mathematics

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Convolution of Decomposition Integrals

Adam Šeliga

Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Radlinského 11, 810 05 Bratislava, Slovakia; adam.seliga@stuba.sk

Abstract: Four different types of convolutions of aggregation functions (the upper, the lower, the super-, and the sub-convolution) are examined in the setting of both sub- and super-decomposition integrals defined on a finite space. Examples of the results of the paper are provided. As a by-product, the super-additive transformation of sub-decomposition integrals and the sub-additive transformation of super-decomposition integrals are fully characterized. Possible applications are indicated.

Keywords: convolution; collection integral; decomposition integral; aggregation functions

MSC: 28B15; 28E10; 91B06

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Citation: Šeliga, A. Convolution of Decomposition Integrals. *Mathematics* **2022**, *10*, 747. https://doi.org/10.3390/math10050747

Academic Editors: Anca Croitoru, Radko Mesiar, Anna Rita Sambucini and Bianca Satco

Received: 28 December 2021
Accepted: 25 February 2022
Published: 26 February 2022

1. Introduction

One may notice that the concept of aggregation functions plays an important role in the decision theory, and the concept of convolution is important in the classical analysis, but also in probability, acoustics, image processing, computer vision, etc. Recall that convolution is a binary operation acting on functions, mostly on n-dimensional real functions. Note that aggregation functions are special n-dimensional functions, and thus, it is no surprise that these two concepts were combined and convolution was introduced for the framework of aggregation functions; for further reference, see, [1]. The mentioned paper introduced four different types of convolutions, namely, the upper convolution, the lower convolution, the super-convolution, and the sub-convolution.

Standard convolutions usually deal with Riemann (or Lebesgue) integral. Inspired by convolutions of aggregation functions proposed in [1], in this paper, we apply these convolutions in the setting of sub-decomposition integrals [2] and super-decomposition integrals [3], a special class of aggregation functions that includes many well-known non-linear integrals, such as the Choquet integral [4], the Shilkret integral [5], the PAN integral [6,7], the concave integral [8], or the convex integral [3]. Note that these integrals contribute to the basics of set-valued analysis, see also, e.g., [9,10]. As a by-product, we obtain the super- and sub-additive transformations [11] of sub-decomposition and super-decomposition integrals, respectively.

The rest of the paper is organized as follows. Section 2 contains some preliminaries and definitions used in the paper. Section 3 examines the upper convolution and super-convolution of sub-decomposition integrals. This section also solves the problem of super-additive transformation of sub-decomposition integrals. Section 4 examines other convolutions for sub-decomposition integrals and analogous results for super-decomposition integrals. The last section, Section 5, concludes the paper with some remarks.

Recall that the upper convolution of sub-collection integrals was examined in the conference paper [12], and this paper extends these results.

2. Preliminaries

Let X be a finite non-empty set referred to as a *space*. Without loss of generality, we may assume that the space X is of the form $\{1, 2, \ldots, n\}$ for some natural number $n \in \mathbb{N}$ that is fixed throughout the paper.

A (non-negative) *function* is a map $X \to [0, \infty[$. Any function f is in one-to-one correspondence with a vector of its values $\big(f(1), f(2), \ldots, f(n)\big)$. Thus, vectors from $\mathbb{F} = [0, \infty[^n$ are referred to as functions and are denoted by bold lower-case letters $\mathbf{x}, \mathbf{y}, \mathbf{z}$, etc. The ith coordinate of the vector $\mathbf{x}$ is denoted by the symbol x_i, for $i = 1, 2, \ldots, n$. An *indicator function* of a set $A \subseteq X$ is denoted by $\mathbf{1}_A$.

A *monotone measure* is a map $\mu: 2^X \to [0, \infty[$ such that μ is grounded, i.e., $\mu(\emptyset) = 0$, and μ is non-decreasing with respect to set inclusion, i.e., the inequality $\mu(A) \le \mu(B)$ holds for $A \subseteq B \subseteq X$. The set of all monotone measures is denoted by $\mathbb{M}$. A monotone measure μ is called an *additive measure* if, and only if, μ is additive, i.e., $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint sets $A, B \in 2^X$. The set of all additive measures is denoted by $\mathbb{M}_+$.

The monograph [6] revisited and revised the terminology used in the generalized measure theory. Note that the term 'monotone measure' is sometimes replaced in the literature with the term 'fuzzy measure'. As remarked in the Preface of the mentioned monograph, monotone measures may not be continuous as in the case of fuzzy measures, and their primary characteristic is the monotonicity, hence the name.

A non-empty set $\mathcal{D} \subseteq 2^X \setminus \{\emptyset\}$ is called a *collection*. A non-empty set of collections is called a *decomposition system*. The set of all collections is denoted by $\mathbb{D}$, and the set of all decomposition systems by $\mathbb{H}$.

Definition 1. *A sub-collection integral [13] with respect to a collection $\mathcal{D} \in \mathbb{D}$ and monotone measure $\mu \in \mathbb{M}$ is an operator $\mathsf{col}^{\mu}_{\mathcal{D}}: \mathbb{F} \to [0, \infty[$ given by*

$$\mathsf{col}^{\mu}_{\mathcal{D}}(\mathbf{x}) = \bigvee \left\{ \sum_{A \in \mathcal{D}} \alpha_A \mu(A) \colon \sum_{A \in \mathcal{D}} \alpha_A \mathbf{1}_A \le \mathbf{x} \text{ where } \alpha_A \ge 0 \right\}$$

for any function $\mathbf{x} \in \mathbb{F}$. A super-collection integral [13] with respect to a collection $\mathcal{D} \in \mathbb{D}$ and monotone measure $\mu \in \mathbb{M}$ is an operator $\mathsf{scol}^{\mu}_{\mathcal{D}}: \mathbb{F} \to [0, \infty]$ given by

$$\mathsf{scol}^{\mu}_{\mathcal{D}}(\mathbf{x}) = \bigwedge \left\{ \sum_{A \in \mathcal{D}} \alpha_A \mu(A) \colon \sum_{A \in \mathcal{D}} \alpha_A \mathbf{1}_A \ge \mathbf{x} \text{ where } \alpha_A \ge 0 \right\}$$

for any function $\mathbf{x} \in \mathbb{F}$ (if necessary, the standard convention $\inf \emptyset = \infty$ is considered).

Definition 2. *A sub-decomposition integral [2] with respect to a decomposition system $\mathcal{H} \in \mathbb{H}$ and a monotone measure $\mu \in \mathbb{M}$ is an operator $\mathsf{dec}^{\mu}_{\mathcal{H}}: \mathbb{F} \to [0, \infty[$ given by*

$$\mathsf{dec}^{\mu}_{\mathcal{D}}(\mathbf{x}) = \bigvee_{\mathcal{D} \in \mathcal{H}} \bigvee \left\{ \sum_{A \in \mathcal{D}} \alpha_A \mu(A) \colon \sum_{A \in \mathcal{D}} \alpha_A \mathbf{1}_A \le \mathbf{x} \text{ where } \alpha_A \ge 0 \right\} = \bigvee_{\mathcal{D} \in \mathcal{H}} \mathsf{col}^{\mu}_{\mathcal{D}}(\mathbf{x})$$

for any function $\mathbf{x} \in \mathbb{F}$. A super-decomposition integral [3] with respect to a decomposition system $\mathcal{H} \in \mathbb{H}$ and a monotone measure $\mu \in \mathbb{M}$ is an operator $\mathsf{dec}^{\mu}_{\mathcal{H}}: \mathbb{F} \to [0, \infty]$ given by

$$\mathsf{sdec}^{\mu}_{\mathcal{D}}(\mathbf{x}) = \bigwedge_{\mathcal{D} \in \mathcal{H}} \bigwedge \left\{ \sum_{A \in \mathcal{D}} \alpha_A \mu(A) \colon \sum_{A \in \mathcal{D}} \alpha_A \mathbf{1}_A \le \mathbf{x} \text{ where } \alpha_A \ge 0 \right\} = \bigwedge_{\mathcal{D} \in \mathcal{H}} \mathsf{scol}^{\mu}_{\mathcal{D}}(\mathbf{x})$$

for any function $\mathbf{x} \in \mathbb{F}$.

Note that sub-collection and sub-decomposition integrals are aggregation functions. An *aggregation function* A is an operator $\mathbb{F} \to [0, \infty[$ such that $\mathsf{A}(\mathbf{0}) = 0$ and $\mathsf{A}(\mathbf{x}) \le \mathsf{A}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{F}$ such that $\mathbf{x} \le \mathbf{y}$. In this setting, super-collection and super-decomposition integrals are not aggregation functions, in general, because the value of ∞ can be attained for some inputs. As an example, take the space $X = \{1, 2\}$, a collection $\mathcal{D} = \{\{1\}\}$ (or the decomposition system $\mathcal{H} = \{\mathcal{D}\}$), and compute the value of the corresponding integral of the function $\mathbf{x} = (0, 1)$.

Definition 3. *Let* $\mathsf{A}, \mathsf{B} \in \mathbb{A}$ *be two aggregation functions. Their upper convolution is an aggregation function* $\mathsf{A} \triangledown \mathsf{B}$ *given by*

$$(\mathsf{A} \triangledown \mathsf{B})(\mathbf{x}) = \bigvee_{\mathbf{0} \le \mathbf{t} \le \mathbf{x}} \big(\mathsf{A}(\mathbf{t}) + \mathsf{B}(\mathbf{x} - \mathbf{t})\big)$$

for all $\mathbf{x} \in \mathbb{F}$*; their lower convolution is an aggregation function* $\mathsf{A} \vartriangle \mathsf{B}$ *given by*

$$(\mathsf{A} \vartriangle \mathsf{B})(\mathbf{x}) = \bigwedge_{\mathbf{0} \le \mathbf{t} \le \mathbf{x}} \big(\mathsf{A}(\mathbf{t}) + \mathsf{B}(\mathbf{x} - \mathbf{t})\big)$$

for all $\mathbf{x} \in \mathbb{F}$*; their super convolution is an aggregation function* $\mathsf{A} \circledtriangledown \mathsf{B}$ *given by*

$$(\mathsf{A} \circledtriangledown \mathsf{B})(\mathbf{x}) = \bigvee \left\{ \sum_{i=1}^{k_1} \mathsf{A}(\mathbf{x}_i) + \sum_{j=1}^{k_2} \mathsf{B}(\mathbf{y}_j) : \mathbf{x}_i, \mathbf{y}_j \ge \mathbf{0}, \sum_{i=1}^{k_1} \mathbf{x}_i + \sum_{j=1}^{k_2} \mathbf{y}_j = \mathbf{x}, k_1, k_2 \in \mathbb{N} \right\}$$

for all $\mathbf{x} \in \mathbb{F}$ *(if defined); and their sub-convolution is an aggregation function* $\mathsf{A} \circledvartriangle \mathsf{B}$ *given by*

$$(\mathsf{A} \circledvartriangle \mathsf{B})(\mathbf{x}) = \bigwedge \left\{ \sum_{i=1}^{k_1} \mathsf{A}(\mathbf{x}_i) + \sum_{j=1}^{k_2} \mathsf{B}(\mathbf{y}_j) : \mathbf{x}_i, \mathbf{y}_j \ge \mathbf{0}, \sum_{i=1}^{k_1} \mathbf{x}_i + \sum_{j=1}^{k_2} \mathbf{y}_j = \mathbf{x}, k_1, k_2 \in \mathbb{N} \right\}$$

for all $\mathbf{x} \in \mathbb{F}$*. These definitions are introduced and examined in [1].*

Notice that the super convolution may not be well defined for some aggregation functions. Consider, for example, one-dimensional aggregation functions $\mathsf{A}(x) = \sqrt{x}$ and $\mathsf{B}(x) = 0$ for all $x \in [0, \infty[$, in which case

$$(\mathsf{A} \circledtriangledown \mathsf{B})(x) = \begin{cases} 0, & \text{if } x = 0, \\ \infty, & \text{otherwise,} \end{cases}$$

i.e., $\mathsf{A} \circledtriangledown \mathsf{B}$ is not an aggregation function.

3. Upper Convolution and Super-Convolution of Sub-Decomposition Integrals

In the conference paper [12], the upper convolution of collection integrals were examined. The following results were obtained: Let $\mathcal{D}, \mathcal{D}_1, \mathcal{D}_2 \in \mathbb{D}$ be collections and let $\mu, \mu_1, \mu_2 \in \mathbb{M}$ be monotone measures. Then

$$\mathsf{col}^{\mu}_{\mathcal{D}_1} \triangledown \mathsf{col}^{\mu}_{\mathcal{D}_2} = \mathsf{col}^{\mu}_{\mathcal{D}_1 \cup \mathcal{D}_2} \quad \text{and} \quad \mathsf{col}^{\mu_1}_{\mathcal{D}} \triangledown \mathsf{col}^{\mu_2}_{\mathcal{D}} = \mathsf{col}^{\mu_1 \vee \mu_2}_{\mathcal{D}}.$$

In the spirit of the first equality, we obtain the following result for decomposition integrals.

Proposition 1. *Let* $\mathcal{H}_1, \mathcal{H}_2 \in \mathbb{H}$ *be two decomposition systems and let* $\mu \in \mathbb{M}$ *be a monotone measure. Then*

$$\mathsf{dec}^{\mu}_{\mathcal{H}_1} \triangledown \mathsf{dec}^{\mu}_{\mathcal{H}_2} = \mathsf{dec}^{\mu}_{\mathcal{H}'},$$

where $\mathcal{H}'$ *is a decomposition system* $\{\mathcal{D}_1 \cup \mathcal{D}_2 : \mathcal{D}_1 \in \mathcal{H}_1, \mathcal{D}_2 \in \mathcal{H}_2\}$.

Proof. Let $\mathbf{x} \in \mathbb{F}$ be a function. Then

$$\big(\mathsf{dec}^{\mu}_{\mathcal{H}_1} \triangledown \mathsf{dec}^{\mu}_{\mathcal{H}_2}\big)(\mathbf{x}) = \bigvee_{\mathbf{0} \le \mathbf{t} \le \mathbf{x}} \big(\mathsf{dec}^{\mu}_{\mathcal{H}_1}(\mathbf{t}) + \mathsf{dec}^{\mu}_{\mathcal{H}_2}(\mathbf{x} - \mathbf{t})\big),$$

i.e., there exists $\bar{\mathbf{t}} \in \mathbb{F}$ such that $\bar{\mathbf{t}} \leq \mathbf{x}$ and

$$\left(\mathsf{dec}^{\mu}_{\mathcal{H}_1} \triangledown \mathsf{dec}^{\mu}_{\mathcal{H}_2}\right)(\mathbf{x}) = \mathsf{dec}^{\mu}_{\mathcal{H}_1}(\bar{\mathbf{t}}) + \mathsf{dec}^{\mu}_{\mathcal{H}_2}(\mathbf{x} - \bar{\mathbf{t}}).$$

Now, there exist collections $\overline{\mathcal{D}}_1 \in \mathcal{H}_1$ and $\overline{\mathcal{D}_2} \in \mathcal{H}_2$ such that

$$\left(\mathsf{dec}^{\mu}_{\mathcal{H}_1} \triangledown \mathsf{dec}^{\mu}_{\mathcal{H}_2}\right)(\mathbf{x}) = \mathsf{col}^{\mu}_{\overline{\mathcal{D}}_1}(\bar{\mathbf{t}}) + \mathsf{col}^{\mu}_{\overline{\mathcal{D}}_2}(\mathbf{x} - \bar{\mathbf{t}})$$

and thus, a sub-decomposition $\sum_{A \in \overline{\mathcal{D}}_1} \alpha_A \mathbf{1}_A$ of $\bar{\mathbf{t}}$ and a sub-decomposition $\sum_{B \in \overline{\mathcal{D}}_2} \beta_B \mathbf{1}_B$ of $\mathbf{x} - \bar{\mathbf{t}}$ such that

$$\left(\mathsf{dec}^{\mu}_{\mathcal{H}_1} \triangledown \mathsf{dec}^{\mu}_{\mathcal{H}_2}\right)(\mathbf{x}) = \sum_{A \in \overline{\mathcal{D}}_1} \alpha_A \mu(A) + \sum_{B \in \overline{\mathcal{D}}_2} \beta_B \mu(B).$$

Note that by summing these two sub-decompositions, we obtain a sub-decomposition of $\mathbf{x}$ from the collection $\overline{\mathcal{D}}_1 \cup \overline{\mathcal{D}}_2$, i.e.,

$$\left(\mathsf{dec}^{\mu}_{\mathcal{H}_1} \triangledown \mathsf{dec}^{\mu}_{\mathcal{H}_2}\right)(\mathbf{x}) \leq \mathsf{col}^{\mu}_{\overline{\mathcal{D}}_1 \cup \overline{\mathcal{D}}_2}(\mathbf{x}) \leq \mathsf{dec}^{\mu}_{\mathcal{H}'}(\mathbf{x}).$$

Thus, following that $\mathbf{x} \in \mathbb{F}$ is arbitrary, $\mathsf{dec}^{\mu}_{\mathcal{H}_1} \triangledown \mathsf{dec}^{\mu}_{\mathcal{H}_2} \leq \mathsf{dec}^{\mu}_{\mathcal{H}'}$. Now, we prove the same inequalities, but with a reversed inequality sign. Let $\mathbf{x} \in \mathbb{F}$. Note that there exists a collection $\mathcal{D}' \in \mathcal{H}'$ (and thus, collections $\mathcal{D}_1 \in \mathcal{H}_1$, $\mathcal{D}_2 \in \mathcal{H}_2$) such that

$$\mathsf{dec}^{\mu}_{\mathcal{H}'}(\mathbf{x}) = \mathsf{col}^{\mu}_{\mathcal{D}_1 \cup \mathcal{D}_2}(\mathbf{x}).$$

From this, the existence of coefficients $\alpha_A \geq 0$ for $A \in \mathcal{D}_1 \cup \mathcal{D}_2$ such that $\sum_{A \in \mathcal{D}_1 \cup \mathcal{D}_2} \alpha_A \mathbf{1}_A$ is a sub-decomposition of $\mathbf{x}$ with

$$\mathsf{dec}^{\mu}_{\mathcal{H}'}(\mathbf{x}) = \sum_{A \in \mathcal{D}_1 \cup \mathcal{D}_2} \alpha_A \mu(A)$$

is guaranteed. Now, set $\beta_A = \alpha_A$ for $A \in \mathcal{D}_1$ and

$$\gamma_A = \begin{cases} \alpha_A, & \text{if } A \in \mathcal{D}_2 \setminus \mathcal{D}_1, \\ 0, & \text{otherwise,} \end{cases}$$

from which

$$\mathsf{dec}^{\mu}_{\mathcal{H}'}(\mathbf{x}) = \sum_{A \in \mathcal{D}_1} \beta_A \mu(A) + \sum_{A \in \mathcal{D}_2} \gamma_A \mu(A).$$

Then, consider $\bar{\mathbf{t}} = \sum_{A \in \mathcal{D}_1} \beta_A \mathbf{1}_A$, and therefore $\sum_{A \in \mathcal{D}_2} \gamma_A \mathbf{1}_A$ is a sub-decomposition of $\mathbf{x} - \bar{\mathbf{t}}$. Thus

$$\begin{aligned} \mathsf{dec}^{\mu}_{\mathcal{H}'}(\mathbf{x}) &\leq \mathsf{col}^{\mu}_{\mathcal{D}_1}(\bar{\mathbf{t}}) + \mathsf{col}^{\mu}_{\mathcal{D}_2}(\mathbf{x} - \bar{\mathbf{t}}) \\ &\leq \bigvee_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{x}} \left(\mathsf{col}^{\mu}_{\mathcal{D}_1}(\mathbf{t}) + \mathsf{col}^{\mu}_{\mathcal{D}_2}(\mathbf{x} - \mathbf{t})\right) = \left(\mathsf{dec}^{\mu}_{\mathcal{H}_1} \triangledown \mathsf{dec}^{\mu}_{\mathcal{H}_2}\right)(\mathbf{x}). \end{aligned}$$

Because, again, $\mathbf{x}$ was arbitrary, $\mathsf{dec}^{\mu}_{\mathcal{H}'} \leq \mathsf{dec}^{\mu}_{\mathcal{H}_1} \triangledown \mathsf{dec}^{\mu}_{\mathcal{H}_2}$. Combining both proved inequalities, we obtain the desired result. □

Remark 1. *The decomposition system $\mathcal{H}'$ from the previous proposition is denoted by $\mathcal{H}_1 \uplus \mathcal{H}_2$. Note that the set of all decomposition systems $\mathbb{H}$ with the operation $\uplus$ forms an Abelian semigroup with annihilator $\mathcal{H}^* = \{2^X \setminus \{\emptyset\}\}$.*

Example 1. *Let $\mu \in \mathbb{M}$ be a monotone measure. Choose a decomposition system $\mathcal{H}_1 = \{\{A\} \colon A \in 2^X \setminus \{\emptyset\}\}$, i.e., $\mathsf{dec}^{\mu}_{\mathcal{H}_1}$ is the Shilkret integral. Then choose $\mathcal{H}_2 = \left\{\{\{x\} \colon x \in X\}\right\}$, i.e., $\mathsf{dec}^{\mu}_{\mathcal{H}_2}$*

is equivalent to the Lebesgue integral for additive measures $\mu \in \mathbb{M}_+$. Then, for example, when $X = \{1,2,3\}$, one obtains

$$\mathcal{H}_1 \uplus \mathcal{H}_2 = \Big\{\{\{1\},\{2\},\{3\}\},\{\{1\},\{2\},\{3\},\{1,2\}\},\{\{1\},\{2\},\{3\},\{1,3\}\} \\ \{\{1\},\{2\},\{3\},\{2,3\}\},\{\{1\},\{2\},\{3\},\{1,2,3\}\}\Big\}.$$

After some algebraic manipulations, one can find that

$$\mathsf{dec}^{\mu}_{\mathcal{H}_1 \uplus \mathcal{H}_2}(\mathbf{x}) = \sum_{i=1}^{n} x_i \mu(\{i\}) + \max_{\substack{A \subseteq X \\ A \neq \emptyset}} \left[\min_{i \in A} x_i \cdot \left(\mu(A) - \sum_{i \in A} \mu(\{i\}) \right) \right].$$

Note that this equality is true for an arbitrary space X.

Following the result [1] of Theorem 5.2, we obtain the following corollary.

Corollary 1. *A sub-decomposition integral $\mathsf{dec}^{\mu}_{\mathcal{H}}$ is super additive for all monotone measures $\mu \in \mathbb{M}$ if and only if $\mathcal{H} \uplus \mathcal{H} = \mathcal{H}$.*

Example 2. *It is easy to notice that singleton decomposition systems $\mathcal{H}$, i.e., all sub-collections integrals, are such that $\mathcal{H} \uplus \mathcal{H} = \mathcal{H}$. In fact, if we consider only minimal decomposition systems, then these are the only ones that generate a super-additive sub-decomposition integral (with the monotone measure not being fixed).*

Now, we can examine the super-convolution of sub-decomposition integrals. Let us start with the super self-convolution.

Proposition 2. *Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system and let $\mu \in \mathbb{M}$ be a monotone measure. Then*

$$\mathsf{dec}^{\mu}_{\mathcal{H}} \circledcirc \mathsf{dec}^{\mu}_{\mathcal{H}} = \mathsf{col}^{\mu}_{\mathcal{D}'},$$

where $\mathcal{D}'$ is a collection given by

$$\mathcal{D}' = \bigcup_{\mathcal{D} \in \mathcal{H}} \mathcal{D}.$$

Proof. From the definition of super self-convolution, it can be noted that it is the same as the limit of the upper self-convolutions applied consecutively over and over. The 'greatest' collection that will appear is the collection $\mathcal{D}'$. All other collections in the decomposition system are subsets of $\mathcal{D}'$. From the properties of sub-decomposition integrals, the smaller collections can be ignored, leaving only the collection $\mathcal{D}'$ in the decomposition system, i.e., it will be the same as the sub-collection integral with respect to the collection $\mathcal{D}'$. □

Note that the corollary of this result, based on the proof of [1] Theorem 5.4, is as follows.

Corollary 2. *Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system and let $\mu \in \mathbb{M}$ be a monotone measure. The super-additive transformation of a sub-decomposition integral $\mathsf{dec}^{\mu}_{\mathcal{H}}$ is the sub-collection integral $\mathsf{col}^{\mu}_{\mathcal{D}'}$, where $\mathcal{D}'$ is a collection given by*

$$\mathcal{D}' = \bigcup_{\mathcal{D} \in \mathcal{H}} \mathcal{D}.$$

Now we examine the super convolution of two different sub-decomposition integrals with respect to the same monotone measure.

Proposition 3. *Let $\mathcal{H}_1, \mathcal{H}_2 \in \mathbb{H}$ be two decomposition systems and let $\mu \in \mathbb{M}$ be a monotone measure. Then*

$$\mathsf{dec}^{\mu}_{\mathcal{H}_1} \circledcirc \mathsf{dec}^{\mu}_{\mathcal{H}_2} = \mathsf{col}^{\mu}_{\mathcal{D}'},$$

where $\mathcal{D}'$ is a collection given by

$$\mathcal{D}' = \bigcup_{\mathcal{D} \in \mathcal{H}_1 \cup \mathcal{H}_2} \mathcal{D}.$$

Proof. This follows directly from [1] Theorem 4.4. □

Example 3. *If we consider decomposition systems $\mathcal{H}_1$ and $\mathcal{H}_2$ from Example 1 of this paper, we can compute super-additive transformations of sub-decomposition integrals $\mathsf{dec}^{\mu}_{\mathcal{H}_1}$ and $\mathsf{dec}^{\mu}_{\mathcal{H}_2}$ (where $\mu \in \mathbb{M}$ is an arbitrary monotone measure). In the first case, we obtain the concave integral (i.e., the sub-collection integral with respect to a collection $2^X \setminus \{\varnothing\}$), and the second one stays unchanged. Additionally, $\mathsf{dec}^{\mu}_{\mathcal{H}_1} \circledcirc \mathsf{dec}^{\mu}_{\mathcal{H}_2}$ is the concave integral [8].*

4. Other Convolutions of Decomposition Integrals

The situation with the lower convolution and sub-decomposition integrals is not so easy. The upper convolution (and also the super-convolution) is closed for sub-decomposition integrals (i.e., the result is again a sub-decomposition integral). In the case of the lower convolution, this is no longer the case. See the following example, where we consider two collections integrals (i.e., decomposition integrals with respect to a singleton decomposition system).

Example 4. *Consider two sub-collection integrals on the space $X = \{1,2\}$ with respect to collections $\mathcal{D}_1 = \big\{\{1\},\{1,2\}\big\}$ and $\mathcal{D}_2 = \big\{\{2\},\{1,2\}\big\}$ (both of these integrals are so-called chain integrals, see, e.g., [13]). Let $\mu \in \mathbb{M}$ be a monotone measure and, for the sake of simplification, we use the following notation: $\mu_1 = \mu\big(\{1\}\big)$, $\mu_2 = \mu\big(\{2\}\big)$, and $\mu_{12} = \mu\big(\{1,2\}\big)$. The value of these integrals is given by*

$$\mathsf{col}^{\mu}_{\mathcal{D}_1}(x_1,x_2) = \begin{cases} x_1\mu_{12}, & \text{if } x_1 \leq x_2, \\ x_2\mu_{12} + (x_1 - x_2)\mu_1, & \text{otherwise,} \end{cases}$$

for the first one, and

$$\mathsf{col}^{\mu}_{\mathcal{D}_2}(x_1,x_2) = \begin{cases} x_1\mu_{12} + (x_2 - x_1)\mu_2, & \text{if } x_1 \leq x_2, \\ x_2\mu_{12}, & \text{otherwise,} \end{cases}$$

for the second one. For the lower convolution of these two integrals we obtain that

$$\begin{aligned}\Big(\mathsf{col}^{\mu}_{\mathcal{D}_1} \triangle \mathsf{col}^{\mu}_{\mathcal{D}_2}\Big)(x_1,x_2) &= \bigvee_{\substack{0 \leq t_1 \leq x_1 \\ 0 \leq t_2 \leq x_2}} \Big(\mathsf{col}^{\mu}_{\mathcal{D}_1}(t_1,t_2) + \mathsf{col}^{\mu}_{\mathcal{D}_2}(x_1 - t_1, x_2 - t_2)\Big) \\ &\leq \mathsf{col}^{\mu}_{\mathcal{D}_1}(0,x_2) + \mathsf{col}^{\mu}_{\mathcal{D}_2}(x_1,0) = 0,\end{aligned}$$

which implies that $\mathsf{col}^{\mu}_{\mathcal{D}_1} \triangle \mathsf{col}^{\mu}_{\mathcal{D}_2} \equiv 0$. There is no decomposition integral $\mathsf{dec}^{\mu}_{\mathcal{H}} \equiv 0$ (with μ being an arbitrary monotone measure).

Remark 2. *Note that the previous example (in the setting of the example) also implies that $\mathsf{col}^{\mu}_{\mathcal{D}_1} \circledast \mathsf{col}^{\mu}_{\mathcal{D}_2} \equiv 0$.*

Similar results, as in the case of the lower convolution and the super convolution for sub-decomposition integrals, can be obtained for the upper convolution and the sub-convolution for super-decomposition integrals. We just need to make sure that we work with those super-decomposition integrals that are aggregation functions, and we refer to those as the well-defined super-decomposition integrals.

Definition 4. *Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system. A super-decomposition integral with respect to the decomposition system $\mathcal{H}$ is called* well-defined *if and only if*

$$\mathsf{sdec}^{\mu}_{\mathcal{H}}(\mathbf{x}) < \infty$$

for all functions $\mathbf{x} \in \mathbb{F}$ and all monotone measures $\mu \in \mathbb{M}$.

We assume that the super-decomposition integrals are well defined for the rest of the paper. Because the proofs of the following statements are completely analogous to proofs in the previous section, we omit them.

Proposition 4. *Let $\mathcal{H}_1, \mathcal{H}_2 \in \mathbb{H}$ be two decomposition systems and let $\mu \in \mathbb{M}$ be a monotone measure. Then*

$$\mathsf{sdec}^{\mu}_{\mathcal{H}_1} \vartriangle \mathsf{sdec}^{\mu}_{\mathcal{H}_2} = \mathsf{sdec}^{\mu}_{\mathcal{H}_1 \uplus \mathcal{H}_2}.$$

Corollary 3. *A super-decomposition integral $\mathsf{sdec}^{\mu}_{\mathcal{H}}$ is sub-additive for all monotone measures $\mu \in \mathbb{M}$ if and only if $\mathcal{H} \uplus \mathcal{H} = \mathcal{H}$.*

Remark 3. *Note that the same decomposition systems generating super-additive sub-decomposition integrals generate sub-additive super-additive integrals and vice versa.*

Proposition 5. *Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system and let $\mu \in \mathbb{M}$ be a monotone measure. Then*

$$\mathsf{sdec}^{\mu}_{\mathcal{H}} \circledast \mathsf{sdec}^{\mu}_{\mathcal{H}} = \mathsf{scol}^{\mu}_{\mathcal{D}'},$$

where $\mathcal{D}'$ is a collection given by

$$\mathcal{D}' = \bigcup_{\mathcal{D} \in \mathcal{H}} \mathcal{D}.$$

Corollary 4. *Let $\mathcal{H} \in \mathbb{H}$ be a decomposition system, and let $\mu \in \mathbb{M}$ be a monotone measure. The sub-additive transformation of a super-decomposition integral $\mathsf{dec}^{\mu}_{\mathcal{H}}$ is the super-collection integral $\mathsf{col}^{\mu}_{\mathcal{D}'}$, where $\mathcal{D}'$ is a collection given by*

$$\mathcal{D}' = \bigcup_{\mathcal{D} \in \mathcal{H}} \mathcal{D}.$$

Proposition 6. *Let $\mathcal{H}_1, \mathcal{H}_2 \in \mathbb{H}$ be two decomposition systems, and let $\mu \in \mathbb{M}$ be a monotone measure. Then*

$$\mathsf{sdec}^{\mu}_{\mathcal{H}_1} \circledast \mathsf{sdec}^{\mu}_{\mathcal{H}_2} = \mathsf{scol}^{\mu}_{\mathcal{D}'},$$

where $\mathcal{D}'$ is a collection given by

$$\mathcal{D}' = \bigcup_{\mathcal{D} \in \mathcal{H}_1 \cup \mathcal{H}_2} \mathcal{D}.$$

5. Concluding Remarks

In the paper, four different types of convolution of aggregation functions in the setting of decomposition integrals, i.e., both the sub-decomposition and super-decomposition integrals, were examined. We solved the problem of the upper convolution and super convolution of sub-decomposition integrals with respect to the same monotone measure and, analogously, the lower convolution and sub-convolution of super-decomposition integrals with respect to the same monotone measure.

Some questions still remain open, both theoretical and practical. For example, is it possible to obtain a result similar to

$$\mathsf{col}^{\mu_1}_{\mathcal{D}} \triangledown \mathsf{col}^{\mu_2}_{\mathcal{D}} = \mathsf{col}^{\mu_1 \vee \mu_2}_{\mathcal{D}}$$

but replacing the sub-collection integrals with sub-decomposition integrals? Or, is it possible to characterize those decomposition systems for which the lower convolution of sub-decomposition integrals is again a sub-decomposition integral (in the spirit of Example 4)? Another interesting question is related to the fact that some decomposition integrals are extensions of the Lebesgue integral (i.e., for additive monotone measures, they coincide with the Lebesgue integral); for more details, see [14]. Now, we have the problem of how our proposed convolutions are related to the standard convolution based on the Lebesgue integral.

Though our work is purely theoretical, we expect several applications of our results in all domains, where particular decomposition integrals and their generalizations were successfully applied. Here, we recall, among others, multi-criteria decision support, image processing, fuzzy ruler-based classification, etc., where generalizations of the Choquet integral were considered; see, for example, [15–18]. In our further research, we will focus on these mentioned problems and possible applications. More, we will aim to focus on algorithms for faster processing of our theoretical results. Observe that we have already proposed some algorithms for the computation of decomposition integrals in [13], where we have also shown that this is not an easy task.

Funding: The author was supported by the Slovak Research and Development Agency under the contract no. APVV-17-0066 and no. APVV-18-0052. In addition, the support of the grant VEGA 1/0006/19 is kindly announced.

Conflicts of Interest: The author declares no conflict of interest.

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