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The Non-Linear Fokker–Planck Equation in Low-Regularity Space

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Abstract: We construct the global existence and exponential time decay rates of mild solutions to the non-linear Fokker–Planck equation near a global Maxwellians with small-amplitude initial data in the low regularity function space $L^1_k L^\infty_T L^2_v$ where the regularity assumption on the initial data is weaker.

Keywords: non-linear Fokker–Planck equation; global existence; exponential time decay rates; low regularity function space

MSC: 35Q20; 35A01

1. Introduction and Main Results

In this paper, we are concerned with the Cauchy problem to the nonlinear Fokker–Planck equation as follows

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = \rho \nabla_v \cdot (\nabla_v F + vF), \\ F(0, x, v) = F_0(x, v), \end{cases} \tag{1}$$

where the non-negative unknown function $F(t, x, v)$ is the spatially periodic density distribution function of particles with position $x = (x_1, x_2, x_3) \in \mathbb{T}^3 := [-\pi, \pi]^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ at time $t \geq 0$, and the density $\rho(t, x)$ is defined as $\rho = \int_{\mathbb{R}^3} F dv$.

In statistical mechanics, nonlinear Fokker–Planck equation is a partial differential equation which describes the Brownian motion of particles. This equation illustrates the evolution of particle probability density function with velocity, time and space position under the influence of resistance or random force. This equation is also widely used in various fields such as plasma physics, astrophysics, nonlinear hydrodynamics, theory of electronic circuitry and laser arrays, population dynamics, human movement sciences and marketing.

The global equilibria for the nonlinear Fokker–Planck Equation (1) is the normalized global Maxwellian

$$\mu = \mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Therefore, we can define the perturbation $f = f(t, x, v)$ by

$$F(t, x, v) = \mu + \mu^{\frac{1}{2}} f(t, x, v),$$

then the Cauchy problem (1) of the nonlinear Fokker–Planck is reformulated as [1]

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \rho L f, \\ f(0, x, v) = f_0(x, v) = \mu^{-\frac{1}{2}} (F_0(x, v) - \mu), \end{cases} \tag{2}$$

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where the density $\rho(t,x)$ and the linear Fokker–Planck operator L are given by

$$\rho = 1 + \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} f dv$$

and

$$Lf = \mu^{-\frac{1}{2}} \nabla_v \cdot \left(\mu \nabla_v (\mu^{-\frac{1}{2}} f)\right) = \Delta_v f + \frac{1}{4}(6 - |v|^2) f.$$

Defining the velocity orthogonal projection

$$\mathbf{P} : L^2(\mathbb{R}^3_v) \to \mathrm{Span}\left\{\mu^{\frac{1}{2}},\ v_i \mu^{\frac{1}{2}} (1 \le i \le 3)\right\},$$

then for any given function $f(t,x,v) \in L^2(\mathbb{R}^3_v)$, one has

$$\mathbf{P}f = a(t,x)\mu^{\frac{1}{2}} + b(t,x) \cdot v\mu^{\frac{1}{2}}, \tag{3}$$

with

$$a = \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} f\, dv, \quad b = \int_{\mathbb{R}^3} v \cdot \mu^{\frac{1}{2}} f\, dv. \tag{4}$$

Therefore, we have the following macro–micro decomposition of solutions $f(t,x,v)$ of the nonlinear Fokker–Planck Equation (1) with respect to the given global Maxwellian μ which was introduced in [2]

$$f(t,x,v) = \mathbf{P}f(t,x,v) + \{\mathbf{I} - \mathbf{P}\} f(t,x,v), \tag{5}$$

where $\mathbf{I}$ denotes the identity operator, $\mathbf{P}f$ and $\{\mathbf{I} - \mathbf{P}\}f$ are called the macroscopic and the microscopic component of $f(t,x,v)$, respectively.

Furthermore, multiplying (2) by $\mu^{\frac{1}{2}}$ and integrating with respect to v over $\mathbb{R}^3$ to obtain

$$\partial_t a + \nabla_x \cdot b = 0,$$

then integrating the equality with respect to x over $\mathbb{T}^3$, we get the conservation of mass

$$\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} f(t,x,v)\, dvdx = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} f(0,x,v)\, dvdx = 0. \tag{6}$$

Let $\nu(v) = 1 + |v|^2$ and denote the norm $|\cdot|_\nu$ by

$$|f|_\nu^2 = \int_{\mathbb{R}^3_v} \left(\nu(v)|f|^2 + |\nabla_v f|^2\right) dv. \tag{7}$$

As is known [3–5], the Fokker–Planck operator L is coercive in the sense that there is a positive constant λ_0 such that

$$-(Lf, f)_{L^2_v} \ge \lambda_0 |\{\mathbf{I} - \mathbf{P}\} f|_\nu^2 + |b|^2. \tag{8}$$

Motivated by [6], we use the low-regularity function space $L^1_k L^\infty_T L^2_v$ equipped with norm

$$\|f\|_{L^1_k L^\infty_T L^2_v} := \int_{\mathbb{Z}^3_k} \sup_{0 \le t \le T} \|\hat{f}(t,k,\cdot)\|_{L^2_v}\, d\Sigma(k) < \infty, \tag{9}$$

where the Fourier transformation of $f(t,x,v)$ with respect to $x \in \mathbb{T}^3$ is defined by

$$\hat{f}(t,k,v) = \mathcal{F}_x f(t,k,v) = \int_{\mathbb{T}^3} e^{-ik\cdot x} f(t,x,v)\, dx, \quad k \in \mathbb{Z}^3.$$

In this paper, we will use $d\Sigma(k)$ to denote the discrete measure in $\mathbb{Z}^3$, i.e.,

$$\int_{\mathbb{Z}^3} g(k)\, d\Sigma(k) = \sum_{k\in\mathbb{Z}^3} g(k).$$

Notations

- $A \lesssim B$ means that there is a constant $C > 0$ such that $A \le CB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.
- Denoting the dot product $(f, g) = f \cdot \overline{g}$ for any complex functions.
- Denoting $(\cdot,\cdot)_{L^2_v}$ the complex inner product over L^2_v, i.e.,

$$(f, g)_{L^2_v} = \int_{\mathbb{R}^3_v} f(v)\overline{g(v)}dv.$$

- The convolution of f and g is defined as

$$(\hat{f} * \hat{g})(k) = \int_{\mathbb{Z}^3_l} \hat{f}(k-l)\hat{g}(l)\, d\Sigma(l).$$

- $\mathbb{R}$ denotes the real part of a complex number.

Based on this preparing work, our main result can be stated as follows.

Theorem 1. *Assume that $f_0(x, v)$ satisfies the conservation of mass*

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \mu^{\frac{1}{2}} f(t, x, v)\, dvdx = \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \mu^{\frac{1}{2}} f(0, x, v)\, dvdx = 0.$$

and $F_0(x, v) = \mu + \mu^{\frac{1}{2}} f_0(x, v) \ge 0$, there is a small sufficiently ϵ_0 such that if

$$\|f_0(x, v)\|_{L^1_k L^2_v} \le \epsilon_0,$$

then the Cauchy problem (2) admits a unique global mild solution $f(t, x, v)$ satisfying $F(t, x, v) = \mu + \mu^{\frac{1}{2}} f(t, x, v) \ge 0$, and it holds that

$$\|f(t, x, v)\|_{L^1_k L^\infty_T L^2_v} + \|f(t, x, v)\|_{L^1_k L^2_T L^2_v} \lesssim \|f_0(x, v)\|_{L^1_k L^2_v}, \tag{10}$$

for any $t > 0$. Moreover, there is a constant $\lambda > 0$ such that the solution also admits the time decay estimate

$$\|f(t, x, v)\|_{L^1_k L^2_v} \lesssim e^{-\lambda t}\|f_0(x, v)\|_{L^1_k L^2_v}, \tag{11}$$

for any $t \ge 0$.

Remark 1. *Compared with the integer Sobolev space H^4_x used in [1], the regularity assumption on the initial data is weaker due to $H^4_x \hookrightarrow L^1_k$.*

There are a lot of results about the global existence and large time behavior of solutions to Fokker–Planck-type equations, such as, for the Fokker–Planck–Boltzmann equation, the global existence and temporal decay estimates of classical solutions are established based on the nonlinear energy method developed in [2], under Grad's angular cut-off in [5,7] and without cut-off in [8,9], respectively. As for the Vlasov–Poisson–Fokker–Planck equation, Duan and Liu [3] obtained the time-periodic small-amplitude solution in the three dimensional whole space by Serrin's method. Hwang and Jang [10], Wang [11] obtained the global existence and the time decay of the solution. For the problem (1), the global existence is proved by combining uniform-in-time energy estimates and the decay rates of the solution is obtained by using the precise spectral analysis of the linearized

Fokker–Planck operator as well as the energy method in [1]. Interested readers can refer to the references [4,12–28] for more related details.

We note that the previous related results that are obtained in Sobolev space involved the v-derivatives or x-derivatives, which required high regularity of the initial data. In order to obtain the global in time solutions in low-regularity function space, Duan–Liu–Sakamoto–Strain [6] introduced the space $L^1_k L^\infty_T L^2_v$ to deal with the Landau and non-cutoff Boltzmann equation, where L^1 corresponds to the Weiner algebra over a torus satisfying $\|fg\|_{L^1_k} \leq \|f\|_{L^1_k}\|g\|_{L^1_k}$. Motivated by this method, we are desired to obtain the global existence of solutions to the Fokker–Planck–Boltzmann equation in low regularity function space.

The rest of this paper is organized as follows. In Section 2, we list some basic lemmas which will be used in the later proof. Sections 3 and 4 are devoted to deducing global existence and exponential time decay rates for the solution to the Cauchy problem of the Fokker–Planck–Boltzmann equation respectively, where the proofs of Theorem 1 is complete.

2. Basic Lemmas

In this section, we give some results concerning the linear Fokker–Planck operator L and the nonlinear term.

Lemma 1 ([3–5])**.** *There is a constant $\lambda_0 > 0$ such that*

$$-\mathbb{R}(L\hat{f},\hat{f})_{L^2_v} \geq \lambda_0 |\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k,\cdot)|^2_\nu + |\hat{b}(t,k)|^2. \tag{12}$$

Lemma 2. *It holds that*

$$\begin{aligned}&\int_{\mathbb{Z}^3_k}\left(\int_0^T\left|\left(\hat{a}*L\hat{f},\hat{f}\right)_{L^2_v}\right|dt\right)^{1/2}d\Sigma(k)\\ &\lesssim\left(\eta+\frac{\|f\|_{L^1_kL^\infty_TL^2_v}}{4\eta}\right)\int_{\mathbb{Z}^3_k}\left(\int_0^T|\hat{f}(t,k,\cdot)|^2_\nu\,dt\right)^{1/2}d\Sigma(k)\end{aligned} \tag{13}$$

for any $T > 0$, where the constant $\eta > 0$ can be arbitrarily small. It also holds that

$$\int_{\mathbb{Z}^3_k}\left(\int_0^T|(|\hat{a}*L\hat{f}|,\mu^{\frac{1}{4}})_{L^2_v}|^2dt\right)^{1/2}d\Sigma(k)\lesssim\|f\|_{L^1_kL^\infty_TL^2_v}\|f\|_{L^1_kL^2_TL^2_\nu}. \tag{14}$$

Proof. By Fubini's theorem, one can obtain

$$\begin{aligned}\left|\left(\hat{a}*\Delta_v\hat{f},\hat{f}\right)_{L^2_v}\right|&=\left|\int_{\mathbb{R}^3_v}\left(\int_{\mathbb{Z}^3_l}\hat{a}(t,k-l)\Delta_v\hat{f}(t,l,v)d\Sigma(l)\right)\hat{f}(t,k,v)dv\right|\\&=\left|\int_{\mathbb{Z}^3_l}\hat{a}(t,k-l)\left(\int_{\mathbb{R}^3_v}\Delta_v\hat{f}(t,l,v)\hat{f}(t,k,v)dv\right)d\Sigma(l)\right|\\&=\left|\int_{\mathbb{Z}^3_l}\hat{a}(t,k-l)\left(\int_{\mathbb{R}^3_v}\nabla_v\hat{f}(t,l,v)\nabla_v\hat{f}(t,k,v)dv\right)d\Sigma(l)\right|\\&\lesssim\int_{\mathbb{Z}^3_l}|\hat{a}(t,k-l)|\|\nabla_v\hat{f}(t,l,\cdot)\|_{L^2_v}\|\nabla_v\hat{f}(t,k,\cdot)\|_{L^2_v}d\Sigma(l).\end{aligned}$$

By applying Cauchy–Schwarz's inequality with respect to $\int_0^T(\cdot)dt$ and using Young's inequality, we have

$$
\begin{aligned}
&\int_{\mathbb{Z}_k^3}\left(\int_0^T\left|\left(\hat{a}*\Delta_v\hat{f},\hat{f}\right)_{L_v^2}\right|dt\right)^{1/2}d\Sigma(k)\\
\lesssim&\int_{\mathbb{Z}_k^3}\left(\int_0^T\int_{\mathbb{Z}_l^3}|\hat{a}(t,k-l)|\|\nabla_v\hat{f}(t,l,\cdot)\|_{L_v^2}\|\nabla_v\hat{f}(t,k,\cdot)\|_{L_v^2}d\Sigma(l)dt\right)^{1/2}d\Sigma(k)\\
\lesssim&\int_{\mathbb{Z}_k^3}\left(\int_0^T\left(\int_{\mathbb{Z}_l^3}|\hat{a}(t,k-l)|\|\nabla_v\hat{f}(t,l,\cdot)\|_{L_v^2}d\Sigma(l)\right)^2dt\right)^{1/4}\\
&\times\left(\int_0^T\|\nabla_v\hat{f}(t,k,\cdot)\|_{L_v^2}^2dt\right)^{1/4}d\Sigma(k)\\
\lesssim&\,\eta\int_{\mathbb{Z}_k^3}\left(\int_0^T\|\nabla_v\hat{f}(t,k,\cdot)\|_{L_v^2}^2dt\right)^{1/2}d\Sigma(k)\\
&+\frac{1}{4\eta}\int_{\mathbb{Z}_k^3}\left(\int_0^T\left(\int_{\mathbb{Z}_l^3}|\hat{a}(t,k-l)|\|\nabla_v\hat{f}(t,l,\cdot)\|_{L_v^2}d\Sigma(l)\right)^2dt\right)^{1/2}d\Sigma(k),
\end{aligned}
\tag{15}
$$

where $\eta > 0$ is a sufficiently small universal constant. For the second term in the above inequality, we can obtain

$$
\begin{aligned}
&\left(\int_0^T\left(\int_{\mathbb{Z}_l^3}|\hat{a}(t,k-l)|\|\nabla_v\hat{f}(t,l,\cdot)\|_{L_v^2}d\Sigma(l)\right)^2dt\right)^{1/2}\\
\leq&\int_{\mathbb{Z}_l^3}\left(\int_0^T|\hat{a}(t,k-l)|^2\|\nabla_v\hat{f}(t,l,\cdot)\|_{L_v^2}^2dt\right)^{1/2}d\Sigma(l),
\end{aligned}
\tag{16}
$$

by the Minkowski's inequality $\|\|\cdot\|_{L_l^1}\|_{L_t^2}\leq\|\|\cdot\|_{L_t^2}\|_{L_l^1}$. By Fubini's theorem and translation invariance with (16), we obtain

$$
\begin{aligned}
&\int_{\mathbb{Z}_k^3}\left(\int_0^T\left(\int_{\mathbb{Z}_l^3}|\hat{a}(t,k-l)|\|\nabla_v\hat{f}(t,l,\cdot)\|_{L_v^2}d\Sigma(l)\right)^2dt\right)^{1/2}d\Sigma(k)\\
\leq&\int_{\mathbb{Z}_k^3}\int_{\mathbb{Z}_l^3}\left(\int_0^T|\hat{a}(t,k-l)|^2\|\nabla_v\hat{f}(t,l,\cdot)\|_{L_v^2}^2dt\right)^{1/2}d\Sigma(l)d\Sigma(k)\\
\leq&\int_{\mathbb{Z}_k^3}\int_{\mathbb{Z}_l^3}\sup_{0\leq t\leq T}|\hat{a}(t,k-l)|\left(\int_0^T\|\nabla_v\hat{f}(t,l,\cdot)\|_{L_v^2}^2dt\right)^{1/2}d\Sigma(l)d\Sigma(k)\\
=&\int_{\mathbb{Z}_l^3}\int_{\mathbb{Z}_k^3}\sup_{0\leq t\leq T}|\hat{a}(t,k-l)|\left(\int_0^T\|\nabla_v\hat{f}(t,l,\cdot)\|_{L_v^2}^2dt\right)^{1/2}d\Sigma(k)d\Sigma(l)\\
=&\int_{\mathbb{Z}_l^3}\left(\int_{\mathbb{Z}_k^3}\sup_{0\leq t\leq T}|\hat{a}(t,k-l)|d\Sigma(k)\right)\left(\int_0^T\|\nabla_v\hat{f}(t,l,\cdot)\|_{L_v^2}^2dt\right)^{1/2}d\Sigma(l)\\
=&\left(\int_{\mathbb{Z}_k^3}\sup_{0\leq t\leq T}|\hat{a}(t,k)|d\Sigma(k)\right)\int_{\mathbb{Z}_l^3}\left(\int_0^T\|\nabla_v\hat{f}(t,l,\cdot)\|_{L_v^2}^2dt\right)^{1/2}d\Sigma(l).
\end{aligned}
\tag{17}
$$

Applying the Hölder inequality, we have

$$
\begin{aligned}
\int_{\mathbb{Z}_k^3}\sup_{0\leq t\leq T}|\hat{a}(t,k)|d\Sigma(k)&=\int_{\mathbb{Z}_k^3}\sup_{0\leq t\leq T}\left|\int_{\mathbb{R}^3}\mu^{\frac{1}{2}}\hat{f}(t,k,v)dv\right|d\Sigma(k)\\
&\leq\int_{\mathbb{Z}_k^3}\sup_{0\leq t\leq T}\|\hat{f}(t,k,\cdot)\|_{L_v^2}d\Sigma(k).
\end{aligned}
\tag{18}
$$

Combining (15), (17) and (18), we obtain

$$\begin{aligned}&\int_{\mathbb{Z}_k^3}\left(\int_0^T\left|\left(\hat{a}*\Delta_v\hat{f},\hat{f}\right)_{L_v^2}\right|dt\right)^{1/2}d\Sigma(k)\\&\lesssim\left(\eta+\frac{\|f\|_{L_k^1L_T^\infty L_v^2}}{4\eta}\right)\int_{\mathbb{Z}_k^3}\left(\int_0^T|\hat{f}(t,k,\cdot)|_\nu^2\,dt\right)^{1/2}d\Sigma(k),\end{aligned}\tag{19}$$

with the fact

$$\int_{\mathbb{Z}_k^3}\left(\int_0^T\|\nabla_v\hat{f}(t,k,\cdot)\|_{L_v^2}^2dt\right)^{1/2}d\Sigma(k)\lesssim\int_{\mathbb{Z}_k^3}\left(\int_0^T|\hat{f}(t,k,\cdot)|_\nu^2\,dt\right)^{1/2}d\Sigma(k).$$

Similarly, we can obtain

$$\begin{aligned}&\int_{\mathbb{Z}_k^3}\left(\int_0^T\left|\left(\hat{a}*\left(\frac{1}{4}(6-|v|^2)\hat{f}\right),\hat{f}\right)_{L_v^2}\right|dt\right)^{1/2}d\Sigma(k)\\&\lesssim\left(\eta+\frac{\|f\|_{L_k^1L_T^\infty L_v^2}}{4\eta}\right)\int_{\mathbb{Z}_k^3}\left(\int_0^T|\hat{f}(t,k,\cdot)|_\nu^2\,dt\right)^{1/2}d\Sigma(k).\end{aligned}\tag{20}$$

By (19) and (20) and the linear Fokker–Planck operator $L\hat{f}=\Delta_v\hat{f}+\frac{1}{4}(6-|v|^2)\hat{f}$, the desired result (13) can be obtained. The proof of (14) is also can be deduced similarly. □

3. Global Existence

Firstly, we need to obtain the estimates of the microscopic dissipation for the solution f in (2).

Proposition 1. *Under the assumptions in Theorem 1, it holds that*

$$\begin{aligned}&\int_{\mathbb{Z}_k^3}\sup_{0\le t\le T}\|\hat{f}(t,k,\cdot)\|_{L_v^2}d\Sigma(k)+\int_{\mathbb{Z}_k^3}\left(\int_0^T|\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k,\cdot)|_\nu^2\,dt\right)^{1/2}d\Sigma(k)\\&+\int_{\mathbb{Z}_k^3}\left(\int_0^T|\hat{b}(t,k)|^2\,dt\right)^{1/2}d\Sigma(k)\\&\lesssim\|f_0\|_{L_k^1L_v^2}+\left(\eta+\frac{\|f\|_{L_k^1L_T^\infty L_v^2}}{4\eta}\right)\int_{\mathbb{Z}_k^3}\left(\int_0^T|\hat{f}(t,k,\cdot)|_\nu^2\,dt\right)^{1/2}d\Sigma(k),\end{aligned}\tag{21}$$

for any $T>0$, where the constant $\eta>0$ can be arbitrarily small.

Proof. Taking Fourier transform of (2) with respect x, we have

$$\partial_t\hat{f}(t,k,v)+iv\cdot\hat{f}(t,k,v)-L\hat{f}(t,k,v)=(\hat{a}*L\hat{f})(t,k,v),$$

where the convolutions are taken with respect to $k\in\mathbb{Z}^3$:

$$(\hat{a}*L\hat{f})(t,k,v)=\int_{\mathbb{Z}_l^3}\hat{a}(t,k-l)L\hat{f}(t,l,v)d\Sigma(l).$$

Taking product with the complex conjugate of $\hat{f}(t,k,v)$ and further taking the real part of the resulting equation, we have

$$\frac{1}{2}\frac{d}{dt}|\hat{f}(t,k,v)|^2-\mathbb{R}(L\hat{f},\hat{f})=\mathbb{R}(\hat{a}*L\hat{f},\hat{f}),$$

integrating the above identity with respect to v and then t, we obtain

$$\frac{1}{2}\|\hat{f}(t,k,\cdot)\|^2_{L^2_v} - \int_0^t \mathbb{R}(L\hat{f},\hat{f})_{L^2_v} d\tau = \frac{1}{2}\|\hat{f}_0(k,\cdot)\|^2_{L^2_v} + \int_0^t \mathbb{R}(\hat{a} * L\hat{f},\hat{f})_{L^2_v} d\tau.$$

Recalling the coercivity estimates of L in Lemma 1, we can obtain

$$\begin{aligned}&\frac{1}{2}\|\hat{f}(t,k,\cdot)\|^2_{L^2_v} + \lambda_0 \int_0^t |\{\mathbf{I}-\mathbf{P}\}\hat{f}(\tau,k,\cdot)|^2_v d\tau + \lambda_0 \int_0^t |\hat{b}(\tau,k)|^2 d\tau \\ \leq &\frac{1}{2}\|\hat{f}_0(k,\cdot)\|^2_{L^2_v} + \int_0^t \left|\mathbb{R}(\hat{a} * L\hat{f},\hat{f})_{L^2_v}\right| d\tau.\end{aligned} \tag{22}$$

Taking the square root on both sides and using the inequality

$$\frac{1}{\sqrt{3}}(A+B+C) \leq \sqrt{A^2+B^2+C^2}, \sqrt{A^2+B^2} \leq A+B, A,B,C \geq 0,$$

we further have

$$\begin{aligned}&\frac{1}{\sqrt{6}}\|\hat{f}(t,k,\cdot)\|_{L^2_v} + \frac{\sqrt{\lambda_0}}{\sqrt{3}}\left(\int_0^t |\{\mathbf{I}-\mathbf{P}\}\hat{f}(\tau,k,\cdot)|^2_v d\tau\right)^{1/2} + \frac{\sqrt{\lambda_0}}{\sqrt{3}}\left(\int_0^t |\hat{b}(\tau,k)|^2 d\tau\right)^{1/2} \\ \leq &\frac{1}{\sqrt{2}}\|\hat{f}_0(k,\cdot)\|_{L^2_v} + \left(\int_0^t \left|\mathbb{R}(\hat{a} * L\hat{f},\hat{f})_{L^2_v}\right| d\tau\right)^{1/2}.\end{aligned} \tag{23}$$

Moreover, taking $\sup_{0\leq t\leq T}$ on both sides of the above inequality and integrating the resulting inequality with respect to $d\Sigma(k)$ over $\mathbb{Z}^3$, we have

$$\begin{aligned}&\int_{\mathbb{Z}^3_k} \sup_{0\leq t\leq T} \|\hat{f}(t,k,\cdot)\|_{L^2_v} d\Sigma(k) + \int_{\mathbb{Z}^3_k}\left(\int_0^T |\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k,\cdot)|^2_v \, dt\right)^{1/2} d\Sigma(k) \\ &+ \int_{\mathbb{Z}^3_k}\left(\int_0^T |\hat{b}(t,k)|^2 \, dt\right)^{1/2} d\Sigma(k) \\ \lesssim &\|f_0\|_{L^1_k L^2_v} + \int_{\mathbb{Z}^3_k}\left(\int_0^T \left|\left(\hat{a} * L\hat{f},\hat{f}\right)_{L^2_v}\right| dt\right)^{1/2} d\Sigma(k) \\ \lesssim &\|f_0\|_{L^1_k L^2_v} + \left(\eta + \frac{\|f\|_{L^1_k L^\infty_T L^2_v}}{4\eta}\right) \int_{\mathbb{Z}^3_k}\left(\int_0^T |\hat{f}(t,k,\cdot)|^2_v \, dt\right)^{1/2} d\Sigma(k),\end{aligned} \tag{24}$$

by (13) in Lemma 2. Thus the desired estimate (21) are obtained. □

Now we give the estimate of the macroscopic component $\hat{a}(t,k)$ by the dual argument.

Proposition 2. *Under the assumptions of Theorem 1, it holds that*

$$\begin{aligned}\int_{\mathbb{Z}^3_k}\left(\int_0^T |\hat{a}(t,k)|^2 \, dt\right)^{1/2} d\Sigma(k) \lesssim &\|f\|_{L^1_k L^\infty_T L^2_v} + \|f_0\|_{L^1_k L^2_v} + \|f\|_{L^1_k L^\infty_T L^2_v}\|f\|_{L^1_k L^2_T L^2_v} \\ &+ \int_{\mathbb{Z}^3_k}\left(\int_0^T |\hat{b}(t,k)|^2 \, dt\right)^{1/2} d\Sigma(k) \\ &+ \int_{\mathbb{Z}^3_k}\left(\int_0^T |\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k,\cdot)|^2_v \, dt\right)^{1/2} d\Sigma(k).\end{aligned} \tag{25}$$

Proof. In order to obtain the estimate of a, we take a test function as

$$\hat{\Phi}(t,k,v) \in C^1\left((0,\infty) \times \mathbb{Z}^3 \times \mathbb{R}^3\right),$$

which will be fixed later. Applying the Fourier transform to (2), taking the inner product of it and $\hat{\Phi}$ in L^2_v, then integrating the resultant over $[0, T]$, one can obtain

$$
\begin{aligned}
&(\hat{f}, \hat{\Phi})|_{t=T} - (\hat{f}, \hat{\Phi})|_{t=0} - \int_0^T (\hat{f}, \partial_t \hat{\Phi}) dt - \int_0^T (\hat{f}, v \cdot ik\hat{\Phi}) dt \\
&= \int_0^T (\hat{a} * L\hat{f}, \hat{\Phi}) dt + \int_0^T (L\hat{f}, \hat{\Phi}) dt,
\end{aligned}
$$

where we have used the notation $(\cdot, \cdot) = (\cdot, \cdot)_{L^2_v}$ and set $T > 0$ be an arbitrary fixed constant. Denoting $(\hat{f}, \hat{\Phi})(T) = (\hat{f}, \hat{\Phi})|_{t=T}$, $(\hat{f}, \hat{\Phi})(0) = (\hat{f}, \hat{\Phi})|_{t=0}$ and plugging in the macro–micro decomposition, we have

$$
\begin{aligned}
-\int_0^T (\mathbf{P}\hat{f}, v \cdot ik\hat{\Phi}) dt = &\underbrace{(\hat{f}, \hat{\Phi})(0) - (\hat{f}, \hat{\Phi})(T) + \int_0^T (\hat{f}, \partial_t \hat{\Phi}) dt + \int_0^T (\{\mathbf{I} - \mathbf{P}\}\hat{f}, v \cdot ik\hat{\Phi}) dt}_{J_1} \\
&+ \underbrace{\int_0^T (\hat{a} * L\hat{f}, \hat{\Phi}) dt}_{J_2} + \underbrace{\int_0^T (L\hat{f}, \hat{\Phi}) dt}_{J_3}.
\end{aligned}
$$

Thanks to (6), we can get the conservation law for mass, i.e.,

$$
\int_{\mathbb{T}^3} a(t, x) dx = \int_{\mathbb{T}^3} a(0, x) dx = 0,
$$

so that $\hat{a}(t, 0) = 0$. Now we choose the test function as

$$
\hat{\Phi}(t, k, v) = (|v|^2 - 10) v \cdot ik\hat{\phi}_a(t, k) \mu^{\frac{1}{2}}, \tag{26}
$$

where $\hat{\phi}_a(t, k)$ is a solution to

$$
|k|^2 \hat{\phi}_a(t, k) = \hat{a}(t, k).
$$

Since $\hat{a}(t, 0) = 0$, we can formally write $\hat{\phi}_a(t, k) = \hat{a}(t, k)/|k|^2$ for any $k \in \mathbb{Z}^3$ where $\hat{\phi}_a(t, 0) = 0$. Using the estimate of a in [6], we can obtain

$$
\begin{aligned}
-\int_0^T (\mathbf{P}\hat{f}, v \cdot ik\hat{\Phi}) dt &= 5 \int_0^T |\hat{a}(t, k)|^2 dt, \\
|J_1| \lesssim &\|\hat{f}(T, k, \cdot)\|^2_{L^2_v} + \|\hat{f}_0(k)\|^2_{L^2_v} + \eta \int_0^T |\hat{a}(t, k)|^2 dt \\
&+ \int_0^T |\hat{b}(t, k)|^2 dt + \int_0^T |\{\mathbf{I} - \mathbf{P}\}\hat{f}(t, k)|^2_v dt, \\
|J_2| \lesssim &\eta \int_0^T |\hat{a}(t, k)|^2 dt + \int_0^T |(|\hat{a} * L\hat{f}|, \mu^{\frac{1}{4}})_{L^2_v}|^2 dt,
\end{aligned}
$$

where $\eta > 0$ is a sufficiently small universal constant. Now we concentrate on the estimate of J_3, since

$$
\begin{aligned}
|J_3| &= \left| \int_0^T (\Delta_v \hat{f} + \frac{1}{4}(6 - |v|^2)\hat{f}, \hat{\Phi}) dt \right| \\
&\leq \underbrace{\left| \int_0^T (\Delta_v \hat{f}, \hat{\Phi}) dt \right|}_{J_{3,1}} + \underbrace{\left| \int_0^T (\frac{1}{4}(6 - |v|^2)\hat{f}, \hat{\Phi}) dt \right|}_{J_{3,2}}.
\end{aligned}
$$

By virtue of the macro–micro decomposition, (5) gives

$$
\begin{aligned}
J_{3,1} \leq& \Big| \int_0^T (\Delta_v \mathbf{P}\hat{f}, \hat{\Phi}) dt \Big| + \Big| \int_0^T (\Delta_v \{\mathbf{I}-\mathbf{P}\}\hat{f}, \hat{\Phi}) dt \Big| \\
\leq& \underbrace{\Big| \int_0^T (\hat{a}\Delta_v \mu^{\frac{1}{2}}, \hat{\Phi}) dt \Big|}_{J_{3,1}^1} + \underbrace{\Big| \int_0^T (\hat{b}\Delta_v (v\mu^{\frac{1}{2}}), \hat{\Phi}) dt \Big|}_{J_{3,1}^2} + \underbrace{\Big| \int_0^T (\Delta_v \{\mathbf{I}-\mathbf{P}\}\hat{f}, \hat{\Phi}) dt \Big|}_{J_{3,1}^3}.
\end{aligned}
$$

Owing to $\Delta_v \mu^{\frac{1}{2}} = (\frac{1}{4}v^2 - \frac{3}{2})\mu^{\frac{1}{2}}$ and (26), we obtain

$$
\begin{aligned}
J_{3,1}^1 =& \Big| \int_0^T \Big(\hat{a}\Delta_v \mu^{\frac{1}{2}}, (|v|^2 - 10)v \cdot ik\hat{\phi}_a(t,k)\mu^{\frac{1}{2}} \Big) dt \Big| \\
=& \Big| \int_0^T \Big(\hat{a}(\frac{1}{4}v^2 - \frac{3}{2})\mu^{\frac{1}{2}}, (|v|^2 - 10)v \cdot ik\hat{\phi}_a(t,k)\mu^{\frac{1}{2}} \Big) dt \Big| = 0,
\end{aligned}
$$

as the integrand function is odd for v. Due to $k \in \mathbb{Z}^3$ and $|k|^2\hat{\phi}_a(t,k) = \hat{a}(t,k)$, it holds that with the Young's inequality

$$
J_{3,1}^2 \lesssim \int_0^T |\hat{b}(t,k)||k||\hat{\phi}_a(t,k)| dt \lesssim \eta \int_0^T |\hat{a}(t,k)|^2 dt + C_\eta \int_0^T |\hat{b}(t,k)|^2 dt.
$$

Similarly, we can obtain

$$
J_{3,1}^3 = \Big| \int_0^T (\{\mathbf{I}-\mathbf{P}\}\hat{f}, \Delta_v \hat{\Phi}) dt \Big| dt \lesssim \eta \int_0^T |\hat{a}(t,k)|^2 dt + C_\eta \int_0^T |\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k)|_\nu^2 dt.
$$

Regarding the estimate of $J_{3,2}$, we can also deduce that

$$
\begin{aligned}
J_{3,2} \lesssim& \Big| \int_0^T (\frac{1}{4}(6-|v|^2)\mathbf{P}\hat{f}, \hat{\Phi}) dt \Big| + \Big| \int_0^T (\frac{1}{4}(6-|v|^2)\{\mathbf{I}-\mathbf{P}\}\hat{f}, \hat{\Phi}) dt \Big| \\
\lesssim& \eta \int_0^T |\hat{a}(t,k)|^2 dt + C_\eta \int_0^T |\hat{b}(t,k)|^2 dt + C_\eta \int_0^T |\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k)|_\nu^2 dt,
\end{aligned}
$$

where

$$
\begin{aligned}
& \int_0^T (\frac{1}{4}(6-|v|^2)\hat{a}\mu^{\frac{1}{2}}, \hat{\Phi}) dt \\
=& \int_0^T (\frac{1}{4}(6-|v|^2)\hat{a}\mu^{\frac{1}{2}}, (|v|^2 - 10)v \cdot ik\hat{\phi}_a(t,k)\mu^{\frac{1}{2}}) dt = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^T (\frac{1}{4}(6-|v|^2)(|v|^2-3)\hat{c}\mu^{\frac{1}{2}}, \hat{\Phi}) dt \\
=& \int_0^T (\frac{1}{4}(6-|v|^2)(|v|^2-3)\hat{c}\mu^{\frac{1}{2}}, (|v|^2 - 10)v \cdot ik\hat{\phi}_a(t,k)\mu^{\frac{1}{2}}) dt = 0,
\end{aligned}
$$

since the integrand function is odd for v. By collecting the above estimates and taking $\eta > 0$ is small enough, we obtain

$$
\begin{aligned}
\int_0^T |\hat{a}(t,k)|^2 dt \lesssim& \|\hat{f}(T,k,\cdot)\|_{L_v^2}^2 + \|\hat{f}_0(k)\|_{L_v^2}^2 + \int_0^T |\hat{b}(t,k)|^2 dt \\
&+ \int_0^T |\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k)|_\nu^2 dt + \int_0^T |(|\hat{a} * L\hat{f}|, \mu^{\frac{1}{4}})_{L_v^2}|^2 dt.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\int_{\mathbb{Z}_k^3}\left(\int_0^T |\hat{a}(t,k)|^2\,dt\right)^{1/2} d\Sigma(k) \lesssim& \|f\|_{L^1_kL^\infty_TL^2_v} + \|f_0\|_{L^1_kL^2_v} + \|f\|_{L^1_kL^\infty_TL^2_v}\|f\|_{L^1_kL^2_TL^2_v} \\
&+ \int_{\mathbb{Z}_k^3}\left(\int_0^T |\hat{b}(t,k)|^2\,dt\right)^{1/2} d\Sigma(k) \\
&+ \int_{\mathbb{Z}_k^3}\left(\int_0^T |\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,k,\cdot)|_v^2\,dt\right)^{1/2} d\Sigma(k),
\end{aligned}
$$

where we have used

$$
\int_{\mathbb{Z}_k^3} \|\hat{f}(T,k,\cdot)\|_{L^2_v} d\Sigma(k) \leq \int_{\mathbb{Z}_k^3} \sup_{0\leq t\leq T} \|\hat{f}(t,k,\cdot)\|_{L^2_v} d\Sigma(k) = \|f\|_{L^1_kL^\infty_TL^2_v},
$$

and

$$
\int_{\mathbb{Z}_k^3} \left(\int_0^T |(|\hat{a}*L\hat{f}|,\mu^{\frac{1}{4}})_{L^2_v}|^2 dt\right)^{1/2} d\Sigma(k) \lesssim \|f\|_{L^1_kL^\infty_TL^2_v}\|f\|_{L^1_kL^2_TL^2_v},
$$

in Lemma 2. Thus the proof the proposition is complete. □

Proposition 3. *Under the assumptions of Theorem 1, it holds that*

$$
\begin{aligned}
&\|f\|_{L^1_kL^\infty_TL^2_v} + \|f\|_{L^1_kL^2_TL^2_v} \\
&\lesssim \|f_0\|_{L^1_kL^2_v} + \|f\|_{L^1_kL^\infty_TL^2_v}\|f\|_{L^1_kL^2_TL^2_v}.
\end{aligned}
\tag{27}
$$

Proof. Taking the linear combination as (21) + $M \times$ (25) with $M > 0$ being small enough and noticing that

$$
\begin{aligned}
\|\mathbf{P}f\|_{L^1_kL^2_TL^2_v} &\sim \|[a,b,c]\|_{L^1_kL^2_T}, \\
\|f\|_{L^1_kL^2_TL^2_v} &\sim \|\{\mathbf{I}-\mathbf{P}\}f\|_{L^1_kL^2_TL^2_v} + \|\mathbf{P}f\|_{L^1_kL^2_TL^2_v},
\end{aligned}
$$

the desired estimate (27) are obtained. □

With the above preparation in hand, we are ready to deduce the global-in-time existence of the solution. Firstly, the local-in-time existence and uniqueness of the solutions to the Cauchy problem (2) can be established by performing the standard arguments as in [6], where we omit its proof for simplicity. To extend the local solution into the global one, we can deduce that

$$
\|f\|_{L^1_kL^\infty_TL^2_v} + \|f\|_{L^1_kL^2_TL^2_v} \lesssim \|f_0\|_{L^1_kL^2_v},
$$

from (27) in Proposition 3 by virtue of the smallness assumption on $\|f_0\|_{L^1_kL^2_v}$. Combining this with the local existence, the global mild solution and uniqueness follows immediately from the standard continuity argument. This completes the proof of the global existence and the uniform estimate (10).

4. Large Time Behavior

To deduce the exponential time decay rates of the solution $f(t,x,v)$, we take

$$
\hat{h} = e^{\lambda t}\hat{f}
$$

with $\lambda > 0$ which will be chosen later. Since f satisfies the system (2), then $\hat{h}$ satisfies

$$
\partial_t\hat{h} + ik\cdot v\hat{h} = e^{-\lambda t}(\hat{a}*L\hat{f}) + \lambda\hat{h} + L\hat{h},
$$

with initial data

$$
\hat{h}(0,k,v) = \hat{h}_0(k,v).
$$

Using the same method to deduce Proposition 3, we can obtain

$$\begin{aligned} \|h\|_{L^1_k L^\infty_T L^2_v} + \|h\|_{L^1_k L^2_T L^2_v} \lesssim & \|h_0\|_{L^1_k L^2_v} + \sqrt{\lambda} \int_{\mathbb{Z}^3_k} \left(\int_0^T ||\hat{h}(t,k,\cdot)||^2_{L^2_v} \, dt \right)^{1/2} d\Sigma(k) \\ \leq & \|h_0\|_{L^1_k L^2_v} + \sqrt{\lambda} \|h\|_{L^1_k L^2_T L^2_v}, \end{aligned} \tag{28}$$

for any $T > 0$. Then, take $\lambda > 0$ to be small enough, yields that

$$\|h\|_{L^1_k L^\infty_T L^2_v} + \|h\|_{L^1_k L^2_T L^2_v} \lesssim \|h_0\|_{L^1_k L^2_v}.$$

By the Minkowski's inequality $\| \| \cdot \|_{L^1_k} \|_{L^\infty_T} \leq \| \| \cdot \|_{L^\infty_T} \|_{L^1_k}$, we can deduce

$$\int_{\mathbb{Z}^3_k} \|\hat{h}(t,k,\cdot)\|_{L^2_v} d\Sigma(k) \lesssim \|h_0\|_{L^1_k L^2_v}.$$

Since that $\hat{h} = e^{\lambda t} \hat{f}$, then one can obtain

$$\|f(t)\|_{L^1_k L^2_v} \lesssim e^{-\lambda t} \|f_0\|_{L^1_k L^2_v},$$

Thus, we have completed the proof of Theorem 1.

5. Conclusions

In this paper, the global existence and exponential time decay rates of mild solutions to the nonlinear Fokker–Planck equation are obtained in the low regularity space $L^1_k L^\infty_T L^2_v$ by the nonlinear energy estimates. Compared with the previous results, the regularity assumption on the initial data is weaker.

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