to \chi \text{ weakly in } L^q(\Omega, L^q(0,T;L^q(\mathbb{R}^n))). \tag{119}$$

By (118) and (119) and Mazur's theorem, we obtain $\chi = (1+\mathrm{i}\mu)|u|^{2\beta}u$ and thus

$$(1+\mathrm{i}\mu)|u_m|^{2\beta}u_m \to (1+\mathrm{i}\mu)|u|^{2\beta}u \text{ weakly in } L^q(\Omega, L^q(0,T;L^q(\mathbb{R}^n))). \tag{120}$$

Similarly, by (106) and (118), we obtain

$$u_m \to u \text{ weakly in } L^p(\Omega, L^p(0,T; L^p(\mathbb{R}^n))). \tag{121}$$

Next, we take the limit of (105) as $m \to \infty$.

Step 3. Limit of approximate equation. Let $\phi \in L^\infty(\Omega, \mathbb{R})$ and $\xi \in V \cap L^p(\mathbb{R}^n)$. Then, by (117) we obtain, for all $t \in [0,T]$,

$$\mathbb{E}((u_m(t), \xi)\phi) \to \mathbb{E}((u(t), \xi)\phi). \tag{122}$$

In addition, for each $t \in [0,T]$, by (117) we obtain

$$\mathbb{E}\left(\phi \int_0^t ((-\Delta)^{\frac{\alpha}{2}} u_m(s), (-\Delta)^{\frac{\alpha}{2}}\xi) ds\right) = \mathbb{E}\left(\int_0^T ((-\Delta)^{\frac{\alpha}{2}} u_m(s), 1_{[0,t]}(s)\phi(-\Delta)^{\frac{\alpha}{2}}\xi) ds\right)$$

$$\to \mathbb{E}\left(\int_0^T ((-\Delta)^{\frac{\alpha}{2}} u(s), 1_{[0,t]}(s)\phi(-\Delta)^{\frac{\alpha}{2}}\xi) ds\right) = \mathbb{E}\left(\phi \int_0^t ((-\Delta)^{\frac{\alpha}{2}} u(s), (-\Delta)^{\frac{\alpha}{2}}\xi) ds\right). \tag{123}$$

Similarly, for each $t \in [0,T]$, by (120), we obtain

$$\begin{aligned} &\mathbb{E}\left(\phi \int_0^t \int_{\mathbb{R}^n} (1+\mathrm{i}\mu)|u_m(s)|^{2\beta} u_m(s)\xi(x) dx ds\right) \\ &\quad \to \mathbb{E}\left(\phi \int_0^t \int_{\mathbb{R}^n} (1+\mathrm{i}\mu)|u(s)|^{2\beta} u(s)\xi(x) dx ds\right). \end{aligned} \tag{124}$$

Since $\sigma_m \to \sigma$ in $L^2(\Omega, L^2(0,T; \mathcal{L}_2(U,H)))$, we obtain, for each $t \in [0,T]$,

$$\mathbb{E}\left(\phi \int_0^t \xi\sigma_m(s) dW(s)\right) \to \mathbb{E}\left(\phi \int_0^t \xi\sigma(s) dW(s)\right). \tag{125}$$

Multiplying Equation (105) by ϕ, taking the expectation, and then letting $m \to \infty$, by (117) and (122)–(125) we obtain, for each $t \in [0,T]$ and $\xi \in V \cap L^p(\mathbb{R}^n)$,

$$\begin{aligned} &\mathbb{E}(\phi(u(t), \xi)) + (1+\mathrm{i}\nu)\mathbb{E}\left(\phi \int_0^t ((-\Delta)^{\frac{\alpha}{2}} u(s), (-\Delta)^{\frac{\alpha}{2}}\xi) ds\right) \\ &+ \mathbb{E}\left(\phi \int_0^t \int_{\mathbb{R}^n} (1+\mathrm{i}\mu)|u(s)|^{2\beta} u(s)\xi(x) dx ds\right) \\ &= \mathbb{E}(\phi(u_0, \xi)) + \rho\mathbb{E}\left(\phi \int_0^t (u(s), \xi) ds\right) + \mathbb{E}\left(\phi \int_0^t (g(s), \xi) ds\right) \\ &+ \mathbb{E}\left(\phi \int_0^t \xi\sigma(s) dW(s)\right). \end{aligned} \tag{126}$$

Since $\phi \in L^\infty(\Omega, \mathbb{R})$ is arbitrary, by (126), we infer that for every $t \in [0,T]$ and $\xi \in V \cap L^p(\mathbb{R}^n)$, there exists a subset Ω_2 (depending on t and ξ) of Ω with $P(\Omega_2) = 0$ such that for all $\omega \in \Omega \backslash \Omega_2$,

$$(u(t), \xi) + (1+\mathrm{i}\nu)\int_0^t ((-\Delta)^{\frac{\alpha}{2}} u(\omega,s), (-\Delta)^{\frac{\alpha}{2}}\xi) ds + \int_0^t \int_{\mathbb{R}^n} (1+\mathrm{i}\mu)|u(\omega,s)|^{2\beta} u(\omega,s)\xi(x) dx ds$$

$$= (u_0(\omega), \xi) + \rho\int_0^t (u(\omega,s), \xi) ds + \int_0^t (g(s), \xi) ds + \int_0^t \xi\sigma(s) dW(s). \tag{127}$$

Note that the subset Ω_2 may depend on $t \in [0,T]$ and $\xi \in V \cap L^p(\mathbb{R}^n)$ in general. However, since every term in (127) is continuous in t and the space $\xi \in V \cap L^p(\mathbb{R}^n)$ is separable, we are able to choose a subset Ω_2 of P-probability zero, which is independent of t and ξ, such that (127) is valid for all $\omega \in \Omega \setminus \Omega_2$, for all $t \in [0, T]$ and $\xi \in V \cap L^p(\mathbb{R}^n)$. By (117) and (121), we have

$$u \in L^2(\Omega, C([0, T], H)) \bigcap L^2(\Omega, L^2(0, T; V)) \bigcap L^p(\Omega, L^p(0, T; L^p(\mathbb{R}^n))). \tag{128}$$

Moreover, taking the limit in (106) with respect to m, by (128), we know that

$$\|u(t)\|^2_{L^2(\Omega,C([0,T],H))} + \|u(t)\|^2_{L^2(\Omega,L^2([0,T],V))} + \|u\|^p_{L^p(\Omega,L^p(0,T;L^p(\mathbb{R}^n)))}$$

$$\leq C_1(\|u_0\|^2_{L^2(\Omega,H)} + \|g\|^2_{L^2(0,T;H)} + \|\sigma\|^2_{L^2(\Omega,L^2(0,T;\mathcal{L}_2(U,H)))}). \tag{129}$$

By (127)–(129) we see that u is a solution of (101) and (102) with the desired estimates.

Step 4. Uniqueness of solutions. Suppose u_1 and u_2 be the solutions of (101) and (102) in the sense of Definition 5 with initial conditions $u_{0,1}$ and $u_{0,2}$, Let $\hat{u} = u_1 - u_2$, then we have, for all $\xi \in V \cap L^p(\mathbb{R}^n)$ and $t \in [0, T]$, P–almost surely,

$$(\hat{u}(t), \xi) + (1 + \mathrm{i}\nu) \int_0^t ((-\Delta)^{\frac{\alpha}{2}} \hat{u}(s), (-\Delta)^{\frac{\alpha}{2}} \xi) ds$$

$$= (u_{0,1} - u_{0,2}, \xi) - \int_0^t \int_{\mathbb{R}^n} ((1 + \mathrm{i}\mu)|u_1|^{2\beta} u_1 - (1 + \mathrm{i}\mu)|u_2|^{2\beta} u_2)\xi(x) dx ds + \rho \int_0^t (\hat{u}(s), \xi) ds. \tag{130}$$

Similar to (91), we obtain that

$$\|\hat{u}(\omega, t)\|^2 \leq e^{c_{14} t} \|\hat{u}(\omega, 0)\|^2,$$

which implies that

$$\mathbb{E}\Big(\|u_1 - u_2\|^2_{C([0,T],H)}\Big) \leq e^{c_{14} T} \mathbb{E}(\|u_{0,1} - u_{0,2}\|^2),$$

and hence the solution is unique. □

5. Existence of Solutions: Globally Lipschitz Noise

In this section, we suppose that $\sigma : \mathbb{R} \times \Omega \times H \to \mathcal{L}_2(U, H)$ is globally Lipschitz continuous in its third argument uniformly for $(t, \omega) \in \mathbb{R} \times \Omega$; namely, there exists a positive number L_0 such that for all $t \in \mathbb{R}, \omega \in \Omega$ and $u_1, u_2 \in H$,

$$\|\sigma(t, \omega, u_1) - \sigma(t, \omega, u_2)\|_{\mathcal{L}_2(U,H)} \leq L_0 \|u_1 - u_2\|. \tag{131}$$

In addition, σ satisfies (4). We suppose that for every fixed $u \in H, \sigma(\cdot, \cdot, u) : \mathbb{R} \times \Omega \to \mathcal{L}_2(\Omega, H)$ is progressively measurable.

Lemma 5. *Suppose* (131) *holds and* $u_0 \in L^2(\Omega, H)$ *is* $\mathcal{F}_0$*-measurable, then problem* (1) *and* (2) *has a unique solution* u *in the sense of Definition* 5*. Moreover, the solution* u *is continuous in* u_0 *from* $L^2(\Omega, H)$ *to* $L^2(\Omega, C([0, T]; H)) \bigcap L^2(\Omega, L^2(\Omega, L^2(0, T; V)))$ *and* u *satisfies the energy equation*

$$\|u(t)\|^2 + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}} u(s)\|^2 ds + 2\int_0^t \int_{\mathbb{R}^n} |u|^{2\beta+2} dx ds$$

$$= \|u_0\|^2 + 2\rho \int_0^t \|u(s, x)\|^2 ds + 2Re \int_0^t (u(s), g(s)) ds + 2Re \int_0^t u(s)\sigma(s, \omega(s)) dW$$

$$+ \int_0^t \|\sigma(s, u(s))\|^2_{\mathcal{L}_2(U,H)} ds,$$

for all $t \in [0,T]$, P–almost surely. In addition,

$$\|u(t)\|^2_{L^2(\Omega,C([0,T],H))} + \|u(t)\|^2_{L^2(\Omega,L^2([0,T],V))} + \|u(t)\|^p_{L^p(\Omega,L^p(0,T;L^p(\mathbb{R}^n)))}$$

$$\leq L_4(T)(\|u_0\|^2_{L^2(\Omega,H)} + \|g\|^2_{L^2(0,T;H)}), \tag{132}$$

where $L_4(T)$ is a positive number only depending on T.

Proof. For an $\mathcal{F}_0$-measurable initial condition $u_0 \in L^2(\Omega, H)$ and a given progressively measurable process $z \in L^2(\Omega, L^2(0,T;H))$, we investigate the following stochastic equation:

$$du(t) + (1+\mathrm{i}\nu)(-\Delta)^\alpha u(t)dt + (1+\mathrm{i}\mu)|u(t)|^{2\beta}u(t)dt = \rho u(t)dt + g(t)dt + \sigma(t,z(t))dW, \tag{133}$$

with initial condition

$$u(0) = u_0. \tag{134}$$

Since $z \in L^2(\Omega, L^2(0,T;H))$ is a progressively measurable process. By (4) and (131), we notice that $\sigma(\cdot, z(\cdot)) \in L^2(\Omega, L^2(0,T;\mathcal{L}_2(U,H)))$ is also progressively measurable. Then, for every $\mathcal{F}_0$-measurable $u_0 \in L^2(\Omega,H)$, by Lemma 4, problem (133) and (134) has a unique solution u in the sense of Definition 5 which satisfies (104). We define a map $\mathcal{G}{:}L^2(\Omega, L^2(0,T;H)) \to L^2(\Omega, L^2(0,T;H))$, for every $z \in L^2(\Omega, L^2(0,T;H))$, $\mathcal{G}(z) = u$, where u is the unique solution of (133) and (134).

Next we prove that $\mathcal{G}$ is a contraction when $L^2(\Omega, L^2(0,T;H))$ is endowed with an equivalent norm using Banach fixed point theorem.

Step 1. Contractility of $\mathcal{G}$. Let z_1, z_2 be progressively measurable in $L^2(\Omega, L^2(0,T;H))$, and u_1, u_2 be the solution of (133) and (134) given by Lemma 4. Let $\hat{u} = u_1 - u_2$ and $\hat{z} = z_1 - z_2$. Then, we have

$$\hat{u}(t) + (1+\mathrm{i}\nu)\int_0^t (-\Delta)^\alpha \hat{u}(s)ds + \int_0^t ((1+\mathrm{i}\mu)|u_1|^{2\beta}u_1 - (1+\mathrm{i}\mu)|u_2|^{2\beta}u_2)ds$$

$$= \rho\int_0^t \hat{u}(s)ds + \int_0^t (\sigma(s,z_1(s)) - \sigma(s,z_2(s)))dW(s) \ in \ \left(V \bigcap L^p(\mathbb{R}^n)\right)^*. \tag{135}$$

Let k_0 be a positive integer such that $k_0 > \frac{(2-q)n}{4q}$. Then we have $W^{2k_0,q}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$. We set that

$$\hat{u}_\varepsilon = (I - \varepsilon\Delta)^{-k_0}\hat{u},$$
$$f_\varepsilon(t) = (I - \varepsilon\Delta)^{-k_0}((1+\mathrm{i}\mu)|u_1|^{2\beta}u_1 - (1+\mathrm{i}\mu)|u_2|^{2\beta}u_2),$$
$$\sigma_\varepsilon(t) = (I - \varepsilon\Delta)^{-k_0}(\sigma(t,z_1) - \sigma(t,z_2)).$$

Hence, we obtain

$$\hat{u}_\varepsilon(t) + (1+\mathrm{i}\nu)\int_0^t (-\Delta)^\alpha \hat{u}_\varepsilon(s)ds + \int_0^t f_\varepsilon(s)ds = \rho\int_0^t \hat{u}_\varepsilon(s)ds + \int_0^t \sigma_\varepsilon(s)dW(s) \ in \ H. \tag{136}$$

Let $\theta \geq 0$ be a fixed constant, $\hat{u}^\theta_\varepsilon = e^{-\theta t}\hat{u}_\varepsilon$, $\hat{u}^\theta = e^{-\theta t}\hat{u}$. By (136), we obtain that

$$\begin{aligned} &\hat{u}^\theta_\varepsilon(t) + \theta\int_0^t \hat{u}^\theta_\varepsilon(s)ds + (1+\mathrm{i}\nu)\int_0^t (-\Delta)^\alpha \hat{u}^\theta_\varepsilon(s)ds + \int_0^t e^{-\mu s} f_\varepsilon(s)ds \\ &= \rho\int_0^t \hat{u}^\theta_\varepsilon(s)ds + \int_0^t e^{-\theta s}\sigma_\varepsilon(s)dW(s) \end{aligned} \tag{137}$$

in H. By (137) and integration by parts of Ito's formula, we obtain

$$\|\hat{u}^\theta_\varepsilon(t)\|^2 + 2\theta\int_0^t \|\hat{u}^\theta_\varepsilon(t)\|^2 ds + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}}\hat{u}^\theta_\varepsilon(s)\|^2 ds + 2\mathrm{Re}\int_0^t e^{-\theta s}(f_\varepsilon(s), \hat{u}^\theta_\varepsilon(s))ds$$

$$= 2\rho\int_0^t \|\hat{u}^\theta_\varepsilon(s)\|^2 ds + 2\mathrm{Re}\int_0^t e^{-\theta s}\hat{u}^\theta_\varepsilon(s)\sigma_\varepsilon(s)dW(s) + \int_0^t \|e^{-\theta s}\sigma_\varepsilon(s)\|^2_{\mathcal{L}_2(U,H)}ds. \tag{138}$$

For all $r \in (1, \infty)$, $h \in L^r(\mathbb{R}^n)$, we have

$$\|(I - \varepsilon\Delta)^{-1}h\|_{L^r(\mathbb{R}^n)} \leq \|h\|_{L^r(\mathbb{R}^n)} \text{ and } \lim_{\varepsilon\to 0}\|(I - \varepsilon\Delta)^{-1}h - h\|_{L^r(\mathbb{R}^n)} = 0. \tag{139}$$

By (139) and the dominated convergence theorem, we obtain that, for every $t \in [0, T]$,

$$\lim_{\varepsilon\to 0}\|\hat{u}^\theta_\varepsilon(t)\|^2 = \|\hat{u}^\theta(t)\|^2, \tag{140}$$

$$\lim_{\varepsilon\to 0}\int_0^t \|\hat{u}^\theta_\varepsilon(s)\|^2 ds = \int_0^t \|\hat{u}^\theta(s)\|^2 ds, \tag{141}$$

$$\lim_{\varepsilon\to 0}\int_0^t \|(-\Delta)^{\frac{\alpha}{2}}\hat{u}^\theta_\varepsilon(s)\|^2 ds = \int_0^t \|(-\Delta)^{\frac{\alpha}{2}}\hat{u}^\theta(s)\|^2 ds, \tag{142}$$

$$\lim_{\varepsilon\to 0}\int_0^t \|e^{-\theta s}\sigma_\varepsilon(s)\|^2_{\mathcal{L}_2(U,H)}ds = \int_0^t \|e^{-\theta s}\sigma(s, z_1(s)) - \sigma(s, z_2(s))\|^2_{\mathcal{L}_2(U,H)}ds, \tag{143}$$

$$\lim_{\varepsilon\to 0}\|f_\varepsilon(s) - ((1 + \mathrm{i}\mu)|u_1(s)|^{2\beta}u_1(s) - (1 + \mathrm{i}\mu)|u_2(s)|^{2\beta}u_2(s))\|_{L^q(\mathbb{R}^n)} = 0,$$

and

$$\lim_{\varepsilon\to 0}\|\hat{u}^\theta_\varepsilon(s) - \hat{u}^\theta(s)\|_{L^p(\mathbb{R}^n)} = 0.$$

Then, we obtain

$$\lim_{\varepsilon\to 0}\|f_\varepsilon(s)\hat{u}^\theta_\varepsilon(s) - ((1 + \mathrm{i}\mu)|u_1(s)|^{2\beta}u_1(s) - (1 + \mathrm{i}\mu)|u_2(s)|^{2\beta}u_2(s))\hat{u}^\theta(s)\|_{L^1(\mathbb{R}^n)} = 0.$$

By (139) and the dominated convergence theorem, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^t e^{-\theta s}\mathrm{Re}(f_\varepsilon(s), \hat{u}^\theta_\varepsilon(s))ds \\ &= \int_0^t e^{-\theta s}\int_{\mathbb{R}^n}\hat{u}^\theta(s)((1 + \mathrm{i}\mu)|u_1(s)|^{2\beta}u_1(s) - (1 + \mathrm{i}\mu)|u_2(s)|^{2\beta}u_2(s))dxds.\end{aligned} \tag{144}$$

To prove the stochastic term in (138), we need to prove the convergence of quadratic variation,

$$\begin{aligned}&\left[\mathrm{Re}\int_0^t e^{-\theta s}\hat{u}^\theta_\varepsilon(s)\sigma_\varepsilon(s)dW(s) - \mathrm{Re}\int_0^t e^{-\theta s}\hat{u}^\theta(\sigma(s, z_1(s)) - \sigma(s, z_2(s))dW(s))\right]_T^{\frac{1}{2}} \\ &= \|\mathrm{Re}(e^{-\theta s}\hat{u}^\theta_\varepsilon(s)\sigma_\varepsilon(s)) - \mathrm{Re}(e^{-\theta s}\hat{u}^\theta(\sigma(s, z_1(s)) - \sigma(s, z_2(s))))\|_{L^2(0,T;\mathcal{L}_2(U,\mathbb{R}))} \\ &\quad\leq \|e^{-\theta s}\hat{u}^\theta_\varepsilon(s)\sigma_\varepsilon(s) - e^{-\theta s}\hat{u}^\theta_\varepsilon(s)(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))\|_{L^2(0,T;\mathcal{L}_2(U,\mathbb{R}))} \\ &\qquad + \|e^{-\theta s}(\hat{u}^\theta_\varepsilon(s) - \hat{u}^\theta(s))(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))\|_{L^2(0,T;\mathcal{L}_2(U,\mathbb{R}))}.\end{aligned} \tag{145}$$

By (139), we have, for $s \in [0, T]$,

$$\|e^{-\theta s}\hat{u}_\varepsilon^\theta(s)\sigma_\varepsilon(s) - e^{-\theta s}\hat{u}_\varepsilon^\theta(s)(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))\|_{\mathcal{L}_2(U,\mathbb{R}))}$$

$$\leq e^{-\theta s}\|\hat{u}^\theta(s)\|\|\sigma_\varepsilon(s) - (\sigma(s, z_1(s)) - \sigma(s, z_2(s)))\|_{\mathcal{L}_2(U,\mathbb{R})} \to 0 \; as \; \varepsilon \to 0. \tag{146}$$

By (139) and (146), we obtain

$$\|e^{-\theta s}\hat{u}_\varepsilon^\theta(s)\sigma_\varepsilon(s) - e^{-\theta s}\hat{u}_\varepsilon^\theta(s)(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))\|_{\mathcal{L}_2(U,\mathbb{R})}$$

$$\leq 2e^{-\theta s}\|\hat{u}^\theta(s)\|\|(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))\|_{\mathcal{L}_2(U,\mathbb{R}))}. \tag{147}$$

It follows from (146) and (147) and the dominated convergence theorem, we have

$$\lim_{\varepsilon \to 0} \|e^{-\theta s}\hat{u}_\varepsilon^\theta(s)\sigma_\varepsilon(s) - e^{-\theta s}\hat{u}_\varepsilon^\theta(s)(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))\|_{L^2(0,T;\mathcal{L}_2(U,\mathbb{R}))} = 0 \tag{148}$$

and

$$\lim_{\varepsilon \to 0} \|e^{-\theta s}(\hat{u}_\varepsilon^\theta(s) - \hat{u}^\theta(s)(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))\|_{L^2(0,T;\mathcal{L}_2(U,\mathbb{R}))} = 0. \tag{149}$$

By (145) and (148) and (149), we obtain

$$\lim_{\varepsilon \to 0} \left[\int_0^t e^{-\theta s}\hat{u}_\varepsilon^\theta(s)\sigma_\varepsilon(s)dW(s) - \int_0^t e^{-\theta s}\hat{u}^\theta(s)(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))dW(s)\right]_T^{\frac{1}{2}} = 0$$

in probability, and hence

$$\lim_{\varepsilon \to 0} \int_0^t e^{-\theta s}\hat{u}_\varepsilon^\theta(s)\sigma_\varepsilon(s)dW(s) = \int_0^t e^{-\theta s}\hat{u}^\theta(s)(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))dW(s), \tag{150}$$

in probability uniformly for $t \in [0, T]$. Letting $\varepsilon \to 0$ in (138). By (140)–(144) and (150), for $t \in [0, T]$ we infer

$$\|\hat{u}^\theta(t)\|^2 + 2\theta \int_0^t \|\hat{u}^\theta(s)\|^2 ds + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}}\hat{u}^\theta\|^2 ds$$

$$+2\mathrm{Re}\int_0^t e^{-\theta s}\int_{\mathbb{R}^n}((1 + \mathrm{i}\mu)|u_1(s)|^{2\beta}u_1(s) - (1 + \mathrm{i}\mu)|u_2(s)|^{2\beta}u_2(s))\hat{u}^\theta(s)dxds$$

$$= 2\rho\int_0^t \|\hat{u}^\theta(s)\|^2 ds + 2\mathrm{Re}\int_0^t e^{-\theta s}\hat{u}^\theta(s)(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))dW(s)$$

$$+ \int_0^t \|e^{-\theta s}(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))\|^2_{\mathcal{L}_2(U,H)}ds. \tag{151}$$

Taking the expectation of (151), and applying in (131), we obtain

$$2\theta\mathbb{E}\left(\int_0^T \|e^{-\theta s}(u_1(s) - u_2(s))\|^2\right)ds$$

$$\leq 2\rho\mathbb{E}\left(\int_0^T \|e^{-\theta s}(u_1(s) - u_2(s))\|^2 ds\right) + 2\mathbb{E}\left(\sup_{0\leq t\leq T} \mathrm{Re}\int_0^t e^{-\theta s}\hat{u}^\theta(s)(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))dW(s)\right)$$

$$+\mathbb{E}\left(\int_0^T \|e^{-\theta s}(\sigma(s, z_1(s)) - \sigma(s, z_2(s)))\|^2_{\mathcal{L}_2(U,H)}ds\right).$$

By the Burkholder–Davis–Gundy inequality and Young's inequality, we obtain

$$2\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t e^{-\theta s}\hat{u}^\theta(s)(\sigma(s,z_1(s))-\sigma(s,z_2(s)))dW(s)\right|\right)$$

$$\le c_{15}\mathbb{E}\left(\int_0^T \|\hat{u}^\theta(s)\|^2\|e^{-\theta s}\sigma(s,z_1(s))-\sigma(s,z_2(s))\|^2_{\mathcal{L}_2(U,H)}\right)^{\frac{1}{2}}$$

$$\le \frac{c_{15}^2}{4}\mathbb{E}\left(\int_0^T \|\hat{u}^\theta(s)\|^2 ds\right)+\mathbb{E}\left(\int_0^T \|e^{-\theta s}(\sigma(s,z_1(s))-\sigma(s,z_2(s)))\|^2 ds\right).$$

Hence, we have

$$2\theta\mathbb{E}\left(\int_0^T \|e^{-\theta s}(u_1(s)-u_2(s))\|^2\right)ds$$

$$\le (2\rho+\frac{c_{15}^2}{4})\mathbb{E}\left(\int_0^T \|e^{-\theta s}(u_1(s)-u_2(s))\|^2\right)ds+2L_0^2\mathbb{E}\left(\int_0^T \|e^{-\theta s}(z_1(s)-z_2(s))\|^2 ds\right). \tag{152}$$

For fixed $\theta \ge 0$, denote by $L^2_\theta(\Omega, L^2(0,T;H))$ the space $L^2(\Omega, L^2(0,T;H))$ equipped with the equivalent norm

$$\|u\|_{L^2_\theta(\Omega,L^2(0,T;H))}=\left(\mathbb{E}\left(\int_0^T \|e^{-\theta s}u(s)\|^2\right)\right)^{\frac{1}{2}}\ for\ u\in L^2(\Omega,L^2(0,T;H))).$$

Then by (152) we obtain, for $\theta > \rho + \frac{c_{15}^2}{8}$,

$$\begin{aligned}\|\mathcal{G}(z_1)-\mathcal{G}(z_2)\|_{L^2_\theta(\Omega,L^2(0,T;H))} &= \|u_1-u_2\|_{L^2_\theta(\Omega,L^2(0,T;H))}\\ &\le \frac{\sqrt{2}L_0}{\sqrt{2\theta-2\rho-\frac{c_{15}^2}{4}}}\|z_1-z_2\|_{L^2_\theta(\Omega,L^2(0,T;H))}.\end{aligned} \tag{153}$$

We choose a positive number θ large enough such that $\frac{\sqrt{2}L_0}{\sqrt{2\theta-2\rho-\frac{c_{15}^2}{4}}}<1$. Then, we obtain that $\mathcal{G}$ is a contraction. Therefore, it has a unique fixed point, which is the unique solution of (133) and (134) in the sense of Definition 5.

Step 2. Continuity of solutions in initial date. Let $u_{0,1}, u_{0,2} \in L^2(\Omega, H)$ be $\mathcal{F}_0$-measurable, $u_1 = \mathcal{G}(u_1), u_2 = \mathcal{G}(u_2)$ be the fixed points of $\mathcal{G}$ corresponding to initial date $u_{0,1}$ and $u_{0,2}$. Denote by $\hat{u} = u_1 - u_2$, $\hat{u}_0 = u_{0,1} - u_{0,2}$. By (151) with $\theta = 0$, for $t \in [0, T]$, we obtain

$$\|\hat{u}(t)\|^2+2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}}\hat{u}(s)\|^2 ds \le \|\hat{u}_0\|^2+2\rho\int_0^t \|\hat{u}(s)\|^2 ds$$

$$+2\mathrm{Re}\int_0^t \hat{u}(s)(\sigma(s,u_1(s))-\sigma(s,u_2(s)))dW(s)+\int_0^t \|\sigma(s,u_1(s))-\sigma(s,u_2(s))\|^2_{\mathcal{L}_2(U,H)}ds. \tag{154}$$

By (154), we find that for all $0 \le t \le T$,

$$\mathbb{E}\left(\sup_{0\le r\le t}\|\hat{u}(r)\|^2\right)\le \mathbb{E}(\|\hat{u}_0\|^2)+2\rho\int_0^t \mathbb{E}\left(\sup_{0\le r\le s}\|\hat{u}(r)\|^2\right)ds$$

$$+2\mathbb{E}\left(\sup_{0\le r\le t}\left|\int_0^r \hat{u}(s)(\sigma(s,u_1(s))-\sigma(s,u_2(s)))dW(s)\right|\right)$$

$$+\mathbb{E}\left(\int_0^T \|\sigma(s,u_1(s))-\sigma(s,u_2(s))\|^2_{\mathcal{L}_2(U,H)}ds\right). \tag{155}$$

By the Burkholder–Davis–Gundy inequality and (131), we deduce, for all $0 \le t \le T$,

$$2\mathbb{E}\left(\sup_{0\le r\le t}\left|\int_0^r \hat{u}(s)(\sigma(s,u_1(s)),\sigma(s,u_2(s)))dW(s)\right|\right)$$

$$\le c_{16}\mathbb{E}\left(\left(\int_0^t \|\hat{u}(s)\|^2\|\sigma(s,u_1(s))-\sigma(s,u_2(s))\|^2_{\mathcal{L}_2(U,H)}ds\right)^{\frac{1}{2}}\right)$$

$$\le c_{16}\mathbb{E}\left(\sup_{0\le s\le t}\|\hat{u}(s)\|\left(\int_0^t \|\sigma(s,u_1(s))-\sigma(s,u_2(s))\|^2_{\mathcal{L}_2(U,H)}ds\right)^{\frac{1}{2}}\right)$$

$$\le \frac{1}{2}\mathbb{E}\left(\sup_{0\le r\le t}\|\hat{u}(r)\|^2\right)+\frac{1}{2}c_{16}^2\mathbb{E}\left(\int_0^t \|\sigma(s,u_1(s))-\sigma(s,u_2(s))\|^2_{\mathcal{L}_2(U,H)}ds\right)$$

$$\le \frac{1}{2}\mathbb{E}\left(\sup_{0\le r\le t}\|\hat{u}(r)\|^2\right)+\frac{1}{2}c_{16}^2L_0^2\mathbb{E}\left(\int_0^t \|u_1(s)-u_2(s)\|^2ds\right)$$

$$\le \frac{1}{2}\mathbb{E}\left(\sup_{0\le r\le t}\|\hat{u}(r)\|^2\right)+\frac{1}{2}c_{16}^2L_0^2\int_0^t \mathbb{E}\left(\sup_{0\le r\le s}\|\hat{u}(r)\|^2\right)ds. \tag{156}$$

By (131), (155) and (156), we obtain, for all $0 \le t \le T$,

$$\mathbb{E}\left(\sup_{0\le r\le t}\|\hat{u}(r)\|^2\right) \le 2\mathbb{E}(\|\hat{u}(0)\|^2)+(c_{16}^2L_0^2+2L_0^2+4\rho)\int_0^t \mathbb{E}\left(\sup_{0\le r\le s}\|\hat{u}(r)\|^2\right)ds. \tag{157}$$

By (157) and the Gronwall inequality, we find that for all $0 \le t \le T$,

$$\mathbb{E}\left(\sup_{0\le r\le t}\|\hat{u}(r)\|^2\right) \le 2e^{(c_{16}^2L_0^2+2L_0^2+4\rho)t}\mathbb{E}(\|\hat{u}(0)\|^2). \tag{158}$$

In addition, by (156), (158) and (154) with $t = T$, we obtain

$$\mathbb{E}\left(\int_0^T \|(-\Delta)^{\frac{\alpha}{2}}\hat{u}(s)\|^2ds\right) \le c_{17}\mathbb{E}(\|\hat{u}_0\|^2), \tag{159}$$

where $c_{17} > 0$ depending only on T. By (158) and (159), we obtain

$$\|u_1-u_2\|^2_{L^2(\Omega,C([0,T],H))}+\|u_1-u_2\|^2_{L^2(\Omega,L^2(0,T;V))} \le c_{20}\|u_{0,1}-u_{0,2}\|^2_{L^2(\Omega,H)}. \tag{160}$$

Therefore, the solution is continuous in initial data.

Step 3. Uniform estimates of solutions. We suppose u is the solution of (1) and (2) with initial data $u_0 \in L^2(\Omega,H)$, Then we have

$$u(t)+(1+i\nu)\int_0^t(-\Delta)^\alpha u(s)ds+\int_0^t(1+i\mu)|u(s)|^{2\beta}u(s)ds = u_0+\rho\int_0^t u(s)ds+\int_0^t g(s)ds$$

$$+\int_0^t \sigma(s,u(s))dW \text{ in } (V\bigcap L^p(\mathbb{R}^n))^*, \tag{161}$$

P–almost surely. We set

$$u_\varepsilon(t)=(I-\varepsilon\Delta)^{-k_0}u(t), u_{0,\varepsilon}=(I-\varepsilon\Delta)^{-k_0}u_0, g_\varepsilon(t)=(I-\varepsilon\Delta)^{-k_0}g(t)$$

and

$$f^{\varepsilon}(t) = (I-\varepsilon\Delta)^{-k_0}(1+i\mu)|u(t)|^{2\beta}u(t), \sigma^{\varepsilon}(t) = (I-\varepsilon\Delta)^{-k_0}\sigma(t,u(t)).$$

Then by (161) we obtain, for $t \in [0,T]$,

$$u_{\varepsilon}(t) + (1+i\nu)\int_0^t (-\Delta)^{\alpha}u_{\varepsilon}(s)ds + \int_0^t f^{\varepsilon}(s)ds,$$

$$= u_{0,\varepsilon} + \rho\int_0^t u_{\varepsilon}(s)ds + \int_0^t g_{\varepsilon}(s)ds + \int_0^t \sigma^{\varepsilon}(s)dW(s) \; in \; H, \tag{162}$$

P–almost surely. By (162) and integration by parts of Ito's formula, we obtain, for every $t \in [0,T]$,

$$\|u_{\varepsilon}(t)\|^2 + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}}u_{\varepsilon}(s)\|^2 ds + 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^n} f^{\varepsilon}(s,x,u(s))\bar{u}_{\varepsilon}(s)dxds$$

$$= \|u_{0,\varepsilon}\|^2 + 2\rho\int_0^t \|u_{\varepsilon}(s,x)\|^2 dxds + 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^n} g_{\varepsilon}(s)\bar{u}_{\varepsilon}(s)dxds + 2\mathrm{Re}\int_0^t u^{\varepsilon}(s)\sigma^{\varepsilon}(s,\omega(s))dW$$

$$+ \int_0^t \|\sigma^{\varepsilon}(s,u(s))\|^2_{\mathcal{L}_2(U,H)}ds, \tag{163}$$

P–almost surely. Taking the limit of (163) as $\varepsilon \to 0$, we obtain, for $t \in [0,T]$,

$$\|u(t)\|^2 + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}}u(s)\|^2 ds + 2\mathrm{Re}\int_0^t \int_{\mathbb{R}^n} (1+i\mu)|u(s)|^{2\beta}u(s)\bar{u}(s)dxds$$

$$= \|u_0\|^2 + 2\rho\int_0^t \|u(s,x)\|^2 dxds + 2\mathrm{Re}\int_0^t (u(s),g(s))ds + 2\mathrm{Re}\int_0^t u(s)\sigma(s,\omega(s))dW$$

$$+ \int_0^t \|\sigma(s,u(s))\|^2_{\mathcal{L}_2(U,H)}ds, \tag{164}$$

P–almost surely. Similar to the proof of Lemma 2 and by (4), we can derive the uniform estimates (132). □

6. Existence of Solutions: Locally Lipschitz Noise

In this section, we prove the existence and uniqueness of solutions to problem (1) and (2) with a locally Lipschitz continuous diffusion term.

Let $\sigma : \mathbb{R}^n \times \Omega \times H \to \mathcal{L}_2(U,H)$ which satisfies condition (3) be locally Lipschitz continuous in its third argument uniformly for $(t,\omega) \in \mathbb{R} \times \Omega$, we introduce a truncation operator $\eta_n : H \to H$ given by

$$\eta_n(u) = \begin{cases} u & if \; \|u\| \le n, \\ \frac{nu}{\|u\|} & if \; \|u\| > n. \end{cases}$$

Then $\eta_n : H \to H$ is globally Lipschitz continuous with unit Lipschitz coefficient,

$$\|\eta_n(u_1) - \eta_n(u_2)\| \le \|u_1 - u_2\|, \;\; for \; all \; u_1, u_2 \in H, \tag{165}$$

and

$$\|\eta_n(u)\| \le n, \;\; for \; all \; u \in H. \tag{166}$$

Given $n \in \mathbb{N}$, we set $\sigma_n(t, \omega, u) = \sigma(t, \omega, \eta_n(u))$ for $t \in \mathbb{R}, \omega \in \Omega$ and $u \in H$. By (3) and (4) and (165) and (166), we infer

$$\|\sigma_n(t, \omega, u_1) - \sigma_n(t, \omega, u_2)\|_{\mathcal{L}_2(U,H)} \leq M_n \|u_1 - u_2\|, \tag{167}$$

and

$$\|\sigma_n(t, \omega, u)\|_{\mathcal{L}_2(U,H)} \leq L(1 + \|u\|). \tag{168}$$

Therefore, we can apply Lemma 5 to approximate σ by globally Lipschitz continuous function σ_n. Given $n \in \mathbb{N}$, we consider the following stochastic equation:

$$\begin{aligned} du_n(t) + (1 + \mathrm{i}\nu)(-\Delta)^\alpha u_n(t)dt + (1 + \mathrm{i}\mu)|u_n(t)|^{2\beta} u_n(t)dt =& \rho u_n(t)dt + g(t, x)dt \\ &+ \sigma_n(t, x, u_n)dW, \end{aligned} \tag{169}$$

with initial condition

$$u_n(0) = u_0. \tag{170}$$

By (167) and (168), for every $\mathcal{F}_0$-measurable $u_0 \in L^2(\Omega, H)$, problem (169) and (170) has a unique solution u_n as given by Lemma 5. In addition, u_n satisfies (132) and the energy equation

$$\begin{aligned} &\|u_n(t)\|^2 + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}} u_n(s)\|^2 ds + 2\int_0^t \int_{\mathbb{R}^n} |u_n(s)|^{2\beta+2} dxds \\ &= \|u_0\|^2 + 2\rho \int_0^t \|u_n(s, x)\|^2 dxds + 2\mathrm{Re} \int_0^t (u_n(s), g(s, x))ds \\ &+ 2\mathrm{Re} \int_0^t u_n(s)\sigma_n(s, x, u_n(s))dW(s) + \int_0^t \|\sigma_n(s, u_n(s))\|^2_{\mathcal{L}_2(U,H)} ds, \end{aligned} \tag{171}$$

for all $t \in [0, T]$, P–almost surely.

Next, we establish the uniform estimates on the sequence $\{u_n\}_{n=1}^\infty$ and prove its limit is a solution of problem (1) and (2).

We define a stopping τ_n (for each $n \in \mathbb{N}$) by

$$\tau_n = \inf\{t \geq 0 : \|u_n(t)\| > n\} \wedge T, \tag{172}$$

where $\inf\{t \geq 0 : \|u_n(t)\| > n\} = +\infty$ if $\{t \geq 0 : \|u_n(t)\| > n\} \neq \emptyset$. We write $u_n^{\tau_n}(t) = u_n(t \wedge \tau_n)$. We will prove $\{u_n^{\tau_n}\}_{n=1}^\infty$ is consistent.

Lemma 6. *Suppose (3) and (4) hold. Let u_n be the solution of (169) and (170) and τ_n be the stopping time given by (172). Then, $u_{n+1}^{\tau_n} = u_n^{\tau_n}$ and $\tau_{n+1} = \tau_n$ a.s. for all $n \in \mathbb{N}$.*

Proof. Let $\hat{u}_n = u_{n+1} - u_n$. Then, we obtain

$$\begin{aligned} d\hat{u}_n + (1 + \mathrm{i}\nu)(-\Delta)^\alpha \hat{u}_n dt + ((1 + \mathrm{i}\mu)|u_{n+1}|^{2\beta} u_{n+1} - (1 + \mathrm{i}\mu)|u_n|^{2\beta} u_n)dt = \rho \hat{u}_n dt \\ + (\sigma_{n+1}(t, u_{n+1}) - \sigma_n(t, u_n))dW. \end{aligned} \tag{173}$$

Similar to (151) with $\theta = 0$, we can obtain that for $t \in [0, T]$,

$$\begin{aligned} &\|\hat{u}_n(t \wedge \tau_n)\|^2 + 2\mathrm{Re} \int_0^{t\wedge\tau_n} \int_{\mathbb{R}^n} ((1 + \mathrm{i}\mu)|u_{n+1}|^{2\beta} u_{n+1} - (1 + \mathrm{i}\mu)|u_n|^{2\beta} u_n)\hat{u}_n(s)dxds \\ &+ 2\int_0^{t\wedge\tau_n} \|(-\Delta)^{\frac{\alpha}{2}} \hat{u}_n\|^2 ds = 2\mathrm{Re} \int_0^{t\wedge\tau_n} \hat{u}_n(s)(\sigma_{n+1}(s, u_{n+1}(s)) - \sigma_n(s, u_n(s)))dW(s) + \end{aligned}$$

$$2\rho\int_0^{t\wedge\tau_n}\|\hat{u}_n(s)\|^2ds+\int_0^{t\wedge\tau_n}\|\sigma_{n+1}(s,u_{n+1}(s))-\sigma_n(s,u_n(s))\|^2_{\mathcal{L}_2(U,H)}ds. \tag{174}$$

By the definition of η_n, we infer from $\sigma_n(s,u_n(s))=\sigma_{n+1}(s,u_n(s))$ for $s\in[0,\tau_n]$, for $t\in[0,T]$,

$$\|\hat{u}_n(t\wedge\tau_n)\|^2=\|u^{\tau_n}_{n+1}(t)-u^{\tau_n}_n(t)\|^2\le 2\rho\int_0^{t\wedge\tau_n}\|u_{n+1}(s)-u_n(s)\|^2ds$$

$$+2\mathrm{Re}\int_0^{t\wedge\tau_n}(u_{n+1}(s)-u_n(s))(\sigma_{n+1}(s,u_{n+1}(s))-\sigma_{n+1}(s,u_n(s)))dW(s)$$

$$+\int_0^{t\wedge\tau_n}\|\sigma_{n+1}(s,u_{n+1}(s))-\sigma_{n+1}(s,u_n(s))\|^2_{\mathcal{L}_2(U,H)}ds.$$

By (167), we imply that for $t\in[0,T]$,

$$\mathbb{E}\left(\sup_{0\le r\le t}\|u^{\tau_n}_{n+1}(r)-u^{\tau_n}_n(r)\|^2\right)\le 2\rho\int_0^t\mathbb{E}\left(\sup_{0\le r\le s}\|u^{\tau_n}_{n+1}(r)-u^{\tau_n}_n(r)\|^2\right)ds$$

$$+2\mathbb{E}\left(\sup_{0\le r\le t\wedge\tau_n}\left|\int_0^r(u_{n+1}(s)-u_n(s))(\sigma_{n+1}(s,u_{n+1}(s))-\sigma_{n+1}(s,u_n(s)))dW(s)\right|\right)$$

$$+M^2_{n+1}\int_0^t\mathbb{E}\left(\sup_{0\le r\le s}\|u^{\tau_n}_{n+1}(r)-u^{\tau_n}_n(r)\|^2\right)ds. \tag{175}$$

For the second term on the right-hand side of (175), we have

$$2\mathbb{E}\left(\sup_{0\le r\le t\wedge\tau_n}\left|\int_0^r(u_{n+1}(s)-u_n(s))(\sigma_{n+1}(s,u_{n+1}(s))-\sigma_{n+1}(s,u_n(s)))dW(s)\right|\right)$$

$$\le c_{18}\mathbb{E}\left(\left(\int_0^{t\wedge\tau_n}\|u_{n+1}(s)-u_n(s)\|^2\|\sigma_{n+1}(s,u_{n+1}(s))-\sigma_{n+1}(s,u_n(s))\|^2_{\mathcal{L}_2(U,H)}ds\right)^{\frac{1}{2}}\right)$$

$$\le c_{18}M_{n+1}\mathbb{E}\left(\sup_{0\le r\le t}\|u^{\tau_n}_{n+1}(r)-u^{\tau_n}_n(r)\|\left(\int_0^t\|u^{\tau_n}_{n+1}(s)-u^{\tau_n}_n(s)\|^2ds\right)^{\frac{1}{2}}\right)$$

$$\le\frac{1}{2}\mathbb{E}\left(\sup_{0\le r\le t}\|u^{\tau_n}_{n+1}(r)-u^{\tau_n}_n(r)\|^2\right)+\frac{1}{2}c_{18}M^2_{n+1}\int_0^t\mathbb{E}\left(\sup_{0\le r\le s}\|u^{\tau_n}_{n+1}(r)-u^{\tau_n}_n(r)\|^2ds\right). \tag{176}$$

It follows from (175) and (176) that for $t\in[0,T]$,

$$\mathbb{E}\left(\sup_{0\le r\le t}\|u^{\tau_n}_{n+1}(r)-u^{\tau_n}_n(r)\|^2\right)$$

$$\le(4\rho+c_{18}M^2_{n+1}+2M^2_{n+1})\int_0^t\mathbb{E}\left(\sup_{0\le r\le s}\|u^{\tau_n}_{n+1}(r)-u^{\tau_n}_n(r)\|^2\right)ds.$$

By Gronwall inequality, we obtain, for $t\in[0,T]$,

$$\mathbb{E}\left(\sup_{0\le r\le t}\|u^{\tau_n}_{n+1}(r)-u^{\tau_n}_n(r)\|^2\right)=0.$$

Therefore, $u^{\tau_n}_{n+1}=u^{\tau_n}_n$. By (172), we can get that for all $n\in\mathbb{N}$, we can infer $\tau_{n+1}\ge\tau_n$. $\square$

Since $\tau_{n+1} \geq \tau_n$ a.s., the stopping time τ is well-defined:

$$\tau = \lim_{n\to\infty} \tau_n = \sup_{n\in\mathbb{N}} \tau_n. \tag{177}$$

Next, we prove $\tau = T$ almost everywhere.

Lemma 7. *Suppose (3) and (4) hold and $u_0 \in L^2(\Omega, H)$ is $\mathcal{F}_0$-measurable. Let τ be the stopping time given by (177). Then, $\tau = T$ almost surely.*

Proof. For all $n \in \mathbb{N}$ these estimates are independent of the Lipschitz coefficient M_n of σ_n in (167), therefore, the solution u_n of (169) and (170) satisfies the estimates given by (132). In addition, by (172) we have $\{\tau_n < T\} \subseteq \{\|u_n\|_{C([0,T],H)} \geq n\}$, applying Chebyshev's inequality and Lemma 5 yields

$$P\{\tau_n < T\} \leq P\{\|u_n\|_{C([0,T],H)} \geq n\} \leq \frac{1}{n^2}\mathbb{E}(\|u_n\|^2_{C([0,T],H)}) \leq \frac{c_{19}}{n^2}.$$

By the Borel-Cantelli lemma, we have

$$P(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{\tau_n < T\}) = 0.$$

As a result, there exists a subset Ω_0 of Ω with $P(\Omega) = 0$ such that for each $\omega \in \Omega \setminus \Omega_0$, there exists $n_0 = n_0(\omega)$ such that $\tau_n(\omega) = T$ for all $n \geq n_0$. Then, $\tau(\omega) = \lim_{n\to\infty} \tau(n) = T$ for all $\omega \in \Omega \setminus \Omega_0$. □

Next we prove the existence and uniqueness of solution (1) and (2).

Theorem 2. *Suppose (3) and (4) hold and $u_0 \in L^2(\Omega, H)$ is $\mathcal{F}_0$-measurable. Then, problem (1) and (2) has a unique solution u in the sense of Definition 5. Moreover, u satisfies the energy equation:*

$$\|u(t)\|^2 + 2\int_0^t \|(-\Delta)^{\frac{\alpha}{2}} u(s)\|^2 ds + 2\int_0^t \int_{\mathbb{R}^n} |u(s)|^{2\beta+2} dxds$$
$$= \|u_0\|^2 + 2\rho\int_0^t \|u(s,x)\|^2 ds + 2Re\int_0^t (u(s), g(s))ds + 2Re\int_0^t u(s)\sigma(s,\omega(s))dW$$
$$+ \int_0^t \|\sigma(s,u(s))\|^2_{\mathcal{L}_2(U,H)} ds, \tag{178}$$

for all $t \in [0,T]$, P–almost surely. In addition,

$$\|u(t)\|^2_{L^2(\Omega,C([0,T],H))} + \|u(t)\|^2_{L^2(\Omega,L^2([0,T],V))} + \|u(t)\|^p_{L^p(\Omega,L^p(0,T;L^p(\mathbb{R}^n)))}$$
$$\leq L_5(T)(\|u_0\|^2_{L^2(\Omega,H)} + \|g\|^2_{L^2(0,T;H)}). \tag{179}$$

where $L_5(T)$ is a positive number depending only on T.

Proof. By Lemma 6 and Lemma 7, we know that there exists a measurable set Ω_0 of Ω such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$,

$$\tau(\omega) = T \text{ and } u_n^{\tau_m}(\omega) = u_m^{\tau_m}(\omega),\ \forall n \geq m. \tag{180}$$

we define a function $u : [0,T] \times \Omega \to H$ by

$$u(t,\omega) = \begin{cases} u_n(t,\omega) & if\ \omega \in \Omega_0\ and\ t \in [0,\tau_n(\omega)]; \\ u_0(\omega) & if\ \omega \in \Omega \backslash \Omega_0\ and\ t \in [0,T]. \end{cases} \tag{181}$$

By above definitions, we can conclude that u is a continuous H-valued $\mathcal{F}_t$-adapted process and the fact:

$$\lim_{n\to\infty} u_n(t,\omega) = u(t,\omega),\ for\ all\ t \in [0,T]\ and\ \omega \in \Omega_0. \tag{182}$$

By Lemma 5, the solution u_n satisfies (132) for all $n \in \mathbb{N}$. By (182) and Fatou's lemma, we conclude that there exists a positive number $L_5(T)$ depending only on T such that

$$\|u(t)\|^2_{L^2(\Omega,C([0,T],H))} + \|u(t)\|^2_{L^2(\Omega,L^2([0,T],V))} + \|u(t)\|^p_{L^p(\Omega,L^p(0,T;L^p(\mathbb{R}^n)))}$$

$$\leq L_5(T)(\|u_0\|^2_{L^2(\Omega,H)} + \|g\|^2_{L^2(0,T;H)}), \tag{183}$$

which gives (179). We are now prove that u satisfies (1) and (2). u_n is the solution of (169) and (170), then we have, for $t \in [0,T]$,

$$u_n(t\wedge\tau_n) + (1+\mathrm{i}\nu)\int_0^{t\wedge\tau_n}(-\Delta)^\alpha u_n(s)ds + \int_0^{t\wedge\tau_n}(1+\mathrm{i}\mu)|u_n(s)|^{2\beta}u_n(s)ds$$

$$= u_0 + \rho\int_0^{t\wedge\tau_n} u_n(s)ds + \int_0^{t\wedge\tau_n} g(s)ds + \int_0^{t\wedge\tau_n}\sigma_n(s,u_n(s))dW(s)\ in\ (V\cap L^p(\mathbb{R}^n))^*, \tag{184}$$

P-almost surely. By (181), we know that $u_n(t\wedge\tau_n) = u(t\wedge\tau_n)$ for $t \in [0,T]$ and $\sigma_n(s,u_n(s)) = \sigma(s,u(s))$, for $s \in [0,\tau_n]$ a.s., then, by (184) we obtain, for all $t \in [0,T]$,

$$u(t\wedge\tau_n) + (1+\mathrm{i}\nu)\int_0^{t\wedge\tau_n}(-\Delta)^\alpha u(s)ds + \int_0^{t\wedge\tau_n}(1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s)ds$$

$$= u_0 + \rho\int_0^{t\wedge\tau_n} u(s)ds + \int_0^{t\wedge\tau_n} g(s)ds + \int_0^{t\wedge\tau_n}\sigma(s,u(s))dW(s)\ in\ (V\bigcap L^p(\mathbb{R}^n))^*, \tag{185}$$

P–almost surely. Letting $n\to\infty$ in (185) we obtain, for all $t \in [0,T]$,

$$u(t) + (1+\mathrm{i}\nu)\int_0^t(-\Delta)^\alpha u(s)ds + \int_0^t(1+\mathrm{i}\mu)|u(s)|^{2\beta}u(s)ds = u_0 + \rho\int_0^t u(s)ds + \int_0^t g(s)ds$$

$$+ \int_0^t \sigma(s,u(s))dW(s)\ in\ (V\cap L^p(\mathbb{R}^n))^*, \tag{186}$$

P-almost surely. We make $t = t\wedge\tau_n$ with (171), letting $n\to\infty$, we can obtain the energy equation (178). □

7. Weak Mean Random Attractors

In this section, we prove the existence and uniqueness of weak pullback mean random attractors. For $\rho > 0$, we consider the following stochastic equation, for $t > \tau$,

$$\begin{aligned} du(t) + (1+\mathrm{i}\nu)(-\Delta)^\alpha u(t)dt + (1+\mathrm{i}\mu)|u(t)|^{2\beta}u(t)dt =& \rho u(t)dt + g(t,x)dt \\ &+ \sigma(t,x,u)dW(s), \end{aligned} \tag{187}$$

with initial condition

$$u(\tau) = u_0. \tag{188}$$

By Theorem (165), we obtain that for every $\tau \in \mathbb{R}$ and every $\mathcal{F}_\tau$-measurable u_0 in $L^2(\Omega, H)$, the problem (187) and (188) has a unique continuous H-valued $\mathcal{F}_t$-adapted solution $u(t, \tau, u_0)$ with initial condition u_0 at τ in the sense of Definition 5. Note that $u(t, \tau, u_0) \in L^2(\Omega, C([\tau, \tau + T], H))$ for all $T > 0$, which implies that $u \in C([\tau, \infty), L^2(\Omega, H))$.

Suppose that Φ is the mean random dynamical system generated by (187) and (188) on $L^2(\Omega, \mathcal{F}; H)$. We will investigate the existence and uniqueness of weak pullback mean random attractors for Φ.

We use $\mathcal{D}$ to denote the collection of all families of nonempty bounded sets satisfying (8), and assume that for all $(t, \omega, u) \in \mathbb{R} \times \Omega \times H$, the function $g(t)$ satisfies that

$$\int_{-\infty}^{\tau} e^{\lambda t} \|g(t)\|^2 dt < \infty, \forall \tau \in \mathbb{R}. \tag{189}$$

The next lemma is concerned with the uniform estimates of the solutions in $L^2(\Omega, H)$.

Lemma 8. *Suppose* (189) *holds. Then, for every $\tau \in \mathbb{R}$ and $D \in \mathcal{D}$, there exists $T = T(\tau, D) > 0$ such that for all $t \geq T$, the solution u of* (187) *and* (188) *satisfies*

$$\mathbb{E}(\|u(\tau, \tau - t, u_0)\|^2) \leq M_1 + M_1 \int_{-\infty}^{\tau} e^{\rho(s-\tau)} \|g(s)\|^2 ds,$$

where $u_0 \in D(\tau - t)$, and M_1 is a positive constant independent of τ and D.

Proof. By the energy Equation (178), we have

$$\frac{d\mathbb{E}(\|u(t)\|^2)}{dt} + 2\mathbb{E}\Big(\|(-\Delta)^{\frac{\alpha}{2}} u(t)\|^2\Big) + 2\mathrm{Re}\mathbb{E}\Big((1 + \mathrm{i}\mu)|u(t)|^{2\beta} u(t), u(t)\Big) = 2\rho\mathbb{E}(\|u(t)\|^2)$$
$$+ 2\mathrm{Re}\mathbb{E}(u(t), g(t)) + \mathbb{E}\Big(\|\sigma(t, u(t))\|^2_{\mathcal{L}_2(U,H)}\Big).$$

By (4) and Young's inequality, we deduce

$$\frac{d\mathbb{E}(\|u(t)\|^2)}{dt} + 2\mathbb{E}(\|u(t)\|^p) + \rho\mathbb{E}(\|u(t)\|^2) \leq (\frac{7\rho}{2} + 2L^2)\mathbb{E}(\|u(t)\|^2) + \frac{2}{\rho}\mathbb{E}(\|g(t)\|^2) + 2L^2.$$

Applying Young's Inequality again, we have

$$\frac{d\mathbb{E}(\|u(t)\|^2)}{dt} + \rho\mathbb{E}(\|u(t)\|^2) \leq \frac{2}{\rho}\mathbb{E}(\|g(t)\|^2) + c(\rho).$$

Applying Gronwall's inequality on the interval $(\tau - t, \tau)$, we get

$$\mathbb{E}(\|u(\tau, \tau - t, u_0)\|^2) \leq e^{-\rho t}\mathbb{E}(\|u_0\|^2) + \frac{c(\rho)}{\rho} + \frac{2}{\rho} \int_{\tau - t}^{\tau} e^{\rho(s-\tau)} \|g(s)\|^2 ds. \tag{190}$$

Due to $u_0 \in D(\tau - t)$ and $D \in \mathcal{D}$, we have

$$e^{-\rho t}\mathbb{E}(\|u_0\|^2) \leq e^{-\rho t}\|D(\tau - t)\|^2 \to 0, \ as\ t \to \infty.$$

which completes the proof. □

Next, we prove the existence of weak $\mathcal{D}$-pullback mean random attractors for Φ.

Theorem 3. *Suppose* (189) *holds, then problem* (187) *and* (188) *has a unique weak $\mathcal{D}$-pullback mean random attractor as Definition* 5 $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ *in* $L^2(\Omega, \mathcal{F}; H)$ *over* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, P)$.

Proof. Given $\tau \in \mathbb{R}$, denote by

$$D(\tau) = \{u \in L^2(\Omega, \mathcal{F}_\tau; H) : \mathbb{E}(\|u\|^2) \leq L(\tau)\},$$

where

$$L(\tau) = M_1 + M_1 \int_{-\infty}^{\tau} e^{\rho(s-\tau)} \|g(s)\|^2 ds.$$

Since $D(\tau)$ is a closed ball in $L^2(\Omega, \mathcal{F}_\tau; H)$ centered at the origin, we know that $D(\tau)$ is weakly compact in $L^2(\Omega, \mathcal{F}_\tau; H)$. On the other hand, by (190), we have

$$\lim_{\tau\to-\infty} e^{\rho\tau} \|D(\tau)\|^2_{L^2(\Omega,\mathcal{F}_\tau;H)} = M_1 \lim_{\tau\to-\infty} e^{\rho\tau} + M_1 \lim_{\tau\to-\infty} \int_{-\infty}^{\tau} e^{\rho s} \|g(s)\|^2 ds = 0,$$

and hence $D = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$. By Lemma 8, we infer that D is a weakly compact $\mathcal{D}$-pullback absorbing set for Φ. According to Theorem 1, we conclude that the existence and uniqueness of weak $\mathcal{D}$-pullback mean random attractors for Φ. □

We emphasize that via Theorem 3, we obtain the long time behavior of the solution of (1) and (2).

8. Conclusions

In this work, we consider the long-time behavior of the stochastic Ginzburg–Landau equation driven by nonlinear noise. The existence and uniqueness of the solution of the equation in the corresponding space is established with detailed discussion in Section 3 (with regular additive noise), Section 4 (with general additive noise), Section 5 (with global Lipschitz continuity noise) and Section 6 (with local Lipschitz continuity noise). Meanwhile, the corresponding estimate of the solution in the corresponding space is obtained respectively. In our analysis for the estimate of the solution, we employ the tools of the Ito's formula, Gronwall's inequality, Young's inequality and Burkholder–Davis–Gundy inequality. We point out that in Section 7, based on the theory of weak pullback attractors established in [36], we obtain the existence of weak pullback random attractors for the mean stochastic dynamical systems constructed through the Equations (1) and (2).

The detailed discussion in current work will naturally lead us to investigate the existence of invariant measures for the distribution of solutions of stochastic fractional Ginzburg–Landau equations. Furthermore, when the parameters, functions and initial values in the Equation (1) are determined and satisfy the corresponding conditions, the specific form of the solution can be studied by numerical methods (see [43] for some discussion).

To end this section, we demonstrate that the method used to study the stochastic Ginzburg–Landau equation is quite different from those employed to analyze the deterministic one. To be specific, for the deterministic equations, one obtains the estimate for solutions, the so-called a priori estimates, by first constructing the energy equation through inner products, then applying suitable inequalities to the resulting energy equations. Based on the discussion, one then can prove the existence of pullback attractor (for more detailed discussion, one may refer to [6]). For the stochastic one, one mainly employ the Ito formula to obtain the estimate of the solutions, and then prove the existence of weak pullback attractors as analyzed in current work.

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References

1. Ginzburg, V.L. Nobel lecture: On superconductivity and superfluidity (what I have and have not managed to do) as well as on the physical minimum at beginning of the XXI century. *Rev. Modern Phys.* **2004**, *76*, 981–998. [CrossRef]
2. Ginzburg, V.L.; Landau, L.D. On the theory of superconductivity. *Zeksp. Teor. Fiz.* **1950**, *20*, 1064–1082.
3. Tarasov, V.E.; Zaslavsky, G.M. Fractional Ginzburg–Landau equation for fractal media. *Phys. Stat. Mech. Its Appl.* **2005**, *354*, 249–261. [CrossRef]
4. Pu, X.; Guo, B. Well-posedness and dynamics for the fractional Ginzburg–Landau equation. *Appl. Anal.* **2011**, *92*, 614601. [CrossRef]
5. Lu, H.; Lü, S.; Feng, Z. Asymptotic Dynamics of 2d Fractional Complex Ginzburg–Landau Equation. *Int. J. Bifur. Chaos* **2013**, *23*, 1350202. [CrossRef]
6. Lu, H.; Bates, P.W.; Lü, S.; Zhang, M. Dynamics of the 3-D fractional complex Ginzburg–Landau equation. *J. Differ. Equ.* **2015**, *259*, 5276–5301. [CrossRef]
7. Lu, H.; Lü, S. Random attractor for fractional Ginzburg–Landau equation with multiplicative noise. *Taiwanese J. Math.* **2014**, *18*, 435–450. [CrossRef]
8. Lu, H.; Bates, P.W.; Lü, S.; Zhang, M. Dynamics of the 3D fractional Ginzburg–Landau equation with multiplicative noise on an unbounded domain. *Com. Math. Sci.* **2016**, *14*, 273–295. [CrossRef]
9. Lu, H.; Zhang, M. Dynamics of non-autonomous fractional Ginzburg–Landau equations driven by colored noise. *Discrete Cont. Dyn. Sys. Series B* **2020**, *25*, 3553–3576. [CrossRef]
10. Shen, T.; Xin, J.; Huang, J. Time-space fractional stochastic Ginzburg–Landau equation driven by Gaussian white noise. *Stoch. Anal. Appl.* **2018**, *36*, 103–113. [CrossRef]
11. Xu, P.; Zou, G.; Huang, J. Time-space fractional stochastic Ginzburg–Landau equation driven by fractional Brownian motion. *Comput. Math. Appl.* **2019**, *78*, 3790–3806. [CrossRef]
12. Bates, P.W.; Lu, K.; Wang, B. Random attractors for stochastic reaction-diffusion equations on unbounded domains. *J. Differ. Equ.* **2009**, *246*, 845–869. [CrossRef]
13. Caraballo, T.; Garrido-Atienza, M.J.; Schmalfuss, B.; Valero, J. Non-autonomous and random attractors for delay random semilinear equations without uniqueness. *Discrete Contin. Dyn. Syst.* **2008**, *21*, 415–443. [CrossRef]
14. Caraballo, T.; Garrido-Atienza, M.J.; Taniguchi, T. The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion. *Nonlinear Anal.* **2011**, **74**, 3671–3684. [CrossRef]
15. Crauel, H.; Debussche, A.; Flandoli, F. Random attractors. *J. Dyn. Differ. Equ.* **1997**, *9*, 307–341. [CrossRef]
16. Chen, Z.; Wang, B. Limit measures and ergodicity of fractional stochastic reaction-diffusion equations on unbounded domains. *Stoc. Dyna.* **2022**, *22*, 2140012. [CrossRef]
17. Flandoli, F.; Schmalfuss, B. Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise. *Stoch. Stoch. Rep.* **1996**, *59*, 21–45. [CrossRef]
18. Garrido-Atienza, M.J.; Schmalfuss, B. Ergodicity of the infinite dimensional fractional Brownian motion. *J. Dyn. Differ. Equ.* **2011**, *23*, 671–681. [CrossRef]
19. Gess, B.; Liu, W.; Rockner, M. Random attractors for a class of stochastic partial differential equations driven by general additive noise. *J. Differ. Equ.* **2011**, *251*, 1225–1253. [CrossRef]
20. Huang, J.; Shen, W. Pullback attractors for nonautonomous and random parabolic equations on non-smooth domains. *Discrete Cont. Dyn. Syst. Ser. A* **2009**, *24*, 855–882. [CrossRef]
21. Schmalfuss, B. Backward cocycles and attractors of stochastic differential equations. In *International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behavior*; Technische Universität: Dresden, Germany, 1992; pp. 185–192.
22. Shen, Z.; Zhou, S.; Shen, W. One-dimensional random attractor and rotation number of the stochastic damped sine-Gordon equation. *J. Differ. Equ.* **2010**, *248*, 1432–1457. [CrossRef]
23. Wang, B. Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$. *Trans. Amer. Math. Soc.* **2011**, *363*, 3639–3663. [CrossRef]
24. Adili, A.; Wang, B. Random attractors for stochastic FitzHugh-Nagumo systems driven by deterministic non-autonomous forcing. *Discrete Cont. Dyn. Syst. Ser. B* **2013**, *18*, 643–666. [CrossRef]

25. Caraballo, T.; Langa, J.A.; Melnik, V.S.; Valero, J. Pullback attractors for nonautonomous and stochastic multivalued dynamical systems. *Set-Valued Anal.* **2003**, *11*, 153–201. [CrossRef]
26. Duan, J.; Schmalfuss, B. The 3D quasigeostrophic fluid dynamics under random forcing on boundary. *Comm. Math. Sci.* **2003**, *1*, 133–151. [CrossRef]
27. Gess, B. Random attractors for singular stochastic evolution equations. *J. Differ. Equ.* **2013**, *255*, 524–559. [CrossRef]
28. Lu, H.; Qi, J.; Wang, B.; Zhang, M. Random attractors for non-autonomous fractional stochastic parabolic equation on unbounded domains. *Discrete Cont. Dyn. Sys.* **2019**, *39*, 683–706. [CrossRef]
29. Lu, H.; Li, J.; Zhang, M. Stochastic dynamics of non-autonomous fractional Ginzburg–Landau equations on R^3. *Disc. Cont. Dyn. Sys. Ser. B* **2022**, *27*, 6943–6968. [CrossRef]
30. Wang, B. Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems. *J. Differ. Equ.* **2012**, *253*, 1544–1583. [CrossRef]
31. Wang, B. Random attractors for non-autonomous stochastic wave equations with multiplicative noise. *Discrete Cont. Dyn. Syst. Ser. A* **2014**, *34*, 269–300. [CrossRef]
32. Wang, B. Well-posedness and long term behavior of supercritical wave equations driven by nonlinear colored noise on R^n. *J. Func. Ana.* **2022**, *283*, 109498. [CrossRef]
33. Wang, X.; Lu, K.; Wang, B. Wong-Zakai approximations and attractors for stochastic reaction-diffusion equations on unbounded domains. *J. Differ. Equ.* **2018**, *264*, 378–424. [CrossRef]
34. Wang, R.; Li, Y.; Wang, B. Random dynamics of fractional nonclassical diffusion equations driven by colored noise. *Disc. Cont. Dyn. Sys.* **2019**, *39*, 4091–4126. [CrossRef]
35. Wang, R.; Wang, B. Asymptotic behavior of non-autonomous fractional p-Laplacian equations driven by additive noise on unbounded domains. *Bull. Math. Sci.* **2021**, *11*, 2050020. [CrossRef]
36. Wang, B. Weak pullback attractors for mean random dynamical systems in Bochner spaces. *J. Dyn. Differ. Equ.* **2019**, *31*, 2177–2204. [CrossRef]
37. Wang, B. Dynamics of fractional stochastic reaction-diffusion equations on unbounded domains driven by nonlinear noise. *J. Differ. Equ.* **2019**, *268*, 1–59. [CrossRef]
38. Nezza, E. Di Palatucci G.; Valdinoci E. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **2012**, *136*, 521–573. [CrossRef]
39. Lv, Y.; Wang, W. Limiting dynamics for stochastic wave equations. *J. Differ. Equ.* **2008**, *244*, 1–23. [CrossRef]
40. Lu, H.; Bates, P.W.; Xin, J.; Zhang, M. Asymptotic behavior of stochastic fractional power dissipative equations on $\mathbb{R}^n$. *Nonlinear Anal. TMA* **2015**, *128*, 176–198. [CrossRef]
41. Arnold, L. *Stochastic Differential Equations: Theory and Applications*; John Wiley, Sons Inc.: New York, NY, USA, 2007.
42. Lions, J.L. *Quelques Methodes de Resolution des Problems aux Limites Non Lineaires*; Dunod: Pairs, France, 1969.
43. Nouri, K.; Ranjbar, H.; Baleanu, D. Torkzadeh, L. Investigation on Ginzburg–Landau equation via a tested approach to benchmark stochastic Davis-Skodje system. *Alex. Eng. J.* **2021**, *60*, 5521–5526. [CrossRef]