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Starlikeness Associated with the Van Der Pol Numbers

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Abstract: In this paper, we define a subclass of starlike functions associated with the Van der Pol numbers. For this class, we derive structural formula, radius of starlikeness of order α, strong starlikeness, and some inclusion results. We also study radii problems for various classes of analytic functions. Furthermore, we investigate some coefficient-related problems which include the sharp initial coefficient bounds and sharp bounds on Hankel determinants of order two and three.

Keywords: analytic functions; Van der Pol numbers; starlike functions; coefficient bounds; Hankel determinants; radii problems

MSC: 30C45; 30C50

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1. Introduction and Preliminaries

Van der Pol [1] studied the sequence $\mathcal{V}_0,\ \mathcal{V}_1,\ \mathcal{V}_2, \cdots$ by using

$$\psi_{\mathcal{V}}(\varsigma) = \frac{\varsigma^3}{6(e^{\varsigma}(\varsigma-2)+(\varsigma+2))} = \sum_{m=0}^{\infty} \frac{\mathcal{V}_m}{m!}\varsigma^m, \tag{1}$$

where the numbers $\mathcal{V}_m$ were later named Van der Pol numbers. These numbers are used in unsmoothing a smoothed function of three variables. The Bernoulli numbers are analogous to $\mathcal{V}_m$ for functions of one variable. The first few of these numbers are $\mathcal{V}_0 = 1$, $\mathcal{V}_1 = \frac{-1}{2}$, $\mathcal{V}_2 = \frac{1}{5}$, $\mathcal{V}_3 = \frac{-1}{120}$; see [2]. The numbers $\mathcal{V}_m$ can be related with the Rayleigh function; see [2]. The Rayleigh functions can be represented in terms of the zeros of the Bessel function; see [3–5]. Howard [6] showed that Euler and Bernoulli polynomials have identical properties to the Van der Pol polynomials.

Geometric function theory is the study of the geometric properties of analytic functions in $\mathbb{D} = \{\varsigma : |\varsigma| < 1,\ \varsigma \in \mathbb{C}\}$. The Riemann mapping theorem is considered as the cornerstone of the theory. The analytic and univalent functions and their generalizations have various applications, such as fluid mechanics [7], image processing, and signal processing [8], while conformal mappings (locally univalent functions) are very useful in cryptography.

Now we give some notions of the theory which will be helpful in our study.

Denote by $\mathcal{A}_m$ the class of functions k which are analytic and have the expansion of the form $k(\varsigma) = \varsigma + d_{m+1}\varsigma^2 + d_{m+2}\varsigma^3 + \cdots$ in $\mathbb{D} = \{\varsigma : |\varsigma| < 1,\ \varsigma \in \mathbb{C}\}$. The class $\mathcal{A}_1 = \mathcal{A}$ is well-known. A function $k \in \mathcal{A}$ is given as

$$k(\varsigma) = \varsigma + \sum_{m=2}^{\infty} d_m \varsigma^m, \quad \varsigma \in \mathbb{D}. \tag{2}$$

Let $\mathcal{S}$ represent a class of univalent functions in $\mathcal{A}$. We denote by $\mathcal{B}$ a family of self maps ω, analytic (holomorphic) in $\mathbb{D}$ with $\omega(0) = 0$. The function ω with such properties is known as a Schwarz function. Consider that functions k and g are both analytic (holomorphic) in $\mathbb{D}$. Then we write mathematically $f \prec g$, read as f is subordinated to g such that $k(\varsigma) = g(\omega(\varsigma))$ for $\omega \in \mathcal{B}$ and $\varsigma \in \mathbb{D}$. If g is univalent (one-to-one) with $k(0) = g(0)$, then $k(\mathbb{D}) \subset g(\mathbb{D})$.

This concept is very useful in studying various problems in function theory. Ma and Minda [9] beautifully utilized this concept to unify various classes of starlike and convex functions. These are defined analytically as $\mathcal{S}^*(\psi) := \{k \in \mathcal{A} : \varsigma k'(\varsigma)/k(\varsigma) \prec \psi(\varsigma)\}$ and $\mathcal{C}(\psi) := \{k \in \mathcal{A} : 1 + \varsigma k''(\varsigma)/k'(\varsigma) \prec \psi(\varsigma)\}$, respectively. The analytic and univalent function ψ satisfies $\psi(0) = 1$ and $Re\{\psi'(\varsigma)\} > 0$, $\varsigma \in \mathbb{D}$ and $\psi(\mathbb{D})$ is a convex set in $\mathbb{C}$. We see that the class $\mathcal{S}^*(\psi)$ generalizes many classes. Some are given as follows:

i. $\mathcal{S}^* = \mathcal{S}^*\left(\frac{1+\varsigma}{1-\varsigma}\right)$.
ii. $\mathcal{S}^*[A, B] := \mathcal{S}^*\left(\frac{1+A\varsigma}{1+B\varsigma}\right)$, $-1 \le B < A \le 1$, see [10].
iii. $\mathcal{S}^*(\alpha) := .\mathcal{S}^*\left(\frac{1+(1-2\alpha)\varsigma}{1-\varsigma}\right)$, $0 \le \alpha < 1$, see [11].
iv. $\mathcal{S}^*_s := \mathcal{S}^*(1 + \sin(\varsigma))$, see [12].
v. $\mathcal{S}^*_L := \mathcal{S}^*(\sqrt{1+\varsigma})$, see [13].
vi. $\mathcal{S}^*_{RL} := \mathcal{S}^*\left(\sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-\varsigma}{1+2(\sqrt{2}-1)\varsigma}}\right)$, see [14].
vii. $\mathcal{S}^*_C := \mathcal{S}^*\left(1 + \frac{4\varsigma}{3} + \frac{2\varsigma^2}{3}\right)$, see [15].
viii. $\mathcal{S}^*_e := \mathcal{S}^*(e^{\varsigma})$, see [16].
ix. $\mathcal{S}^*_{\cos} := \mathcal{S}^*(\cos(\varsigma))$, see [17].
x. $\mathcal{S}^*_l := \mathcal{S}^*(\sqrt{1+\varsigma^2} + \varsigma)$, see [18].
xi. $\mathcal{BS}(\alpha) := \mathcal{S}^*(1 + \frac{\varsigma}{1-\alpha\varsigma^2})$, $0 \le \alpha \le 1$, see [19].
xii. $\mathcal{S}^*_{\lim} := \mathcal{S}^*\left(1 + \sqrt{2}\varsigma + \frac{\varsigma^2}{2}\right)$, see [20,21].

The geometry of analytic functions related with some familiar sequences of numbers has been explored by some researchers working in the theory. The class $\mathcal{SL}$ related with Fibonacci numbers was introduced and investigated by Sokół [22]. The class $\mathcal{S}^*_B$ related with Bell numbers was introduced by Cho et al. [23] and Kumar et al. [24], whereas functions related with generalized Telephone numbers were utilized by Deniz [25] to introduce a subclass of $\mathcal{S}^*$. The generating function for Euler numbers was recently used to introduce a subclass of starlike functions (see [26]), while the generating function for Bernoulli numbers is considered in [27] to investigate a subclass of $\mathcal{S}^*$.

Motivated by the above contributions, we study starlike functions related with Van der Pol numbers.

The function $\psi_{\mathcal{V}}(\varsigma)$ defined in (1) is analytic in $\mathbb{D}$ and maps $\mathbb{D}$ onto a convex set and $Re\psi_{\mathcal{V}}(\varsigma) > 0$. We define the class $\mathcal{S}^*_{\mathcal{V}}$ of starlike functions by using the generating function of Van der Pol numbers as follows:

$$\mathcal{S}^*_{\mathcal{V}} := \left\{ k \in \mathcal{A} : \frac{\varsigma k'(\varsigma)}{k(\varsigma)} \prec \frac{\varsigma^3}{6(e^{\varsigma}(\varsigma-2)+(\varsigma+2))} \right\}.$$

From the above definition, $k \in \mathcal{S}^*_{\mathcal{V}}$ if and only if $h(\varsigma) \prec \psi_{\mathcal{V}}(\varsigma) = \frac{\varsigma^3}{6(e^{\varsigma}(\varsigma-2)+(\varsigma+2))}$ and

$$k(\varsigma) = \varsigma\, exp\left(\int_0^{\varsigma} \frac{h(u)-1}{u} du \right), \tag{3}$$

where h is analytic in $\mathbb{D}$. The set $\mathcal{S}^*_{\mathcal{V}}$, is non-empty; we present some examples for functions in it. Consider $h_i : \mathbb{D} \to \mathbb{C}$, $(i = 1,2,3,4)$ given by

$$h_1(\varsigma) = 1 + \frac{2}{5}\varsigma, \quad h_2(\varsigma) = \frac{4+2\varsigma}{4+\varsigma}, \quad h_3(\varsigma) = 1 + \frac{\cos(1)}{2}\varsigma, \quad h_4(\varsigma) = e^{\frac{\varsigma}{3}}.$$

The function $\psi_{\mathcal{V}}(\varsigma)$ is univalent in $\mathbb{D}$. Furthermore, $h_i(0) = h_0(0) = 1$ and $h_i(\mathbb{D}) \subset h_0(\mathbb{D})$, $(i = 1,2,3,4)$. This implies $h_i(\varsigma) \prec \psi_{\mathcal{V}}(\varsigma)$. Therefore, from (3), we obtain the functions $k_i \in \mathcal{S}^*_{\mathcal{V}}$, $(i = 1,2,3,4)$ with $k_i(0) = k_i'(0) - 1 = 0$ to every h_i, respectively, as follows:

$$k_1(\varsigma) = \varsigma e^{\frac{2\varsigma}{5}}, \;\; k_2(\varsigma) = \varsigma + \frac{\varsigma^2}{4}, \quad k_3(\varsigma) = \varsigma e^{\frac{\cos(1)}{2}\varsigma}, \quad k_4(\varsigma) = \varsigma\, exp\left(\int_0^{\varsigma} \frac{e^{\frac{t}{3}}-1}{t} dt \right).$$

We have the following layout of our work.

In Section 2, we find the growth result and some inclusion results for the class $\mathcal{S}^*_{\mathcal{V}}$. In Section 3, we give sharp radii problems for various classes of analytic functions. In the last section, we derive coefficient bounds and Hankel determinants for the class $\mathcal{S}^*_{\mathcal{V}}$.

2. Inclusion Results

Firstly, we study the order of starlikeness and strong starlikeness for the class $\mathcal{S}^*_{\mathcal{V}}$.

Lemma 1. *Let* $\psi_{\mathcal{V}}(\varsigma) = \frac{\varsigma^3}{6(e^{\varsigma}(\varsigma-2)+(\varsigma+2))}$. *Then for* $\varsigma = te^{i\varphi}$, $t \in (0,1)$,

$$\min_{|\varsigma|=t} Re\psi_{\mathcal{V}}(\varsigma) = \psi_{\mathcal{V}}(t) = \min_{|\varsigma|=t} |\psi_{\mathcal{V}}(\varsigma)|$$

and

$$\max_{|\varsigma|=t} Re\psi_{\mathcal{V}}(\varsigma) = \psi_{\mathcal{V}}(-t).$$

Proof. For $\varphi \in [0, 2\pi)$, $x = t\cos(\varphi)$ and $y = t\sin(\varphi)$; we have $Re\psi_{\mathcal{V}}(\varsigma) = \frac{u}{r}$, where

$$\begin{aligned} u &= t^3\cos(3\varphi)(e^x(t\cos(y+\varphi) - 2\,(y)) + x + 2) \\ &\quad + t^3\sin(3\varphi)(e^x(t\sin(y+\varphi) - 2\,\sin(y)) + y), \end{aligned} \tag{4}$$

$$r = 6\,(e^x(t\cos(y+\varphi) - 2\,(y)) + x + 2)^2 + 6\,e^x(t\sin(y+\varphi) - 2\,(y)) + y^2 \tag{5}$$

Let $g(\varphi) = \frac{u}{r}$. Then $g'(\varphi) = 0$ has 0 and π roots. Furthermore, we see that $g''(0) > 0$ and $g''(\pi) < 0$ for $t \in (0,1)$. Therefore, g has minima at $\varphi = 0$ and maxima at $\varphi = \pi$. Hence,

$$\min_{|\varsigma|=t} Re\psi_{\mathcal{V}}(\varsigma) = \psi_{\mathcal{V}}(t) = \frac{t^3}{6(t(e^t+1) - 2(e^t-1))}$$

and

$$\max_{|\varsigma|=t} Re\psi_{\mathcal{V}}(\varsigma) = \psi_{\mathcal{V}}(-t) = -\frac{t^3}{6(-t(e^{-t}+1) - 2(e^{-t}-1))}.$$

Similarly,

$$|\psi_{\mathcal{V}}(\varsigma)|^2 = \left(\frac{u}{r}\right)^2 + \left(\frac{v}{r}\right)^2 = g_1(\varphi),$$

where

$$\begin{aligned} v &= -t^3\cos(3\,\varphi)(e^x(t\sin(t\sin(\varphi)+\varphi) - 2\,\sin(y)) + y) \\ &\quad + t^3\sin(3\varphi)(e^x(t\cos(y+\varphi) - 2\,\cos(y)) + x + 2) \end{aligned}$$

and u and r are given in (4) and (5), respectively. Some computations show that g_1 has a minimum value at $\varphi = 0$. Hence, we conclude that

$$\min_{|\varsigma|=t} |\psi_{\mathcal{V}}(\varsigma)| = \psi_{\mathcal{V}}(t) = \frac{t^3}{6(t(e^t+1) - 2(e^t-1))}.$$

□

Theorem 1. *The class $\mathcal{S}^*_{\mathcal{V}}$ satisfies the following relations:*
(i) $\mathcal{S}^*_{\mathcal{V}} \subset \mathcal{S}^*(\alpha)$, for $0 \le \alpha \le \frac{1}{6(3-e)}$.
(ii) $\mathcal{S}^*_{\mathcal{V}} \subset \mathcal{M}(\alpha)$ for $\alpha \ge \frac{e}{6(3-e)}$,
(iii) $\mathcal{S}^*_{\mathcal{V}} \subset \mathcal{SS}^*(\beta)$, where $\beta \ge 2h(\varphi_1)/\pi \approx 0.3199041635$,
where $\varphi_1 \approx 4.811266810$ is the root of the equation $h'(\varphi) = 0$ and h is defined in (7).

Proof. (i) Let $k \in \mathcal{S}^*_{\mathcal{V}}$. Then, it is easy to see that

$$\frac{\varsigma k'(\varsigma)}{k(\varsigma)} \prec \frac{\varsigma^3}{6(e^{\varsigma}(\varsigma-2) + (\varsigma+2))}.$$

Therefore,

$$\min_{|\varsigma|=1} Re\left(\frac{\varsigma^3}{6(e^{\varsigma}(\varsigma-2)+(\varsigma+2))}\right) < Re\frac{\varsigma k'(\varsigma)}{k(\varsigma)} < \max_{|\varsigma|=1} Re\left(\frac{\varsigma^3}{6(e^{\varsigma}(\varsigma-2)+(\varsigma+2))}\right).$$

Hence, by using Lemma 1, we conclude that

$$\frac{1}{6(3-e)} < Re\frac{\varsigma k'(\varsigma)}{k(\varsigma)} < \frac{e}{6(3-e)}. \tag{6}$$

Thus, $\mathcal{S}^*_{\mathcal{V}} \subset \mathcal{S}^*(\alpha)$, where $0 \le \alpha \le \frac{1}{6(3-e)}$.

(ii) Result follows from (6).

(iii) Let $k \in \mathcal{S}^*_{\mathcal{V}}$. Then

$$\left|\arg\frac{\varsigma k'(\varsigma)}{k(\varsigma)}\right| < \max_{|\varsigma|=1} \arg\frac{\varsigma^3}{6(e^{\varsigma}(\varsigma-2)+(\varsigma+2))} = \max_{|\varsigma|=1} h(\varphi),$$

where

$$h(\varphi) = \tan^{-1}\left(\frac{\begin{array}{c}-t^3\cos(3\varphi)(-2\sin(y) + e^x(t\sin(y+\varphi)) + y) \\ +t^3\sin(3\varphi)(-2\cos(y) + e^x(t\cos(y+\varphi)) + x + 2)\end{array}}{\begin{array}{c}t^3\cos(3\varphi)(-2\cos(y) + e^x(t\cos(y+\varphi)) + x + 2) \\ +t^3\sin(3\varphi)(-2\sin(y) + e^x(t\sin(y+\varphi)) + y)\end{array}}\right). \tag{7}$$

It is easy to see that $h'(\varphi) = 0$ has two roots in $[0, 2\pi]$, namely

$$\varphi_0 = 1.471918497 \text{ and } \varphi_1 = 4.811266810.$$

Furthermore, we see that $h''(\varphi_1) = -0.5177687016$. Hence, $\max(h(\varphi)) = h(\varphi_1) = 0.5025042849$. Thus,

$$k \in \mathcal{SS}^*\left(\frac{2}{\pi}h(\varphi_1)\right).$$

□

Theorem 2. *The $\mathcal{S}^*(\alpha)$-radii, for $\mathcal{S}^*_{\mathcal{V}}$ is t_0, where t_0 is the solution of $t^3 - 6\alpha\left[e^t(t-2) + (t+2)\right] = 0$ and $1/6(3-e) \leq \alpha < 1$.*

Proof. Let $k \in \mathcal{S}^*_{\mathcal{V}}$. Then from Lemma 1, we can write

$$\frac{t^3}{6(e^t(t-2) + (t+2))} \leq Re\left(\frac{\varsigma k'(\varsigma)}{k(\varsigma)}\right) \leq \frac{t^3}{6(e^{-t}(t+2) + (t-2))}.$$

Hence,

$$Re\left(\frac{\varsigma k'(\varsigma)}{k(\varsigma)}\right) \geq \frac{t^3}{6(e^t(t-2) + (t+2))} \geq \alpha$$

for $t^3 - 6\alpha\left[e^t(t-2) + (t+2)\right] = 0 > 0$. Thus, the radius of $\mathcal{S}^*(\alpha)$, for $\mathcal{S}^*_{\mathcal{V}}$ is the smallest root $t_0 \in (0,1)$ of $t^3 - 6\alpha\left[e^t(t-2) + (t+2)\right] = 0$. □

3. Radius Problems

Consider the class

$$\mathcal{P}_m(\alpha) := \left\{p(\varsigma) = 1 + \sum_{k=m}^{\infty} c_m\varsigma^m : Rep(\varsigma) > 0\right\}.$$

Furthermore, $\mathcal{P}_m := \mathcal{P}_m(0)$. Let

$$\mathcal{S}^*_{\mathcal{V},m} := \mathcal{A}_m \cap \mathcal{S}^*_{\mathcal{V}},\ \mathcal{S}^*_m(\alpha) := \mathcal{A}_m \cap \mathcal{S}^*(\alpha).$$

Ali et al. [28] investigated the class $\mathcal{S}_m := \{k \in \mathcal{A}_m : \frac{k(\varsigma)}{\varsigma} \in \mathcal{P}_m\}$. Now consider the following useful results to prove our results.

Lemma 2 ([29])**.** *If $p \in \mathcal{P}_m(\alpha)$, then, for $|\varsigma| = t$,*

$$\left|\frac{\varsigma p'(\varsigma)}{p(\varsigma)}\right| \leq \frac{2(1-\alpha)mt^m}{(1-t^m)(1+(1-2\alpha)t^m)}.$$

Lemma 3 ([30])**.** *If $p \in \mathcal{P}_m(\alpha)$, then, for $|\varsigma| = t$,*

$$\left|p(\varsigma) - \frac{(1+(1-2\alpha))t^{2m}}{1-t^{2m}}\right| \leq \frac{2(1-\alpha)t^m}{1-t^{2m}}.$$

The main purpose of the next result is to obtain the disks of maximum radius and minimum radius centered at $(a, 0)$ such that $\Delta_{\mathcal{V}} := \psi_{\mathcal{V}}(\mathbb{D})$, where $\psi_{\mathcal{V}}(\varsigma) = \frac{\varsigma^3}{6(\varsigma(e^\varsigma+1) - 2(e^\varsigma - 1))}$ is contained in the smallest disk and contains the largest disk.

Lemma 4. *Let $\frac{1}{6(3-e)} < a < \frac{e}{6(3-e)}$. Then, the following inclusions hold:*

$$\{\omega \in \mathbb{C} : |\omega - a| < t_a\} \subset \Delta_{\mathcal{V}} \subset \{\omega \in\colon |\omega - a| < T_a\},$$

where

$$t_a = \begin{cases} a - \frac{1}{6(3-e)} & \frac{1}{6(3-e)} < a \leq a^\circ, \\ \frac{e}{6(3-e)} - a, & a^\circ \leq a < \frac{e}{6(3-e)}, \end{cases}$$

and

$$T_a = \begin{cases} \frac{e}{6(3-e)} - a & \frac{1}{6(3-e)} < a \leq a^*, \\ \sqrt{h(\varphi_a)}, & a^* < a \leq a^{**}, \\ a - \frac{1}{6(3-e)}, & a^{**} < a < \frac{e}{6(3-e)}, \end{cases}$$

where φ_a *is the zero of* $h'(\varphi)$*. The function* h *is given by* (8) *with* $a^\circ \approx 1.099999$, $a^* \approx 1.0683509192$*, and* $a^{**} \approx 1.1944463972$.

Proof. Let $\psi_{\mathcal{V}}(\varsigma) = \frac{\varsigma^3}{6(e^\varsigma(\varsigma-2)+(\varsigma+2))}$. Then

$$\psi_{\mathcal{V}}\left(e^{i\varphi}\right) = \frac{e^{3i\varphi}}{6\left(e^{e^{i\varphi}}\left(e^{i\varphi}-2\right)+\left(e^{i\varphi}+2\right)\right)}$$

represents the boundary of $\psi_{\mathcal{V}}(\mathbb{D})$. Let $x = \cos(\varphi)$ and $y = \sin(\varphi)$. Then the square of the distance of $(a,0)$ to the boundary of $\Delta_{\mathcal{V}}$ is given by the function

$$h(\varphi) = \left(\frac{u_1}{r_1} - a\right)^2 + \left(\frac{v_1}{r_1}\right)^2, \tag{8}$$

where

$$\begin{aligned} u_1 &= \cos(3\varphi)(e^x(-2\cos(y)+\cos(y+\varphi))+x+2) \\ &\quad +\sin(3\varphi)(-2\sin(y)+e^x(\sin(y+\varphi))+y), \\ v_1 &= -\cos(3\varphi)(-2\sin(y)+e^x(\sin(y+\varphi))+y) \\ &\quad +\sin(3\varphi)(-2\cos(y)+e^x(\cos(y+\varphi))+x+2), \\ r_1 &= 6(-2\cos(y)+e^x(\cos(y+\varphi))+x+2)^2 \\ &\quad +6(-2\sin(y)+e^x(\sin(y+\varphi))+y)^2. \end{aligned}$$

To prove that $|\omega - a| < t_a$ is a disk with maximum radius contained in $\Delta_{\mathcal{V}}$, we have to show that $\min_{0\leq\varphi\leq\pi}\sqrt{h(\varphi)} = t_a$. Since $h(\varphi) = h(-\varphi)$, we consider $0 \leq \varphi \leq \pi$ only. We suppose that

$$\frac{1}{6(3-e)} < a \leq a^*,$$

where $a^* \approx 1.0683509192$. We see that the equation $h'(\varphi) = 0$ has the roots 0 and π. The graph of the function $h'(\varphi)$ is positive in the interval $[0,\pi]$. Hence, it is increasing; therefore,

$$\min_{0\leq\varphi\leq\pi}\sqrt{h(\varphi)} = \sqrt{h(0)} = a - \frac{1}{6(3-e)}.$$

Now, we consider $a^* < a \leq a^{**}$, where $a^{**} \approx 1.1944463972$. Then $h'(\varphi) = 0$ has 0, $\varphi_a \in (0, \pi)$, and π roots. The root φ_a depends upon a. We notice that $h'(\varphi) > 0$ in $(0,\varphi_a)$ and $h'(\varphi) < 0$ in (φ_0,π). We also see that $h(0) < h(\pi)$ for $a^* < a \leq a^\circ$, where $a^\circ \approx 1.099999$. Hence,

$$\min_{0\leq\varphi\leq\pi}\sqrt{h(\varphi)} = \sqrt{h(0)} = a - \frac{1}{6(3-e)}.$$

Similarly, we see that $h(\pi) < h(0)$ for $1.099999 < a < a^{**}$. Therefore,

$$\min_{0\leq\varphi\leq\pi}\sqrt{h(\varphi)} = \sqrt{h(\pi)} = \frac{e}{6(3-e)} - a.$$

For $a^{**} < a < \frac{e}{6(3-e)}$, we notice that $h'(\varphi) < 0$ in $(0, \pi)$. Hence,

$$\min_{0\leq\varphi\leq\pi} \sqrt{h(\varphi)} = \sqrt{h(\pi)} = \frac{e}{6(3-e)} - a.$$

For the case of the minimum radius of a circle centered at $(a, 0)$ which contains $\psi_{\mathcal{V}}(\mathbb{D}) = \frac{\varsigma^3}{6(\varsigma(e^{\varsigma}+1)-2(e^{\varsigma}-1))}$, we calculate the maximum distance of $(a, 0)$ to a point on the boundary of $\Delta_{\mathcal{V}} = \psi_{\mathcal{V}}(\mathbb{D})$. We notice that h is increasing for $\frac{1}{6(3-e)} < a \leq a^*$. Therefore,

$$\max_{0\leq\varphi\leq\pi} \sqrt{h(\varphi)} = \sqrt{h(\pi)} = \frac{e}{6(3-e)} - a.$$

When $a^* < a \leq a^{**}$, the function $h'(\varphi)$ has 0, φ_a, and π. The root φ_a depends on a. The graph of $h'(\varphi)$ indicates that $h'(\varphi) > 0$ when $\varphi \in (0, \varphi_a)$ and $h'(\varphi) < 0$ when $\varphi \in (\varphi_a, \pi)$. We conclude that $\max\limits_{0\leq\varphi\leq\pi} \sqrt{h(\varphi)} = \sqrt{h(\varphi_a)}$. Furthermore, h is decreasing when $a^{**} < a < \frac{e}{6(3-e)}$ and

$$\max_{0\leq\varphi\leq\pi} \sqrt{h(\varphi)} = \sqrt{h(0)} = a - \frac{1}{6(3-e)}.$$

Hence, we obtain the required result. □

Example 1. *(a) The function $k(\varsigma) = \varsigma + d_2\varsigma^2$ is in $\mathcal{S}^*_{\mathcal{V}}$, if and only if*

$$|d_2| \leq \frac{17-6e}{35-12e} \approx 0.29480.$$

(b) The function $k(\varsigma) = \frac{\varsigma}{1-\lambda\varsigma^2}$ is in $\mathcal{S}^*_{\mathcal{V}}$, if and only if

$$|\lambda| \leq \frac{7e-18}{18-5e} \approx 0.22540.$$

Proof. (a) We know that $k(\varsigma) = \varsigma + d_2\varsigma^2 \in \mathcal{S}^*$ if and only if $|d_2| \leq \frac{1}{2}$. Since $\mathcal{S}^*_{\mathcal{V}} \subset \mathcal{S}^*$, we have $|d_2| \leq \frac{1}{2}$, whenever $k \in \mathcal{S}^*_{\mathcal{V}}$. The function

$$\omega(\varsigma) = \frac{\varsigma k'(\varsigma)}{k(\varsigma)} = \frac{1+2d_2\varsigma}{1+d_2\varsigma},$$

maps $\mathbb{D}$ onto

$$\left|\omega - \frac{1-2|d_2|^2}{1-|d_2|^2}\right| < \frac{|d_2|}{1-|d_2|^2}.$$

Since $\frac{1-2|d_2|^2}{1-|d_2|^2} \leq 1$,

$$\frac{1}{6(3-e)} \leq \frac{1-2|d_2|^2}{1-|d_2|^2} \quad \text{and} \quad \frac{|d_2|}{1-|d_2|^2} \leq \frac{1-2\,|d_2|^2}{1-|d_2|^2} - \frac{1}{6(3-e)}.$$

The above two inequalities give us

$$|d_2| \leq \sqrt{\frac{17-6e}{35-12e}} \text{ and } |d_2| \leq \frac{17-6e}{35-12e}$$

respectively. Thus, we have

$$|d_2| \leq \min\left\{\sqrt{\frac{17-6e}{35-12e}}, \frac{17-6e}{35-12e}\right\} = \frac{17-6e}{35-12e}.$$

(b) Logarithmic differentiation of the function $k(\varsigma) = \frac{\varsigma}{1-\lambda\varsigma^2}$ yields

$$\omega(\varsigma) = \frac{\varsigma k'(\varsigma)}{k(\varsigma)} = \frac{1+\lambda\varsigma}{1-\lambda\varsigma}.$$

The function ω maps $\mathbb{D}$ onto

$$\left|\frac{\varsigma k'(\varsigma)}{k(\varsigma)} - \frac{1+|\lambda|^2}{1-|\lambda|^2}\right| \leq \frac{2|\lambda|}{1-|\lambda|^2}.$$

Hence, by using Lemma 4, it is contained in $\Delta_{\mathcal{V}}$ provided

$$\frac{1+|\lambda|^2}{1-|\lambda|^2} \leq \frac{e}{6(3-e)} \quad \text{and} \quad \frac{2|\lambda|}{1-\ |\lambda|^2} \leq \frac{e}{6(3-e)} - \frac{1+|\lambda|^2}{1-|\lambda|^2}.$$

Thus,

$$|\lambda| < \sqrt{\frac{7e-18}{18-5e}} \quad \text{and} \quad |\lambda| \leq \frac{7e-18}{18-5e},$$

respectively. Thus, we have

$$|\lambda| \leq \min\left\{\sqrt{\frac{7e-18}{18-5e}}, \frac{7e-18}{18-5e}\right\} = \frac{7e-18}{18-5e}.$$

Hence, we obtain the result. □

Theorem 3. *The $\mathcal{S}^*_{\mathcal{V},m}$-radius for $\mathcal{S}_m$ is*

$$R_{\mathcal{S}^*_{\mathcal{V},m}}(\mathcal{S}_m) = \left(\frac{17-6e}{6m(3-e) + \sqrt{36m^2(3-e)^2 + (17-6e)^2}}\right)^{\frac{1}{m}}.$$

Proof. Let $k \in \mathcal{S}_m$. Consider the function $h : \mathbb{D} \longrightarrow \mathbb{C}$ defined by

$$h(\varsigma) = \frac{k(\varsigma)}{\varsigma}, \quad h \in \mathcal{P}_m.$$

Taking logarithmic differentiation, it follows that

$$\frac{\varsigma k'(\varsigma)}{k(\varsigma)} - 1 = \frac{\varsigma h'(\varsigma)}{h(\varsigma)}.$$

By applying Lemma 2, we obtain

$$\left|\frac{\varsigma k'(\varsigma)}{k(\varsigma)} - 1\right| = \left|\frac{\varsigma h'(\varsigma)}{h(\varsigma)}\right| \leq \frac{2mt^m}{1-t^{2m}}.$$

By using Lemma 4, the image of $|\varsigma| \leq t$ under $\frac{\varsigma k'(\varsigma)}{k(\varsigma)}$ is contained in $\Delta_{\mathcal{V}}$, if

$$\frac{2mt^m}{1-t^{2m}} \leq \frac{17-6e}{6(3-e)}.$$

This implies that

$$t^{2m}(17-6e)+12m(3-e)t^m-(17-6e)\leq 0.$$

Hence, the $\mathcal{S}^*_{\mathcal{V},m}$-radius of $\mathcal{S}_m$ is the root of $t^{2m}(17-6e)+12m(3-e)t^m-(17-6e)=0$ in $(0,1)$. Consider the function $k_0(\varsigma)=\frac{\varsigma(1+\varsigma^m)}{1-\varsigma^m}$. Then $Re\frac{k(\varsigma)}{\varsigma}>0$ in $\mathbb{D}$. Thus, $k_0\in\mathcal{S}_m$ and $\frac{\varsigma k_0'(\varsigma)}{k(\varsigma)}=1+\frac{2m\varsigma^m}{1-\varsigma^{2m}}$. Furthermore, k_0 gives a sharp result, since at $\varsigma=R_{\mathcal{S}^*_{\mathcal{B},m}}(\mathcal{S}_m)$, we have

$$\frac{\varsigma k_0'(\varsigma)}{k_0(\varsigma)}-1=\frac{2m\varsigma^m}{1-\varsigma^{2m}}=\frac{17-6e}{6(3-e)}.$$

This completes the proof. □

Consider the class F defined as
$F=\left\{k\in\mathcal{A}_m : Re\left(\frac{k(\varsigma)}{g(\varsigma)}\right)>0 \text{ and } Re\left(\frac{g(\varsigma)}{\varsigma}\right)>0,\ g\in\mathcal{A}_m\right\}$,

Theorem 4. *The sharp $\mathcal{S}^*_{\mathcal{V},m}$-radius for F is*

$$R_{\mathcal{S}^*_{\mathcal{V},m}}(F)=\left(\frac{17-6e}{6m(3-e)+\sqrt{(17-6e)^2+36m^2(3-e)^2}}\right)^{\frac{1}{m}}.$$

Proof. (1) Let $k\in F$ and define $p,\ h:\mathbb{D}\to\mathbb{C}$ by $p(\varsigma)=\frac{g(\varsigma)}{\varsigma}$ and $h(\varsigma)=\frac{k(\varsigma)}{g(\varsigma)}$. Then, clearly $p,\ h\in\mathcal{P}_m$. Since $k(\varsigma)=\varsigma p(\varsigma)h(\varsigma)$, by Lemma 2 it implies that

$$\left|\frac{\varsigma k'(\varsigma)}{k(\varsigma)}-1\right|\leq\frac{4mt^m}{1-t^{2m}}\leq 1-\frac{1}{6(3-e)}$$

for $t\leq\left(\frac{17-6e}{6m(3-e)+\sqrt{36m^2(3-e)^2+(17-6e)^2}}\right)^{\frac{1}{m}}=R_{\mathcal{S}^*_{\mathcal{V},m}}(F)$. Consider

$$k_0(\varsigma)=\varsigma\left(\frac{1+\varsigma^m}{1-\varsigma^m}\right)^2 \text{ and } g_0(\varsigma)=\varsigma\left(\frac{1+\varsigma^m}{1-\varsigma^m}\right).$$

Thus, clearly

$$Re\left(\frac{k_0(\varsigma)}{g_0(\varsigma)}\right)>0 \text{ and } Re\left(\frac{g_0(\varsigma)}{\varsigma}\right)>0$$

and hence, $k\in F$. A computation shows that at $\varsigma=R_{\mathcal{S}^*_{\mathcal{V},m}}(F)e^{\frac{i\pi}{m}}$

$$\frac{\varsigma k'(\varsigma)}{k(\varsigma)}=1+\frac{4m\varsigma^m}{1-\varsigma^{2m}}=2-\frac{1}{6(3-e)}.$$

This confirms the sharpness.
□

Theorem 5. *The sharp $\mathcal{S}^*_{\mathcal{V}}$-radii for the classes $\mathcal{S}^*_L$, $\mathcal{S}^*_C$, $\mathcal{S}^*_e$, and $\mathcal{S}^*_{\lim}$ are*
(1) $R_{\mathcal{S}^*_{\mathcal{V}}}(\mathcal{S}^*_L)=\frac{(17-6e)(19-6e)}{36(3-e)^2}\approx 0.65109$,
(2) $R_{\mathcal{S}^*_{\mathcal{V}}}(\mathcal{S}^*_C)=\frac{1}{2}\left(-6+2e+\sqrt{-15+11e-2e^2}\right)/(3-e)\approx 0.37751$,
(3) $R_{\mathcal{S}^*_{\mathcal{B}}}(\mathcal{S}^*_e)=\ln(\frac{1}{18-6e})+1\approx 0.47528$,
(4) $R_{\mathcal{S}^*_{\mathcal{V}}}(\mathcal{S}^*_{\lim})=\left(\frac{-9\sqrt{2}+3e\sqrt{2}+\sqrt{9-3e}}{3(-3+e)}\right)\approx 0.32624$.

Proof. (1) Let $k \in \mathcal{S}_L^*$. Then, we have $\frac{\varsigma k'(\varsigma)}{k(\varsigma)} \prec \sqrt{1+\varsigma}$. Thus, for $|\varsigma| \leq t < R_{\mathcal{S}_{\mathcal{V}}^*}(\mathcal{S}_L^*)$, we have

$$\left|\frac{\varsigma k'(\varsigma)}{k(\varsigma)} - 1\right| = \left|\sqrt{1+\varsigma} - 1\right| \leq 1 - \sqrt{1-t} \leq 1 - \frac{1}{6(3-e)}.$$

By using Lemma 4, we obtain the hypothesis. Consider the function

$$k_0(\varsigma) = \frac{4\varsigma \exp\left[2(\sqrt{1+\varsigma} - 1)\right]}{(1+\sqrt{1+\varsigma})^2}.$$

Since $\frac{\varsigma k_0'(\varsigma)}{k_0(\varsigma)} = \sqrt{1+\varsigma}$, $k_0 \in \mathcal{S}_L^*$. Furthermore, $\frac{\varsigma k_0'(\varsigma)}{k_0(\varsigma)} = \sqrt{1+\varsigma} = \frac{1}{6(3-e)}$ at $\varsigma = \frac{(17-6e)(19-6e)}{36(3-e)^2}$; hence, the sharpness of the result is verified.

(2) Let $k \in \mathcal{S}_C^*$. Then $\frac{\varsigma k'(\varsigma)}{k(\varsigma)} \prec 1 + \frac{4\varsigma}{3} + \frac{2\varsigma^2}{3}$. Thus, for $|\varsigma| = t$, we obtain

$$\begin{aligned} \left|\frac{\varsigma k'(\varsigma)}{k(\varsigma)} - 1\right| &= \left|1 + \frac{4\varsigma}{3} + \frac{2\varsigma^2}{3} - 1\right| \\ &\leq 1 - \left(1 + \frac{4t}{3} + \frac{2t^2}{3}\right) \leq 1 - \frac{1}{6(3-e)} \end{aligned}$$

for $t \leq \frac{1}{2}\left(-6 + 2e + \sqrt{15e - 3e^2 - 18}\right)/(3-e)$. Consider the function k_1 given by

$$k_1(\varsigma) = \varsigma exp\left\{\frac{4\varsigma + \varsigma^2}{3}\right\}.$$

Since $\frac{\varsigma k_1'(\varsigma)}{k_1(\varsigma)} = 1 + \frac{4\varsigma}{3} + \frac{2\varsigma^2}{3}$, it follows that $k_1 \in \mathcal{S}_C^*$ and at $\varsigma = R_{\mathcal{S}_{\mathcal{V}}^*}(\mathcal{S}_C^*)$, we have

$$\frac{\varsigma k_1'(\varsigma)}{k_1(\varsigma)} = \frac{1}{6(3-e)}.$$

Hence, the result is sharp.

(3) For $k \in \mathcal{S}_e^*$, we have

$$\left|\frac{\varsigma k'(\varsigma)}{k(\varsigma)} - 1\right| = |e^{\varsigma} - 1| \leq e^t - 1 \leq \frac{e}{6(3-e)} - 1.$$

Sharpness is guaranteed by k_2 such that $\frac{\varsigma k_2'(\varsigma)}{k_2(\varsigma)} = e^{\varsigma}$.

(4) Suppose $k \in \mathcal{S}_{\lim}^*$. Then, $\frac{\varsigma k'(\varsigma)}{k(\varsigma)} \prec 1 + \sqrt{2}\varsigma + \frac{\varsigma^2}{2}$ (see [21]). Thus, for $|\varsigma| = t$, we obtain

$$\begin{aligned} \left|\frac{\varsigma k'(\varsigma)}{k(\varsigma)} - 1\right| &= \left|1 + \sqrt{2}\varsigma + \frac{\varsigma^2}{2} - 1\right| \\ &\leq 1 - \left(1 + \sqrt{2}t + \frac{t^2}{2}\right) \leq 1 - \frac{1}{6(3-e)} \end{aligned}$$

for $t \leq -\sqrt{2} + \sqrt{\frac{e^2+1}{e}}$. For sharpness, consider k_3 given by

$$k_3(\varsigma) = \varsigma exp\left\{\frac{4\sqrt{2}\varsigma + \varsigma^2}{4}\right\}.$$

Since $\frac{\varsigma k_3'(\varsigma)}{k_3(\varsigma)} = 1+\sqrt{2}\varsigma + \frac{\varsigma^2}{2}$, it follows that $k_3 \in \mathcal{S}^*_{lim}$ and at $\varsigma = R_{\mathcal{S}^*_{\vartheta}}(\mathcal{S}^*_{lim})$, we have

$$\frac{\varsigma k_3'(\varsigma)}{k_3(\varsigma)} = \frac{1}{6(3-e)}.$$

Hence, the result is sharp. $\square$

4. Coefficient Estimates

Pommerenke [31] introduced the qth Hankel determinant for analytic functions. It is given as

$$H_{q,m}(k) := \begin{vmatrix} d_m & d_{m+1} & \dots & d_{m+q-1} \\ d_{m+1} & d_{m+2} & \dots & d_{m+q} \\ \vdots & \vdots & \dots & \vdots \\ d_{m+q-1} & d_{m+q} & \dots & d_{m+2q-2} \end{vmatrix}, \tag{9}$$

where $m \geq 1$ and $q \geq 1$. We note that

$$H_{2,1}(k) = d_3 - d_2^2, \quad H_{2,2}(k) = d_2 d_4 - d_3^2$$

and

$$H_{3,1}(k) = 2d_2d_3d_4 - d_3^3 - d_4^2 + d_3d_5 - d_2^2 d_5. \tag{10}$$

To find the sharp upper bound of $H_{3,1}$ for subclasses of analytic function is much difficult. Only a few papers [32–37] are devoted to finding a sharp bound for $H_{3,1}$. In this section, we find the sharp coefficient bound and sharp results for the Hankel determinants $H_{2,1}$, $H_{2,2}$, and $H_{3,1}$.

In order to prove our theorems, we will use the following useful results related to the functions in the class $\mathcal{P}$.

Let $\mathcal{P}$ represent the class of functions p which are analytic and defined for $\varsigma \in \mathbb{D}$ given by

$$p(\varsigma) = 1 + \sum_{m=1}^{\infty} c_m \varsigma^m \tag{11}$$

having positive real part in $\mathbb{D}$.

Lemma 5 ([9])**.** *Let $h \in \mathcal{P}$ be given by* (11)*. Then*

$$\left|c_2 - \xi c_1^2\right| \leq \begin{cases} -4\xi + 2, & \xi < 0, \\ 2, & 0 \leq \xi \leq 1, \\ 4\xi - 2, & \xi > 1. \end{cases}$$

Lemma 6. *Let $h \in \mathcal{P}$ and of the form* (11)*. Then*

$$\left|c_2 - \xi c_1^2\right| \leq 2\max\{1, |2\xi - 1|\}.$$

Lemma 7 ([38,39])**.** *If $h \in \mathcal{P}$ of the form* (11) *with $c_1 > 0$, then*

$$c_2 = \frac{1}{2}[c_1^2 + (4 - c_1^2)x], \tag{12}$$

$$c_3 = \frac{1}{4}[c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)y], \tag{13}$$

$$\begin{aligned} c_4 = {} & \frac{1}{8}[c_1^4 + 3c_1^2(4-c_1^2)x + (4-3c_1^2)(4-c_1^2)x^2 + c_1^2(4-c_1^2)x^3 \\ & +4(4-c_1^2)(1-|x|^2)(c_1 y - c_1 x y - \overline{x} y^2) \\ & +4(4-c_1^2)(1-|x|^2)(1-|y|^2)z] \end{aligned} \tag{14}$$

for some $x,\ y,\ z \in \overline{\mathbb{D}} := \{\varsigma, |\varsigma| \leq 1\}$.

Lemma 8 ([40])**.** *Let* $h \in \mathcal{P}$ *given by* (11)*. Let* $0 \leq J \leq 1$ *and* $J(2J-1) \leq K \leq J$*. Then*

$$|c_3 - 2J\,c_1c_2 + Kc_1^3| \leq 2.$$

Lemma 9 ([41])**.** *Let* $h \in \mathcal{P}$ *be given by* (11)*,* $0 < j < 1$*,* $0 < k < 1$ *and let*

$$8j(1-j)\{(kl-2m)^2 + (k(j+k)-k)^2\} + k(1-k)(l-2jk)^2 \leq 4k^2j(1-k)^2(1-j).$$

Then

$$|mc_1^4 + jc_2^2 + 2kc_1c_3 - \frac{3}{2}lc_1^2c_2 - c_4| \leq 2.$$

Lemma 10 ([42])**.** *Let* $\overline{\mathbb{D}} := \{\varsigma \in \mathbb{C} : |\varsigma| \leq 1\}$*, and* J*,* K*,* L *are real numbers; let*

$$Y(J,K,L) := \max\Big\{|J + K\varsigma + L\varsigma^2| + 1 - |\varsigma|^2 : \varsigma \in \overline{\mathbb{D}}\Big\}.$$

If $JL \geq 0$*, then*

$$Y(J,K,L) = \begin{cases} |J| + |K| + |L|, & |K| \geq 2(1-|L|), \\ 1 + |J| + \dfrac{K^2}{4(1-|L|)}, & |K| < 2(1-|L|). \end{cases}$$

Theorem 6. *Let* $k \in \mathcal{S}^*_{\mathcal{V}}$ *be of the form* (2)*. Then*

$$|d_m| \leq \frac{1}{2(m-1)}, \quad m = 2,3,4,5.$$

These bounds are sharp.

Proof. Let $k \in \mathcal{S}^*_{\mathcal{V}}$. Then

$$\frac{\varsigma k'(\varsigma)}{k(\varsigma)} = \frac{(\omega(\varsigma))^3}{6\big(\omega(\varsigma)\big(e^{\omega(\varsigma)}+1\big) - 2\big(e^{\omega(\varsigma)}-1\big)\big)}, \tag{15}$$

where $\omega \in \mathcal{B}$ in $\mathbb{D}$. Now for $h \in \mathcal{P}$ and of the form (11), we can write

$$\omega(\varsigma) = \frac{h(\varsigma)-1}{h(\varsigma)+1} = \frac{\sum_{m=1}^{\infty} c_m\varsigma^m}{2 + \sum_{m=1}^{\infty} c_m\varsigma^m}.$$

Now

$$\begin{aligned} &\frac{(\omega(\varsigma))^3}{6\big(\omega(\varsigma)\big(e^{\omega(\varsigma)}+1\big) - 2\big(e^{\omega(\varsigma)}-1\big)\big)} \\ = \;& 1 - \frac{1}{4}c_1\varsigma + \left(\frac{-1}{4}c_2 + \frac{3}{20}c_1^2\right)\varsigma^2 + \left(\frac{-1}{4}c_3 + \frac{3}{10}c_1c_2 - \frac{17}{192}c_1^3\right)\varsigma^3 \\ & + \left(\frac{6929}{134,400}c_1^4 - \frac{17}{64}c_1^2c_2 + \frac{3}{10}c_1c_3 - \frac{1}{4}c_4 + \frac{3}{20}c_2^2\right)\varsigma^4 + \cdots. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \frac{\varsigma k'(\varsigma)}{k(\varsigma)} &= 1 + d_2\varsigma + \left(2d_3 - d_2^2\right)\varsigma^2 + \left(3d_4 - 3d_2d_3 + d_2^3\right)\varsigma^3 \\ & \quad + \left(4d_5 - 4d_2d_4 - 2d_3^2 + 4d_3d_2^2 - d_2^4\right)\varsigma^4 + \cdots. \end{aligned} \tag{16}$$

Substituting in (15) and comparing the coefficients, we obtain

$$d_2 = \frac{-1}{4}c_1, \tag{17}$$

$$d_3 = -\frac{1}{8}c_2 + \frac{17}{160}c_1^2, \tag{18}$$

$$d_4 = -\frac{293}{5760}c_1^3 + \frac{21}{160}c_1c_2 - \frac{1}{12}c_3, \tag{19}$$

$$d_5 = \frac{82531}{3225600}c_1^4 - \frac{67}{640}c_2c_1^2 + \frac{23}{240}c_1c_3 + \frac{29}{640}c_2^2 - \frac{1}{16}c_4. \tag{20}$$

The bound for $|d_2|$ can easily be obtained by using the well-known coefficient bounds for class $\mathcal{P}$. The bound for $|d_3|$ is obtained by using Lemma 5 for $\xi = 17/20$. For $|d_4|$, we may write (19) as follows:

$$|d_4| = \frac{1}{12}\left|c_3 - \frac{63}{40}c_1c_2 + \frac{293}{480}c_1^3\right| = \frac{1}{12}\left|c_3 - 2Jc_1c_2 + Kc_1^3\right|,$$

where $J = \frac{63}{80}$ and $K = \frac{293}{480}$. It is easy to verify that $0 \le J \le 1$ and $J(2J-1) \le K \le J$. Then by using Lemma 8, we have the required result. For d_5, we can rewrite (20) as

$$\begin{aligned} |d_5| &= \frac{1}{16}\left|\frac{82531}{201600}c_1^4 + \frac{29}{40}c_2^2 + \frac{23}{15}c_1c_3 - \frac{67}{40}c_2c_1^2 - c_4\right| \\ &= \frac{1}{16}\left|mc_1^4 + jc_2^2 + 2kc_1c_3 - \frac{3}{2}lc_2c_1^2 - c_4\right|. \end{aligned}$$

By using Lemma 9 with $m = \frac{82531}{201600}$, $j = \frac{29}{40}$, $k = \frac{23}{30}$, and $l = \frac{67}{60}$, we have

$$\begin{aligned} &8j(1-j)\{(kl-2m)^2 + (k(j+k)-k)^2\} + k(1-k)(l-2jk)^2 - 4k^2j(1-k)^2(1-j) \\ \le\; &\frac{-44977769161}{2032128000000}. \end{aligned}$$

Therefore,

$$|d_5| \le \frac{1}{8}.$$

For sharpness, consider the function $k_m : \mathbb{D} \to \mathbb{C}$ given by

$$k_m(\varsigma) = \varsigma \exp \int_0^{\varsigma} \frac{1}{t}\left(\frac{t^{3m}}{6(t^m(e^{t^m}+1) - 2(e^{t^m}-1))} - 1\right)dt, \quad m = 1,2,3,4.$$

Then

$$\frac{\varsigma k_m'(\varsigma)}{k_m(\varsigma)} = \frac{t^{3m}}{6(t^m(e^{t^m}+1) - 2(e^{t^m}-1))}, \quad m = 1,2,3,4.$$

Hence, $k_m \in \mathcal{S}^*_{\mathcal{V}}$, and

$$
\begin{aligned}
k_1(\varsigma) &= \varsigma \exp \int_0^{\varsigma} \frac{1}{t}\left(\frac{t^3}{6(t(e^t+1)-2(e^t-1))}-1\right)dt \\
&= \varsigma - \frac{1}{2}\varsigma^2 + \frac{7}{40}\varsigma^3 - \frac{7}{144}\varsigma^4 + \cdots, \qquad (21) \\
k_2(\varsigma) &= \varsigma \exp \int_0^{\varsigma} \frac{1}{t}\left(\frac{t^6}{6\left(t^2\left(e^{t^2}+1\right)-2\left(e^{t^2}-1\right)\right)}-1\right) \\
&= \varsigma - \frac{1}{4}\varsigma^3 + \frac{9}{160}\varsigma^5 + \cdots, \qquad (22) \\
k_3(\varsigma) &= \varsigma \exp \int_0^{\varsigma} \frac{1}{t}\left(\frac{t^9}{6\left(t^3\left(e^{t^3}+1\right)-2\left(e^{t^3}-1\right)\right)}-1\right) \\
&= \varsigma - \frac{1}{6}\varsigma^4 + \frac{11}{360}\varsigma^7 + \cdots, \qquad (23) \\
k_4(\varsigma) &= \varsigma \exp \int_0^{\varsigma} \frac{1}{t}\left(\frac{t^{12}}{6\left(t^4\left(e^{t^4}+1\right)-2\left(e^{t^4}-1\right)\right)}-1\right) \\
&= \varsigma - \frac{1}{8}\varsigma^5 + \frac{13}{640}\varsigma^9 + \cdots. \qquad (24)
\end{aligned}
$$

□

Next we investigate the Hankel determinant problems; the first two results study Fekete–Szego functional, which is a generalized form of $H_{2,1}$.

Theorem 7. *Let $k \in \mathcal{S}^*_{\mathcal{V}}$ be given by (2). Then*

$$
|d_3 - \mu d_2^2| \leq \frac{1}{8}\begin{cases} \frac{7-10\mu}{5}, & \mu \leq \frac{-3}{10}, \\ 2, & -\frac{3}{10} \leq \mu \leq \frac{17}{10}, \\ \frac{-7+10\mu}{5}, & \mu > \frac{17}{10}. \end{cases}
$$

This result is sharp.

Proof. If $k \in \mathcal{S}^*_{\mathcal{V}}$, then from (17) and (18), we have

$$
\left|d_3 - \mu d_2^2\right| = \frac{1}{8}\left|c_2 - \frac{1}{20}(17-10\mu)c_1^2\right|.
$$

Then, by using Lemma 5 for $\xi = \frac{1}{20}(17-10\mu)$, this completes the result. □

Theorem 8. *Let $k \in \mathcal{S}^*_{\mathcal{V}}$ be given by (2). Then*

$$
|d_3 - \mu d_2^2| \leq \frac{1}{4}\max\left\{1, \frac{1}{10}|7-10\mu|\right\}, \quad \mu \in \mathbb{C}.
$$

Sharpness is obtained by k_2 and k_3 given in (21) and (22), respectively.

Corollary 1. *Let $k \in \mathcal{S}^*_{\mathcal{V}}$ and of the form (2). Then*

$$
|H_{2,1}(k)| = |d_3 - d_2^2| \leq \frac{3}{40}.
$$

This inequality is sharp for the function k_3 defined by (22).

Theorem 9. *Let $k \in \mathcal{S}^*_{\mathcal{V}}$ and of the form (2). Then*

$$|H_{2,2}(k)| \leq \frac{1}{16}.$$

This inequality is sharp for the function k_3 defined by (22).

Proof. *From (17)–(19) , we obtain*

$$H_{2,2}(k) = \frac{329c_1^4}{230400} - \frac{c_1^2 c_2}{60} - \frac{c_2^2}{64} + \frac{c_1 c_3}{48}. \tag{25}$$

Now we can write

$$H_{2,2}(k) = \frac{1}{230400}\phi, \tag{26}$$

where

$$\phi = 329c_1^4 - 1440c_1^2 c_2 + 4800c_3 c_1 - 3600c_2^2.$$

The class $\mathcal{S}^*_{\mathcal{V}}$ as well as the functional $H_{2,2}(k)$ are invariant (rotationally); we suppose that $c := c_1$, such that $0 \leq c \leq 2$. Then from (12) and (13) and by simplifying, we have

$$\phi = -91c^4 - 120(4-c^2)xc^2 - 300(4-c^2)\left(c^2+12\right)x^2 + 2400c(4-c^2)(1-|x|^2)y,$$

where x and y are such that $|x| \leq 1$, $|y| \leq 1$.

First assume that $c = 2$. Then

$$|\phi| \leq 1456,$$

From (26), we obtain

$$|H_{2,2}(k)| \leq \frac{91}{14400},$$

and when $c = 0$,

$$|\phi| = 14400|x|^2 \leq 14400,$$

so that

$$|H_{2,2}(k)| \leq \frac{1}{16}.$$

Next assume that $c \in (0,2)$. Using triangle inequality, we obtain

$$|\phi| \leq 2400c(4-c^2)\Psi(J,K,L),$$

where

$$\Psi(J,K,L) = \left|J + Kx + Lx^2\right| + 1 - |x|^2, \qquad x \in \overline{\mathbb{D}},$$

with $J = \dfrac{-91c^3}{2400(4-c^2)}$, $K = \frac{-c}{20}$, and $L = -\dfrac{(c^2+12)}{8c}$. So clearly

$$JL = \frac{91c^2(c^2+12)}{2400(4-c^2)} > 0, \qquad \text{for} \quad c \in (0,2).$$

Note now that

$$|K| - 2(1-|L|) = \frac{3c^2 - 20c + 30}{10c} > 0, \quad c \in (0,2),$$

which shows that $|K| \geq 2(1-|L|)$.

Using Lemma 10, we have

$$|\phi| \leq 2400c(4-c^2)(|J|+|K|+|L|) := g(c),$$

where

$$g(c) = -329c4 - 1920c^2 + 14,400.$$

Since $g'(c) < 0$ for $c \in (0,2)$, $\max g(c) = g(0) = 14,400$, and hence from (26), we obtain the result.

It is sharp for k_2 given in (22). This completes the proof. □

Theorem 10. *Let $k \in \mathcal{S}^*_{\mathcal{V}}$ and of the form (2). Then*

$$|d_2d_3 - d_4| \leq \frac{1}{6}.$$

This result is sharp.

Proof. From (17)–(19), we obtain

$$|d_2d_3 - d_4| = \frac{48}{576}\left|c_3 - c_1c_2 + \frac{11}{48}c_1^3\right| = \frac{48}{576}\left|c_3 - 2Jc_1c_2 + Kc_1^3\right|,$$

where $J = \frac{1}{2}$ and $K = \frac{11}{48}$. It is clear that $0 \leq J \leq 1$ and $J(2J-1) \leq K \leq J$. By the application of Lemma 8, we obtain the result. It is sharp for k_4 defined by (23). □

Theorem 11. *Let $k \in \mathcal{S}^*_{\mathcal{V}}$ and of the form (2). Then*

$$|H_{3,1}(k)| \leq \frac{1}{36}.$$

This bound is sharp.

Proof. Using (17)–(20), we obtain

$$\begin{aligned} H_{3,1}(k) =& \frac{1}{4644864000}(161371c_1^6 - 17236800c_2^3 + 21772800c_1c_2c_3 - 1597860c_1^4c_2 \\ & +658560c_1^3c_3 + 4,944,240c_1^2c_2^2 - 12700800c_1^2c_4 + 36288000c_2c_4 - 32256000c_3^2). \end{aligned}$$

Using Lemma 7 and after simplification we obtain

$$H_{3,1}(k) = \frac{1}{4644864000}\Big(v_1(c,x) + v_2(c,x)y + v_3(c,x)y^2 + \psi(c,x,y)z\Big),$$

where $x,\ y, z \in \overline{\mathbb{D}}$ and

$$\begin{aligned} v_1(c,x) :=& -5459c^6 + (4-c^2)((4-c^2)(252,000x^4c^2 - 1044540c^2x^2 + 453600x^3 + 693000x^3c^2) \\ & + 2721600c^2x^2 - 51330c^4x + 680400c^4x^3 - 895440c^4x^2), \\ v_2(c,x) :=& -6720c(4-c^2)(1-|x|^2)(30(4-c^2)(8x+5x^2) - 64c^2 + 405xc^2), \\ v_3(c,x) :=& -100800(4-c^2)(1-|x|^2)(10(4-c^2)(x^2+8) + 27c^2\bar{x}), \\ \psi(c,x,y) :=& 907200(4-c^2)(1-|x|^2)(1-|y|^2)(3c^2 + 10x(4-c^2)). \end{aligned}$$

Now, by using $|x| = x$, $|y| = y$ and $|z| \leq 1$, we obtain

$$\begin{aligned} H_{3,1}(k) \leq & \frac{1}{4644864000}\Big(|v_1(c,x)| + |v_2(c,x)|y + |v_3(c,x)|y^2 + |\psi(c,x,y)|\Big) \\ \leq & G(c,x,y), \end{aligned}$$

where

$$G(c,x,y) := \frac{1}{4644864000}\Big(g_1(c,x) + g_2(c,x)y + g_3(c,x)y^2 + g_4(c,x)(1-y^2)\Big),$$

with

$$\begin{aligned} g_1(c,x) :=& 5459c^6 + (4-c^2)((4-c^2)(252000x^4c^2 + 1044540c^2x^2 + 453600x^3 + 693000x^3c^2) \\ &+ 2721600c^2x^2 + 51330c^4x + 680400c^4x^3 + 895440c^4x^2), \\ g_2(c,x) :=& 6720c(4-c^2)(1-x^2)(30(4-c^2)(8x+5x^2) + 64c^2 + 405xc^2), \\ g_3(c,x) :=& 100800(4-c^2)(1-x^2)(10(4-c^2)(x^2+8) + 27c^2x), \\ g_4(c,x) :=& 907200(4-c^2)(1-x^2)(3c^2 + 10x(4-c^2)). \end{aligned}$$

To prove the result, we maximize $G(c,x,y)$ over $\Lambda : [0,2] \times [0,1] \times [0,1]$. We discuss all the cases one by one.

I. Firstly, we prove that interior of Λ has no critical point.
Let $(c,x,y) \in (0,2) \times (0,1) \times (0,1)$. Then

$$\begin{aligned} \frac{\partial G}{\partial y} = & \frac{1}{691200}(4-c^2)(1-x^2)[30y(x-1)(10(4-c^2)(x-8) + 27c^2) \\ & + c(30x(4-c^2)(8+5x) + c^2(405x+64))]. \end{aligned}$$

So $\frac{\partial G}{\partial y} = 0$ when

$$y = \frac{c(30x(4-c^2)(8+5x) + c^2(405x+64))}{30(1-x)(10(4-c^2)(x-8) + 27c^2)} := y_0.$$

If y_0 is in Λ, a critical point, then $y_0 \in (0,1)$, and

$$c^3(405x+64) + 30cx(8+5x)(4-c^2) + 300(x-1)(x-8)(4-c^2) < 810(1-x)c^2 \tag{27}$$

and

$$c^2 > \frac{40(x-8)}{10x-107}. \tag{28}$$

Suppose $g(x) := 40(8-x)/(107-10x)$. Now $g'(x) < 0$ for $(0,1)$. This implies that $g(x)$ is decreasing in $(0,1)$. Hence, $c^2 > 280/97$. We see that (27) is satisfied for $c > 1.760524723$ and $x < \frac{341}{810}$. Now we prove that $G(c,x,y) < \frac{1}{36}$ in $(1.760524723, 2) \times (0, \frac{341}{810}) \times (0,1)$. We see that $1 - x^2 < 1$ for $x < \frac{341}{810}$; we may write

$$\begin{aligned} g_1(c,x) &\leq 5459c^6 + (4-c^2)\left(\frac{34138222033916}{23914845}c^2 + \frac{4441003952}{32805} - \frac{326949173771}{23914845}c^4\right) := \Phi_1(c), \\ g_2(c,x) &\leq 6720c(4-c^2)\left(\frac{1116434}{2187} + \frac{233743}{2187}c^2\right) := \Phi_2(c), \\ g_3(c,x) &\leq 100800(4-c^2)\left(\frac{10730162}{32805} - \frac{2309657}{32805}c^2\right) := \Phi_3(c), \\ g_4(c,x) &\leq 907200(4-c^2)\left(-\frac{98}{81}c^2 + \frac{1364}{81}\right) := \Phi_4(c). \end{aligned}$$

Therefore

$$G(c,x,y) \leq \frac{1}{1194393600}\left[\Phi_1(c)+\Phi_4(c)+\Phi_2(c)y+[\Phi_3(c)-\Phi_4(c)]y^2\right] := \psi(c,y).$$

Now

$$\frac{\partial \psi}{\partial y} = \frac{1}{1194393600}[\Phi_2(c)+2[\Phi_3(c)-\Phi_4(c)]y]$$

and

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{1}{1194393600}[\Phi_3(c)-\Phi_4(c)].$$

Since $\Phi_3(c)-\Phi_4(c) \leq 0$ for $c \in (1.760524723, 2)$, $\frac{\partial^2 \psi}{\partial y^2} \leq 0$ for $y \in (0,1)$. This shows that $\frac{\partial \psi}{\partial y}$ is decreasing. Hence, for $y \in (0,1)$,

$$\frac{\partial \psi}{\partial y} \leq \frac{\partial \psi}{\partial y}|_{y=0} = \phi_2(c) \geq 0.$$

Therefore,

$$\psi(c,y) \leq \psi(c,1) = \frac{1}{1194393600}[\phi_1(c)+\phi_2(c)+\phi_3(c)] := \kappa(c).$$

We see that κ takes its maximum value 0.02473632401 at $c = 1.760524723$. Thus,

$$G(c,x,y) < \frac{1}{36} \approx 0.027778, \quad (c,x,y) \in (1.760524723, 2) \times \left(0, \frac{341}{810}\right) \times (0,1).$$

Hence, $G(c,x,y) < \frac{1}{36}$. Therefore, G has no optimal solution in the interior of Λ.

II. Next we obtain the maxima inside the six faces of Λ.

On the face $c = 0$, we have

$$j_1(x,y) := G(0,x,y) = \frac{20(1-x^2)(x-1)(x-8)y^2 - x(171x^2-180)}{5760}, \; x, \; y \in (0,1).$$

As j_1 has no point of maxima in $(0,1)\times(0,1)$ since $x, \; y \in (0,1)$,

$$\frac{\partial j_1}{\partial y} = \frac{(1-x^2)(x-1)(x-8)y}{144} \neq 0.$$

On the face $c = 2$, we write

$$G(2,x,y) = \frac{5459}{72,576,000}, \; x, \; y \in (0,1).$$

On the face $x = 0$, $G(c,x,y)$ reduces to $G(c,0,y)$, given by

$$j_2(c,y) = \frac{100800(4-c^2)(320-107c^2)y^2 + 430080c^3(4-c^2)y + 5459c^6 - 2721600c^4 + 10886400c^2}{4644864000},$$

where $c \in (0,2)$ and $y \in (0,1)$. We solve $\frac{\partial j_2}{\partial y} = 0$ and $\frac{\partial j_2}{\partial c} = 0$ to obtain the required result. On solving $\frac{\partial j_2}{\partial y} = 0$, we obtain

$$y = \frac{32c^3}{15(107c^2-320)} =: y_1. \tag{29}$$

For $y_1 \in (0,1)$, which is possible only if $c > c_0$, $c_0 \approx 1.72935$. The equation $\frac{\partial j_2}{\partial c} = 0$ implies

$$(-25132800 + 7190400c^2)y^2 + (860160c - 358400c^3)y + 5459c^4 - 1814400c^2 + 3628800 = 0. \quad (30)$$

By substituting Equation (29) in Equation (30) and simplifying, we obtain

$$-2313362944c^6 + 1490403c^8 + 18418671360c^4 - 48254976000c^2 + 41287680000 = 0. \quad (31)$$

After some simplifications, we have a solution $c \approx 1.40960$ of (31) in $(0,2)$. This value does not satisfy (29). Thus we conclude that j_2 has no point of maxima in $(0,2) \times (0,1)$.
On $x = 1$, we have

$$j_3(c,y) := G(c,1,y) = \frac{367829c^6 - 11675640c^4 + 39090240c^2 + 7257600}{4644864000}, \quad c \in (0,2).$$

Solving $\frac{\partial j_3}{\partial c} = 0$, we obtain $c := c_0 \approx 1.35379$ as a critical point. We see that j_3 has maxima approximately equal to 0.00903 at c_0.

On $y = 0$, $G(c,x,y)$ can be written as

$$\begin{aligned} j_4(c,x) \quad : &= G(c,x,0) \\ &= \frac{1}{4644864000}\left(\begin{array}{l} 5459c^6 + (4-c^2)((4-c^2)(252000x^4c^2 - 8618400x^3 \\ +693000x^3c^2 + 9072000x + 1044540c^2x^2) + 51330c^4x \\ +680400c^4x^3 + 895440c^4x^2 + 2721600c^2) \end{array}\right). \end{aligned}$$

We see that by using the numerical method, the system $\frac{\partial j_4}{\partial x} = 0$ and $\frac{\partial j_4}{\partial c} = 0$ has no solution in $(0,2) \times (0,1)$.

On $y = 1$, $G(c,x,y)$ reduces to

$$\begin{aligned} j_5(c,x) \quad : &= G(c,x,1) \\ &= \frac{1}{4644864000}\left(\begin{array}{l} 5459c^6 + (4-c^2)((4-c^2)(1044540c^2x^2 + 1008000cx^2 \\ +252000x^4c^2 + 1612800cx - 1008000cx^4 + 693000x^3c^2 \\ -1612800cx^3 - 7056000x^2 + 453600x^3 - 1008000x^4 \\ +8064000) + 51,330c^4x - 2721600c^3x^3 - 430080c^3x^2 \\ +2721600c^2x^2 - 2721600x^3c^2 + 680400c^4x^3 + 430080c^3 \\ +895440c^4x^2 + 2721600c^3x + 2721600c^2x) \end{array}\right). \end{aligned}$$

Similarly, $\frac{\partial j_5}{\partial x} = 0$ and $\frac{\partial j_5}{\partial c} = 0$ has no solution in $(0,2) \times (0,1)$.

III. On the vertices of Λ, we have

$$G(0,0,0) = 0, \quad G(0,0,1) = \frac{1}{36}, \quad G(0,1,1) = \frac{1}{640}, \quad G(0,1,0) = \frac{1}{640},$$
$$G(2,1,0) = G(2,0,0) = G(2,1,1) = G(2,0,1) = \frac{5459}{72576000}.$$

IV. Lastly, we find points of maxima of $G(c,x,y)$ on the 12 edges of Λ.

$$\begin{aligned} G(c,0,0) &= \frac{5459c^6 - 2721600c^4 + 10886400c^2}{4644864000} \le G(\lambda_1,0,0) \\ &= \frac{1992069}{238405448}\sqrt{1549387} - \frac{1239527205}{119202724} \approx 0.00235, \quad c \in (0,2). \end{aligned}$$

where

$$c =: \lambda_1 = \frac{12}{5459}\sqrt{34391700 - 27295\sqrt{1549387}} \approx 1.41851.$$

$$
\begin{aligned}
G(c,0,1) &= \frac{5459c^6 - 430080c^5 + 8064000c^4 + 1720320c^3 - 64512000c^2 + 129024000}{4644864000} \leq G(0,0,1) \\
&= \frac{1}{36} \approx 0.02778, \quad c \in (0,2).
\end{aligned}
$$

$$
\begin{aligned}
G(c,1,0) &= \frac{367829c^6 - 11675640c^4 + 39090240c^2 + 7257600}{4644864000} \leq G(\lambda_2,1,0) \\
&= \frac{16177950997}{61371251382117600}\sqrt{161779509970} - \frac{2381135977308821}{24548500552847040} \approx 0.00903, \quad c \in (0,2),
\end{aligned}
$$

where

$$c := \lambda_2 = \frac{2}{367,829}\sqrt{357886582130 - 735658\sqrt{161779509970}} \approx 1.35379.$$

$$
\begin{aligned}
G(0,x,0) &= \frac{x(20-19x^2)}{640} \leq G(0,\frac{2}{57}\sqrt{285},0) = \frac{\sqrt{285}}{1368} \approx 0.01234, \quad x \in (0,1) \\
G(0,x,1) &= \frac{-20x^4 + 9x^3 - 140x^2 + 160}{5760} \leq G(0,0,1) = \frac{1}{36}, \quad x \in (0,1).
\end{aligned}
$$

$$
\begin{aligned}
G(2,x,0) &= \frac{5459}{72576000}, \quad x \in (0,1). \\
G(2,x,1) &= \frac{5459}{72576000}, \quad x \in (0,1). \\
G(0,0,y) &= \frac{1}{36}y^2 \leq \frac{1}{36}, \quad y \in (0,1). \\
G(0,1,y) &= \frac{1}{640} \approx 0.00156, \quad y \in (0,1). \\
G(2,0,y) &= \frac{5459}{72576000}, \quad y \in (0,1). \\
G(2,1,y) &= \frac{5459}{72576000}, \quad y \in (0,1).
\end{aligned}
$$

Since all cases have been dealt with, we have the required result. The result is sharp for k_3 given in (23), which is equivalent to choosing $d_2 = d_3 = d_5 = 0$ and $d_4 = \frac{1}{6}$, which from (10), gives $|H_{3,1}(k)| = \frac{1}{36}$. □

5. Conclusions

We have defined and studied the starlike functions associated with Van der Pol numbers. We have studied certain geometrical characteristics of the said functions which include the derivation of structural formula, finding the radius of starlikeness of order α and strong starlikeness, and establishing some inclusion results. We have also studied the radii problems for various classes of analytic functions. Furthermore, we have investigated some coefficient-related problems which include the sharp initial coefficient bounds and sharp bounds of Hankel determinants of order two and three. This work would be helpful in finding the bounds of the fourth Hankel determinant, Toelpitz determinants, bounds of logarithmic coefficients and their related Hankel determinants for the functions of defined class $\mathcal{S}^*_{\mathcal{V}}$ and their associated convex functions.

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