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Asymptotic Stability for the 2D Navier–Stokes Equations with Multidelays on Lipschitz Domain

Ling-Rui Zhang [1], Xin-Guang Yang [1,*] and Ke-Qin Su [2]

1 Department of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China
2 College of Information and Management Science, Henan Agricultural University, Zhengzhou 450046, China
* Correspondence: yangxinguang@hotmail.com or 011137@htu.edu.cn

Abstract: This paper is concerned with the asymptotic stability derived for the two-dimensional incompressible Navier–Stokes equations with multidelays on Lipschitz domain, which models the control theory of 2D fluid flow. By a new retarded Gronwall inequality and estimates of stream function for Stokes equations, the complete trajectories inside pullback attractors are asymptotically stable via the restriction on the generalized Grashof number of fluid flow. The results in this presented paper are some extension of the literature by Yang, Wang, Yan and Miranville in 2021, as well as also the preprint by Su, Yang, Miranville and Yang in 2022

Keywords: Navier–Stokes equations; multidelays; Lipschitz domain

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1. Introduction

The 2D incompressible Navier–Stokes equations govern the conservation law of fluid flow for momentum and mass on a bounded domain with smooth boundary, which can be described by

$$\begin{cases} \frac{\partial}{\partial t}u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = F(t,x), \\ \mathrm{div} u = 0, \end{cases} \tag{1}$$

where u and p are the velocity field and pressure for incompressible fluid flow such as water, $\nu > 0$ denotes the viscosity of fluid and $F(t,x)$ is the external force.

A bounded domain $\Omega \subset \mathbb{R}^d$ is said to be Lipschitz if $\partial\Omega$ can be covered by finite many balls $B_i = B(Q_i, r_0)$ with $Q_i \in \partial\Omega$, such that for any ball B_i there is a rectangular coordinate system and a Lipschitz function $\Psi_i : \mathbb{R}^{d-1} \to \mathbb{R}$ with

$$B(Q_i, 3r_0) \cap \Omega = \{(x_1, x_2, \cdots, x_d) | x_d > \Psi_i(x_1, x_2, \cdots, x_{d-1})\} \cap \Omega,$$

which can be seen in [1]. The 2D incompressible Navier–Stokes equations defined on the Lipschitz domain have been studied in Brown, Perry and Shen [1], which presented the well-posedness and finite fractal dimensional global attractor for an autonomous system, which has been extended to a non-autonomous case in [2] and some related literature.

The delay on differential equations originates from the controller on boundary in engineering, which can be described by evolutionary partial differential equations with delayed term, and were first investigated for ordinary differential equations, such as in [3]. The Navier–Stokes equations with delay have also become interesting topics in the recent two decades, which are important dominant physical models for fluid mechanics, such as the wind tunnel model. The research on the well-posedness and dynamics of Navier–Stokes equations with delay can be seen in [4–12] and the literature therein. For the Navier–Stokes system with time-varying delay, the tempered pullback dynamics are obtained by energy equation approach to achieve compactness, such as in Caraballo and Real [5–7], García-Luengo and Marín-Rubio [9] and Yang, Wang, Yan and Miranville [12]. Recently, Su, Yang, Miranville and Yang [11] considered (2) and derived the well-posedness, regularity, pullback dynamics and robustness. Since stability, observability and controllability are crucial in the control theory and applications in engineering, the asymptotic stability of complete trajectories is an important basis for the research on controllability and dynamic systems. To the best of our knowledge, there are fewer results on the asymptotic stability and reduction in trajectories inside pullback attractors of 2D incompressible Navier–Stokes equations defined on Lipschitz domain which are non-smooth, this is our motivation for this presented research.

This paper investigates the asymptotic stability of trajectories inside pullback attractors for the two-dimensional incompressible Navier–Stokes equations with multidelays on Lipschitz domain $\Omega \subset \mathbb{R}^2$ with inhomogeneous boundary, which reads as

$$\begin{cases} \frac{\partial}{\partial t}u - \nu\Delta u + (u(t-\rho(t))\cdot\nabla)u + \nabla p = f(t, u(t-\rho(t))) + g(t,x), \ (t,x) \in \Omega_\tau, \\ divu = 0, \ (t,x) \in \Omega_\tau, \\ u(t,x) = \varphi, \ \ \varphi\cdot\mathbf{n} = 0, \ (t,x) \in \partial\Omega_\tau, \\ u(\tau,x) = u(\tau), \ x \in \Omega, \\ u(\tau+\theta,x) = \phi(\theta,x), \ (\theta,x) \in \Omega_h, \end{cases} \tag{2}$$

where $\Omega_\tau = (\tau, +\infty) \times \Omega$, $\partial\Omega_\tau = (\tau, +\infty) \times \partial\Omega$, $\Omega_h = [-h, 0] \times \Omega$, $\tau \in \mathbb{R}$ is the initial time and $h > 0$ is a positive constant. ν is the kinematic viscosity of the fluid, $u = (u_1(t,x), u_2(t,x))$ is the unknown velocity field of the fluid, p denotes the unknown pressure and ν is the kinematic viscosity of the fluid. The non-autonomous external forces contain $g(t,x)$ and continuous delay $f(t, u(t-\rho(t)))$, where $\rho(t)$ is the delay in $[0, h]$. The function ϕ denotes the initial state in $[-h, 0]$ with $u(\tau) = \phi(0)$. The forcing boundary condition $\varphi \in L^\infty(\partial\Omega)$, where $\mathbf{n}$ is the outward unit normal to the boundary $\partial\Omega$.

Originated from [13–16], based on the results in [11], the asymptotic stability of trajectories inside pullback attractors for (2) are investigated in this presented paper with features and difficulties as follows.

(I) The problem (2) contains an inhomogeneous boundary on a Lipschitz-like domain; using the stream function ψ for the corresponding Stokes equations subject to the same boundary condition, the inhomogeneous problem (2) can be transformed into an equivalent homogeneous system (10). For the model (2), the delays on external force $f(\cdot,\cdot)$ and convective term $(u(\cdot)\nabla)u(\cdot)$ can be different as $\rho_1(t)$ and $\rho_2(t)$, which have the same difficulty under some appropriate hypotheses in Section 2.2. For simplicity, we assume they are the same as the case $\rho_1(t) = \rho_2(t) = \rho(t)$.
Based on the global well-posedness of weak solutions and pullback attractors in [11], the asymptotic stability of complete trajectories inside pullback attractor $\mathcal{A}^{M_H}$ of (18) has been achieved by using a new retard Gronwall inequality and some estimates on stream function for Stokes equations. Since there are two delays contained in (2), the energy estimates cannot be obtained by using the technique as in [15,17] to achieve the desired estimate for using differential Gronwall inequalities, which is the main difficulty here. By introducing a new retard Gronwall inequality in [13], and using the iteration technique, one sufficient condition (12) on generalized Grashof number guarantees our asymptotic stability; see Theorem 5 in Section 2.4.

(II) The results in this presented paper are a further research of [15], which is a special case of (2). The asymptotic stability of (2) is an extension of the recent work [11]. Our work also implies the exponentially attracting property for the existence of invariant manifold although the inertial manifolds for 2D Navier–Stokes equation is still open.

The outline is organized as follows. The main results are stated in Section 2 and proved in the third part, which is based on the preliminary in Section 3.

2. Main Results

2.1. Preliminary

Let $E := \{u | u \in (\mathfrak{D}(\Omega))^2, \mathrm{div} u = 0\}$, H and V are the closure of E in $(L^2(\Omega))^2$ and $(H^1(\Omega))^2$ topology, respectively; the norm and inner product of H is defined as

$$\|u\|_H^2 = (u, u),\ (u, v) = \sum_{j=1}^{2} \int_\Omega u_j(x) v_j(x) dx$$

for $u, v \in H$, and for V as $\|u\|^2 = ((u, u))$ and

$$((u, v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx$$

for $u, v \in V$. It is easy to check that H and V are Hilbert spaces, $V \hookrightarrow H \equiv H' \hookrightarrow V'$, and the injections are dense and continuous. $\| \cdot \|_*$ and $\langle \cdot, \cdot \rangle$ denote the norm in V' and the dual product between V and V', respectively, and also H to itself.

Let P_L be the Helmholz–Leray orthogonal projection in $(L^2(\Omega))^2$ onto H, and $A := -P_L \Delta$ the Stokes operator. The bilinear and trilinear operators are defined as $B(u, v) := P_L((u \cdot \nabla)v)$, $b(u, v, w) = \langle B(u, v), w \rangle$, which satisfies $b(u, v, v) = 0$, $b(u, v, w) = -b(u, w, v)$, and hence

$$|b(u, v, w)| \leq C \|u\|_H^{\frac{1}{2}} \|u\|^{\frac{1}{2}} \|v\| \|w\|_H^{\frac{1}{2}} \|w\|^{\frac{1}{2}},\ \forall\, u, v, w \in V.$$

For any $t \in (\tau, T)$, we define $u : (\tau - h, T) \to (L^2(\Omega))^2$, and the delayed functional space as follows

$$C_X = C([-h, 0]; X),\ \|u\|_{C_X} = \sup_{\theta \in [-h, 0]} \|u(t + \theta)\|_X,\ X = H,\ V,$$

which are Banach spaces. Moreover, the p-power delayed integrable space can be defined as $L_X^p = L^p(-h, 0; X)$, $1 < p \leq +\infty$, and the norm is similar as the general Lebesgue space in delayed interval $[-h, 0]$. Moreover, the product space is defined well as $M_H = H \times (C_H \cap L_V^2)$ with norm

$$\|(u(t), u_t)\|_{M_H}^2 = \|u(t)\|_H^2 + \|u_t\|_{C_H}^2 + \|u_t\|_{L_V^2}^2.$$

2.2. Hypotheses

For the well-posedness and pullback dynamics of (2), we force assumptions on $\rho(t)$ and $f(\cdot, \cdot)$ as follows.

(H-a) There exists $m > 0$ such that the external force $g(\cdot, \cdot) \in L_{loc}^2(\mathbb{R}, V')$ satisfies

$$\int_{-\infty}^{t} e^{ms} \|g(s, \cdot)\|_{V'}^2 ds < \infty,\ \ \forall t \in \mathbb{R}. \tag{3}$$

(H-b) The function $f(\cdot, u) : [\tau, +\infty) \to H$ is measurable for all $u \in H$, and $f(t, \cdot) : C_H \to H$ is continuous for all $t \geq \tau$. The delay $\rho \in C^1([0, +\infty); [0, h])$, and there exists a positive constant $\rho^* < 1$ such that $|\frac{d\rho}{dt}| \leq \rho^*$.

(H-c) There exist functions α, $\beta : [\tau, +\infty) \to [0, +\infty)$, where $\alpha(\cdot) \in L^\infty(\tau, T)$ and $\beta(\cdot) \in L^1(\tau, T)$ with $\limsup_{\tau \to -\infty} \dfrac{1}{t - \tau} \displaystyle\int_\tau^t \beta(s) ds = \tilde{\beta}_0 \in (0, +\infty)$, such that $|f(t, u)|^2 \leq \alpha(t)|u|^2 + \beta(t)$, $\forall\, t \geq \tau$.

In addition, there exists a constant $L(r) > 0$ such that $|f(t, w_1) - f(t, w_2)| \leq L(r) \gamma^{1/2}(t)$ $|w_1 - w_2|$ for $\|w_1\|_H \leq r$, $\|w_2\|_H \leq r$ with $\tilde{\gamma}(t) \in L^\infty(\tau, T)$.

(H-d) $\nu - \frac{\|\alpha(t)\|_{L^\infty(\tau,T)}}{\nu \lambda_1 (1 - \rho^*)} > 0$.

(H-e) Denoting

$$\limsup_{\tau \to -\infty} \frac{1}{t-\tau} \int_\tau^t \alpha(r)dr = \alpha_0 \in [0, +\infty), \tag{4}$$

for arbitrary $t \in \mathbb{R}$, then there exists some $\delta > 0$ such that

$$\frac{e^{\nu\lambda_1 h}\alpha_0}{\nu\lambda_1(1-\rho^*)} + \frac{Ce^{\frac{\nu\lambda_1 h}{2}}\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu(1-\rho^*)} + \delta < \nu\lambda_1. \tag{5}$$

(H-f) Assume that

$$\kappa_\delta(t,s) = \Big(\nu\lambda_1 - \frac{Ce^{\frac{\nu\lambda_1 h}{2}}\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu(1-\rho^*)} - \delta\Big)(t-s) - \frac{e^{\nu\lambda_1 h}}{\nu\lambda_1(1-\rho^*)} \int_s^t \alpha(r)dr, \tag{6}$$

where

$$\kappa_\delta(0,t) - \kappa_\delta(0,s) = -\kappa_\delta(t,s) \tag{7}$$

and

$$\kappa_\delta(0,r) \le \kappa_\delta(0,t) + \Big(\nu\lambda_1 - \frac{Ce^{\frac{\nu\lambda_1 h}{2}}\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu(1-\rho^*)} - \delta\Big)h \tag{8}$$

if $\nu\lambda_1 - \frac{Ce^{\frac{\nu\lambda_1 h}{2}}\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu(1-\rho^*)} - \delta > 0$ for $r \in [t-h, t]$.

The function $\beta(\cdot)$ satisfies the pullback tempered condition

$$\int_{-\infty}^t e^{-\kappa_\delta(t,s)}\beta(s)ds < +\infty. \tag{9}$$

2.3. Well-Posedness and Pullback Dynamics

The problem (2) can be transformed into the following equivalent homogeneous system in abstract form

$$\begin{cases} \frac{\partial v}{\partial t} + \nu Av + B(v(t-\rho(t)), v) + B(v(t-\rho(t)), \psi) + B(\psi, v) & \\ \qquad\qquad = P_L(g(t,x) + f(t, u(t-\rho(t))) + \nu F) - B(\psi), & (t,x) \in \Omega_\tau, \\ \mathrm{div} v = 0, & (t,x) \in \Omega_\tau, \\ v = 0, & (t,x) \in \partial\Omega_\tau, \\ v(\tau, x) = v(\tau), & x \in \Omega, \\ v(\theta) = \eta(\theta, x) = \eta(\theta), & (\theta, x) \in \Omega_h. \end{cases} \tag{10}$$

Theorem 1. *(Global weak solution) Let $(v(\tau), \eta) \in M_H$, and the hypotheses (H-a)-(H-d) hold. Then, there exists at least one global weak solution $v(t,x)$ to system (10) on $[\tau - h, T]$.*

Proof. See, e.g., the details in Su, Yang, Miranville and Yang [11]. □

Theorem 2. *(Uniqueness) Assume the hypotheses in Theorem 1 hold. Moreover, we assume that for any $r > 0$, there exists a constant $L(r) > 0$ such that*

$$|f(t,w_1) - f(t,w_2)| \le L(r)\gamma^{1/2}(t)|w_1 - w_2|, \ \forall\, t \ge \tau, \ \|w_1\|_H \le r, \ \|w_2\|_H \le r, \tag{11}$$

where $\gamma \in L^\infty(\tau, T) : [\tau, T) \to [0, +\infty)$. Then, the global weak solution in Theorem 1 is unique, which generates a continuous process $\{S(t,\tau)\}$ in M_H.

Proof. See, e.g., the details in Su, Yang, Miranville and Yang [11]. □

Remark 1. *Originated from the idea to deal with uniform attractors in Chepyzhov and Vishik [18], based on global well-posedness in the phase space M_H, the global solution generates a process $S(t,\tau) : M_H \to M_H$, which has the similar property of skew product flow as in [18].*

The existence of a minimal family of pullback attractors for problem (18) can be stated as follows.

Theorem 3. *(Tempered pullback dynamics) Suppose that $f : \mathbb{R} \times C_H \to H$ satisfies the hypotheses (H-a)–(H-d); let the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfy (H-e)–(H-f). Then, for any $(v(\tau), \eta) \in M_H$, the process $(S(t,\tau); M_H)$ generated by the global weak solutions of problem (10) possesses a minimal family of tempered pullback attractors $\mathcal{A}_{\kappa_\delta}$ in $H \times C_H$, for all $\kappa_\delta \in (0, \kappa_\delta(t,\tau)]$. Moreover, if we choose fixed κ_δ^F for fixed universe to achieve pullback attractors as $\mathcal{A}_{\kappa_\delta^F}$, then we have the relation $\mathcal{A}_{\kappa_\delta^F} \subset \mathcal{A}_{\kappa_\delta} \subset \mathcal{A}_{\kappa_\delta(t,\tau)} \subset \mathcal{D}^{H\times C_H}_{\kappa_\delta(t,\tau)}$.*

Proof. See, e.g., the details in Su, Yang, Miranville and Yang [11]. □

Theorem 4. *Assume $(v(\tau), \eta) \in M_H$ and $\frac{\nu}{4} > \frac{Ce^{\frac{\nu\lambda_1}{2}h}\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu(1-\rho^*)} + \frac{4e^{\frac{\nu\lambda_1}{2}h}\|\alpha(t)\|_{L^\infty(\tau,T)}}{\nu(1-\rho^*)}$, the process $S(t,\tau) : M_H \to M_H$ generated by the system (10) possesses a minimal family of $\mathcal{D}$-pullback attractors $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ in M_H.*

Proof. See, e.g., the details in Su, Yang, Miranville and Yang [11]. □

2.4. Asymptotic stability

Definition 1. *The pullback attractors are asymptotically stable if the trajectories inside the attractors reduce to a single orbit as $\tau \to -\infty$, which also demonstrates the exponentially tracking property.*

Based on the global well-posedness and the existence of tempered and $\mathcal{D}$-pullback attractors for problems (2) and (18) in [11], we present our main result as the following theorem.

Theorem 5. *Assume the external force $g \in L^2_{loc}(\mathbb{R}; V')$ and the hypothesis (H-a)-(H-d) hold, the initial data $(u(\tau), \phi) \in M_H$. Then, the trajectories' pullback attractor $\mathcal{A} = \{A(t)\}|_{t\geq\tau}$ is asymptotically stable if $G(t) + K_0 \leq \frac{2\nu}{7(1+\gamma)}$, where $G^2(t) = \frac{\langle\|g\|^2_{V'}\rangle|_{\leq t}}{\nu^2\lambda_1}$ is a generalized Grashof number for the fluid flow, and*

$$
\begin{aligned}
\frac{2}{7(1+\gamma)}K_0 \quad = \quad & \Big[\frac{C\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu} + \frac{C}{\nu\lambda_1}\|\alpha(t)\|_{L^\infty}\Big]\frac{C|\Omega|}{\nu\lambda_1}\|\beta\|_{L^1(\tau,T)} \\
& + \frac{C|\Omega|}{\nu\lambda_1}\tilde{\beta}_0 + \frac{C|\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu\lambda_1}\|\alpha(t)\|_{L^\infty(\tau,T)} \\
& + \Big[\frac{C\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu^2\lambda_1} + \frac{C}{\nu^2\lambda_1^2}\|\alpha(t)\|_{L^\infty} + 1\Big]\Big[\frac{C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu}\Big],
\end{aligned} \tag{12}
$$

where $\gamma > 0$ is defined by the retard Gronwall inequality determined by the parameters in our problem.

3. The Proof of Theorem 5

3.1. A Retarded Gronwall Inequality

Lemma 1. *(See [13]) Considering the following retarded integral inequalities for*

$$y(t) \leq E(t,\tau)\|y_\tau\|_X + \int_\tau^t K_1(t,s)\|y_s\|_X ds + \int_t^\infty K_2(t,s)\|y_s\|_X ds + \rho, \quad \forall\, t \geq \tau \geq 0, \tag{13}$$

where E, K_1 and K_2 are non-negative measurable functions on $\mathbb{R}^2$, and $\rho \geq 0$ denotes a constant. Let X be a Banach space with a spatial variable, then we use $\|\cdot\|$ to denote the norm of space $C([-h,0];X)$ for some $h \geq 0$, $y(t) \geq 0$ is a continuous function defined on $C([-h,T];X)$, $y_t(s) = y(t+s)$ for $s \in [-h,0]$. Let

$$\mathcal{L}(E,K_1,K_2,\rho) = \{y \in C([-h,T];X) | y \geq 0 \text{ and satisfies the inequality (13)}\},$$

and

$$\kappa(K_1,K_2) = \sup_{t\geq\tau}\Big(\int_\tau^t K_1(t,s)ds + \int_t^\infty K_2(t,s)ds\Big)$$

with $\kappa(K_1,K_2) < +\infty$. Assume that $\lim_{t\to+\infty} E(t+s,s) = 0$ uniformly with respect to $s \in \mathbb{R}^+$, and denote $\vartheta = \sup_{t\geq s\geq\tau} E(t,s)$ and $\kappa = \kappa(K_1,K_2)$, then we have the following estimates:

(1) If $\kappa < 1$, then for any R, $\varepsilon > 0$, there exists $\tilde{T} > 0$ such that

$$\|y_t\|_X < \mu\rho + \varepsilon, \tag{14}$$

for $t > \tilde{T}$ and all bounded functions $y \in \mathcal{L}(E,K_1,K_2,\rho)$ with $\|y_0\| \leq R$, where $\mu = \frac{1}{1-\kappa}$.

(2) If $\kappa < \frac{1}{1+\vartheta}$, then there exist parameters $M > 0$ and $\lambda > 0$, which are independent on ρ such that

$$\|y_t\|_X \leq M\|y_0\|_X e^{-\lambda t} + \gamma\rho, \quad t \geq \tau \tag{15}$$

for all bounded functions $y \in \mathcal{L}(E,K_1,K_2,\rho)$, where $\gamma = \frac{\mu+1}{1-\kappa c}$ and $c = \max\{\frac{\vartheta}{1-\kappa}, 1\}$.

3.2. The Stokes Problem on Lipschitz Domains

From [1], the stream function ψ solves the following Stokes system on the Lipschitz domain

$$\begin{cases} -\Delta u + \nabla q = 0, \ x \in \Omega, \\ \mathrm{div} u = 0, \ x \in \Omega, \\ u = \varphi \text{ a.e. } x \in \partial\Omega \text{ in the sense of non-tangential convergence.} \end{cases} \tag{16}$$

Assume that $u = (u_1, u_2)$ is the solution to (16) with $\varphi \in L^\infty(\partial\Omega)$ and $\varphi \cdot n = 0$, then we define the stream function ψ satisfying (16) and

$$\begin{cases} \|\psi\|_{L^\infty(\Omega)} \leq C\|\varphi\|_{L^\infty(\partial\Omega)}, \\ \sup_{x\in\Omega}|\psi(x)| + \sup_{x\in\Omega}|\nabla\psi(x)|\mathrm{dist}(x,\partial\Omega) \leq C\|\varphi\|_{L^\infty(\partial\Omega)}, \\ \||\nabla\psi|\mathrm{dist}(\cdot,\partial\Omega)^{1-\frac{1}{p}}\|_{L^p(\Omega)} \leq C\|\varphi\|_{L^p(\partial\Omega)}, \ 2 \leq p \leq \infty. \end{cases}$$

In addition, the stream function ψ can be written as the following form $\Delta\psi = \nabla(q\eta_\varepsilon) + F$, where $\mathrm{supp}F \subset \{x \in \Omega;\ C_1'\varepsilon \leq \mathrm{dist}(x,\partial\Omega) \leq C_2'\varepsilon\}$ and $|F| \leq \frac{C}{\varepsilon^{3/2}}\|\varphi\|_{L^2(\partial\Omega)}$. The above estimate is based on the singular operator and Hardy's inequality as

$$\int_\Omega \frac{|u(x)|^2}{[\mathrm{dist}(x,\partial\Omega)]^2}dx \leq C\int_\Omega |\nabla u(x)|^2 dx, \ \forall u \in V. \tag{17}$$

3.3. Proof of Main Results

Proof. By an equivalent system as (18) and stationary equation as (16), the trajectories in pullback attractors of systems (2) and (18) are synchronous, which implies we only need to consider the asymptotic stability of trajectories inside the pullback attractor for (18). The proofs are divided into the following steps.

Step 1: Some estimates of differencing equations

Setting $v = u - \psi$ and $(v(\tau), \eta) \in H \times (C_H \cap L^2_V)$, then (2) can be transformed into the following equivalent abstract functional evolutionary differential equations with homogeneous boundary condition

$$\begin{cases} \frac{\partial v}{\partial t} + \nu A v + B(v(t-\rho(t)), v) + B(v(t-\rho(t)), \psi) + B(\psi, v) \\ \qquad\qquad\qquad = P_L(f(t, u(t-\rho(t))) + g(t,x) + \nu F) - B(\psi), \\ \mathrm{div} v = 0, \\ v|_{\partial\Omega} = 0, \\ v(\tau, x) = v(\tau), \\ v(\theta) = \eta(\theta),\ \theta \in [\tau - h, \tau]. \end{cases} \tag{18}$$

Let $v(t)$ and $\tilde{v}(t)$ be two global solution orbits for problem (18) inside the $\mathcal{D}$-pullback attractor with initial data

$$v(\tau + \theta)|_{\theta \in [-h,0]} = \eta(\theta), \quad v|_{t=\tau} = v(\tau),$$
$$\tilde{v}(\tau + \theta)|_{\theta \in [-h,0]} = \tilde{\eta}(\theta), \quad \tilde{v}|_{t=\tau} = \tilde{v}(\tau),$$

respectively.

By the procedure in achieving the $\mathcal{D}$-pullback attractors in [11], the global weak solution for (18) generates a continuous process $S(t,\tau)$ in $M_H = H \times (C_H \cap L^2_V)$ as

$$(v, v_t) = S(t,\tau)(v(\tau), \eta) \ \text{and} \ (\tilde{v}, \tilde{v}_t) = S(t,\tau)(\tilde{v}(\tau), \tilde{\eta}), \tag{19}$$

which are also two trajectories inside the pullback attractors $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ in M_H, here, $v_t = v(t+s)$ for $s \in [-h, 0]$.

Denoting $w = v(t) - \tilde{v}(t)$ and $w_t = v_t - \tilde{v}_t$ by some simple computation, it is easy to check that w satisfies the following initial and boundary value problem for functional evolutionary partial differential equations as

$$\begin{cases} \frac{\partial w}{\partial t} + \nu A w + B(w(t-\rho(t)), v) + B(\tilde{v}(t-\rho(t)), w) + B(w(t-\rho(t)), \psi) + B(\psi, w) \\ \qquad\qquad\qquad\qquad = P_L(f(v(t-\rho(t))) + \psi) - f(\tilde{v}(t-\rho(t)) + \psi)), \\ \mathrm{div} w = 0, \\ w|_{\partial\Omega} = 0, \\ w(t=\tau) = v(\tau) - \tilde{v}(\tau), \\ w(\tau+\theta) = \eta(\theta) - \tilde{\eta}(\theta),\ \theta \in [-h, 0]. \end{cases} \tag{20}$$

Multiplying (20) by w at both sides, using Poincaré's inequality, noting the property of the trilinear operator $(B(\tilde{v}(t-\rho(t)), w), w) = 0$ and $(B(\psi, w), w) = 0$, we derive that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|_H^2 + \nu\|w\|^2 \quad &\leq \quad \Big|(B(w(t-\rho(t)), v) + B(w(t-\rho(t)), \psi), w)\Big| \\ &\quad + \Big|(P_L(f(v(t-\rho(t)) + \psi) - f(\tilde{v}(t-\rho(t)) + \psi)), w)\Big|. \end{aligned} \tag{21}$$

Using the Hardy and Hölder inequalities, we have

$$\begin{aligned} \Big|(B(w(t-\rho(t)), v, w)\Big| \quad &\leq \quad \int_\Omega |(w(t-\rho(t))||\nabla w||v| dx \\ &\leq \quad \frac{1}{2}\|v\|^2\|w(t)\|^2 + \frac{C}{2}\|w(t-\rho(t))\|_H^2 \end{aligned} \tag{22}$$

and

$$\begin{aligned} \Big|(B(w(t-\rho(t), \psi), w)\Big| \quad &\leq \quad C\|\varphi\|_{L^\infty(\partial\Omega)} \int_{\mathrm{dist}(x,\partial\Omega) \leq C_2'\varepsilon} \frac{|w(t)||w(t-\rho(t))|}{\mathrm{dist}(x, \partial\Omega)} dx \\ &\leq \quad \frac{\nu}{4}\|w\|^2 + \frac{C\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu}\|w(t-\rho(t))\|_H^2 \end{aligned} \tag{23}$$

and

$$\begin{aligned} & |(f(t, v(t-\rho(t))+\psi)-f(t, \tilde{v}(t-\rho(t))+\psi), w)| \\ \leq \; & \frac{\nu}{4}\|w(t)\|^2+\frac{1}{\nu \lambda_1} L^2(r) \tilde{\gamma}(t)\|w(t-\rho(t))\|_H^2 . \end{aligned} \tag{24}$$

We can use the Poincaré and Gronwall inequalities to achieve the hypothesis in Lemma 1 for the asymptotic stability of trajectories inside $\mathcal{D}$-pullback attractors $\mathcal{A}$ in [11], provided that

$$\nu \lambda_1-\|v\|_V^2>0, \tag{25}$$

then, we can obtain

$$\begin{aligned} \|w\|_H^2 \leq \; & e^{\int_\tau^t -(\nu \lambda_1-\|v\|_V^2) ds}\|v(\tau)-\tilde{v}(\tau)\|_H^2+ \\ & +\Big[\frac{C}{2}+\frac{C\|\varphi\|_{L^{\infty}(\partial \Omega)}}{\nu}+\frac{1}{\nu \lambda_1} L^2(r)\|\tilde{\gamma}\|_{L^{\infty}(\tau, T)}\Big] \int_\tau^t e^{-\int_s^t(\nu \lambda_1-\|v\|_V^2) d \sigma}\|w_t\|_H^2 d s . \end{aligned} \tag{26}$$

Denoting

$$\begin{aligned} & E(t, \tau)=e^{-\int_\tau^t(\nu \lambda_1-\|v\|_V^2) d s}, \\ & K_1(t, s)=\Big[\frac{C}{2}+\frac{C\|\varphi\|_{L^{\infty}(\partial \Omega)}}{\nu}+\frac{1}{\nu \lambda_1} L^2(r)\|\tilde{\gamma}(t)\|_{L^{\infty}}\Big] e^{-\int_s^t(\nu \lambda_1-\|v\|_V^2) d \sigma}, \\ & \Theta=\sup_{t \geq s \geq \tau} E(t, s), \quad \kappa(K_1, 0)=\sup_{t \geq \tau} \int_\tau^t K_1(t, s) d s, \end{aligned}$$

noting the assumption and inequality in Lemma 1, choosing $\kappa(K_1, 0)<\frac{1}{1+\Theta}$. In fact, since $v \in L^{\infty}(\tau, T ; H) \cap L^2(\tau, T ; V)$, we have

$$\sup_{t \geq \tau} \int_\tau^t K_1(t, s) d s \leq \sup_{t \geq \tau} \frac{M\Big(\frac{C}{2}+\frac{C\|\varphi\|_{L^{\infty}(\partial \Omega)}}{\nu}+\frac{1}{\nu \lambda_1} L^2(r)\|\tilde{\gamma}(t)\|_{L^{\infty}}\Big)}{\nu \lambda_1}[1-e^{-\nu \lambda_1(t-\tau)}] \tag{27}$$

and there exists a pullback time $\bar{\tau}<<\tau$ such that $\kappa(K_1, 0)<\frac{1}{2}$, which implies the assumption in Lemma 1 holds.

Hence, from Lemma 1, there exist $\bar{M}>0$ and $\mu>0$, such that we can obtain the following estimate

$$\|w(t-\rho(t))\|_H^2 \leq M\Big[\|v(\tau)-\tilde{v}(\tau)\|_H^2+\|\eta(\theta)-\tilde{\eta}(\theta)\|_{L_V^2}^2\Big] e^{-\mu(t-\tau)} . \tag{28}$$

Substituting (28) into (21), we can conclude the following estimate

$$\begin{aligned} \|w\|_H^2 \leq \; & e^{\int_\tau^t -(\nu \lambda_1-\|v\|_V^2) d s}\|v(\tau)-\tilde{v}(\tau)\|_H^2+ \\ & +M\Big[\|v(\tau)-\tilde{v}(\tau)\|_H^2+\|\eta(\theta)-\tilde{\eta}(\theta)\|_{L_V^2}^2\Big] e^{-\mu(t-\tau)} \\ & \times\Big[\frac{C}{2}+\frac{C\|\varphi\|_{L^{\infty}(\partial \Omega)}}{\nu}+\frac{1}{\nu \lambda_1} L^2(r)\|\tilde{\gamma}(t)\|_{L^{\infty}}\Big] \int_\tau^t e^{-\int_s^t(\nu \lambda_1-\|v\|_V^2) d \sigma} d s . \end{aligned} \tag{29}$$

Step 2: The sufficient condition on asymptotic stability via generalized Grashof number

Combining (28) with (29), considering the trajectories represented by (19) for fixing initial data, and letting $\tau \rightarrow-\infty$, we can then conclude that the trajectories inside pullback attractors reduce to a single one, which implies the asymptotic stability provided that $\nu\lambda_1 > \langle \|v\|_V^2\rangle_{\le t}$, where $\langle h\rangle_{\le t}$ is defined as

$$\langle h\rangle_{\le t} = \limsup_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t h(r)dr. \tag{30}$$

Since v and $\tilde{v}$ are two global weak solutions for (18), we use Lemma 1 for a iteration procedure and some delicate estimates to present a sufficient condition for asymptotic stability of trajectories inside the pullback attractors by virtue of the uniform boundedness of stream function.

Taking the inner product of (18) with u in H, this yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v\|_H^2 + \nu\|v\|^2\\
\le\ & |(P_L(f(t,v(t-\rho(t))+\psi)+\nu F),v)| + |(B(\psi,\psi),v)|\\
&+|(B(v(t-\rho(t)),\psi),v)| + |\langle g,v\rangle|.
\end{aligned}\tag{31}$$

Using the Hardy and Hölder inequalities, by virtue of estimates for stream function in Section 3.2 and $\|\varphi\|_{L^2(\partial\Omega)} \le C|\partial\Omega|^{1/2}\|\varphi\|_{L^\infty(\partial\Omega)}$ from [1], we obtain

$$\begin{aligned}
|b(v(t-\rho(t)),\psi,v)| &\le C\|\varphi\|_{L^\infty(\partial\Omega)}\int_{\mathrm{dist}(x,\partial\Omega)\le C_2'\varepsilon}\frac{|v(t)||v(t-\rho(t))|}{[\mathrm{dist}(x,\partial\Omega)]}dx\\
&\le \frac{\nu}{14}\|v\|^2 + \frac{C\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu}\|v(t-\rho(t))\|_H^2, &(32)\\
|(B(\psi,\psi),v)| &\le C\|\varphi\|_{L^\infty(\partial\Omega)}\int_\Omega\frac{|v|}{\mathrm{dist}(x,\partial\Omega)}|\psi|dx\\
&\le C\varepsilon^{1/2}\|\varphi\|^2_{L^\infty(\partial\Omega)}|\partial\Omega|^{1/2}\|v\|\\
&\le \frac{\nu}{14}\|v\|^2 + \frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu} &(33)
\end{aligned}$$

and

$$\begin{aligned}
\nu|\langle F,v\rangle| &\le \frac{C\nu}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\partial\Omega)}\|v\| \le \frac{\nu}{14}\|v\|^2 + \frac{C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}, &(34)\\
|\langle g,v\rangle| &\le \frac{\nu}{14}\|v\|^2 + \frac{7/2}{\nu}\|g(t)\|^2_{V'}. &(35)
\end{aligned}$$

By hypotheses (H-a)-(H-d), the estimates of stream function and the Minkowski inequality, we can derive that

$$\begin{aligned}
&(f(t,v(t-\rho(t))+\psi),v(t))\\
\le\ & \alpha^{\frac{1}{2}}(t)\|v(t-\rho(t))\|_H\|v(t)\|_H + \alpha^{\frac{1}{2}}(t)|\psi|\|v(t)\|_H + \beta^{\frac{1}{2}}(t)\|v(t)\|_H\\
\le\ & \frac{C}{\nu\lambda_1}\alpha(t)\|v(t-\rho(t))\|_H^2 + \frac{\nu}{14}\|v(t)\|^2 + \frac{C|\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu\lambda_1}\alpha(t) + \frac{C|\Omega|}{\nu\lambda_1}\beta(t).
\end{aligned}\tag{36}$$

Combining (31)–(36), we obtain

$$\begin{aligned}
&\frac{d}{dt}\|v\|^2 + \nu\|v\|^2\\
\le\ & \left[\frac{C\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu} + \frac{C}{\nu\lambda_1}\|\alpha(t)\|_{L^\infty}\right]\|v(t-\rho(t))\|_H^2 + \frac{C|\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu\lambda_1}\alpha(t) + \frac{C|\Omega|}{\nu\lambda_1}\beta(t)\\
&+\frac{C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu} + \frac{7/2}{\nu}\|g(t)\|^2_{V'}.
\end{aligned}\tag{37}$$

By using the Poincaré inequality and Lemma 1, we can conclude that

$$
\begin{aligned}
\|v\|_H^2 \quad \leq \quad & e^{-\nu\lambda_1(t-\tau)}\|v(\tau)\|_H^2 + \frac{C|\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}}{\nu\lambda_1}\int_\tau^t e^{-\nu\lambda_1(t-s)}\alpha(s)ds \\
& + \frac{C|\Omega|}{\nu\lambda_1}\int_\tau^t e^{-\nu\lambda_1(t-s)}\beta(s)ds \\
& + \Big[\frac{C\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu} + \frac{C}{\nu\lambda_1}\|\alpha(t)\|_{L^\infty}\Big]\int_\tau^t e^{-\nu\lambda_1(t-s)}\|v(s-\rho(s))\|_H^2 ds \\
& + \frac{1}{\nu\lambda_1}\Big[\frac{C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu}\Big](1-e^{-\nu\lambda_1(t-\tau)}) \\
& + \frac{7/2}{\nu}\int_\tau^t e^{-\nu\lambda_1(t-s)}\|g\|^2_{V'}ds.
\end{aligned} \tag{38}
$$

Denoting

$$
\begin{aligned}
& E(t,\tau) = e^{-\nu\lambda_1(t-\tau)}, \\
& K_1(t,s) = \Big[\frac{C\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu} + \frac{C}{\nu\lambda_1}\|\alpha(t)\|_{L^\infty}\Big]e^{-\nu\lambda_1(t-s)}, \\
& \rho = \frac{C|\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}}{\nu\lambda_1}\int_\tau^t e^{-\nu\lambda_1(t-s)}\alpha(s)ds + \frac{C|\Omega|}{\nu\lambda_1}\int_\tau^t e^{-\nu\lambda_1(t-s)}\beta(s)ds \\
& \quad + \frac{1}{\nu\lambda_1}\Big[\frac{C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu}\Big](1-e^{-\nu\lambda_1(t-\tau)}) \\
& \quad + \frac{7/2}{\nu}\int_\tau^t e^{-\nu\lambda_1(t-s)}\|g\|^2_{V'}ds, \\
& \Theta = \sup_{t\geq s\geq\tau} E(t,s), \quad \kappa(K_1,0) = \sup_{t\geq\tau}\int_\tau^t K_1(t,s)ds,
\end{aligned}
$$

choosing a small enough $\tilde{\tau} << \tau$ such that $\kappa(K_1,0) < \frac{1}{1+\Theta}$, then by using Lemma 1, there exist parameters $\hat{M} > 0$, $\gamma > 0$ and $\hat{\mu} > 0$, such that we can obtain the estimate

$$
\begin{aligned}
\|v(t-\rho(t))\|_H^2 \quad \leq \quad & \hat{M}\Big[\|v(\tau)\|_H^2 + \|\eta\|^2_{L^2_V}\Big]e^{-\hat{\mu}(t-\tau)} + \gamma\Big[\frac{7/2}{\nu}\int_\tau^t e^{-\nu\lambda_1(t-s)}\|g\|^2_{V'}s \\
& + \frac{C|\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}}{\nu\lambda_1}\int_\tau^t e^{-\nu\lambda_1(t-s)}\alpha(s)ds + \frac{C|\Omega|}{\nu\lambda_1}\int_\tau^t e^{-\nu\lambda_1(t-s)}\beta(s)ds \\
& + \frac{1}{\nu\lambda_1}\Big(\frac{C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu}\Big)(1-e^{-\nu\lambda_1(t-\tau)})\Big]
\end{aligned} \tag{39}
$$

Substituting (39) into (38), integrating (37) over $[\tau, t]$, we can obtain

$$
\begin{aligned}
\|v\|_H^2 \quad \leq \quad & e^{-\nu\lambda_1(t-\tau)}\|v(\tau)\|_H^2 + C_1\Big[\|v(\tau)\|_H^2 + \|\eta\|^2_{L^2_V}\Big]e^{-\hat{\mu}(t-\tau)} \\
& + C_2\Big[\|g(t)\|^2_{L^2(\tau,T;V')} + \|\alpha(t)\|_{L^\infty(\tau,T)} + \|\beta(t)\|_{L^1(\tau.T)} + 1\Big]
\end{aligned} \tag{40}
$$

and

$$\begin{aligned}\frac{\nu}{t-\tau}\int_{\tau}^{t}\|v(r)\|_V^2dr \leq\ & \Big[\frac{C\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu}+\frac{C}{\nu\lambda_1}\|\alpha(t)\|_{L^\infty}\Big]\Big\{\hat{M}\Big[\|v(\tau)\|_H^2+\|\eta\|^2_{L^2_V}\Big]e^{-\tilde{\mu}(t-\tau)}\\ &+\gamma\Big[\frac{C|\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}}{\nu\lambda_1}\int_\tau^t e^{-\nu\lambda_1(t-s)}\alpha(s)ds+\frac{C|\Omega|}{\nu\lambda_1}\int_\tau^t e^{-\nu\lambda_1(t-s)}\beta(s)ds\\ &+\frac{1}{\nu\lambda_1}\Big(\frac{C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu}\Big)\Big]\Big\}\\ &+\frac{C|\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}}{2\nu\lambda_1}\|\alpha(t)\|_{L^\infty(\tau,T)}+\frac{C|\Omega|}{\nu\lambda_1}\tilde{\beta}_0\\ &+\frac{C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu}+\frac{7(1+\gamma)}{2\nu}\|g(t)\|^2_{V'}. \end{aligned} \tag{41}$$

Combining (37)–(41), we conclude the asymptotic stability holds, provided that

$$\begin{aligned}\langle\|v\|_V^2\rangle|_{\leq t} \leq\ & \Big[\frac{C\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu}+\frac{C}{\nu\lambda_1}\|\alpha(t)\|_{L^\infty}\Big]\frac{C|\Omega|}{\nu^2\lambda_1}\|\beta\|_{L^1(\tau,T)}+\frac{C|\Omega|}{\nu^2\lambda_1}\tilde{\beta}_0\\ &+\frac{C|\Omega|\|\varphi\|^2_{L^\infty(\partial\Omega)}}{\nu^2\lambda_1}\|\alpha(t)\|_{L^\infty(\tau,T)}+\frac{7(1+\gamma)}{2\nu^2}\langle\|g\|^2_{V'}\rangle|_{\leq t}\\ &+\Big[\frac{C\|\varphi\|_{L^\infty(\partial\Omega)}}{\nu^3\lambda_1}+\frac{C}{\nu^3\lambda_1^2}\|\alpha(t)\|_{L^\infty}+1\Big]\Big[\frac{C\nu|\partial\Omega|}{\varepsilon}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{C\varepsilon\|\varphi\|^4_{L^\infty(\partial\Omega)}|\partial\Omega|}{\nu}\Big]\\ \leq\ & \nu\lambda_1. \end{aligned} \tag{42}$$

If the generalized Grashof number is defined as $G(t)=\Big(\frac{\langle\|g\|^2_{V'}\rangle|_{\leq t}}{\nu^2\lambda_1}\Big)^{1/2}$, then we can deduce a sufficient condition for the asymptotic stability of trajectories inside pullback attractors as

$$G(t)+K_0\leq\frac{2\nu}{7(1+\gamma)}, \tag{43}$$

which completes the proof for our work. □

4. Conclusions and Further Research

Based on the well-posedness and pullback dynamics for 2D Navier–Stokes equations with double time-varying delays defined on a Lipschitz-like domain in [11], this presented work investigated the asymptotic stability of complete trajectories inside a pullback attractor of problem (2), which is an extension of [11,12]. However, when the delay is infinite, the dynamics and asymptotic stability are still open, which is our interest in the future.

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