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Future Failure Time Prediction Based on a Unified Hybrid Censoring Scheme for the Burr-X Model with Engineering Applications

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Abstract: Industries are constantly seeking ways to avoid corrective maintenance in order to reduce costs. Performing regular scheduled maintenance can help to mitigate this problem, but not necessarily in the most efficient way. In many real life applications, one wants to predict the future failure time of equipment or devices that are expensive, or with long lifetimes, to save costs and/or time. In this paper, statistical prediction was studied using the classical and Bayesian approaches based on a unified hybrid censoring scheme. Two prediction schemes were used: (1) a one-sample prediction scheme that predicted the unobserved future failure times of devices that did not complete the lifetime experiments; and (2) a two-sample prediction scheme to predict the ordered values of a future independent sample based on past data from a certain distribution. We chose to apply the results of the paper to the Burr-X model, due to the importance of this model in many fields, such as engineering, health, agriculture, and biology. Point and interval predictors of unobserved failure times under one- and two-sample prediction schemes were computed based on simulated data sets and two engineering applications . The results demonstrate the ability of predicting the future failure of equipment using a statistical prediction branch based on collected data from an engineering system.

Keywords: Burr-X distribution; maximum likelihood prediction; Bayesian prediction; one and two-sample prediction schemes; unified hybrid censoring; Gibbs sampler and Metropolis–Hastings algorithm

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1. Introduction

Industries are constantly seeking ways to avoid corrective maintenance in order to reduce costs. Performing regular scheduled maintenance can help to mitigate this problem, but not necessarily in the most efficient way, see [1–3]. In condition-based maintenance, the main goal is to come up with ways to treat and transform data from an engineering system, so that it can be used to build a data set to make statistical predictions about how the equipment will act in the future and when it will break down.

In many practical situations, one desires to predict future observations from the same population of previous data. This may be done by constructing an interval that will include future observations with a certain probability.

Predictive interval accuracy depends on sample size; full testing is impractical in real testing, owing to the advancement of industrial design and technology, which results in very reliable products with long lifespans. Censoring has been implemented in this case for a variety of reasons, including a lack of available resources and the need to save costs. In general, only a small percentage of failure times are recorded when a CS is engaged in a test environment.

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the ordered failure times of n identical units placed on a life-test, from a certain distribution with PDF, $f(x;\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the vector of parameters and RF, $R(x;\boldsymbol{\theta})$. For fixed $k, r \in \{1, 2, \ldots, n\}$ and $T_1 < T_2 \in (0, \infty)$ with $k < r$ and upon the relation between T_1, T_2, X_k and X_r, an $UHCS$ is defined by Balakrishnan with six decisions, as follows:

(1) Stopping the experiment at T_1 if $0 < X_{k:n} < X_{r:n} < T_1 < T_2$;
(2) Stopping the experiment at $X_{r:n}$ if $0 < X_{k:n} < T_1 < X_{r:n} < T_2$;
(3) Stopping the experiment at T_2 if $0 < X_{k:n} < T_1 < T_2 < X_{r:n}$;
(4) Stopping the experiment at $X_{r:n}$ if $0 < T_1 < X_{k:n} < X_{r:n} < T_2$;
(5) Stopping the experiment at T_2 if $0 < T_1 < X_{k:n} < T_2 < X_{r:n}$;
(6) Stopping the experiment at $X_{k:n}$ if $0 < T_1 < T_2 < X_{k:n} < X_{r:n}$.

Let d_i denote the number of failures until time T_i, $i = 1, 2$. Then, the LF of this $UHCS$ censored sample is as follows:

$$L(\boldsymbol{\theta}; data) = \begin{cases} \frac{n!}{(n-d)!}[\prod_{i=1}^{d} f(x_i;\boldsymbol{\theta})][R(T_1;\boldsymbol{\theta})]^{(n-d)}, d_1 = d_2 = d = r, \ldots, n, \\ \frac{n!}{(n-r)!}[\prod_{i=1}^{r} f(x_i;\boldsymbol{\theta})][R(x_r;\boldsymbol{\theta})]^{(n-r)}, d_1 = k, \ldots, r-1,\ d_2 = r, \\ \frac{n!}{(n-d_2)!}[\prod_{i=1}^{d_2} f(x_i;\boldsymbol{\theta})][R(T_2;\boldsymbol{\theta})]^{(n-d_2)}, d_1 = k, \ldots, r-1,\ d_2 = k, \ldots, r-1,\ d_1 \le d_2 \\ \frac{n!}{(n-r)!}[\prod_{i=1}^{r} f(x_i;\boldsymbol{\theta})][R(x_r;\boldsymbol{\theta})]^{(n-r)}, d_1 = 0, 1, \ldots, k-1,\ d_2 = r, \\ \frac{n!}{(n-d_2)!}[\prod_{i=1}^{d_2} f(x_i;\boldsymbol{\theta})][R(T_2;\boldsymbol{\theta})]^{(n-d_2)}, d_1 = 0, \ldots, k-1,\ d_2 = k, \ldots, r-1, \\ \frac{n!}{(n-k)!}[\prod_{i=1}^{k} f(x_i;\boldsymbol{\theta})][R(x_k;\boldsymbol{\theta})]^{(n-k)}, d_2 = 0, \ldots, k-1. \end{cases} \tag{1}$$

Many well-known censoring schemes can be considered as special cases from the studied $UHCS$, such as generalized type-I HCS, see [4] when $T_1 \to 0$, generalized type-II HCS, see [4] when $k = 1$, type-I HCS, see [5], when $T_1 \to 0$ and $k = 1$, type-II HCS, see [5], when $T_2 \to \infty$ and $k = 1$, type-I censoring, see [6], when $T_1 \to 0$, $k = 1$, $r = n$ and type-II censoring, see [6], when $T_1 \to 0$, $T_2 \to \infty$ and $k = 1$.

Among the advantages of $UHCS$ is that it is more flexible than the generalized type-I HCS and generalized type-II HCS; moreover, it guarantees us more observations, which will increase the accuracy of the predictive intervals.

Ref. [7] proposes the Burr-X distribution as a member of the Burr distribution family. This model is extremely useful in the fields of statistics and operations research. Engineering, health, agriculture, and biology are just some of the fields where it can be used to great effect.

A random variable X is said to have a Burr-X with a vector of parameters $\boldsymbol{\theta} = (\alpha, \beta)$ if the PDF is given by

$$f(x;\alpha,\ \beta) = 2\,\alpha\,\beta\,x\,e^{-\beta\,x^2}\,(1 - e^{-\beta\,x^2})^{\alpha-1},\ x > 0, (\alpha > 0, \beta > 0). \tag{2}$$

The corresponding CDF and RF are given, respectively, as:

$$F(x;\alpha,\ \beta) = (1 - e^{-\beta\,x^2})^{\alpha},\ x > 0, (\alpha > 0, \beta > 0), \tag{3}$$

$$R(x;\alpha,\ \beta) = 1 - (1 - e^{-\beta\,x^2})^{\alpha},\ x > 0, (\alpha > 0, \beta > 0). \tag{4}$$

For more details about some Burr models with related inferences using classical and Bayesian approaches, see [8–18].

Many contributions found in this paper, such as: studying the prediction problem in a $UHCS$ using the classical and Bayesian approaches with making some comparisons between the two approaches, analyzing two engineering real data sets using Burr-X distribution and applying the obtained results on these real data sets as illustrative examples.

This paper is organized as follows: the point and interval prediction problems under one- and two-sample prediction schemes were studied using the classical and Bayesian approaches in Sections 2 and 3, respectively. In Section 4, the obtained results were applied on simulated and real data sets. Our conclusions are summarized in Section 5.

2. One-Sample Prediction

Assume that n items are placed in a life-time experiment and that this experiment will be terminated at a fixed time T^* and the number of failures until this time is D. The previous ordered failures denoted by $\boldsymbol{x} = (x_{1:n}, x_{2:n}, \ldots, x_{D:n})$, which can be written for simplicity as $\boldsymbol{x} = (x_1, x_2, \ldots, x_D)$, called (***Informative sample***). In Balakrishnan's $UHCS$, T^* will equal T_1 in the first case, x_r in the second case, T_2 in the third case, x_r in the fourth case, T_2 in the fifth case and x_k in the sixth case. Moreover, D will equal d_1 in the first case, r in the second case, d_2 in the third case, r in the fourth case, d_2 in the fifth case, and k in the sixth case. In the one-sample prediction scheme, the future failure time $x_{D+s} \equiv y_s, s = 1, 2, \ldots, n-D$ will be predicted based on the informative sample.

In this section, the PPs and IPs of the future unknown failure time y_s will be computed using classical and Bayesian methods.

First, the conditional PDF of the future failure time y_s given the vector of parameters $\boldsymbol{\theta}$ should be derived as follows:

Based on the informative sample $\boldsymbol{x} = (x_1, x_2, \ldots, x_D)$, the PDF of y_s given $\boldsymbol{\theta}$ will be the PDF of the sth ordered value from $n - D$ ordered values after T^*, which can be written as (see [15,19–21]):

$$g_1(y_s;\boldsymbol{\theta}) \propto [R(T^*;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\, [R(y_s;\boldsymbol{\theta})]^{n-D-s}\, [R(T^*;\boldsymbol{\theta})]^{-(n-D)}\, f(y_s;\boldsymbol{\theta}),\ y_s > T^*. \tag{5}$$

Using this PDF, the conditional PDF of the future failure time y_s given $\boldsymbol{\theta}$ based on all cases of Balakrishnan's $UHCS$ is:

$$g_1(y_s;\boldsymbol{\theta}) \propto \begin{cases} [R(T_1;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\, [R(y_s;\boldsymbol{\theta})]^{n-d-s}\, [R(T_1;\boldsymbol{\theta})]^{-(n-d)}\, f(y_s;\boldsymbol{\theta}),\ y_s > T_1, \\ d_1 = d_2 = d = r, \ldots, n, \\ [R(x_r;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\, [R(y_s;\boldsymbol{\theta})]^{n-r-s}\, [R(x_r;\boldsymbol{\theta})]^{-(n-r)}\, f(y_s;\boldsymbol{\theta}),\ y_s > x_r, \\ d_1 = k, \ldots, r-1,\ d_2 = r, \\ [R(T_2;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\, [R(y_s;\boldsymbol{\theta})]^{n-d_2-s}\, [R(T_2;\boldsymbol{\theta})]^{-(n-d_2)}\, f(y_s;\boldsymbol{\theta}),\ y_s > T_2, \\ d_1 = k, \ldots, r-1,\ d_2 = k, \ldots, r-1,\ d_1 \le d_2 \\ [R(x_r;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\, [R(y_s;\boldsymbol{\theta})]^{n-r-s}\, [R(x_r;\boldsymbol{\theta})]^{-(n-r)}\, f(y_s;\boldsymbol{\theta}),\ y_s > x_r, \\ d_1 = 0, 1, \ldots, k-1,\ d_2 = r, \\ [R(T_2;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\, [R(y_s;\boldsymbol{\theta})]^{n-d_2-s}\, [R(T_2;\boldsymbol{\theta})]^{-(n-d_2)}\, f(y_s;\boldsymbol{\theta}),\ y_s > T_2, \\ d_1 = 0, \ldots, k-1,\ d_2 = k, \ldots, r-1, \\ [R(x_k;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\, [R(y_s;\boldsymbol{\theta})]^{n-k-s}\, [R(x_k;\boldsymbol{\theta})]^{-(n-k)}\, f(y_s;\boldsymbol{\theta}),\ y_s > x_k,\ d_2 = 0, \ldots, k-1. \end{cases} \tag{6}$$

2.1. Classical Method (Maximum Likelihood Prediction)

In this subsection, the PPs and IPs of y_s were obtained using the following PLF (see [22]):

$$g_1^*(y_s;\boldsymbol{\theta},\boldsymbol{x}) \propto\ L(\boldsymbol{\theta};\ \boldsymbol{x})\, g_1(y_s;\boldsymbol{\theta}),\ y_s > T^*. \tag{7}$$

Substituting from (1) and (6) in (7), we have

$$g_1^*(y_s;\boldsymbol{\theta},\boldsymbol{x}) \propto \begin{cases} [\prod_{i=1}^{d} f(x_i;\boldsymbol{\theta})]\ [R(T_1;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\ [R(y_s;\boldsymbol{\theta})]^{n-d-s}\ f(y_s;\boldsymbol{\theta}),\ y_s > T_1, \\ d_1 = d_2 = d = r, \ldots, n, \\ [\prod_{i=1}^{r} f(x_i;\boldsymbol{\theta})]\ [R(x_r;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\ [R(y_s;\boldsymbol{\theta})]^{n-r-s}\ f(y_s;\boldsymbol{\theta}),\ y_s > x_r, \\ d_1 = k, \ldots, r-1,\ d_2 = r, \\ [\prod_{i=1}^{d_2} f(x_i;\boldsymbol{\theta})]\ [R(T_2;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\ [R(y_s;\boldsymbol{\theta})]^{n-d_2-s}\ f(y_s;\boldsymbol{\theta}),\ y_s > T_2, \\ d_1 = k, \ldots, r-1,\ d_2 = k, \ldots, r-1,\ d_1 \le d_2 \\ [\prod_{i=1}^{r} f(x_i;\boldsymbol{\theta})]\ [R(x_r;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\ [R(y_s;\boldsymbol{\theta})]^{n-r-s}\ f(y_s;\boldsymbol{\theta}),\ y_s > x_r, \\ d_1 = 0, 1, \ldots, k-1,\ d_2 = r, \\ [\prod_{i=1}^{d_2} f(x_i;\boldsymbol{\theta})]\ [R(T_2;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\ [R(y_s;\boldsymbol{\theta})]^{n-d_2-s}\ f(y_s;\boldsymbol{\theta}),\ y_s > T_2, \\ d_1 = 0, \ldots, k-1,\ d_2 = k, \ldots, r-1, \\ [\prod_{i=1}^{k} f(x_i;\boldsymbol{\theta})][R(x_k;\boldsymbol{\theta}) - R(y_s;\boldsymbol{\theta})]^{s-1}\ [R(y_s;\boldsymbol{\theta})]^{n-k-s}\ f(y_s;\boldsymbol{\theta}),\ y_s > x_k, \\ d_2 = 0, \ldots, k-1. \end{cases} \quad (8)$$

Substituting from (2)–(4) in (8), we have

$$g_1^*(y_s;\alpha,\beta,\boldsymbol{x}) \propto \begin{cases} \alpha^{d+1}\,\beta^{d+1}\,y_s\,e^{-\beta y_s^2}\,(1-e^{-\beta y_s^2})^{\alpha-1}\,[\prod_{i=1}^{d} x_i\,e^{-\beta x_i^2}\,(1-e^{-\beta x_i^2})^{\alpha-1}]\times \\ [(1-e^{-\beta y_s^2})^{\alpha} - (1-e^{-\beta T_1^2})^{\alpha}]^{s-1}\,[1-(1-e^{-\beta y_s^2})^{\alpha}]^{n-d-s},\ y_s > T_1, \\ d_1 = d_2 = d = r, \ldots, n, \\ \alpha^{r+1}\,\beta^{r+1}\,y_s\,e^{-\beta y_s^2}\,(1-e^{-\beta y_s^2})^{\alpha-1}\,[\prod_{i=1}^{r} x_i\,e^{-\beta x_i^2}\,(1-e^{-\beta x_i^2})^{\alpha-1}]\times \\ [(1-e^{-\beta y_s^2})^{\alpha} - (1-e^{-\beta x_r^2})^{\alpha}]^{s-1}\,[1-(1-e^{-\beta y_s^2})^{\alpha}]^{n-r-s},\ y_s > x_r, \\ d_1 = k, \ldots, r-1,\ d_2 = r, \\ \alpha^{d_2+1}\,\beta^{d_2+1}\,y_s\,e^{-\beta y_s^2}\,(1-e^{-\beta y_s^2})^{\alpha-1}\,[\prod_{i=1}^{d_2} x_i\,e^{-\beta x_i^2}\,(1-e^{-\beta x_i^2})^{\alpha-1}]\times \\ [(1-e^{-\beta y_s^2})^{\alpha} - (1-e^{-\beta T_2^2})^{\alpha}]^{s-1}\,[1-(1-e^{-\beta y_s^2})^{\alpha}]^{n-d_2-s},\ y_s > T_2, \\ d_1 = k, \ldots, r-1,\ d_2 = k, \ldots, r-1,\ d_1 \le d_2 \\ \alpha^{r+1}\,\beta^{r+1}\,y_s\,e^{-\beta y_s^2}\,(1-e^{-\beta y_s^2})^{\alpha-1}\,[\prod_{i=1}^{r} x_i\,e^{-\beta x_i^2}\,(1-e^{-\beta x_i^2})^{\alpha-1}]\times \\ [(1-e^{-\beta y_s^2})^{\alpha} - (1-e^{-\beta x_r^2})^{\alpha}]^{s-1}\,[1-(1-e^{-\beta y_s^2})^{\alpha}]^{n-r-s},\ y_s > x_r, \\ d_1 = 0, 1, \ldots, k-1,\ d_2 = r, \\ \alpha^{d_2+1}\,\beta^{d_2+1}\,y_s\,e^{-\beta y_s^2}\,(1-e^{-\beta y_s^2})^{\alpha-1}\,[\prod_{i=1}^{d_2} x_i\,e^{-\beta x_i^2}\,(1-e^{-\beta x_i^2})^{\alpha-1}]\times \\ [(1-e^{-\beta y_s^2})^{\alpha} - (1-e^{-\beta T_2^2})^{\alpha}]^{s-1}\,[1-(1-e^{-\beta y_s^2})^{\alpha}]^{n-d_2-s},\ y_s > T_2, \\ d_1 = 0, \ldots, k-1,\ d_2 = k, \ldots, r-1, \\ \alpha^{k+1}\,\beta^{k+1}\,y_s\,e^{-\beta y_s^2}\,(1-e^{-\beta y_s^2})^{\alpha-1}\,[\prod_{i=1}^{k} x_i\,e^{-\beta x_i^2}\,(1-e^{-\beta x_i^2})^{\alpha-1}]\times \\ [(1-e^{-\beta y_s^2})^{\alpha} - (1-e^{-\beta x_k^2})^{\alpha}]^{s-1}\,[1-(1-e^{-\beta y_s^2})^{\alpha}]^{n-k-s},\ y_s > x_k, \\ d_2 = 0, \ldots, k-1. \end{cases} \quad (9)$$

2.1.1. Point Predictor

In this subsection, the *PPs* of y_s will be obtained using two methods.

Method (1): obtaining the values of α, β, and y_s, which maximize the logarithm of the *PLF*, and will be denoted by α^*, β^*, and y_s^*, respectively. The values α^* and β^* will be called the *PMLEs* and the value y^* will be called the *MLP* of y_s. To maximize the logarithm of the *PLF*, we will differentiate $log(g_1^*(y_s;\alpha,\beta,\boldsymbol{x}))$ with respect to α, β, and y_s, set the resulting derivatives to zero and solve the resulting nonlinear equations. The solution of the resulting nonlinear equations will be α^*, β^*, and y^*.

Method (2): first, the *MLEs* of the parameters α and β, denoted by $\hat{\alpha}$ and $\hat{\beta}$, will be obtained, then replace α and β by $\hat{\alpha}$ and $\hat{\beta}$, respectively, in the *PLF*, to obtain the *MLPF* in the form: $g_1^{**}(y_s;\boldsymbol{x}) = g_1^*(y_s;\hat{\alpha},\hat{\beta},\boldsymbol{x})$, and finally the *MLP* of y_s will be equal to $\int_{T^*}^{\infty} y_s\, g_1^{**}(y_s;\boldsymbol{x})\, dy_s$, which represents the mathematical expectation of the random variable Y_s. To obtain the ($MLEs$) of α and β, we will differentiate the logarithm of the LF then set the resulting equations to zero and solve the resulting nonlinear equations. The solution of the resulting nonlinear equations will be $\hat{\alpha}$ and $\hat{\beta}$.

Based on the studied $UHCS$, $g_1^{**}(y_s;\boldsymbol{x})$ can be written in the form:

$$g_1^{**}(y_s;\boldsymbol{x}) = A\, y_s\, e^{-\hat{\beta}\, y_s^2}\,(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}-1}\,[(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}} - (1-e^{-\hat{\beta}\, T^{*2}})^{\hat{\alpha}}]^{s-1}[1-(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}}]^{n-D-s},\ y_s > T^*, \tag{10}$$

where A is a normalizing constant and has the value

$$A = \frac{1}{\int_{T^*}^{\infty}\left(y_s\, e^{-\hat{\beta}\, y_s^2}\,(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}-1}\,[(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}} - (1-e^{-\hat{\beta}\, T^{*2}})^{\hat{\alpha}}]^{s-1}[1-(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}}]^{n-D-s}\right)dy_s}. \tag{11}$$

So, the MLP of y_s will be

$$\begin{aligned} y_s^* = E[Y_s] = \int_{T^*}^{\infty} y_s\, g_1^{**}(y_s;\boldsymbol{x})\, dy_s = \\ \frac{\int_{T^*}^{\infty}\left(y_s^2\, e^{-\hat{\beta}\, y_s^2}\,(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}-1}\,[(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}} - (1-e^{-\hat{\beta}\, T^{*2}})^{\hat{\alpha}}]^{s-1}[1-(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}}]^{n-D-s}\right)dy_s}{\int_{T^*}^{\infty}\left(y_s\, e^{-\hat{\beta}\, y_s^2}\,(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}-1}\,[(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}} - (1-e^{-\hat{\beta}\, T^{*2}})^{\hat{\alpha}}]^{s-1}[1-(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}}]^{n-D-s}\right)dy_s}, \end{aligned} \tag{12}$$

where

$$\begin{cases} D = d,\ y_s > T^*,\ T^* = T_1,\ d_1 = d_2 = d = r, \ldots, n, \\ D = r,\ y_s > T^*,\ T^* = x_r,\ d_1 = k, \ldots, r-1,\ d_2 = r, \\ D = d_2,\ y_s > T^*,\ T^* = T_2,\ d_1 = k, \ldots, r-1,\ d_2 = k, \ldots, r-1,\ d_1 \le d_2 \\ D = r,\ y_s > T^*,\ T^* = x_r,\ d_1 = 0, 1, \ldots, k-1,\ d_2 = r, \\ D = d_2,\ y_s > T^*,\ T^* = T_2,\ d_1 = 0, \ldots, k-1,\ d_2 = k, \ldots, r-1, \\ D = k,\ y_s > T^*,\ T^* = x_k,\ d_2 = 0, \ldots, k-1. \end{cases} \tag{13}$$

2.1.2. Interval Predictor

A $(1-\tau)\times 100\%$ $MLPI$ (LM_1, UM_1) of the future failure time y_s can be obtained by solving the following two nonlinear equations:

$$\begin{cases} \int_{LM_1}^{\infty} g_1^{**}(y_s;\boldsymbol{x})\, dy_s = 1 - \frac{\tau}{2}, \\ \int_{UM_1}^{\infty} g_1^{**}(y_s;\boldsymbol{x})\, dy_s = \frac{\tau}{2}. \end{cases} \tag{14}$$

From (10) and (11) in (14), the two nonlinear equations in (14) can be rewritten to be of the form

$$\begin{cases} \dfrac{\int_{LM_1}^{\infty}\left(y_s\, e^{-\hat{\beta}\, y_s^2}\,(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}-1}\,[(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}} - (1-e^{-\hat{\beta}\, T^{*2}})^{\hat{\alpha}}]^{s-1}[1-(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}}]^{n-D-s}\right)dy_s}{\int_{T^*}^{\infty}\left(y_s\, e^{-\hat{\beta}\, y_s^2}\,(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}-1}\,[(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}} - (1-e^{-\hat{\beta}\, T^{*2}})^{\hat{\alpha}}]^{s-1}[1-(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}}]^{n-D-s}\right)dy_s} = 1 - \frac{\tau}{2}, \\ \dfrac{\int_{UM_1}^{\infty}\left(y_s\, e^{-\hat{\beta}\, y_s^2}\,(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}-1}\,[(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}} - (1-e^{-\hat{\beta}\, T^{*2}})^{\hat{\alpha}}]^{s-1}[1-(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}}]^{n-D-s}\right)dy_s}{\int_{T^*}^{\infty}\left(y_s\, e^{-\hat{\beta}\, y_s^2}\,(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}-1}\,[(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}} - (1-e^{-\hat{\beta}\, T^{*2}})^{\hat{\alpha}}]^{s-1}[1-(1-e^{-\hat{\beta}\, y_s^2})^{\hat{\alpha}}]^{n-D-s}\right)dy_s} = \frac{\tau}{2}. \end{cases} \tag{15}$$

By solving the previous system, the $MLPI$ of y_s, (LM_1, UM_1), can be computed.

2.2. Bayesian Method (Bayesian Prediction)

Using the following bivariate prior suggested by [23,24]:

$$\pi(\alpha,\beta) \propto \alpha^{c_1+c_3-1}\,\beta^{c_3-1}e^{-\alpha\,(\beta+c_2)}, \alpha>0,\ \beta>0,\ (c_1>0,\ c_2>0,\ c_3>0), \tag{16}$$

where c_1, c_2, and c_3 are the prior parameters (also known as hyperparameters) and LF (1) replace $f(x_i;\boldsymbol{\theta})$ and $R(x_i;\boldsymbol{\theta})$ by its definitions from (2) and (4), the posterior PDF of α and β can be written as:

$$\pi^*(\alpha,\beta;\boldsymbol{x}) \propto \begin{cases} \alpha^{d+c_1+c_3-1}\,\beta^{d+c_3-1}\ e^{-\alpha\,(\beta+c_2)}\,[\prod_{i=1}^{d} x_i\,e^{-\beta\,x_i^2}\,(1-e^{-\beta\,x_i^2})^{\alpha-1}]\times \\ [1-(1-e^{-\beta\,T_1^2})^{\alpha}]^{n-d-s},\ d_1=d_2=d=r,\ldots,n, \\ \alpha^{r+c_1+c_3-1}\,\beta^{r+c_3-1}\ e^{-\alpha\,(\beta+c_2)}\,[\prod_{i=1}^{r} x_i\,e^{-\beta\,x_i^2}\,(1-e^{-\beta\,x_i^2})^{\alpha-1}]\times \\ [1-(1-e^{-\beta\,x_r^2})^{\alpha}]^{n-r-s},\ d_1=k,\ldots,r-1,\ d_2=r, \\ \alpha^{d_2+c_1+c_3-1}\,\beta^{d_2+c_3-1}\ e^{-\alpha\,(\beta+c_2)}\,[\prod_{i=1}^{d_2} x_i\,e^{-\beta\,x_i^2}\,(1-e^{-\beta\,x_i^2})^{\alpha-1}]\times \\ [1-(1-e^{-\beta\,T_2^2})^{\alpha}]^{n-d_2-s},\ d_1=k,\ldots,r-1,\ d_2=k,\ldots,r-1,\ d_1\le d_2 \\ \alpha^{r+c_1+c_3-1}\,\beta^{r+c_3-1}\ e^{-\alpha\,(\beta+c_2)}\,[\prod_{i=1}^{r} x_i\,e^{-\beta\,x_i^2}\,(1-e^{-\beta\,x_i^2})^{\alpha-1}]\times \\ [1-(1-e^{-\beta\,x_r^2})^{\alpha}]^{n-r-s},\ d_1=0,1,\ldots,k-1,\ d_2=r, \\ \alpha^{d_2+c_1+c_3-1}\,\beta^{d_2+c_3-1}\ e^{-\alpha\,(\beta+c_2)}\,[\prod_{i=1}^{r} x_i\,e^{-\beta\,x_i^2}\,(1-e^{-\beta\,x_i^2})^{\alpha-1}]\times \\ [1-(1-e^{-\beta\,T_2^2})^{\alpha}]^{n-d_2-s},\ d_1=0,\ldots,k-1,\ d_2=k,\ldots,r-1, \\ \alpha^{k+c_1+c_3-1}\,\beta^{k+c_3-1}\ e^{-\alpha\,(\beta+c_2)}\,[\prod_{i=1}^{k} x_i\,e^{-\beta\,x_i^2}\,(1-e^{-\beta\,x_i^2})^{\alpha-1}]\times \\ [1-(1-e^{-\beta\,x_k^2})^{\alpha}]^{n-k-s},\ d_2=0,\ldots,k-1. \end{cases} \tag{17}$$

Using the previous posterior PDF and the conditional PDF of y_s given α and β, (6), after using the definition of $f(x_i;\boldsymbol{\theta})$ and $R(x_i;\boldsymbol{\theta})$ from (2) and (4), the Bayesian predictive PDF of y_s given $\boldsymbol{x}$ will be as follows (see [22]):

$$h_1^*(y_s;\boldsymbol{x}) = \int_0^\infty \int_0^\infty h_1(y_s;\alpha,\beta,\boldsymbol{x})d\beta\,d\alpha, \tag{18}$$

where

$$h_1(y_s;\alpha,\beta,\boldsymbol{x}) = \pi^*(\alpha,\beta;\boldsymbol{x})\,g_1(y_s;\alpha,\beta) = $$

$$\begin{cases} A_1\,y_s\,\alpha^{d+c_1+c_3}\,\beta^{d+c_3}\ e^{-(\alpha\,(\beta+c_2)+\beta\,y_s^2)}\,(1-e^{-\beta\,y_s^2})^{\alpha-1}\times \\ [\prod_{i=1}^{d} x_i\,e^{-\beta\,x_i^2}\,(1-e^{-\beta\,x_i^2})^{\alpha-1}]\,[(1-e^{-\beta\,y_s^2})^{\alpha}-(1-e^{-\beta\,T_1^2})^{\alpha}]^{s-1}\times \\ [1-(1-e^{-\beta\,y_s^2})^{\alpha}]^{n-d-s},\ y_s>T_1, d_1=d_2=d=r,\ldots,n, \\ A_2\,y_s\,\alpha^{r+c_1+c_3}\,\beta^{r+c_3}\ e^{-(\alpha\,(\beta+c_2)+\beta\,y_s^2)}\,(1-e^{-\beta\,y_s^2})^{\alpha-1}\times \\ [\prod_{i=1}^{r} x_i\,e^{-\beta\,x_i^2}\,(1-e^{-\beta\,x_i^2})^{\alpha-1}]\,[(1-e^{-\beta\,y_s^2})^{\alpha}-(1-e^{-\beta\,x_r^2})^{\alpha}]^{s-1}\times \\ [1-(1-e^{-\beta\,y_s^2})^{\alpha}]^{n-r-s},\ y_s>x_r, d_1=k,\ldots,r-1,\ d_2=r, \\ A_3\,y_s\,\alpha^{d_2+c_1+c_3}\,\beta^{d_2+c_3}\ e^{-(\alpha\,(\beta+c_2)+\beta\,y_s^2)}\,(1-e^{-\beta\,y_s^2})^{\alpha-1}\times \\ [\prod_{i=1}^{d_2} x_i\,e^{-\beta\,x_i^2}\,(1-e^{-\beta\,x_i^2})^{\alpha-1}]\,[(1-e^{-\beta\,y_s^2})^{\alpha}-(1-e^{-\beta\,T_2^2})^{\alpha}]^{s-1}\times \\ [1-(1-e^{-\beta\,y_s^2})^{\alpha}]^{n-d_2-s},\ y_s>T_2, d_1=k,\ldots,r-1,\ d_2=k,\ldots,r-1,\ d_1\le d_2 \\ A_4\,y_s\,\alpha^{r+c_1+c_3}\,\beta^{r+c_3}\ e^{-(\alpha\,(\beta+c_2)+\beta\,y_s^2)}\,(1-e^{-\beta\,y_s^2})^{\alpha-1}\times \\ [\prod_{i=1}^{r} x_i\,e^{-\beta\,x_i^2}\,(1-e^{-\beta\,x_i^2})^{\alpha-1}]\,[(1-e^{-\beta\,y_s^2})^{\alpha}-(1-e^{-\beta\,x_r^2})^{\alpha}]^{s-1}\times \\ [1-(1-e^{-\beta\,y_s^2})^{\alpha}]^{n-r-s},\ y_s>x_r, d_1=0,1,\ldots,k-1,\ d_2=r, \\ A_5\,y_s\,\alpha^{d_2+c_1+c_3}\,\beta^{d_2+c_3}\ e^{-(\alpha\,(\beta+c_2)+\beta\,y_s^2)}\,(1-e^{-\beta\,y_s^2})^{\alpha-1}\times \\ [\prod_{i=1}^{d_2} x_i\,e^{-\beta\,x_i^2}\,(1-e^{-\beta\,x_i^2})^{\alpha-1}]\,[(1-e^{-\beta\,y_s^2})^{\alpha}-(1-e^{-\beta\,T_2^2})^{\alpha}]^{s-1}\times \\ [1-(1-e^{-\beta\,y_s^2})^{\alpha}]^{n-d_2-s},\ \ y_s>T_2, d_1=0,\ldots,k-1,\ d_2=k,\ldots,r-1, \\ A_6\,y_s\,\alpha^{k+c_1+c_3}\,\beta^{k+c_3}\ e^{-(\alpha\,(\beta+c_2)+\beta\,y_s^2)}\,(1-e^{-\beta\,y_s^2})^{\alpha-1}\times \\ [\prod_{i=1}^{k} x_i\,e^{-\beta\,x_i^2}\,(1-e^{-\beta\,x_i^2})^{\alpha-1}]\,[(1-e^{-\beta\,y_s^2})^{\alpha}-(1-e^{-\beta\,x_k^2})^{\alpha}]^{s-1}\times \\ [1-(1-e^{-\beta\,y_s^2})^{\alpha}]^{n-k-s},\ y_s>x_k, d_2=0,\ldots,k-1, \end{cases} \tag{19}$$

where $A_i, i = 1, 2, \ldots, 6$ are normalizing constants.

The BP of y_s will equal to (see [22]):

$$y_s^{**} = E[Y_s] = \int_{T^*}^{\infty} y_s h_1^*(y_s; \boldsymbol{x}) dy_s, \tag{20}$$

and the $(1-\tau) \times 100\%$ BPI, (LB_1, UB_1), of y_s can be obtained by solving the following two nonlinear equations:

$$\begin{cases} \int_{LB_1}^{\infty} h_1^*(y_s; \boldsymbol{x})\, dy_s = 1 - \frac{\tau}{2}, \\ \int_{UB_1}^{\infty} h_1^*(y_s; \boldsymbol{x})\, dy_s = \frac{\tau}{2}. \end{cases} \tag{21}$$

It is clear that the previous system contains double integration on α and β, which will make the problem of finding the solution for this system very complicated. In this situation, the Gibbs sampler and Metropolis–Hastings algorithm were used to generate a random sample $(\alpha^{(1)}, \beta^{(1)}), (\alpha^{(2)}, \beta^{(2)}), \ldots, (\alpha^{(K)}, \beta^{(K)})$ from the posterior PDF (17); the the system (21) will be of the form

$$\begin{cases} \frac{\sum_{i=1}^{K} \int_{LB_1}^{\infty} h_1(y_s; \alpha^{(i)}, \beta^{(i)}, \boldsymbol{x})\, dy_s}{\sum_{i=1}^{K} \int_{T^*}^{\infty} h_1(y_s; \alpha^{(i)}, \beta^{(i)}, \boldsymbol{x})\, dy_s} = 1 - \frac{\tau}{2}, \\ \frac{\sum_{i=1}^{K} \int_{UB_1}^{\infty} h_1(y_s; \alpha^{(i)}, \beta^{(i)}, \boldsymbol{x})\, dy_s}{\sum_{i=1}^{K} \int_{T^*}^{\infty} h_1(y_s; \alpha^{(i)}, \beta^{(i)}, \boldsymbol{x})\, dy_s} = \frac{\tau}{2}. \end{cases} \tag{22}$$

By solving this system, the BPI, (LB_1, UB_1), for y_s will be obtained.

For more details about the Gibbs sampler and Metropolis–Hastings algorithms, see, for example [25–28].

3. Two-Sample Prediction

Assume that $\boldsymbol{x} = (x_1, x_2, \ldots, x_D)$ and $\boldsymbol{z} = (z_1, z_2, \ldots, z_m)$ represent the informative sample, from the studied $UHCS$ and a future ordered sample of size m, respectively. It is assumed that the two samples are independent.

In this section, PPs and IPs of the observation $z_s, s = 1, 2, \ldots, m$ will be obtained using the classical and Bayesian methods. The conditional PDF of the observation z_s given the vector of parameters $\boldsymbol{\theta}$ is the PDF of the sth ordered value from the m ordered values, which can be written as (see [15,22]):

$$g_2(z_s; \boldsymbol{\theta}) \propto [1 - R(z_s; \boldsymbol{\theta})]^{s-1}\, [R(z_s; \boldsymbol{\theta})]^{m-s}\, f(z_s; \boldsymbol{\theta}),\ z_s > 0. \tag{23}$$

Using the definitions of $R(x; \boldsymbol{\theta})$ and $f(x; \boldsymbol{\theta})$ from (2) and (4) in (23), the conditional PDF of the observation z_s given $\boldsymbol{\theta}$ will be:

$$g_2(z_s; \alpha,\, \beta) \propto \alpha\, \beta\, z_s\, e^{-\beta\, z_s^2}\, (1 - e^{-\beta\, z_s^2})^{s\,\alpha - 1}\, [1 - (1 - e^{-\beta\, z_s^2})^{\alpha}]^{m-s},\ z_s > 0. \tag{24}$$

Based on the two-sample scheme and the same prior (16), the IPs and PPs of z_s can be summarized as follows in the following subsections.

3.1. Maximum Likelihood Prediction (Point and Interval Predictors)

The $MLPF$ can be obtained from (24) after replacing each parameter by its MLE to be of the form

$$g_2^{**}(z_s; \boldsymbol{x}) = B\, z_s\, e^{-\hat{\beta}\, z_s^2}\, (1 - e^{-\hat{\beta}\, z_s^2})^{s\,\hat{\alpha} - 1}\, [1 - (1 - e^{-\hat{\beta}\, z_s^2})^{\hat{\alpha}}]^{m-s},\ z_s > 0, \tag{25}$$

where B is a normalizing constant has the value

$$B = \frac{1}{\int_0^\infty \left(z_s \, e^{-\hat{\beta}\, z_s^2} \, (1 - e^{-\hat{\beta}\, z_s^2})^{s\,\hat{\alpha}-1} \, [1 - (1 - e^{-\hat{\beta}\, z_s^2})^{\hat{\alpha}}]^{m-s} \right) dz_s}. \tag{26}$$

So, the MLP of z_s will be

$$z_s^* = E[Z_s] = \int_0^\infty z_s \, g_2^{**}(z_s; \boldsymbol{x}) \, dz_s = \frac{\int_0^\infty \left(z_s^2 \, e^{-\hat{\beta}\, z_s^2} \, (1 - e^{-\hat{\beta}\, z_s^2})^{s\,\hat{\alpha}-1} \, [1 - (1 - e^{-\hat{\beta}\, z_s^2})^{\hat{\alpha}}]^{m-s} \right) dz_s}{\int_0^\infty \left(z_s \, e^{-\hat{\beta}\, z_s^2} \, (1 - e^{-\hat{\beta}\, z_s^2})^{s\,\hat{\alpha}-1} \, [1 - (1 - e^{-\hat{\beta}\, z_s^2})^{\hat{\alpha}}]^{m-s} \right) dz_s}, \tag{27}$$

A $(1-\tau) \times 100\%$ $MLPI$ (LM_2, UM_2) of z_s can be obtained by solving the following two nonlinear equations:

$$\begin{cases} \int_{LM_2}^\infty g_2^{**}(z_s; \boldsymbol{x}) \, dz_s = 1 - \frac{\tau}{2}, \\ \int_{UM_2}^\infty g_2^{**}(z_s; \boldsymbol{x}) \, dz_s = \frac{\tau}{2}. \end{cases} \tag{28}$$

From (25) and (26) in (28), the two nonlinear equations in (28) can be rewritten, to be of the form

$$\begin{cases} \dfrac{\int_{LM_2}^\infty \left(z_s \, e^{-\hat{\beta}\, z_s^2} \, (1 - e^{-\hat{\beta}\, z_s^2})^{s\,\hat{\alpha}-1} \, [1 - (1 - e^{-\hat{\beta}\, z_s^2})^{\hat{\alpha}}]^{m-s} \right) dz_s}{\int_0^\infty \left(z_s \, e^{-\hat{\beta}\, z_s^2} \, (1 - e^{-\hat{\beta}\, z_s^2})^{s\,\hat{\alpha}-1} \, [1 - (1 - e^{-\hat{\beta}\, z_s^2})^{\hat{\alpha}}]^{m-s} \right) dz_s} = 1 - \frac{\tau}{2}, \\ \dfrac{\int_{UM_2}^\infty \left(z_s \, e^{-\hat{\beta}\, z_s^2} \, (1 - e^{-\hat{\beta}\, z_s^2})^{s\,\hat{\alpha}-1} \, [1 - (1 - e^{-\hat{\beta}\, z_s^2})^{\hat{\alpha}}]^{m-s} \right) dz_s}{\int_0^\infty \left(z_s \, e^{-\hat{\beta}\, z_s^2} \, (1 - e^{-\hat{\beta}\, z_s^2})^{s\,\hat{\alpha}-1} \, [1 - (1 - e^{-\hat{\beta}\, z_s^2})^{\hat{\alpha}}]^{m-s} \right) dz_s} = \frac{\tau}{2}. \end{cases} \tag{29}$$

By solving the previous system, the $MLPI$ of z_s, (LM_2, UM_2), can be computed.

3.2. Bayesian Prediction (Point and Interval Predictors)

The Bayesian predictive PDF of z_s given $\boldsymbol{x}$ will be as follows:

$$h_2^*(z_s; \boldsymbol{x}) = \int_0^\infty \int_0^\infty h_2(z_s; \alpha, \beta, \boldsymbol{x}) d\beta \, d\alpha, \tag{30}$$

where

$$h_2(z_s;\alpha,\beta,\boldsymbol{x}) = \pi^*(\alpha,\beta;\boldsymbol{x})\, g_2(z_s;\alpha,\beta) =$$
$$\begin{cases} B_1\, z_s\, \alpha^{d+c_1+c_3}\, \beta^{d+c_3}\; e^{-(\alpha\, (\beta+c_2)+\beta\, z_s^2)}\, (1-e^{-\beta\, z_s^2})^{s\,\alpha-1} \times \\ [\prod_{i=1}^{d} x_i\, e^{-\beta\, x_i^2}\, (1-e^{-\beta\, x_i^2})^{\alpha-1}]\, [1-(1-e^{-\beta\, z_s^2})^{\alpha}]^{m-s} \times \\ [1-(1-e^{-\beta\, T_1^2})^{\alpha}]^{n-d},\; z_s > 0, d_1 = d_2 = d = r,\ldots,n, \\ B_2\, z_s\, \alpha^{r+c_1+c_3}\, \beta^{r+c_3}\; e^{-(\alpha\, (\beta+c_2)+\beta\, z_s^2)}\, (1-e^{-\beta\, z_s^2})^{s\,\alpha-1} \times \\ [\prod_{i=1}^{r} x_i\, e^{-\beta\, x_i^2}\, (1-e^{-\beta\, x_i^2})^{\alpha-1}]\, [1-(1-e^{-\beta\, z_s^2})^{\alpha}]^{m-s} \times \\ [1-(1-e^{-\beta\, x_r^2})^{\alpha}]^{n-r},\; z_s > 0, d_1 = k,\ldots,r-1,\, d_2 = r, \\ B_3\, z_s\, \alpha^{d_2+c_1+c_3}\, \beta^{d_2+c_3}\; e^{-(\alpha\, (\beta+c_2)+\beta\, z_s^2)}\, (1-e^{-\beta\, z_s^2})^{s\,\alpha-1} \times \\ [\prod_{i=1}^{d_2} x_i\, e^{-\beta\, x_i^2}\, (1-e^{-\beta\, x_i^2})^{\alpha-1}]\, [1-(1-e^{-\beta\, z_s^2})^{\alpha}]^{m-s} \times \\ [1-(1-e^{-\beta\, T_2^2})^{\alpha}]^{n-d_2},\; z_s > 0, d_1 = k,\ldots,r-1,\, d_2 = k,\ldots,r-1,\, d_1 \le d_2 \\ B_4\, z_s\, \alpha^{r+c_1+c_3}\, \beta^{r+c_3}\; e^{-(\alpha\, (\beta+c_2)+\beta\, z_s^2)}\, (1-e^{-\beta\, z_s^2})^{s\,\alpha-1} \times \\ [\prod_{i=1}^{r} x_i\, e^{-\beta\, x_i^2}\, (1-e^{-\beta\, x_i^2})^{\alpha-1}]\, [1-(1-e^{-\beta\, z_s^2})^{\alpha}]^{m-s} \times \\ [1-(1-e^{-\beta\, x_r^2})^{\alpha}]^{n-r},\; z_s > 0,, d_1 = 0,1,\ldots,k-1,\, d_2 = r, \\ B_5\, z_s\, \alpha^{d_2+c_1+c_3}\, \beta^{d_2+c_3}\; e^{-(\alpha\, (\beta+c_2)+\beta\, z_s^2)}\, (1-e^{-\beta\, z_s^2})^{s\,\alpha-1} \times \\ [\prod_{i=1}^{d_2} x_i\, e^{-\beta\, x_i^2}\, (1-e^{-\beta\, x_i^2})^{\alpha-1}]\, [1-(1-e^{-\beta\, z_s^2})^{\alpha}]^{m-s} \times \\ [1-(1-e^{-\beta\, T_2^2})^{\alpha}]^{n-d_2},\; z_s > 0, d_1 = 0,\ldots,k-1,\, d_2 = k,\ldots,r-1, \\ B_6\, z_s\, \alpha^{k+c_1+c_3}\, \beta^{k+c_3}\; e^{-(\alpha\, (\beta+c_2)+\beta\, z_s^2)}\, (1-e^{-\beta\, z_s^2})^{s\,\alpha-1} \times \\ [\prod_{i=1}^{k} x_i\, e^{-\beta\, x_i^2}\, (1-e^{-\beta\, x_i^2})^{\alpha-1}]\, [1-(1-e^{-\beta\, z_s^2})^{\alpha}]^{m-s} \times \\ [1-(1-e^{-\beta\, x_k^2})^{\alpha}]^{n-k},\; z_s > 0,, d_2 = 0,\ldots,k-1, \end{cases} \tag{31}$$

where $B_i, i = 1,2,\ldots,6$ are normalizing constants.

The BP of z_s will equal

$$z_s^{**} = E[Z_s] = \int_0^\infty z_s h_2^*(z_s;\boldsymbol{x}) dz_s, \tag{32}$$

and the $(1-\tau)\times 100\%$ BPI, (LB_2, UB_2), of z_s can be obtained by solving the following two nonlinear equations:

$$\begin{cases} \int_{LB_2}^{\infty} h_2^*(z_s;\boldsymbol{x})\, dz_s = 1-\frac{\tau}{2}, \\ \int_{UB_2}^{\infty} h_2^*(z_s;\boldsymbol{x})\, dz_s = \frac{\tau}{2}. \end{cases} \tag{33}$$

Using $(\alpha^{(1)},\beta^{(1)}), (\alpha^{(2)},\beta^{(2)}),\ldots,(\alpha^{(K)},\beta^{(K)})$, which are generated from the posterior PDF (15), then the system (33) will be of the form

$$\begin{cases} \frac{\sum_{i=1}^{K} \int_{LB_2}^{\infty} h_2(z_s;\alpha^{(i)},\beta^{(i)},\boldsymbol{x})\, dz_s}{\sum_{i=1}^{K} \int_0^{\infty} h_2(z_s;\alpha^{(i)},\beta^{(i)},\boldsymbol{x})\, dz_s} = 1-\frac{\tau}{2}, \\ \frac{\sum_{i=1}^{K} \int_{UB_2}^{\infty} h_2(z_s;\alpha^{(i)},\beta^{(i)},\boldsymbol{x})\, dz_s}{\sum_{i=1}^{K} \int_0^{\infty} h_2(z_s;\alpha^{(i)},\beta^{(i)},\boldsymbol{x})\, dz_s} = \frac{\tau}{2}. \end{cases} \tag{34}$$

By solving this system, the BPI, (LB_2, UB_2), for z_s will be obtained.

From the results of the second and third sections, it is clear that the classical method of prediction and inference in general, called the maximum likelihood approach, depends only on an informative sample from the studied distribution under a suggested censoring scheme, and does not depend on any additional information about the parameters of the population. However, for the Bayes method, it depends on the same informative sample, but in addition to additional information about the parameters of the population represented in the prior distribution of the parameters. This obviously leads to better results. The results obtained based on the samples in the next section will verify this fact.

In case of absence of information on the population parameters, we have two choices. The first is to use the Bayes approach under a vague prior and the second is to use the classical method.

4. Results

In this section, one- and two-sample PPs and IPs using the classical and Bayesian approaches were obtained based on simulated and real data sets.

4.1. Simulated Results

The predictive process is a process that takes in historical data to predict which areas and parts of an asset will fail and at what time. The technician can receive relevant and accurate data points, remotely. The collected data are then analyzed and predictive algorithms to determine which part are more likely to fail. This information is communicated to workers via collaboration tools and data visualization, with which they can perform maintenance work only on the parts that require it. By implementing a predictive maintenance solution (Figure 1), organizations will know when to schedule a specific part replacement and be alerted to future degradations due to faulty parts.

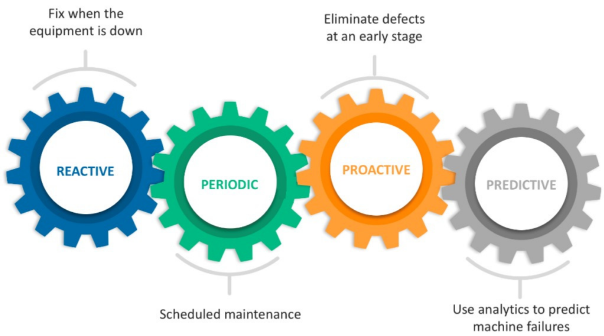


Figure 1. Reactive periodic proactive predictive four stage engineering process.

In this section, the PPs and IPs of future failure times are computed, in one- and two-sample schemes, using the classical and Bayesian methods based on a generated $UHCS$ informative sample for different values of r, k, T_1, and T_2 as follows:

1. For a given set of prior parameters c_1, c_2, and c_3, the population parameters α and β are generated from the joint prior (16).
2. Making use of α and β obtained in step 1, a sample of size n of upper ordered values from Burr-X is generated.
3. For different values of n, r, k, T_1, and T_2, a $UHCS$ informative sample is generated from the complete sample in step 2.
4. For different values of n, r, k, T_1, and T_2, the PPs and IPs of the future failure times are computed using classical and Bayes methods in a one-sample scheme, as explained in Section 2.
5. The same is done in a two-sample scheme, as explained in Section 3.
6. For each future failure time, the PP, IP, length of the IP, and the CP of the IP are computed.
7. The results are summarized in Tables 1 and 2.
   From Tables 1 and 2, observe the following:
   (a) For fixed r, k, T_1, and T_2, the length and the CP of the IP increase by increasing s because the element y_s or z_s will be larger, which will widen its predictive interval and, therefore, its CP.
   (b) In all six cases of the studied $UHCS$:

i. The length and the CP of the IP decreases by increasing the ratio $\frac{D}{n}$, which means that the results will be better by increasing the available information.
ii. In the cases with constant ratio $\frac{D}{n}$ and fixed r, T_1, and T_2, the length and the CP of the IP decrease by increasing k, which show us that the results will be better by increasing k.

(c) In all cases, the lengths of the IPs are shorter in case of the Bayesian method than that computed by the classical method, which means that the Bayesian method is better than the classical method.
(d) In all cases, the Bayesian CPs are less than that computed by the classical method, which is also a criterion indicating that the results obtained by using the Bayes method is better than that obtained using the classical method.
(e) The values r, k, T_1, and T_2 have been chosen so as to give all six cases of the studied $UHCS$.

4.2. Data Analysis

In this section, two real data sets are introduced; they were analyzed using Burr-X. The studied real data sets are from [8]. The first data set represents the failure times in the hours of 15 devices, and the second represents the first failure times in the months of 20 electronic cards. These real data sets are:

Data I: 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.76, 4.85, 6.5, 7.35, 8.01, 8.27, 12.06 and 31.75.
Data II: 0.9, 1.5, 2.3, 3.2, 3.9, 5.0, 6.2, 7.5, 8.3, 10.4, 11.1, 12.6, 15.0, 16.3, 19.3, 22.6, 24.8, 31.5, 38.1 and 53.0.

In Table 3, the $MLEs$ of the parameters α and β and the corresponding Kolmogorov–Smirnov ($K-S$) test statistic were computed under the Burr-X model.

Under the significance level (0.05) and using the Kolmogorov–Smirnov table, the critical value for the $K-S$ test statistic is 0.33760, which is greater than the computed $K-S$ test statistics for the two real data sets under the *Burr-X* model. This means that the studied model fits the two biological data sets well.

PPs and IPs of the remaining future failure times ($y_s, s = 1,2,3,4$) and of the first four observations ($z_s, s = 1,2,3,4$) from an independent ordered sample based on a generated Balakrishnan $UHCS$ informative sample from the given real data sets, were computed; they are summarized in Tables 4–7.

Table 1. PPs and IPs of the future failure time $y_s, s = 1,2,3,4$, based on the generated Balakrishnan $UHCS$ informative sample. $(\alpha = 3.1811, \beta = 1.5779)$, $(c_1 = 4.8, c_2 = 2.5, c_3 = 4.5)$.

Values of (T_1, T_2)		(1.5, 1.6)			
(r,k) (n,D)	*Method*	*Generated* y_1 PP of y_1 IP of y_1 *Length* CP	*Generated* y_2 PP of y_2 IP of y_2 *Length* CP	*Generated* y_3 PP of y_3 IP of y_3 *Length* CP	*Generated* y_4 PP of y_4 IP of y_4 *Length* CP
$(10,5)$ $(20,12)$	*ML*	1.03809 0.91676 (0.87340,1.12306) 0.24966 0.9176	1.11608 1.0914 (0.82512,1.13621) 0.31109 0.92177	1.57305 1.4711 (1.10036,1.61367) 0.51331 0.93001	1.65624 1.6674 (1.42708,1.97736) 0.55028 0.93816
	Bayes	1.03809 0.91053 (0.87103,0.98636) 0.11523 0.9037	1.11608 1.1537 (1.02104,1.22278) 0.20174 0.9165	1.57305 1.5018 (1.16172,1.47264) 0.31092 0.9275	1.65624 1.6214 (1.29082,1.70564) 0.41482 0.9310
$(10,7)$ $(20,14)$	*ML*	1.03809 1.18077 (1.09166,1.22885) 0.13719 0.9001	1.11608 1.32147 (1.21106,1.42664) 0.21558 0.91130	1.57305 1.59917 (1.38102,1.68120) 0.30018 0.92719	1.65624 1.79101 (1.49016,1.88918) 0.39902 0.92991
	Bayes	1.03809 1.17914 (1.11057,1.23958) 0.12901 0.8997	1.11608 1.34221 (1.20184,1.4052) 0.20336 0.91055	1.57305 1.62017 (1.42061,1.69377) 0.27316 0.9206	1.65624 1.77082 (1.58062,1.94235) 0.36173 0.9227
Values of (T_1, T_2)		(1.3, 1.5)			
$(25,5)$ $(30,25)$	*ML*	0.66209 0.58144 (0.40068,0.79311) 0.39243 0.95161	0.89038 0.9102 (0.60106,1.09158) 0.49052 0.96173	0.91005 0.88152 (0.74061,1.27075) 0.53014 0.96106	1.75215 1.70119 (1.12105,1.85211) 0.73106 0.97015
	Bayes	0.662089 0.64825 (0.50151,0.86203) 0.36052 0.9483	0.890378 0.86239 (0.76175,1.16289) 0.40114 0.9553	0.910053 0.94931 (0.95276,1.46868) 0.51592 0.9581	1.75215 1.69046 (1.39173,1.90765) 0.61058 0.98813
$(25,10)$ $(30,25)$	*ML*	0.662089 0.66131 (0.49106,0.78122) 0.29016 0.94814	0.890378 0.86561 (0.73083,1.11254) 0.38171 0.95231	0.910053 0.94072 (0.807016,1.26818) 0.46117 0.95618	1.75215 1.78334 (1.24608,1.94780) 0.70172 0.96131
	Bayes	0.662089 0.67013 (0.58043,0.79278) 0.21225 0.9381	0.890378 0.88172 (0.72063,1.07077) 0.35041 0.9481	0.910053 0.90157 (0.83803,1.24985) 0.41182 0.9511	1.75215 1.74105 (1.14473,1.82615) 0.68142 0.9591

Table 1. *Cont.*

Values of (T_1, T_2)		(1.25, 1.4)			
(25, 15) (30, 21)	*ML*	1.42077 1.39354 (1.31075,1.56622) 0.25547 0.9695	1.50944 1.48593 (1.32194,1.86571) 0.27377 0.9726	1.54494 1.51207 (1.34528,1.70186) 0.35658 0.9799	1.63888 1.65472 (1.45619,1.84651) 0.45032 0.9801
	Bayes	1.42077 1.43406 (1.40089,1.52250) 0.12161 0.9217	1.50944 1.49225 (1.40916,1.59487) 0.18571 0.9317	1.54494 1.53821 (1.42534,1.67038) 0.24504 0.9502	1.63888 1.64152 (1.50294,1.77342) 0.27048 0.9573
Values of (T_1, T_2)		(0.8, 2.5)			
(25, 20) (30, 25)	*ML*	1.42077 1.40593 (1.32194,1.56571) 0.24377 0.9504	1.50944 1.49593 (1.32194,1.58239) 0.26045 0.9551	1.54494 1.57207 (1.34528,1.68186)) 0.33658 0.9623	1.63888 1.61472 (1.25619,1.70651) 0.45032 0.9708
	Bayes	1.42077 1.41948 (1.35285,1.47209) 0.11924 0.9495	1.50944 1.51021 (1.40421,1.65515) 0.25094 0.9525	1.54494 1.54843 (1.48797,1.81620) 0.32823 0.9605	1.63888 1.63451 (1.52675,1.95603) 0.42928 0.9693
Values of (T_1, T_2)		(0.8, 1.1)			
(30, 20) (40, 23)	*ML*	1.42061 1.41092 (1.32901,1.59942) 0.27041 0.9601	1.52062 1.49332 (1.40162,1.69674) 0.29512 0.9664	1.63815 1.61337 (1.48054,1.78922) 0.34116 0.9718	1.64518 1.66319 (1.50512,2.14725) 0.64213 0.9804
	Bayes	1.42061 1.41941 (1.34162,1.56357) 0.22195 9594	1.52062 1.51804 (1.44076,1.69407) 0.25331 0.9614	1.63815 1.64183 (1.52184,1.82301) 0.30117 0.9695	1.64518 1.65184 (1.59042,2.00790) 0.41748 0.9748
Values of (T_1, T_2)		(0.5, 0.8)			
(30, 25) (40, 25)	*ML*	1.51436 1.49201 (1.40162,1.663278) 0.23116 0.9584	1.60734 1.58801 (1.45281,1.74086) 0.28805 0.9615	1.71813 1.77419 (1.59042,1.91318) 0.32276 0.9697	1.72479 1.75118 (1.66492,2.25535) 0.59043 0.978
	Bayes	1.51436 1.50184 (1.41062,1.62090) 0.21028 0.9533	1.60734 1.58294 (1.50372,1.74565) 0.24193 0.9557	1.71813 1.70174 (1.60152,1.88221) 0.28069 0.9614	1.72479 1.73182 (1.64107,1.98290) 0.34183 0.9736

Table 2. PPs and IPs of the future failure time $z_s, s = 1,2,3,4$, based on the generated Balakrishnan $UHCS$ informative sample. $(\alpha = 3.1811, \beta = 1.5779)$, $(c_1 = 4.8, c_2 = 2.5, c_3 = 4.5)$.

Values of (T_1, T_2)		(1.5, 1.6)			
(r,k) (n,D)	*Method*	*Generated* z_1 / PP of z_1 / IP of z_1 / *Length* / CP	*Generated* z_2 / PP of z_2 / IP of z_2 / *Length* / CP	*Generated* z_3 / PP of z_3 / IP of z_3 / *Length* / CP	*Generated* z_4 / PP of z_4 / IP of z_4 / *Length* / CP
$(10,5)$ $(20,12)$	*ML*	0.41303	0.78654	1.09878	1.31352
		0.37017	0.69154	0.95124	1.24012
		(0.25175,0.46227)	(0.43129,0.80322)	(0.71182,1.30162)	(1.00273,1.59152)
		0.21052	0.37193	0.4898	0.58879
		0.88153	0.9152	0.9205	0.9317
	Bayes	0.41303	0.78654	1.09878	1.31352
		0.38718	0.71032	0.97182	1.27104
		(0.33152,0.53268)	(0.51037,0.82988)	(0.81094,1.20606)	(1.17213,1.65357)
		0.20116	0.31951	0.39512	0.48144
		0.8781	0.9013	0.9114	0.9226
$(10,7)$ $(20,14)$	*ML*	0.41303	0.78654	1.09878	1.31352
		0.38053	0.71108	0.98155	1.30153
		(0.35102,0.54184)	(0.51005,0.82108)	(0.81106,1.22128)	(1.10924,1.63025)
		0.19082	0.31103	0.41022	0.52101
		0.8771	0.9010	0.9113	0.9215
	Bayes	0.41303	0.78654	1.09878	1.31352
		0.39012	0.73318	1.1192	1.34417
		(0.38065,0.56341)	(0.54194,0.83207)	(0.79168,1.16883)	(1.01845,1.50039)
		0.18276	0.29013	0.37715	0.48194
		0.8771	0.8917	0.9016	0.9106
Values of (T_1, T_2)		(1.3, 1.5)			
$(25,5)$ $(30,25)$	*ML*	0.41303	0.78654	1.09878	1.31352
		0.38026	0.82165	1.16271	1.41152
		(0.15243,0.42406)	(0.40072,0.90224)	(0.87932,1.44074)	(1.27194,1.88367)
		0.27163	0.50152	0.56152	0.61173
		0.9201	0.9332	0.9441	0.9505
	Bayes	0.41303	0.78654	1.09878	1.31352
		0.39012	0.73515	0.92166	1.28061
		(0.27251,0.52427)	(0.50041,0.96225)	(0.88015,1.40234)	(1.18145,1.78310)
		0.25176	0.46184	0.52219	0.60165
		0.9115	0.9271	0.9396	0.9471
$(25,10)$ $(30,25)$	*ML*	0.41303	0.78654	1.09878	1.31352
		0.40712	0.0.81026	0.99172	1.28017
		(0.35143,0.59310)	(0.53183,0.99795)	(0.80384,1.29498)	(0.93317,1.51480)
		0.24167	0.46612	0.0.49114	0.0.58163
		0.9195	0.9307	0.9421	0.9497
	Bayes	0.41303	0.78654	1.09878	1.31352
		0.41005	0.79015	1.01823	1.31561
		(0.37041,0.60357)	(0.55272,0.97563)	(0.79043,1.24077)	(0.93962,1.49109)
		0.23316	0.42291	0.45037	0.55147
		0.9061	0.9298	0.9402	0.9488

Table 2. *Cont.*

Values of (T_1, T_2)		(1.25, 1.4)			
(25, 15) (30, 21)	*ML*	0.41303	0.78654	1.09878	1.31352
		0.40815	0.77804	0.1.10926	1.28915
		(0.31629,0.57135)	(0.56052,0.86964)	(0.80057,1.24265)	(1.01748,1.55254)
		0.25506	0.30912	0.44208	0.53506
		0.9479	0.95514	0.9609	0.9716
	Bayes	0.41303	0.78654	1.09878	1.31352
		0.41105	0.78052	1.08805	1.30615
		(0.30225,0.547738)	(0.52817,0.82633)	(0.93183,1.3335)	(1.13052,1.64131)
		0.24513	0.29816	0.40167	0.0.51079
		0.9397	0.9520	0.9593	0.9675
Values of (T_1, T_2)		(0.8, 2.5)			
(25, 20) (30, 25)	*ML*	0.41303	0.78654	1.09878	1.31352
		0.41201	0.80152	1.11026	1.28961
		(0.43172,0.65335)	(0.60332,0.85448)	(0.82184,1.20366)	(1.10286,1.56910)
		0.22163	0.25116	0.38182	0.46624
		0.9418	0.9536	0.9592	0.9663
	Bayes	0.41303	0.78654	1.09878	1.31352
		0.41052	0.79316	0.1.12052	1.29164
		(0.45132,0.66194)	(0.65293,0.89786)	(0.87281,1.23444)	(1.17148.1.60341)
		0.21062	0.24493	0.0.36163	0.43193
		0.9406	0.9513	0.9554	0.9614
Values of (T_1, T_2)		(0.8, 1.1)			
(30, 20) (40, 23)	*ML*	0.41303	0.78654	1.09878	1.31352
		0.41903	0.79016	0.99173	0.1.3201
		(0.32183,0.58324)	(0.53148,0.83165)	(0.81064,1.22247)	(1.10573,1.68725)
		0.26141	0.30017	0.41183	0.58152
		0.9726	0.9775	0.9802	0.9892
	Bayes	0.41303	0.78654	1.09878	1.31352
		0.41525	0.77812	0.1.03124	1.31902
		(0.35149,0.59301)	(0.56028,0.84226)	(0.79104,1.18118)	(0.97823,1.51955)
		0.24152	0.28198	0.39014	0.54132
		0.9618	0.9693	0.9715	0.9801
Values of (T_1, T_2)		(0.5, 0.8)			
(30, 25) (40, 25)	*ML*	0.41303	0.78654	1.09878	1.31352
		0.4111	0.7902	0.1.1058	0.1.3111
		(0.31047,0.56566)	(0.55061,0.84878)	(0.81718,1.20735)	(1.03081,1.59187)
		0.25519	0.29817	0.39017	0.56106
		0.9594	0.9615	0.9694	0.9772
	Bayes	0.41303	0.78654	1.09878	1.31352
		0.40183	0.79163	1.10815	1.29284
		(0.28071,0.52488)	(0.51148,0.80667)	(0.79208,1.15819)	(1.10273,1.62478)
		0.24417	0.29519	0.36611	0.52205
		0.9523	0.9594	0.9611	0.9731

Table 3. $MLEs$ of the parameters and the associated $K-S$ based on the real data sets I and II.

Data Set No.	$\boldsymbol{MLEs}$	$\boldsymbol{K-S}$
I	$\hat{\alpha}=0.436377$, $\hat{\beta}=0.0218652$	0.193849
II	$\hat{\alpha}=0.611733$, $\hat{\beta}=0.140011$	0.247625

Table 4. PPs and IPs of the future failure time $y_s, s=1,2,3,4$ based on a generated Balakrishnan $UHCS$ informative sample from real data set I.

Values of (T_1,T_2)		(1.1, 7.5)			
True y_s (r,k) (n,D) *Case No.*	*Method*	y_1 PP of y_1 IP of y_1 *Length*	y_2 PP of y_2 IP of y_2 *Length*	y_3 PP of y_3 IP of y_3 *Length*	y_4 PP of y_4 IP of y_4 *Length*
(12, 5) (15, 11) 5	*ML*	8.01 7.5715 (6.77191,9.97962) 3.20771	8.27 8.1129 (7.17305,10.60674) 3.43369	12.06 11.3816 (10.48435,14.55529) 4.07094	31.75 33.8818 (25.66194,39.64581) 13.98387
	Bayes	8.01 7.70925 (7.21062,9.51244) 2.30182	8.27 8.19027 (7.50592,10.42419) 2.91827	12.06 11.69016 (10.88017,14.53119) 3.65102	31.75 32.20172 (27.44018,38.34811) 10.90793

Table 5. PPs and IPs of the future failure time $y_s, s=1,2,3,4$, based on a generated Balakrishnan's $UHCS$ informative sample from real data set II.

Values of (T_1,T_2)		(1.3, 1.8)			
True y_s (r,k) (n,D) *Case No.*	*Method*	y_1 PP of y_1 IP of y_1 *Length*	y_2 PP of y_2 IP of y_2 *Length*	y_3 PP of y_3 IP of y_3 *Length*	y_4 PP of y_4 IP of y_4 *Length*
(12, 5) (15, 5) 6	*ML*	5.0 4.6827 (3.51081,6.01779) 2.50698	6.2 6.0196 (4.89192,7.91478) 3.02286	7.5 7.3891 (6.09047,9.82138) 3.73091	8.3 8.9017 (6.12081,10.55201) 4.43120
	Bayes	5.0 4.7908 (3.78207,5.66372) 1.88165	6.2 6.3005 (5.03927,7.07429) 2.03502	7.5 7.4213 (6.24718,8.57770) 2.33052	8.3 8.5105 (6.74039,10.55102) 3.81063

From previous tables and figures, we can observe (for fixed r, k, T_1, and T_2):

1. Increase the length of the predictive intervals by increasing s, because, as mentioned previously, the element y_s or z_s will be larger, which will widen its predictive interval.
2. The length of the predictive intervals computed by the Bayesian method is less that that computed by the classical method, which means that Bayes technique is better than the other technique.
3. For Bayes and classical approaches, and for all values of s, the exact value of y_s lies in its predictive interval.
4. From Figures 2A,B, 3A,B, 4A,B and 5A,B, we can observe that:
   (a) The red broken refracted line, which represents the true value of the observation to be predicted, is located between the two broken lines that represent the lower and upper bounds of the predictive internals, which confirms with 3.

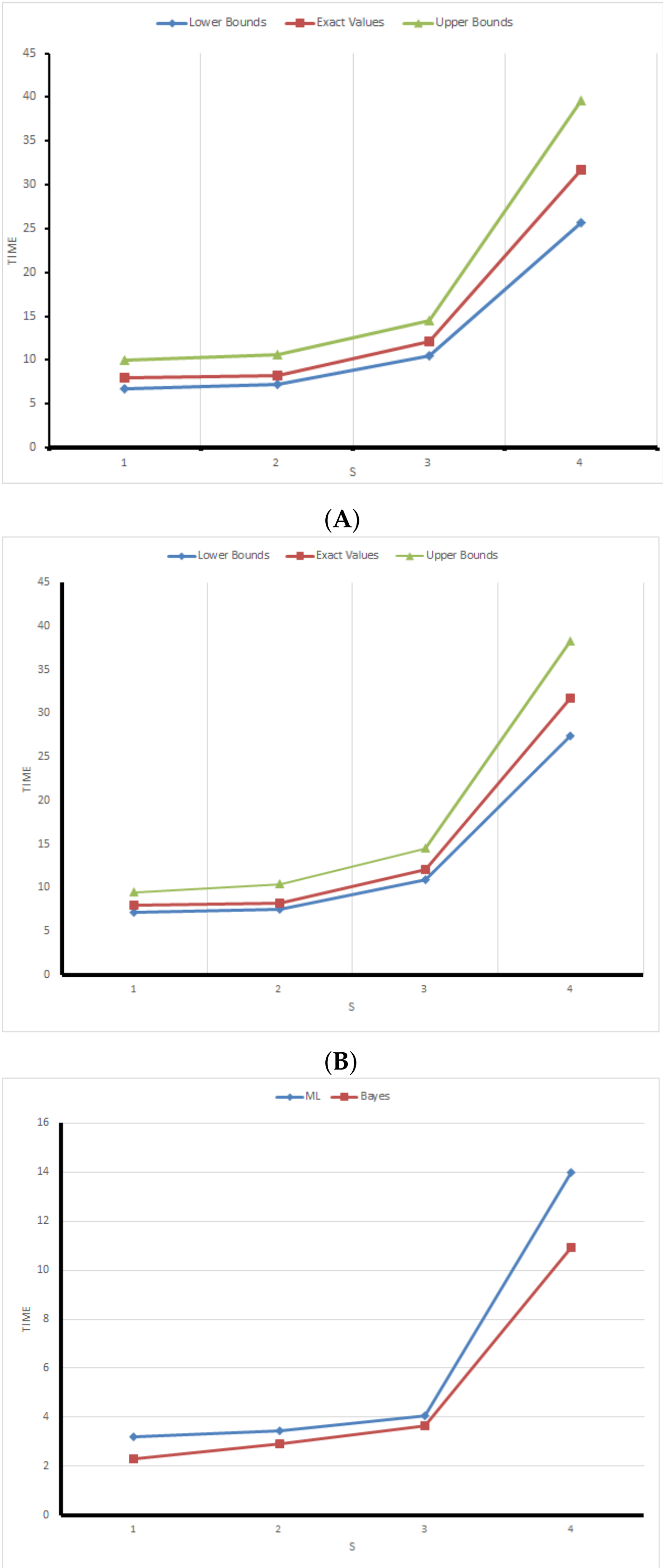
   (b) The lower bounds increase by increasing s.
   (c) The upper bounds also increase by increasing s.
5. From Figures 2C, 3C, 4C and 5C, we can observe:
   (a) The length of the predictive interval increase by increasing s, which confirms with 1.
   (b) The lengths of the predictive intervals obtained using the Bayes approach are less than that obtained by the classical approach, which confirms with 2.

Table 6. PPs and IPs of the future failure time $z_s, s = 1,2,3,4$, based on a generated Balakrishnan $UHCS$ informative sample from real data set I.

Values of (T_1, T_2)		$(1.1, 7.5)$			
Generated z_s (r,k) (n,D) *Case No.*	*Method*	z_1 PP of z_1 IP of z_1 *Length*	z_2 PP of z_2 IP of z_2 *Length*	z_3 PP of z_3 IP of z_3 *Length*	z_4 PP of z_4 IP of z_4 *Length*
$(12,5)$ $(15,11)$ 5	*ML*	0.38819 0.42115 (0.34905,0.51106) 0.16201	0.49016 0.50284 (0.36373,0.54428) 0.18055	0.52015 0.51082 (0.39017,0.60024) 0.21007	0.60823 0.58807 (0.48165,0.7626) 0.28095
	Bayes	0.38819 0.41066 (0.36105,0.49267) 0.13162	0.49016 0.48107 (0.40552,0.56075) 0.15523	0.52015 0.52713 (0.48174,0.65461) 0.17287	0.60823 0.60153 (0.52315,0.76499) 0.24184

Table 7. PPs and IPs of the future failure time $z_s, s = 1,2,3,4$, based on a generated Balakrishnan $UHCS$ informative sample from real data set II.

Values of (T_1, T_2)		$(1.3, 1.8)$			
Generated z_s (r,k) (n,D) *Case No.*	*Method*	z_1 PP of z_1 IP of z_1 *Length*	z_2 PP of z_2 IP of z_2 *Length*	z_3 PP of z_3 IP of z_3 *Length*	z_4 PP of z_4 IP of z_4 *Length*
$(12,5)$ $(15,5)$ 6	*ML*	0.28003 0.30119 (0.20275,0.37427) 0.171752	0.41052 0.38826 (0.25594,0.4890) 0.23306	0.48105 0.41918 (0.30142,0.71214) 0.41072	0.50185 0.46817 (0.35107,0.82210) 0.47103
	Bayes	0.28003 0.29014 (0.26023,0.42308) 0.16285	0.41052 0.39082 (0.27684,0.48890) 0.21206	0.48105 0.45119 (0.33206,0.61621) 0.28415	0.50185 0.48017 (0.39082,0.70595) 0.31513

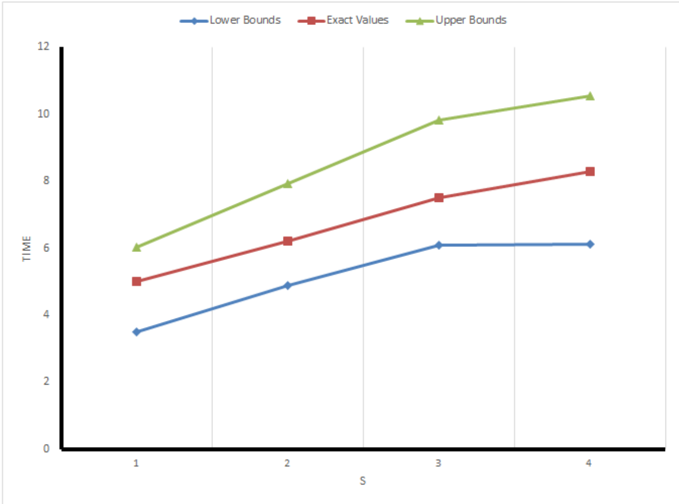


(**A**)

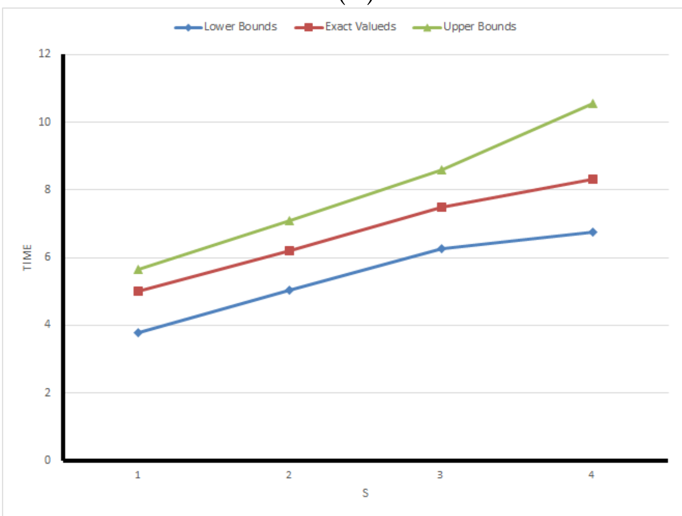
(**B**)

(**C**)

Figure 2. (**A**) ML one-sample predictive intervals based on sample I; (**B**) Bayesian one-sample predictive intervals based on sample I; (**C**) lengths of the one-sample predictive intervals based on sample I.



(**A**)



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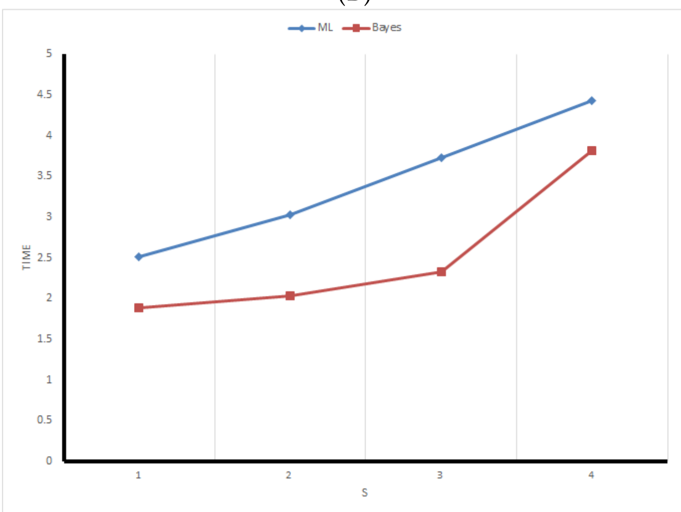
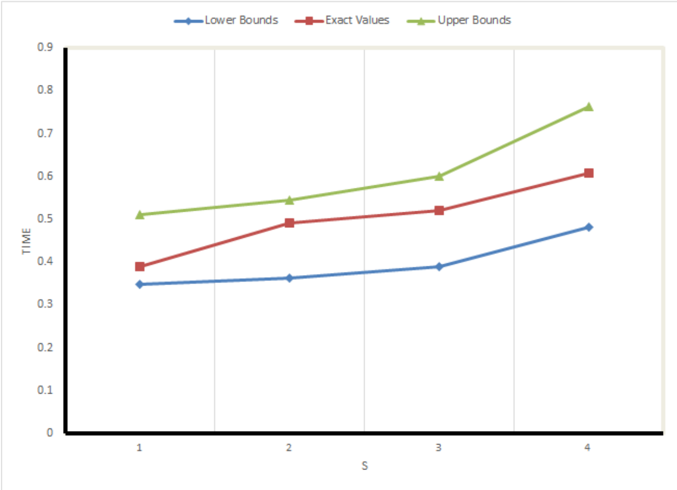
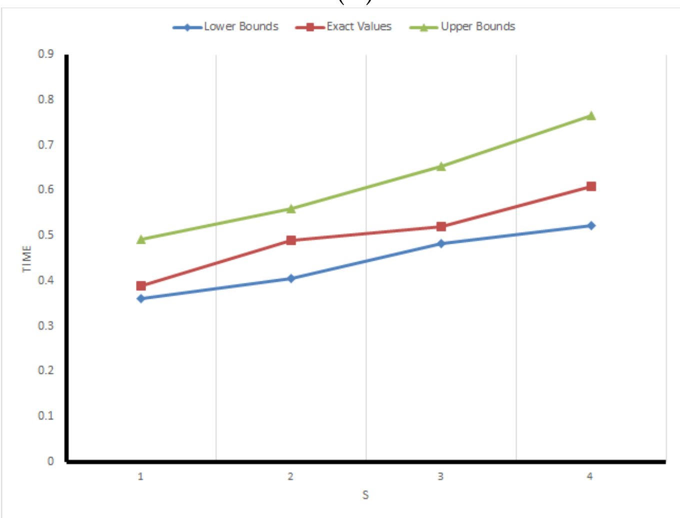


(**C**)

Figure 3. (**A**) ML one-sample predictive intervals based on sample II; (**B**) Bayesian one-sample predictive intervals based on sample II; (**C**) lengths of one-sample predictive intervals based on sample II.



(**A**)



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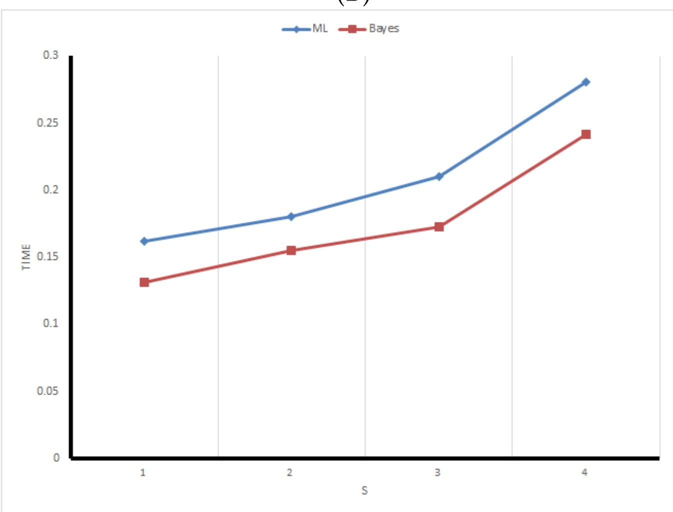
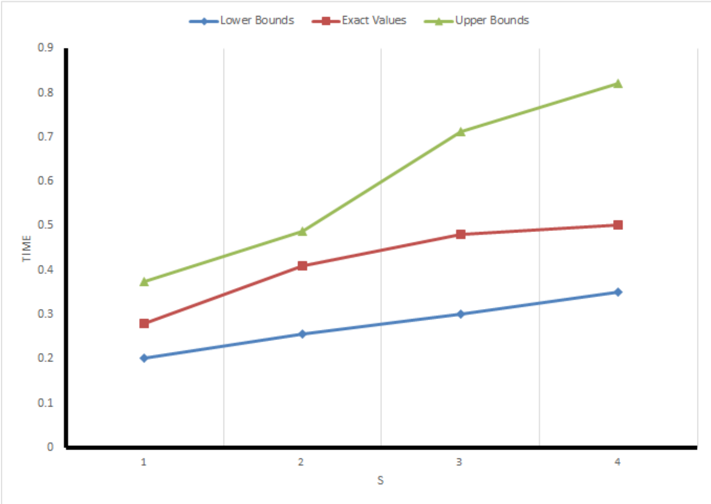
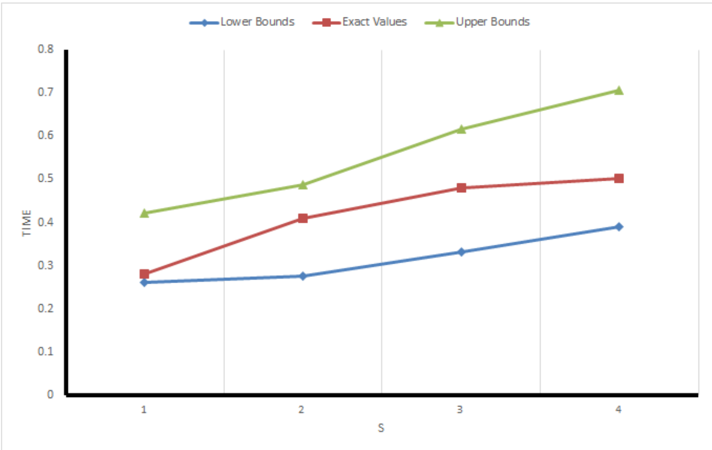


(**C**)

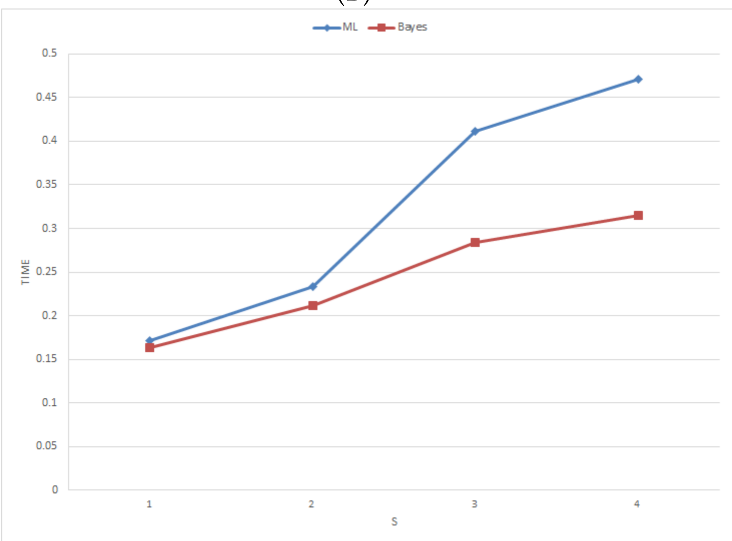
Figure 4. (**A**) ML two-sample predictive intervals based on sample I; (**B**) Bayesian two-sample predictive intervals based on sample I; (**C**) lengths of two-sample predictive intervals based on sample I.



(**A**)



(**B**)



(**C**)

Figure 5. (**A**) ML two-sample predictive intervals based on sample II; (**B**) Bayesian two-sample predictive intervals based on sample II; (**C**) lengths of two-sample predictive intervals based on sample II.

5. Conclusions

In this paper, the PPs and IPs of the future failure times from Burr-X were computed based on a $UHCS$ (suggested by Balakrishnan et al. (2008)) informative sample using different values of r, k, T_1, and T_2, using classical and Bayesian approaches, making some comparisons between the two approaches. Two engineering real data sets were introduced and analyzed using the Burr-X model to emphasize that the studied model fits the given real data sets well. Based on a generated $UHCS$ informative sample from the given real data sets, the PPs and IPs of the future failure times under one- and two-sample schemes were computed using classical and Bayesian approaches; it was found that the predictive intervals using the Bayesian approach were shorter than those computed by the classical approach, which means that the Bayesian approach is better than the other approach. In addition to the tabular description of the results related to the real data sets, graphical descriptions were also introduced. The results of the work confirm that it is possible to use statistical prediction to perform predictive tasks in relation to the conditions of industrial equipment.

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Abbreviations

The following abbreviations are used in this manuscript:

BP	Bayesian predictor
BPI	Bayesian predictive interval
TLA	Three letter acronym
CS	Censoring scheme
CDF	Cumulative distribution function
CP	Coverage probability
HCS	Hybrid censoring scheme
IPs	Interval predictors
LF	Likelihood function
ML	Maximum likelihood
MLEs	Maximum likelihood estimates
MLP	Maximum likelihood predictor
$MLPF$	Maximum likelihood predictive function
$MLPI$	Maximum likelihood predictive interval
PDF	Probability density function
PLF	Predictive likelihood function
$PMLEs$	Predictive maximum likelihood estimates
PPs	predictors
RF	Reliability function
$UHCS$	Unified hybrid censoring scheme

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