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Existence and Hyers–Ulam Stability for a Multi-Term Fractional Differential Equation with Infinite Delay

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Abstract: This paper is devoted to investigating one type of nonlinear two-term fractional order delayed differential equations involving Caputo fractional derivatives. The Leray–Schauder alternative fixed-point theorem and Banach contraction principle are applied to analyze the existence and uniqueness of solutions to the problem with infinite delay. Additionally, the Hyers–Ulam stability of fractional differential equations is considered for the delay conditions.

Keywords: fixed-point theorem; infinite delay; fractional differential equation; stability; existence and uniqueness; Caputo fractional derivative

MSC: 34A08; 26A33; 45M10

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1. Introduction

In recent decades, relevant theories and applications of fractional differential equations [1–6] have developed rapidly. Generally, fractional differential equations are derived from the research of solid mechanics [7], chemistry [8], physics [9,10], electromechanics [11], finance [12], and so on. Abundant theoretical achievements have been made in the study of the existence and uniqueness of fractional differential equations by applying the fixed-pointed theorem, such as [12–16]. However, there are few articles in the research and application of fractional differential equations with time delay. The delay factor has an important influence on the solution to the fractional differential system. The change of the system solution not only depends on the present state but also is constrained by the past state. Therefore, it is of great significance to consider the delay effect on a fractional differential system. In [17], the authors discussed the stability of fractional differential equations with delay evolution inclusion. Li et al. [18] derived a comparison principle for functional differential equations with infinite delays. Additionally, note that the Hyers–Ulam stability property of delay differential equations can be mainly considered by the Gronwall inequality. It is worth mentioning that the mentioned method can be applied for the stability study of Caputo fractional delay differential equations (see, for example, in [19–21]).

In [22], Qixiang Dong et al. investigated a kind of weighted fractional differential equations with infinite delay, which can be expressed by

$$\begin{cases} D^\alpha y(t) = f(t, \widetilde{y}_t), & t \in (0, b], \\ \widetilde{y}_0 = \phi \in \mathscr{B}, \end{cases}$$

where $\alpha \in (0, 1]$, $\widetilde{y}(t) = t^{1-\alpha} y(t)$, D^α represents the Riemann–Liouville fractional derivative, $f : (0, b] \times \mathscr{B} \to \mathscr{B}$ is a given function satisfying some assumptions, and $\mathscr{B}$ is the phase space. A method named weighted delay is applied by the authors to study the properties of solutions to fractional differential equations whose initial value is not zero.

On the basis of these contents, we study the related properties of solutions to a class of nonlinear fractional differential equations with infinite delay, namely

$$\begin{cases} {}_cD^{\alpha}y(t) - a_cD^{\beta}y(t) = f(t, y_t), & t \in J = [0, b], \\ y(t) = \phi(t), & t \in (-\infty, 0], \end{cases} \tag{1}$$

where ${}_cD^{\alpha}$ and ${}_cD^{\beta}$ are Caputo fractional derivatives with $0 < \beta < \alpha \leqslant 1$, a is a certain constant, $f : J \times \mathscr{B} \to \mathbb{R}$ is a given function satisfying some assumptions that will be specified later, function $\phi \in \mathscr{B}$, and $\mathscr{B}$ is called a phase space, as defined later. Function y_t, which is an element $\mathscr{B}$, is defined as any function y on $(-\infty, b]$ as follows:

$$y_t(s) = y(t+s), \quad s \in (-\infty, 0], \quad t \in J. \tag{2}$$

Here, $y_t(\cdot)$ represents the preoperational state from time $-\infty$ up to time t. The notion of the phase space $\mathscr{B}$ plays an important role in the study of both qualitative and quantitative theories for functional differential equations. A common choice is the seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [23].

Our approach is largely based on the alternative of Leray–Schauder and Banach fixed-point theorem. Due to the characteristic of delay equations, we need to give the proper form of the solutions when discussing the existence and uniqueness, which is one of the key and difficult points to solve the problem. Generally, delay differential equations can be transformed into integral equations. Under the definition of phase space, the solutions of the integral equations can be appropriately extended, and the constructed equations are still continuous at the point $x = 0$. Additionally, we study the Hyers–Ulam stability of fractional differential Equation (1) with infinite delay $y(t) = \phi(t)$. Due to the limitation of delay conditions, the research of the Hyers–Ulam stability becomes more complicated. In this paper, we verify the Hyers–Ulam stability of delay differential Equation (1) by using the related properties of phase space and obtain the stability conclusion by means of a class of Gronwall inequalities.

This paper is organized as follows. In Section 2, some basic mathematical tools are introduced that are used throughout the article. Section 3 is devoted to our main conclusions. The stability analysis is discussed in Section 4. Two examples are given at the end of the article to illustrate the conclusions.

2. Preliminaries and Lemmas

In order to facilitate readers in reading the following contents, we introduce some basic definitions and lemmas which are used throughout this paper in this section. First and foremost, we denote $C([a, b], \mathbb{R})$ the Banach space of all continuous functions $y : [a, b] \to \mathbb{R}$ with the norm $\|y\| = \sup\{|y(t)|, t \in [a, b]\}$. Additionally, we denote by $C^m([0, b]; \mathbb{R})$ the Banach space of all continuously differentiable functions, with the norm defined as usual.

Definition 1 ([24])**.** *The Riemann–Liouville integral with order $\alpha > 0$ of the given function $h : [a, b] \to \mathbb{R}$ is defined as*

$$J_a^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}h(s)ds, \quad t \in [a, b],$$

provided the other side is point-wisely defined, where $\Gamma(\cdot)$ is the Euler's gamma function; i.e, $\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt$.

Definition 2 ([24])**.** *The Caputo derivative with order $\alpha > 0$ of the given function $h : [a, b] \to \mathbb{R}$ is defined as*

$${}_cD_a^{\alpha}h(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1}h^{(m)}(s)ds, \quad t \in [a, b],$$

provided the other side is point-wisely defined, where m is a positive integer satisfying $m - 1 < \alpha \leqslant m$. Incidentally, ${}_cD_a^{\alpha}$ is called the Caputo fractional differential operator as well.

Lemma 1 ([24]). *Let $\alpha > 0$ and $m = [\alpha] + 1$. Then, the general solution to the fractional differential equation ${}_cD^\alpha u(t) = 0$ is given by*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1} t^{m-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \cdots, m-1$ are some constants. Further, assuming that $u \in C^m([0,b];\mathbb{R})$, we can get

$$J^\alpha {}_cD^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1} t^{m-1},$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, \cdots, m-1$.

Definition 3 ([25]). *Let $\mathbb{X}$ be a Banach space; a linear topological space of functions from $(-\infty, 0]$ into $\mathbb{X}$, with the seminorm $\|\cdot\|_{\mathscr{B}}$, is called an admissible phase space if $\mathscr{B}$ has the following properties:*

(A1) There exists a positive constant H and functions $K(\cdot), M(\cdot) : [0, +\infty) \to [0, +\infty)$, with K continuous and M locally bounded, such that for any constant $a, b \in \mathbb{R}$ and $b > a$, if the function $x : (-\infty, b] \to \mathbb{X}, x_a \in \mathscr{B}$ and function $x(\cdot)$ is continuous on $[a,b]$, then for every $t \in [a,b]$, the following conditions (i)–(iii) hold:

(i) $x_t \in \mathscr{B}$;

(ii) $\|x(t)\| \leqslant H\|x_t\|_{\mathscr{B}}$ for some $H > 0$;

(iii) $\|x_t\|_{\mathscr{B}} \leqslant K(t-a)\sup_{a\leqslant s\leqslant t}\|x(s)\| + M(t-a)\|x_a\|_{\mathscr{B}}$.

(A2) For the function $x(\cdot)$ in $(A1)$, $t \mapsto x_t$ is a $\mathscr{B}$-valued continuous function for $t \in [a,b]$.

(B1) The space $\mathscr{B}$ is complete.

Lemma 2 ([26] Leray-Schauder alternative). *Let $\mathbb{X}$ be a Banach space, $\mathcal{C} \subset \mathbb{X}$ be a closed, convex subset of $\mathbb{X}$, $\mathcal{U}$ is an open subset of $\mathcal{C}$ and $0 \in \mathcal{U}$. Suppose $\mathcal{T} : \overline{\mathcal{U}} \to \mathcal{C}$ is a continuous, compact (in other words, $\mathcal{T}(\mathcal{U})$ is a relatively compact subset of $\mathcal{C}$) map. Then, either*

(i) $\mathcal{T}$ has a fixed point in $\mathcal{U}$, or

(ii) there is a $u \in \partial\mathcal{U}$ and $\lambda \in (0,1)$ with $u = \lambda\mathcal{T}(u)$.

In general, Gronwall inequality plays a vital role in the study of Hyers–Ulam stability of differential equations. Next, we introduce an integral inequality which can be considered as a generalization of the Gronwall inequality.

Lemma 3 ([27]). *Suppose $\alpha > 0$, $a > 0$, $g(t,s)$ is a nonnegative continuous function defined on $[0,T] \times [0,T]$ with $g(t,s) \leqslant M$, and $g(t,s)$ is nondecreasing w.r.t. the first variable and nonincreasing w.r.t. the second variable. Assume that function $u(t)$ is nonnegative and integrable on $[0,T]$ with*

$$u(t) \leqslant a + \int_0^t g(t,s)(t-s)^{\alpha-1} u(s) ds, \quad t \in [0,T].$$

Then, we have

$$u(t) \leqslant a + a\int_0^t \sum_{n=1}^{\infty} \frac{(g(t,s)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} ds,$$

where the notion "w.r.t." means "with respect to".

Lemma 4 ([22]). *Suppose $\alpha > 0$ and function $f \in C[0,b]$ is nonnegative and nondecreasing. Then, function $F(t) = J_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s) ds$ is nondecreasing on $[0,b]$.*

Based on the Lemma 4 introduced above, the following inequality is proved to verify the Hyers–Ulam stability in Section 4.

Lemma 5. *For any nonnegative function $\omega \in C[a,b]$ and any $t \in [a,b]$, we have the following integral inequality*

$$\sup_{0\leqslant\tau\leqslant t}\int_0^\tau (\tau-s)^{\alpha-1}\omega(s)ds \leqslant \int_0^t (t-s)^{\alpha-1}\sup_{0\leqslant\sigma\leqslant s}\omega(\sigma)ds.$$

Proof of Lemma 5. Since function $\omega(\cdot)$ is nonnegative, $\sup_{0\leqslant\sigma\leqslant s}\omega(\sigma)$ is nondecreasing , which implies that the function $\int_0^t (t-s)^{\alpha-1}\sup_{0\leqslant\sigma\leqslant s}\omega(\sigma)ds$ is also nondecreasing , by Lemma 4. Now, fix $t \in [a,b]$. Then, for any $\tau \in [0,t]$, we have

$$\begin{aligned}\int_0^\tau (\tau-s)^{\alpha-1}\omega(s)ds &\leqslant \int_0^\tau (\tau-s)^{\alpha-1}\sup_{0\leqslant\sigma\leqslant s}\omega(s)ds\\ &\leqslant \int_0^t (t-s)^{\alpha-1}\sup_{0\leqslant\sigma\leqslant s}\omega(s)ds,\end{aligned}$$

which indicates that

$$\sup_{0\leqslant\tau\leqslant t}\int_0^\tau (\tau-s)^{\alpha-1}\omega(s)ds \leqslant \int_0^t (t-s)^{\alpha-1}\sup_{0\leqslant\sigma\leqslant s}\omega(\sigma)ds.$$

Thus, the Lemma is proved. □

3. Existence Results

In this section, we prove the existence results for problem (1) by using the alternative of Leray–Schauder theorem. Further, our results for the unique solution are based on the Banach contraction principle. Let us start by defining what we mean by a solution of problem (1). Define the space:

$$\Omega' = \{y : (-\infty,b] \to \mathbb{R} : y\mid_{(-\infty,0]} \in \mathscr{B} \text{ and } y\mid_{[0,b]} \text{ is contiunous}\}. \tag{3}$$

It can be easily verified that a function $y \in \Omega'$ is said to be a solution of (1) if y satisfies (1). For the existence results on (1), we need the following Lemma.

Lemma 6. *The solution y of the fractional differential Equation (1) has the following form:*

$$y(t) = aJ^{\alpha-\beta}y(t) + J^\alpha f(t,y_t) + \theta(t), \qquad t \in J = [0,b],$$

where $\theta(t) = c_0\left(\frac{at^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - 1\right)$ is a polynomial type function, and c_0 is a certain constant.

Proof of Lemma 6. The proof is an immediate consequence of the Lemma 1. □

The following assumptions are essential to the results of existence.

Assumption 1. *$f : [0,b] \times \mathscr{B} \to \mathbb{R}$ is continuous, and there exists a bounded set $W_0 \subset \mathscr{B}$ such that $f : [0,b] \times W_0$ uniformly continuous.*

Assumption 2. *There exist function $g, l \in C(J,\mathbb{R}^+)$ such that $|f(t,u)| \leqslant g(t) + l(t)\|u\|_{\mathscr{B}}$ for $t \in J$ and every $u \in \mathscr{B}$.*

Assumption 3. *There exists a nonnegative function $\eta \in L^p[0,b]$ with $p > \frac{1}{\alpha}$ and a continuously non-decreasing function $\Omega : [0,+\infty) \to [0,+\infty)$ such that $|f(t,u)| \leqslant \eta(t)\Omega(\|u\|_{\mathscr{B}})$ for $t \in J$ and every $u \in \mathscr{B}$.*

Assumption 4. *There exists a constant L such that $|f(t,u) - f(t,v)| \leqslant L\|u-v\|_{\mathscr{B}}$ for $t \in J$ and every $u, v \in \mathscr{B}$.*

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Additionally, assume that*

$$\frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{b^{\alpha}K_b}{\Gamma(\alpha+1)}\|l\|<1 \tag{4}$$

holds. Then, the Equation (1) has at least one solution on $(-\infty,b]$.

Proof of Theorem 1. According to the content discussed above, we know that y is a solution to (1) if and only if y satisfies

$$y(t)=\begin{cases} aJ^{\alpha-\beta}y(t)+J^{\alpha}f(t,y_t)+\theta(t), & t\in[0,b],\\ \phi(t), & t\in(-\infty,0]. \end{cases}$$

For any given function $\phi:(-\infty,0]$ that belongs to $\mathscr{B}$, let $\widetilde{\phi}$ be a function defined by

$$\widetilde{\phi}(t)=\begin{cases} \phi(0), & t\in[0,b],\\ \phi(t), & t\in(-\infty,0]. \end{cases}$$

For each $z\in C([0,b],\mathbb{R})$, we denote by $\widetilde{z}$ the function defined by

$$\widetilde{z}(t)=\begin{cases} z(t)-\phi(0), & t\in[0,b],\\ 0, & t\in(-\infty,0]. \end{cases}$$

It can be easily seen that if $y(\cdot)$ satisfies the following integral equation

$$y(t)=aJ^{\alpha-\beta}y(t)+J^{\alpha}f(t,y_t)+\theta(t),$$

we can decompose $y(\cdot)$ as $y(t)=\widetilde{\phi}(t)+\widetilde{z}(t)$, $t\in[0,b]$, which implies that $y_t=\widetilde{\phi}_t+\widetilde{z}_t$, for every $t\in[0,b]$, and the function $z(\cdot)$ satisfies

$$z(t)=aJ^{\alpha-\beta}z(t)+J^{\alpha}f(t,\widetilde{z}_t+\widetilde{\phi}_t)+\theta(t).$$

Set $C_0=\{z\in C\big([0,b],\mathbb{R}\big):z(0)=\phi(0)\}$. Then C_0 is closed, and hence completed. Define an operator $P:C_0\to C_0$ by

$$(Pz)(t)=aJ^{\alpha-\beta}z(t)+J^{\alpha}f(t,\widetilde{z}_t+\widetilde{\phi}_t)+\theta(t). \tag{5}$$

where $t\in[0,b]$. According to the Schauder's fixed point theorem, we show that the operator P is continuous and completely continuous in the following four steps.

Step 1. P is continuous.

Let $\{z_n\}$ be a sequence such that $z_n\to z$ in C_0. Then, we have for each $t\in[0,b]$

$$\begin{aligned}|Pz_n(t)-Pz(t)|\leqslant{}&\frac{|a|}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}|z_n(s)-z(s)|ds\\&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,\widetilde{(z_n)}_s+\widetilde{\phi}_s)-f(s,\widetilde{z}_s+\widetilde{\phi}_s)|ds.\end{aligned}$$

Set $W_0=\{(z_n)_s:s\in[0,b],\ n\geqslant1\}\subset\mathscr{B}$. It can be easily known from Assumption 1 that function f is uniformly continuous in $s\in[0,t]$, which implies that $\forall\varepsilon>0$, $\exists\delta>0$, $s.t\ \forall z_1,z_2\in W_0$, $|z_1-z_2|<\delta$, we have $|f(s,z_1)-f(s,z_2)|<\varepsilon$. Since $z_n\to z$, then $\exists N>0$, $s.t\ \forall n>N$, we have $|z_n-z|<\delta$. Hence, for any $s\in[0,t]$, we can claim that $|f(s,z_n)-f(s,z)|<\varepsilon$. According to the definition $z(t)=\widetilde{z}(t)+\widetilde{\phi}(t)$ introduced above, it follows that $|f(s,\widetilde{(z_n)}_s+\widetilde{\phi}_s)-f(s,\widetilde{z}_s+\widetilde{\phi}_s)|<\varepsilon$, so we get

$$|Pz_n(t)-Pz(t)|\leqslant\frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|z_n-z\|+\frac{b^{\alpha}}{\Gamma(\alpha+1)}\|f(s,\widetilde{(z_n)}_s+\widetilde{\phi}_s)-f(s,\widetilde{z}_s+\widetilde{\phi}_s)\|.$$

Hence, $|Pz_n(t) - Pz(t)| \to 0$ as $z_n \to z$, and P is continuous.

Step 2. P maps bounded sets into bounded sets in C_0.

Indeed, it is enough to show that for any $r > 0$ there exists a positive constant ξ such that for each $z \in B_r = \{z \in C_0 : \|z\| \leqslant r\}$ one has $\|Pz(t)\| \leqslant \xi$. Let $z \in B_r$. Since f is a continuous function, we have for each $t \in [0, b]$

$$\begin{aligned}
|Pz(t)| &\leqslant \frac{|a|}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} |z(s)| ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, \widetilde{z}_s + \widetilde{\phi}_s)| ds + |\theta(b)| \\
&\leqslant \frac{|a| b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \|z\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(g(s) + l(s) \|\widetilde{z}_s + \widetilde{\phi}_s\|_{\mathscr{B}}\big) ds + |\theta(b)|.
\end{aligned}$$

According to Definition 3, we can conclude that

$$\begin{aligned}
\|\widetilde{z}_s + \widetilde{\phi}_s\|_{\mathscr{B}} &\leqslant \|\widetilde{z}_s\|_{\mathscr{B}} + \|\widetilde{\phi}_s\|_{\mathscr{B}} \\
&\leqslant K(s) \sup_{0 \leqslant \tau \leqslant s} \|\widetilde{z}(\tau)\| + M(s) \|\widetilde{z}_0\|_{\mathscr{B}} + K(s) \sup_{0 \leqslant \tau \leqslant s} \|\widetilde{\phi}(\tau)\| + M(s) \|\widetilde{\phi}_0\|_{\mathscr{B}} \\
&\leqslant K_b \sup_{0 \leqslant \tau \leqslant s} \|z(\tau) - \phi(0)\| + K_b \|\phi(0)\| + M_b \|\phi\|_{\mathscr{B}} \\
&\leqslant K_b r + K_b \|\phi(0)\| + K_b \|\phi(0)\| + M_b \|\phi\|_{\mathscr{B}} \\
&\leqslant K_b r + 2K_b H \|\phi\|_{\mathscr{B}} + M_b \|\phi\|_{\mathscr{B}} \\
&= K_b r + \big(2K_b H + M_b\big) \|\phi\|_{\mathscr{B}} \\
&:= r_0,
\end{aligned}$$

where $M_b = \sup\{|M(t)| \colon t \in [a, b]\}$, $K_b = \sup\{|K(t)| \colon t \in [a, b]\}$ and H is a positive constant. So we have

$$|Pz(t)| \leqslant \frac{|a| b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} r + \frac{b^\alpha \|g\|}{\Gamma(\alpha+1)} + \frac{b^\alpha \|l\|}{\Gamma(\alpha+1)} \Big(K_b r + \big(2K_b H + M_b\big) \|\phi\|_{\mathscr{B}}\Big) + |\theta(b)| := \xi.$$

Hence, $|Pz(t)| \leqslant \xi$, which implies P maps bounded subsets into bounded subsets in C_0.

Step 3. P maps bounded sets into equicontinuous sets of C_0.

Let $t_1, t_2 \in [0, b],\ t_1 < t_2$, and B_r be a bounded set of C_0 as in Step 2. Let $z \in B_r$. Then, for each $t \in [0, b]$, we have

$$\begin{aligned}
&|(Pz)(t_2) - (Pz)(t_1)| \\
&\leqslant \frac{|a| \|z\|}{\Gamma(\alpha-\beta)} \left| \int_0^{t_1} \Big((t_2-s)^{\alpha-\beta-1} - (t_1-s)^{\alpha-\beta-1}\Big) ds + \int_{t_1}^{t_2} (t-s)^{\alpha-\beta-1} ds \right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \Big((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\Big) f(s, \widetilde{z}_s + \widetilde{\phi}_s) ds \right. \\
&\quad \left. + \int_{t_1}^{t_2} (t-s)^{\alpha-\beta-1} f(s, \widetilde{z}_s + \widetilde{\phi}_s) ds \right| + |\theta(t_2) - \theta(t_1)| \\
&\leqslant \frac{|a| \|z\|}{\Gamma(\alpha-\beta+1)} \Big({t_2}^{\alpha-\beta} - {t_1}^{\alpha-\beta} + 2(t_2-t_1)^{\alpha-\beta}\Big) \\
&\quad + \frac{\|g\| + \|l\| r_0}{\Gamma(\alpha+1)} ({t_2}^\alpha - {t_1}^\alpha + 2(t_2-t_1)^\alpha) + |\theta(t_2) - \theta(t_1)|.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero, and the equicontinuity for the cases that $t_1 < t_2 \leqslant 0$ and $t_1 \leqslant 0 \leqslant t_2$ is obvious.

As a consequence of Steps 1–3, together with the Arzela–Ascoli theorem, we can conclude that $P : C_0 \to C_0$ is a completely continuous mapping.

Step 4. (A priori bounds). There exists an open set $U \subseteq C_0$ with $z \neq \lambda P(z)$ for $\lambda \in (0,1)$ and $z \in \partial U$.

According to the condition $\frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{b^{\alpha}K_b}{\Gamma(\alpha+1)}\|l\| < 1$, we can deduce that there exists a constant $N > 0$ such that

$$\frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}N + \frac{b^{\alpha}K_b}{\Gamma(\alpha+1)}\|l\|N + \frac{b^{\alpha}}{\Gamma(\alpha+1)}\Big(\|g\| + \|l\|(2K_bH + M_b)\|\phi\|_{\mathscr{B}}\Big) + |\theta(b)| < N.$$

Define the set $\mathcal{E} = \{z \in C_0 : \|z\| < N\}$. Thus, the operator $P : \overline{\mathcal{E}} \to C_0$ satisfies the complete continuity. Assume the equation

$$z = \lambda Pz$$

holds for some $z \in \overline{\mathcal{E}}$ and $\lambda \in (0,1)$. Then, we obtain

$$\begin{aligned}|z(t)| &= |\lambda Pz(t)| \leqslant |Pz(t)| \\ &\leqslant \frac{|a|b^{\alpha-\beta}\|z\|}{\Gamma(\alpha-\beta+1)} + \frac{b^{\alpha}\|g\|}{\Gamma(\alpha+1)} + \frac{b^{\alpha}\|l\|}{\Gamma(\alpha+1)}\Big(K_b\|z\| + (2K_bH + M_b)\|\phi\|_{\mathscr{B}}\Big) + |\theta(b)|.\end{aligned}$$

Hence, the following inequality

$$\begin{aligned}\|z\| \leqslant{}& \frac{|a|b^{\alpha-\beta}\|z\|}{\Gamma(\alpha-\beta+1)} + \frac{b^{\alpha}K_b\|l\|\|z\|}{\Gamma(\alpha+1)} + \frac{b^{\alpha}}{\Gamma(\alpha+1)}\Big(\|g\| + \|l\|(2K_bH + M_b)\|\phi\|_{\mathscr{B}}\Big) + |\theta(b)| \\ &< N\end{aligned}$$

holds, which contradicts to $N = \|z\|$. Thus, we get

$$z \neq \lambda Pz$$

for any $z \in \overline{\mathcal{E}}$ and λ. By the Leray–Schauder alternative, we infer that there exists at least one fixed point z of P, and $y = \widetilde{z} + \widetilde{\phi}$ is a solution to problem (1). The proof is thus complete. □

Remark 1. *In infinite dimensional space, continuous functions are not uniformly continuous in a bounded closed region. In order to verify the continuity of the operator P in the step 1, we give Assumption 1. The conclusion of continuity of the map P can be directly obtained by using the Lebesgue Dominated Convergence Theorem.*

Theorem 2. *Suppose that Assumptions 1 and 3 hold. Additionally, assume that*

$$\frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{b^{(\alpha-1)q+1}\|\eta\|_p}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}} \limsup_{r\to\infty} \frac{\Omega(r)}{r} < 1 \tag{6}$$

holds. Then, the Equation (1) has at least one solution on $(-\infty, b]$.

Proof of Theorem 2. Let $P : C_0 \to C_0$ be defined as in (5). The conclusion can be verified analogously in the following four steps as well.

Step 1. P is continuous.

Similar to the proof of Theorem 1, it is not difficult to verify that P is continuous by Assumption 3 and the Lebesgue dominated convergence theorem.

Step 2. P maps bounded sets into bounded sets in C_0.

Let $B_r = \{z \in C_0 : \|z\| \leqslant r\}$. Then, for any $z \in B_r$ and $t \in [0, b]$, we have

$$\begin{aligned}|Pz(t)| &\leqslant \frac{|a|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|z(s)|ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,\widetilde{z}_s+\widetilde{\phi}_s)|ds + |\theta(b)| \\ &\leqslant \frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|z\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\Omega(\|\widetilde{z}_s+\widetilde{\phi}_s\|_{\mathscr{B}})ds + |\theta(b)|.\end{aligned}$$

Since

$$\|\widetilde{z}_s+\widetilde{\phi}_s\|_{\mathscr{B}} \leqslant \|\widetilde{z}_s\|_{\mathscr{B}} + \|\widetilde{\phi}_s\|_{\mathscr{B}} \leqslant K_b r + \big(2K_bH + M_b\big)\|\phi\|_{\mathscr{B}} := r_0,$$

where $M_b = \sup\{|M(t)|: t \in [a,b]\}$, $K_b = \sup\{|K(t)|: t \in [a,b]\}$ and H is a positive constant. It follows from Holder's inequality and Assumption 3 that

$$\begin{aligned}|Pz(t)| &\leqslant |\theta(b)| + \frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|z\| \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)ds\,\Omega\Big(K_b r + \big(2K_bH+M_b\big)\|\phi\|_{\mathscr{B}}\Big) \\ &\leqslant |\theta(b)| + \frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|z\| \\ &\quad + \frac{1}{\Gamma(\alpha)}\Omega\Big(K_b r + \big(2K_bH+M_b\big)\|\phi\|_{\mathscr{B}}\Big)\Big(\int_0^t (t-s)^{(\alpha-1)q}\Big)^{\frac{1}{q}}\|\eta\|_p \\ &\leqslant |\theta(b)| + \frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|z\| \\ &\quad + \Omega\Big(K_b r + \big(2K_bH+M_b\big)\|\phi\|_{\mathscr{B}}\Big)\frac{b^{(\alpha-1)q+1}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\|\eta\|_p \\ &:= \xi,\end{aligned}$$

where $\|\eta\|_p = (\int_0^b |\eta(s)|^p ds)^{\frac{1}{p}}$ and $\frac{1}{p} + \frac{1}{q} = 1$, $(\alpha-1) > -1$. Therefore, $\|Pz\| \leqslant \xi$ for every $z \in B_r$, which implies that P maps bounded subsets into bounded subsets in C_0.

Step 3. P maps bounded sets into equicontinuous sets of C_0.

Let $t_1, t_2 \in [0,b]$, $t_1 < t_2$, and let B_r be a bounded set of C_0 as in the Step 2. Let $z \in B_r$. Then for each $t \in [0,b]$, we have

$$\begin{aligned}&|(Pz)(t_2) - (Pz)(t_1)| \\ &\leqslant \frac{|a|\|z\|}{\Gamma(\alpha-\beta)}\left|\int_0^{t_1}\Big((t_2-s)^{\alpha-\beta-1} - (t_1-s)^{\alpha-\beta-1}\Big)ds + \int_{t_1}^{t_2}(t-s)^{\alpha-\beta-1}ds\right| \\ &\quad + \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\Big((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\Big)f(s,\widetilde{z}_s+\widetilde{\phi}_s)ds\right. \\ &\quad \left. + \int_{t_1}^{t_2}(t-s)^{\alpha-\beta-1}f(s,\widetilde{z}_s+\widetilde{\phi}_s)ds\right| + |\theta(t_2)-\theta(t_1)| \\ &\leqslant \frac{|a|\|z\|}{\Gamma(\alpha-\beta+1)}\Big({t_2}^{\alpha-\beta} - {t_1}^{\alpha-\beta} + 2(t_2-t_1)^{\alpha-\beta}\Big) \\ &\quad + \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\Big((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\Big)\eta(s)\Omega(\|\widetilde{z}_s+\widetilde{\phi}_s\|_{\mathscr{B}})ds\right. \\ &\quad \left. + \int_{t_1}^{t_2}(t-s)^{\alpha-\beta-1}\eta(s)\Omega(\|\widetilde{z}_s+\widetilde{\phi}_s\|_{\mathscr{B}})ds\right| + |\theta(t_2)-\theta(t_1)| \\ &\leqslant \frac{|a|\|z\|}{\Gamma(\alpha-\beta+1)}\Big({t_2}^{\alpha-\beta} - {t_1}^{\alpha-\beta} + 2(t_2-t_1)^{\alpha-\beta}\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Omega(r_0)}{\Gamma(\alpha)}\left(\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})^q ds\right)^{\frac{1}{q}}\left(\int_0^{t_1}\eta^p(s)ds\right)^{\frac{1}{p}}\\
&+\frac{\Omega(r_0)}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-1)q}ds\right)^{\frac{1}{q}}\left(\int_{t_1}^{t_2}\eta^p(s)ds\right)^{\frac{1}{p}}+|\theta(t_2)-\theta(t_1)|\\
\leqslant{}&\frac{|a|\|z\|}{\Gamma(\alpha-\beta+1)}\left({t_2}^{\alpha-\beta}-{t_1}^{\alpha-\beta}+2(t_2-t_1)^{\alpha-\beta}\right)\\
&+\frac{\Omega(r_0)\|\eta\|_p}{\Gamma(\alpha)r_1}\left({t_2}^{r_2}-{t_1}^{r_2}+2(t_2-t_1)^{r_2}\right)+|\theta(t_2)-\theta(t_1)|,
\end{aligned}
$$

where $r_0=K_b r+\big(2K_bH+M_b\big)\|\phi\|_{\mathscr{B}}$, $r_1=(1+(\alpha-1)q)^{\frac{1}{q}}$, $r_2=[(\alpha-1)q+1]/q>0$. As $t_1\to t_2$ the right-hand side of the above inequality tends to zero, and the equicontinuity for the cases that $t_1<t_2\leqslant 0$ and $t_1\leqslant 0\leqslant t_2$ is obvious.

As a consequence of Steps 1–3, together with the Arzela–Ascoli theorem, we can conclude that $P: C_0\to C_0$ is a completely continuous mapping.

Step 4. (A priori bounds). There exists an open set $U\subseteq C_0$ with $z\neq\lambda P(z)$ for $\lambda\in(0,1)$ and $z\in\partial U$.

According to the condition $\frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{b^{(\alpha-1)q+1}\|\eta\|_p}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\lim_{r\to\infty}\sup\frac{\Omega(r)}{r}<1$, we can deduce that there exists a constant $N>0$ such that

$$
\frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}N+\frac{b^{(\alpha-1)q+1}\|\eta\|_p}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\Omega(N)+|\theta(b)|<N.
$$

Define the set $\mathcal{E}=\{z\in C_0:\|z\|<N\}$. So the operator $P:\overline{\mathcal{E}}\to C_0$ satisfies the complete continuity. Assume the equation

$$
z=\lambda Pz
$$

holds for some $z\in\overline{\mathcal{E}}$ and $\lambda\in(0,1)$. Then we obtain

$$
\begin{aligned}
|z(t)|&=|\lambda Pz(t)|\leqslant|Pz(t)|\\
&\leqslant\frac{|a|b^{\alpha-\beta}\|z\|}{\Gamma(\alpha-\beta+1)}+\frac{b^{(\alpha-1)q+1}\|\eta\|_p}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\Omega\Big(K_b\|z\|+\big(2K_bH+M_b\big)\|\phi\|_{\mathscr{B}}\Big)+|\theta(b)|\\
&<N.
\end{aligned}
$$

Hence, the following inequality

$$
\|z\|<N
$$

holds, which contradicts $N=\|z\|$. Thus, we get

$$
z\neq\lambda Pz
$$

for any $z\in\overline{\mathcal{E}}$ and λ. By the Leray–Schauder alternative, we infer that there exists at least one fixed point z of P, and $y=\widetilde{z}+\widetilde{\phi}$ is a solution to problem (1). The proof is thus complete. □

Theorem 3. *Suppose that Assumptions 1 and 4 hold. Additionally, assume that*

$$
0<\frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{Lb^{\alpha}K_b}{\Gamma(\alpha+1)}<1 \tag{7}
$$

holds, Then, Equation (1) has a unique solution on $(-\infty,b]$.

Proof of Theorem 3. Let $P : C_0 \to C_0$ be defined as in (5). The operator P has a fixed point. which is equivalent to Equation (1) having a unique solution, and we turn to proving that P has a fixed point. We shall show that $P : C_0 \to C_0$ is a contraction map. Indeed, consider any $u, v \in C_0$. Then for each $t \in [0, b]$, we have

$$
\begin{aligned}
|(Pu)(t) - (Pv)(t)| &\leqslant \frac{|a|}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} |u(s) - v(s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, \widetilde{u}_s + \widetilde{\phi}_s) - f(s, \widetilde{v}_s + \widetilde{\phi}_s)| ds \\
&\leqslant \frac{|a| b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \|u - v\| + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\widetilde{u}(s) - \widetilde{v}(s)\|_{\mathscr{B}} ds.
\end{aligned}
$$

Since

$$
\begin{aligned}
\|\widetilde{u}(s) - \widetilde{v}(s)\|_{\mathscr{B}} &\leqslant K(s) \sup_{0 \leqslant \tau \leqslant s} \|\tilde{u}(\tau) - \tilde{v}(\tau)\| + M(s) \|\tilde{u}_0 - \tilde{v}_0\|_{\mathscr{B}} \\
&\leqslant K_b \sup_{0 \leqslant \tau \leqslant s} \|u(\tau) - \phi(0) - v(\tau) + \phi(0)\| \\
&\leqslant K_b \|u - v\|,
\end{aligned}
$$

where $K_b = \sup\{|K(t)|: t \in [a, b]\}$, we get

$$
\|Pu - Pv\| \leqslant \left(\frac{|a| b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{L b^{\alpha} K_b}{\Gamma(\alpha+1)} \right) \|u - v\|,
$$

and P is a contraction. Therefore, P has a unique fixed point by applying the Banach contraction principle. □

4. Stability Analysis

In this section, the analysis of Hyers–Ulam stability of the fractional differential Equation (1) with infinite delay is presented. First and foremost, the definition given below is crucial to the proof of Hyers–Ulam stability.

Definition 4. *The problem (1) is said to be Hyers–Ulam stable if there exists a positive real number c such that for each $\varepsilon > 0$ and for each solution $u(\cdot)$ of the inequalities*

$$
\begin{cases}
\left| {}_cD^{\alpha} u(t) - a_c D^{\beta} u(t) = f(t, u_t) \right| \leqslant \varepsilon, & t \in J = [0, b], \\
u(t) = \phi(t), & t \in (-\infty, 0],
\end{cases} \tag{8}
$$

there exists a solution $v(\cdot)$ of the problem (1) with

$$
|u(t) - v(t)| \leqslant c\varepsilon, \quad t \in J = [0, b].
$$

Theorem 4. *Further, assume that the conditions of Theorem 3 are satisfied and the inequality (8) has at least one solution. Then, the problem (1) is Hyers–Ulam stable.*

Proof of Theorem 4. For each $\varepsilon > 0$, and each function u that satisfies the following inequalities

$$
\left| {}_cD^{\alpha} u(t) - a_c D^{\beta} u(t) - f(t, u_t) \right| \leqslant \varepsilon, \quad t \in [0, b],
$$

a function $g(t) = {}_cD^{\alpha} u(t) - a_c D^{\beta} u(t) - f(t, u_t)$ can be found; then, we have $|g(t)| \leqslant \varepsilon$, which implies that

$$
u(t) = \theta(t) + a J^{\alpha-\beta} u(t) + J^{\alpha} f(t, u_t) + J^{\alpha} g(t),
$$

where $\theta(t)$ is a polynomial function which is given in Lemma 6. According to Theorem 3, it has been verified that there is a unique solution $v(t)$ of problem (1), then function v can be expressed as

$$v(t) = \theta(t) + aJ^{\alpha-\beta}v(t) + J^{\alpha}f(t, v_t),$$

so we have

$$\begin{aligned} |u(t) - v(t)| \leqslant{}& \frac{|a|}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1}|u(s) - v(s)|ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|f(s,u_s) - f(s,v_s)|ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|g(s)|ds. \end{aligned}$$

Since

$$|f(s,u_s) - f(s,v_s)| \leqslant L\|u_s - v_s\|_{\mathscr{B}},$$

together with Definition 3, we get

$$\begin{aligned} \|u_s - v_s\|_{\mathscr{B}} &= \|(\widetilde{u}_s + \widetilde{\phi}_s) - (\widetilde{v}_s + \widetilde{\phi}_s)\|_{\mathscr{B}} = \|\widetilde{u}_s - \widetilde{v}_s\|_{\mathscr{B}} \\ &\leqslant K(s) \sup_{0\leqslant\tau\leqslant s} \|\tilde{u}(\tau) - \tilde{v}(\tau)\| + M(s)\|\tilde{u}_0 - \tilde{v}_0\|_{\mathscr{B}} \\ &\leqslant K_b \sup_{0\leqslant\tau\leqslant s} \|u(\tau) - \phi(0) - v(\tau) + \phi(0)\| \\ &= K_b \sup_{0\leqslant\tau\leqslant s} |u(\tau) - v(\tau)|, \end{aligned}$$

where $K_b = \sup\{|K(t)|: t \in [a,b]\}$, it indicates that

$$\begin{aligned} |u(t) - v(t)| \leqslant{}& \frac{|a|}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1}|u(s) - v(s)|ds \\ &+ \frac{LK_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{0\leqslant\sigma\leqslant s} |u(\sigma) - v(\sigma)|ds + \frac{b^{\alpha}}{\Gamma(\alpha+1)}\varepsilon. \end{aligned}$$

According to Lemma 5, it immediately follows that

$$\begin{aligned} \sup_{0\leqslant\tau\leqslant t} |u(\tau) - v(\tau)| \leqslant{}& \frac{|a|}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \sup_{0\leqslant\sigma\leqslant s} |u(\sigma) - v(\sigma)|ds \\ &+ \frac{LK_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{0\leqslant\sigma\leqslant s} |u(\sigma) - v(\sigma)|ds + \frac{b^{\alpha}}{\Gamma(\alpha+1)}\varepsilon \\ ={}& \int_0^t \left[|a|\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + LK_b\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right] \sup_{0\leqslant\sigma\leqslant s} |u(\sigma) - v(\sigma)|ds \\ &+ \frac{b^{\alpha}}{\Gamma(\alpha+1)}\varepsilon, \end{aligned}$$

let $\varphi(t) := \sup_{0\leqslant\tau\leqslant t} |u(\tau) - v(\tau)|$, $M := \frac{b^{\alpha}}{\Gamma(\alpha+1)}$, and $g(t,s) := |a|\frac{1}{\Gamma(\alpha-\beta)} + LK_b\frac{(t-s)^{\beta}}{\Gamma(\alpha)}$, we can get

$$\varphi(t) \leqslant M\varepsilon + \int_0^t g(t,s)(t-s)^{\alpha-\beta-1}\varphi(s)ds.$$

It is not difficult to note that $g(t,s) \leqslant |a|\frac{1}{\Gamma(\alpha-\beta)} + LK_b\frac{b^{\beta}}{\Gamma(\alpha)} (:= M_0)$. Hence, in view of Lemma 3,

$$\begin{aligned} \varphi(t) &\leqslant M\varepsilon + M\varepsilon \int_0^t \sum_{n=1}^{\infty} \frac{(g(t,s)\Gamma(\alpha-\beta))^n}{\Gamma(n(\alpha-\beta))}(t-s)^{n(\alpha-\beta)-1}ds \\ &\leqslant M\varepsilon + M\varepsilon \int_0^t \sum_{n=1}^{\infty} \frac{(M_0\Gamma(\alpha-\beta))^n}{\Gamma(n(\alpha-\beta))}(t-s)^{n(\alpha-\beta)-1}ds \end{aligned}$$

$$\leqslant M\varepsilon + M\varepsilon \sum_{n=1}^{\infty} \frac{(M_0\Gamma(\alpha-\beta))^n}{\Gamma(n(\alpha-\beta)+1)} b^{n(\alpha-\beta)}$$
$$\leqslant M\varepsilon E_{\alpha-\beta}\big(M_0 b^{(\alpha-\beta)}\Gamma(\alpha-\beta)\big),$$

let $c := ME_{\alpha-\beta}\big(M_0 b^{(\alpha-\beta)}\Gamma(\alpha-\beta)\big)$, then the inequality

$$\varphi(t) \leqslant c\varepsilon$$

holds, which implies that Hyers-Ulam stability of problem (1) is proved. □

5. Examples

Two examples are presented in this section to illustrate the conclusions. To begin with, let $\gamma > 0$ be a real constant and

$$E_\gamma = \{y \in C\big((-\infty,0],\mathbb{R}\big) : \lim_{\theta\to-\infty} e^{\gamma\theta} y(\theta) \ exists\ in\ \mathbb{R}\}.$$

Accordingly, the norm of E_γ is given by

$$|y|_\gamma = \sup_{-\infty<\theta\leqslant 0} e^{\gamma\theta}|y(\theta)|.$$

By [28], E_γ satisfies the conditions in Definition 3 with $K = M = H = 1$. It can be easily claimed that E_γ is a phase space.

Example 1. *Consider the following nonlinear Caputo-type fractional differential equation with infinite delay of the form*

$$ {}_cD^{0.8}y(t) - \frac{1}{2}{}_cD^{0.4}y(t) = \frac{e^{-\gamma t}}{10}(|y_t| + \frac{1}{2}cost), \quad t \in J = [0,1], \tag{9}$$
$$y(t) = \phi(t) \in E_\gamma, \quad t \in (-\infty,0]. \tag{10}$$

According to the given data, it can be easily found that Assumptions 1 and 2 are satisfied with function $l(t) = \frac{e^{-\gamma t}}{10}$. *Furthermore, we have* $\frac{|a|b^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{b^\alpha K_b}{\Gamma(\alpha+1)}\|l\| < \frac{1}{2\Gamma(1.4)} + \frac{1}{10\Gamma(1.8)} \approx 0.6707 < 1$. *Therefore, all the conditions of Theorem 1 hold true, and consequently the problems (9) and (10) with* $f(t,y_t)$ *given by the equation* $f(t,y_t) = \frac{e^{-\gamma t}}{10}(|y_t| + \frac{1}{2}cost)$ *have at least one solution on* $(-\infty,1]$.

Example 2. *We can investigate the following nonlinear delayed fractional differential equation*

$$ {}_cD^{0.7}y(t) - \frac{1}{3}{}_cD^{0.5}y(t) = \frac{\widetilde{c}e^{-\gamma t+t}|y|_\gamma}{(e^t+e^{-t})(1+|y|_\gamma)}, \quad t \in J = [0,1], \tag{11}$$
$$y(t) = \phi(t) \in E_\gamma, \quad t \in (-\infty,0], \tag{12}$$

where $\widetilde{c}$ *is a given positive constant. Set*

$$f(t,x) = \frac{e^{-\gamma t+t}x}{\widetilde{c}(e^t+e^{-t})(1+x)}, \ (t,x) \in [0,1]\times\mathbb{R}^+.$$

Then, for any $x, y \in E_\gamma$, *we have*

$$|f(t,x) - f(t,y)| = \frac{e^{-\gamma t+t}}{\widetilde{c}(e^t+e^{-t})}\Big|\frac{x}{1+x} - \frac{y}{1+y}\Big|$$
$$\leqslant \frac{e^{-\gamma t+t}|x-y|}{\widetilde{c}(e^t+e^{-t})(1+x)(1+y)}$$

$$\leqslant \frac{e^t|x-y|}{\widetilde{c}(e^t+e^{-t})}$$
$$\leqslant \frac{1}{\widetilde{c}}|x-y|.$$

Hence, the condition Assumption 4 holds. Since $K=1$, assume that $\widetilde{c} > \frac{3\Gamma(1.2)}{\Gamma(1.7)(3\Gamma(1.2)-1)} \approx 1.7269$, and Equation (7) holds. Thus, it can be verified that problems (11) and (12) have a unique solution on $(-\infty, 1]$ by applying Theorem 3.

On the basis of the conclusions, we further discuss the Hyers–Ulam stability of problem (11) and (12). For any $\varepsilon > 0$ and each function y that satisfies the following inequalities

$$\left|{}_cD^{0.7}y(t) - \frac{1}{3}{}_cD^{0.5}y(t) - \frac{\widetilde{c}e^{-\gamma t+t}|y|_\gamma}{(e^t+e^{-t})(1+|y|_\gamma)}\right| \leqslant \varepsilon, \quad t \in J = [0,1],$$

let $g(t)$ represent the right side of the inequality above. Additionally, let $x(t)$ be the unique solution of problem (11) and (12); then, we have

$$\sup_{0\leqslant\tau\leqslant t}|y(\tau)-x(\tau)| \leqslant \int_0^t \left(\frac{(t-s)^{-0.8}}{3\Gamma(0.2)} + \frac{(t-s)^{-0.3}}{\widetilde{c}\Gamma(0.7)}\right)\sup_{0\leqslant\sigma\leqslant s}|y(\sigma)-x(\sigma)|ds + \frac{1}{\Gamma(1.7)}\varepsilon,$$

let $\varphi(t) := \sup_{0\leqslant\tau\leqslant t}|y(\tau)-x(\tau)|$, $g(t,s) := \frac{1}{3\Gamma(0.2)} + \frac{(t-s)^{0.5}}{\widetilde{c}\Gamma(0.7)}$ and $M := \frac{1}{\Gamma(1.7)}$, then it is easy to get that $g(t,s) \leqslant \frac{1}{3\Gamma(0.2)} + \frac{1}{\widetilde{c}\Gamma(0.7)}(:= M_0)$, and in view of Lemma 3,

$$\begin{aligned}
\varphi(t) &\leqslant M\varepsilon + \int_0^t g(t,s)(t-s)^{-0.8}\varphi(s)ds \\
&\leqslant M\varepsilon + M\varepsilon\int_0^t \sum_{n=1}^{\infty} \frac{(g(t,s)\Gamma(0.2))^n}{\Gamma(0.2n)}(t-s)^{0.2n-1}ds \\
&\leqslant M\varepsilon + M\varepsilon\int_0^t \sum_{n=1}^{\infty} \frac{(M_0\Gamma(\alpha-\beta))^n}{\Gamma(0.2n)}(t-s)^{0.2n-1}ds \\
&\leqslant M\varepsilon + M\varepsilon\sum_{n=1}^{\infty} \frac{(M_0\Gamma(0.2))^n}{\Gamma(0.2n+1)} \\
&\leqslant M\varepsilon E_{0.2}\big(M_0\Gamma(0.2)\big),
\end{aligned}$$

let $c := ME_{0.2}\big(M_0\Gamma(0.2)\big) = \frac{1}{\Gamma(1.7)}E_{0.2}\left(\frac{1}{3} + \frac{\Gamma(0.2)}{\widetilde{c}\Gamma(0.7)}\right)$, it follows that $\varphi(t) \leqslant c\varepsilon$, which implies that the problem (11) and (12) is Hyers-Ulam stable.

6. Conclusions

This paper mainly discusses and investigates a class of nonlinear fractional differential equations with infinite time delay. Based on the properties of Green's function, we give the form of a solution to the differential equations. In addition to applying the fixed point theorem and Gronwall inequality, the related properties of the phase space are explored to investigate the nature and Hyers–Ulam stability of the solutions of fractional order differential equations under time delay conditions. Generally, various types of Gronwall inequalities can be utilized to explore the stability of fractional differential equations. However, we have found that only applying Gronwall inequalities is not enough to get stability conclusions in this paper. Therefore, we prove a comparative property of fractional calculus as an auxiliary tool to verify the stability of solutions. Furthermore, two examples are listed to confirm the conclusions.

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