mathematics

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Optical Solitons and Modulation Instability Analysis with Lakshmanan–Porsezian–Daniel Model Having Parabolic Law of Self-Phase Modulation

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Abstract: This paper seeks to find optical soliton solutions for Lakshmanan–Porsezian–Daniel (LPD) model with the parabolic law of nonlinearity. The spatiotemporal dispersion is included to the model, as it can contribute to handling the problem of internet bottleneck. This study was performed analytically using the traveling wave hypothesis to reduce the model to an integrable form. Then, the resulting equation was handled with two approaches, namely, the auxiliary equation method and the Bernoulli subordinary differential equation (sub-ODE) method. With an intentional focus on hyperbolic function solutions, abundant optical soliton waves including W-shaped, bright, dark, kink-dark, singular, kink, and antikink solitons were derived with the existing conditions. Furthermore, the behaviors of some optical solitons are illustrated. The spatiotemporal dispersion was found to significantly affect the pulse propagation dynamics. Finally, the modulation instability (MI) of the LPD model is explained in detail along with the extraction of the expression of MI gain.

Keywords: Lakshmanan–Porsezian–Daniel model; optical solitons; parabolic law; modulation instability

MSC: 78A60

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Citation: Al-Kalbani, K.K.; Al-Ghafri, K.S.; Krishnan, E.V.; Biswas, A. Optical Solitons and Modulation Instability Analysis with Lakshmanan–Porsezian–Daniel Model Having Parabolic Law of Self-Phase Modulation. *Mathematics* **2023**, *11*, 2471. https://doi.org/10.3390/math11112471

Academic Editor: Panayiotis Vafeas

Received: 20 April 2023
Revised: 17 May 2023
Accepted: 25 May 2023
Published: 27 May 2023

1. Introduction

The concept of an optical soliton has been a significant subject in many physical and engineering studies such as those of electronic telecommunication system and social media networks [1–3]. Optical solitons play a crucial role since they represent the nature of the propagation of pulse in various nonlinear media [4]. In nonlinear optics, the existence of optical solitons is based on the delicate balance between the group velocity dispersion and the nonlinearity of self-phase modulation (SPM) of the pulse propagation. SPM is a dominant effect in nonlinear optical media, and it has an important role in optical systems that use short, intense pulses of light, such as lasers and optical fiber communications systems. Over the last three decades, various forms of models that are considered as a generalization of nonlinear Schrödinger equation (NLSE) have been developed to characterize the dynamic behaviors of optical solitons in fiber medium. Among these attractive models are the Fokas–Lenells equation [5–7], the Gerdjikov–Ivanov equation [8–10], and the Schrödinger–Hirota equation [11–14] equation, which have been dealt with by many authors. Chirped and chirped-free optical solitons of these models are scrutinized comprehensively so as to contribute to diagnosing the physical properties of optical fiber and its applications. Moreover, NLSE with nonlocal characteristics can generate a variety of new solitons such as fundamental solitons, in-phase and out-of-phase bound-state solitons, multipeak solitons, spatial solitons, and breathers [15–17]. The literature describes various mathematical tools that have been implemented to create solitons and other wave structures for mathematical models. Some of these techniques are Sardar subequation scheme [18], $\exp(-\varphi(\xi))$-expansion approach [19,20], the modified extended tanh-function method [21], and the unified method [22].

The high-level usage of the internet leads to slow internet flow and hence causes network congestion whenmany users attempt to access specific content. In addition, the network demands increases and grows with time, generating the pressures on networks. Thus, all users try to exploit the bandwidth at the same time creating an internet traffic jam. This crisis is known as the internet bottleneck, and it is a growing problem in the telecommunications industry. Researchers across the world have proposed various forms of techniques to overcome this problem and to facilitate the transfer of data smoothly. One of these mechanisms is to introduce the spatiotemporal dispersion (STD) into the medium. The internet bottleneck can be manipulated through reducing the level of internet traffic flow in one direction and enabling full flow in the cross-direction. For this reason, this study was conducted in the presence of STD and chromatic dispersion (CD) in order to address and control the effect of internet bottleneck.

In this paper, we are interested in investigating a model that also describes the nature of optical soliton transmission through optical waveguides—the Lakshmanan–Porsezian–Daniel model (LPD) equation. The first description of this equation appeared in 1988 in the context of the Heisenburg Spain chain equation [23]. Since that time, it has been studied in the content of fiber optics extensively. The dimensionless form of the LPD model that includes different physical effects such as higher order dispersion, full nonlinearity, and spatiotemporal dispersion is addressed as [24]

$$\begin{aligned} iQ_t + aQ_{xx} + bQ_{xt} + c\mathcal{F}\left(|Q|^2\right)Q = \sigma Q_{xxxx} + \alpha(Q_x)^2 Q^* \\ +\beta|Q_x|^2 Q + \gamma|Q|^2 Q_{xx} + \lambda Q^2 Q^*_{xx} + \delta|Q|^4 Q, \end{aligned} \tag{1}$$

where $Q(x,t)$ stands for the complex valued wave function in space x and time t. On the left hand side of Equation (1), the first term is the time evolution, the second term with the coefficient a is the group velocity dispersion, and the third term with the coefficient b is the spatiotemporal dispersion. The last term on this side of the equation represents the effect of nonlinearity given by the function $\mathcal{F}$. On the right hand side of Equation (1), the term with the coefficient σ accounts for the fourth order dispersion whereas the term with coefficient δ defines a two-photon absorption. The reset of terms containing the coefficients α, β, γ, and λ represent perturbation terms with nonlinear forms of dispersion. The symbol $*$ indicates the complex conjugate of the function $Q(x,t)$ and $i = \sqrt{-1}$.

The LPD model (1) was discussed in the past by some experts to investigate the dynamics of solitons under the influence of various forms of nonlinearity. These previous studies were carried out using different mathematical methods, including the improved tanh-expansion technique [25], method of undetermined coefficients [26], the semi-inverse variational principle [27], and many others [28–35].

The present work focuses on deriving distinct structures of optical solitons for the LPD model with a parabolic law of nonlinearity. The effect of spatiotemporal dispersion on the soliton behaviors is also examined. The study was implemented with the aid of two integration schemes known as the auxiliary equation method and the Bernoulli sub-ODE method. To shed light on the behaviors of obtained solitons, the graphical representations of some solutions are displayed. In addition, the modulation instability of LPD model was executed via a standard linear stability analysis.

2. Governing Model

In particular, we consider here the LPD model (1) with the parabolic law of nonlinearity in which the function $\mathcal{F}$ is expressed as $\mathcal{F}(r) = c_1 r + c_2 r^2$. Thus, Equation (1) takes the form [24,36]

$$iQ_t + aQ_{xx} + bQ_{xt} + \left(c_1|Q|^2 + c_2|Q|^4\right)Q = \sigma Q_{xxxx} + \alpha(Q_x)^2 Q^* \\ +\beta|Q_x|^2 Q + \gamma|Q|^2 Q_{xx} + \lambda Q^2 Q^*_{xx} + \delta|Q|^4 Q. \tag{2}$$

To overcome the complexity and to reach an integrable form for this equation, assume that Equation (2) has the traveling wave solution of the form

$$Q(x,t) = q(\xi)\, e^{i\varphi(x,t)}, \quad \xi = x - vt. \tag{3}$$

The function $q(\xi)$ is the amplitude component of the soliton, and $\xi = x - vt$ is the wave variable, where v is the velocity of the soliton. The phase component is denoted by $\varphi(x,t)$ which is defined as

$$\varphi(x,t) = -kx + \omega t + \theta, \tag{4}$$

where k, w, and θ describe the soliton frequency, wave number, and phase constant, respectively. Substituting (3) into Equation (2) and splitting the resulting equation into real and imaginary parts, we obtain the following equations:

$$\sigma q'''' - (6\sigma k^2 - bv + a)q'' - (b\omega k - \sigma k^4 - ak^2 - \omega)q - [c_1 + k^2(\alpha - \beta + \gamma + \lambda)]q^3 \\ -(c_2 - \delta)q^5 + (\alpha + \beta)qq'^2 + (\gamma + \lambda)q^2 q'' = 0, \tag{5}$$

$$(b\omega + bkv - v - 2ak - 4\sigma k^3)q' + 2k(\alpha + \gamma - \lambda)q^2 q' + 4\sigma k q''' = 0. \tag{6}$$

Now, taking the coefficients of the linearly independent functions and equating them to zero gives the following:

$$\sigma = 0, \tag{7}$$

$$\alpha + \beta = 0, \tag{8}$$

$$\gamma + \lambda = 0, \tag{9}$$

$$\alpha + \gamma - \lambda = 0, \tag{10}$$

$$b\omega + bkv - v - 2ak = 0. \tag{11}$$

From Equation (11), one can deduce that

$$v = \frac{2ak - b\omega}{bk - 1}, \tag{12}$$

which represents the wave speed provided that $bk \neq 1$. With the constraints (7)–(10), Equation (5) becomes

$$L_0 q'' + L_1 q + L_2 q^3 + L_3 q^5 = 0, \tag{13}$$

which can be integrated after multiplying by q' to arrive at

$$L_0 q'^2 + L_1 q^2 + \frac{L_2}{2} q^4 + \frac{L_3}{3} q^6 + 2L_4 = 0, \tag{14}$$

where L_4 is an arbitrary constant of integration, and L_0, L_1, L_2, L_3 are given by

$$L_0 = a - bv, \tag{15}$$

$$L_1 = b\omega k - ak^2 - \omega, \tag{16}$$

$$L_2 = c_1 - 4\gamma k^2, \tag{17}$$

$$L_3 = c_2 - \delta. \tag{18}$$

For convenience, we employ the variable transformation, presented as

$$q^2 = P, \tag{19}$$

From this, Equation (14) is converted into the following form:

$$L_0 P'' + 4L_1 P + 3L_2 P^2 + \frac{8}{3} L_3 P^3 + 4L_4 = 0. \tag{20}$$

3. Optical Soliton Solutions

This section is dedicated to creating optical soliton solutions for the LPD model (2) through tackling Equation (20) with two powerful approaches: the auxiliary equation method and the Bernoulli sub-ODE method. To apply these techniques, we firstly set

$$P(\xi) = W(\zeta), \quad \zeta = \Omega\,\xi, \tag{21}$$

from which Equation (20) becomes

$$\Omega^2 L_0 W'' + 4L_1 W + 3L_2 W^2 + \frac{8}{3} L_3 W^3 + 4L_4 = 0, \tag{22}$$

where the prime represents the derivative with respect to ζ.

3.1. Auxiliary Equation Method

Herein, we assume that Equation (22) has solutions in the form

$$W(\Omega) = a_0 + a_1 F(\zeta) + a_2 F^2(\zeta), \tag{23}$$

where a_0, a_1 and a_2 are constants to be determined. The function $F(\zeta)$ satisfies

$$\left(\frac{dF}{d\zeta}\right)^2 = h_0 + h_2 F^2(\zeta) + h_4 F^4(\zeta), \tag{24}$$

where h_0, h_2 and h_4 are constants to be obtained. Substituting (23) and (24) into Equation (22) gives a polynomial in $F^i(\zeta), i = 0, 1, 2, \ldots, 6$. Equating the coefficients of various powers of $F^i(\zeta)$ in this polynomial to zero gives the following system of algebraic equations.

$$\begin{aligned}
F^0 &: \quad 2\Omega^2 L_0 a_2 h_0 + 4L_1 a_0 + 3L_2 a_0^2 + \frac{8}{3} L_3 a_0^3 + 4L_4 = 0,\\
F^1 &: \quad \Omega^2 a_1 h_2 L_0 + 8a_0^2 a_1 L_3 + 6a_0 a_1 L_2 + 4a_1 L_1 = 0,\\
F^2 &: \quad 4\Omega^2 a_2 h_2 L_0 + 8a_0^2 a_2 L_3 + 8a_0 a_1^2 L_3 + 6a_0 a_2 L_2 + 3a_1^2 L_2 + 4a_2 L_1 = 0,\\
F^3 &: \quad 6L_2 a_1 a_2 + \frac{8}{3} L_3 a_1^3 + 2\Omega^2 L_0 a_1 h_4 + 16L_3 a_0 a_1 a_2 = 0,\\
F^4 &: \quad 6\Omega^2 a_2 h_4 L_0 + 8a_0 a_2^2 L_3 + 8a_1^2 a_2 L_3 + 3a_2^2 L_2 = 0,\\
F^5 &: \quad 8L_3 a_1 a_2^2 = 0,\\
F^6 &: \quad \frac{8}{3} L_3 a_2^3 = 0.
\end{aligned} \tag{25}$$

The solution of the above system constructs two sets of values for the constants a_0, a_1 and a_2 under specific conditions. Each set yields many cases of abundant solutions to Equation (2).

- Set 1

$$a_0 = \frac{-2}{3L_2}\left(L_1 + h_2\sqrt{\frac{L_1^2 - 3L_2L_4}{h_2^2 - 3h_0h_4}}\right),\ a_1 = 0,\ a_2 = \frac{-2h_4}{L_2}\sqrt{\frac{L_1^2 - 3L_2L_4}{h_2^2 - 3h_0h_4}},$$
$$\Omega = \left(\frac{L_1^2 - 3L_2L_4}{L_0^2(h_2^2 - 3h_0h_4)}\right)^{\frac{1}{4}},\ L_3 = 0. \tag{26}$$

From (26), one can retrieve the general form of the Jacobi elliptic function (JEF) solutions of Equation (20) as

$$P(\xi) = \frac{-2}{3L_2}\left(L_1 + \sqrt{\frac{L_1^2 - 3L_2L_4}{h_2^2 - 3h_0h_4}}\left\{h_2 + 3h_4\,F^2(\Omega\xi)\right\}\right), \tag{27}$$

provided that $L_2 \neq 0$ and $h_2^2 \neq 3h_0h_4$. Hence, the relations (3) and (19) lead to the general solution of Equation (2) as

$$Q(x,t) = \pm\left[\frac{-2}{3(c_1 - 4\gamma k^2)}\left(b\omega k - ak^2 - \omega + \sqrt{\frac{(b\omega k - ak^2 - \omega)^2 - 3(c_1 - 4\gamma k^2)L_4}{h_2^2 - 3h_0h_4}}\right.\right.$$
$$\left.\left.\times\left\{h_2 + 3h_4\,F^2(\Omega[x - vt])\right\}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{28}$$

where $\Omega = \left[\frac{(b\omega k - ak^2 - \omega)^2 - 3(c_1 - 4\gamma k^2)L_4}{(a - bv)^2(h_2^2 - 3h_0h_4)}\right]^{\frac{1}{4}}$ provided that $c_1 \neq 4\gamma k^2$ and $a \neq bv$.

Making use of some of Jacobi elliptic functions, we can obtain the following types of solutions:

Case 1. If $h_0 = 1,\ h_2 = -(1 + m^2),\ h_4 = m^2$, then $F(\xi) = \mathrm{sn}(\xi)$. Subsequently, we extract JEF solutions of Equation (2) in the form

$$Q(x,t) = \pm\left[\frac{-2}{3(c_1 - 4\gamma k^2)}\left(b\omega k - ak^2 - \omega - \sqrt{\frac{(b\omega k - ak^2 - \omega)^2 - 3(c_1 - 4\gamma k^2)L_4}{m^4 - m^2 + 1}}\right.\right.$$
$$\left.\left.\times\{(m^2 + 1) - 3m^2\,\mathrm{sn}^2(\Omega[x - vt])\}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{29}$$

where $\Omega = \left[\frac{(b\omega k - ak^2 - \omega)^2 - 3(c_1 - 4\gamma k^2)L_4}{(a - bv)^2(m^4 - m^2 + 1)}\right]^{\frac{1}{4}}$. As $m \to 1$, solution (29) degenerates to soliton solutions given by

$$Q(x,t) = \pm\left[\frac{-2}{3(c_1 - 4\gamma k^2)}\left(b\omega k - ak^2 - \omega - \sqrt{(b\omega k - ak^2 - \omega)^2 - 3(c_1 - 4\gamma k^2)L_4}\right.\right.$$
$$\left.\left.\times\left\{2 - 3\tanh^2(\Omega[x - vt])\right\}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{30}$$

where $\Omega = \left[\frac{(b\omega k - ak^2 - \omega)^2 - 3(c_1 - 4\gamma k^2)L_4}{(a - bv)^2}\right]^{\frac{1}{4}}$.

Case 2. If $h_0 = 1 - m^2,\ h_2 = 2m^2 - 1,\ h_4 = -m^2$, then $F(\xi) = \mathrm{cn}(\xi)$. Accordingly, the JEF solutions of Equation (2) are written as

$$Q(x,t) = \pm\left[\frac{-2}{3(c_1-4\gamma k^2)}\left(b\omega k - ak^2 - \omega + \sqrt{\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{m^4-m^2+1}}\right.\right.$$
$$\left.\left.\times\left\{(2m^2-1) - 3m^2\,\mathrm{cn}^2(\Omega[x-vt])\right\}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{31}$$

where $\Omega = \left[\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{(a-bv)^2(m^4-m^2+1)}\right]^{\frac{1}{4}}$. As $m \to 1$, solution (31) gives rise to soliton solutions given by

$$Q(x,t) = \pm\left[\frac{-2}{3(c_1-4\gamma k^2)}\left(b\omega k - ak^2 - \omega + \sqrt{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}\right.\right.$$
$$\left.\left.\times\left\{1 - 3\,\mathrm{sech}^2(\Omega[x-vt])\right\}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{32}$$

where $\Omega = \left[\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{(a-bv)^2}\right]^{\frac{1}{4}}$.

Case 3. If $h_0 = h_4 = \frac{1}{4}$, $h_2 = \frac{1-2m^2}{2}$, then $F(\xi) = \frac{\mathrm{sn}(\xi)}{1+\mathrm{cn}(\xi)}$. Thus, we end up with JEF solutions of Equation (2) in the form

$$Q(x,t) = \pm\left[\frac{-2}{3(c_1-4\gamma k^2)}\left(b\omega k - ak^2 - \omega + \sqrt{\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{16m^4-16m^2+1}}\right.\right.$$
$$\left.\left.\times\left\{(2-4m^2) + 3\left(\frac{\mathrm{sn}(\Omega[x-vt])}{1+\mathrm{cn}(\Omega[x-vt])}\right)^2\right\}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{33}$$

where $\Omega = 2\left[\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{(a-bv)^2(16m^4-16m^2+1)}\right]^{\frac{1}{4}}$. As $m \to 1$, solution (33) reduces to soliton solutions given by

$$Q(x,t) = \pm\left[\frac{-2}{3(c_1-4\gamma k^2)}\left(b\omega k - ak^2 - \omega + \sqrt{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}\right.\right.$$
$$\left.\left.\times\left\{-2 + 3\left(\frac{\tanh(\Omega[x-vt])}{1+\mathrm{sech}(\Omega[x-vt])}\right)^2\right\}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{34}$$

where $\Omega = 2\left[\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{(a-bv)^2}\right]^{\frac{1}{4}}$.

Case 4. If $h_0 = 1-m^2$, $h_2 = 2-m^2$, $h_4 = 1$, then $F(\xi) = \mathrm{cs}(\xi)$. In consequence, we obtain JEF solutions of Equation (2) as

$$Q(x,t) = \pm\left[\frac{-2}{3(c_1-4\gamma k^2)}\left(b\omega k - ak^2 - \omega + \sqrt{\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{m^4-m^2+1}}\right.\right.$$
$$\left.\left.\times\left\{(2-m^2) + 3\,\mathrm{cs}^2(\Omega[x-vt])\right\}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{35}$$

where $\Omega = \left[\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{(a-bv)^2(m^4-m^2+1)}\right]^{\frac{1}{4}}$. As $m \to 1$, solution (35) is converted into singular soliton solutions given by

$$Q(x,t) = \pm\Bigg[\frac{-2}{3(c_1-4\gamma k^2)}\Bigg(b\omega k - ak^2 - \omega + \sqrt{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4} \times\Big\{1 + 3\,\operatorname{csch}^2(\Omega[x-vt])\Big\}\Bigg)\Bigg]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{36}$$

where $\Omega = \left[\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{(a-bv)^2}\right]^{\frac{1}{4}}$.

Case 5. If $h_0 = m^2,\ h_2 = -(1+m^2),\ h_4 = 1$, then $F(\xi) = \operatorname{ns}(\xi)$. Therefore, we obtain JEF solutions of Equation (2) presented as

$$Q(x,t) = \pm\left[\frac{-2}{3(c_1-4\gamma k^2)}\left(b\omega k - ak^2 - \omega - \sqrt{\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{m^4-m^2+1}} \times\{(m^2+1) - 3\,\operatorname{ns}^2(\Omega[x-vt])\}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{37}$$

where $\Omega = \left[\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{(a-bv)^2(m^4-m^2+1)}\right]^{\frac{1}{4}}$. As $m \to 1$, solution (37) changes to soliton solutions having a profile of singular wave given by

$$Q(x,t) = \pm\Bigg[\frac{-2}{3(c_1-4\gamma k^2)}\Bigg(b\omega k - ak^2 - \omega - \sqrt{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4} \times\Big\{2 - 3\coth^2(\Omega[x-vt])\Big\}\Bigg)\Bigg]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{38}$$

where $\Omega = \left[\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{(a-bv)^2}\right]^{\frac{1}{4}}$.

Case 6. If $h_0 = 4m(1+m)^2,\ h_2 = -(4m+(1+m)^2),\ h_4 = 1$, then $F(\xi) = m\,\operatorname{sn}(\xi) + \operatorname{ns}(\xi)$. Consequently, we obtain JEF solutions of Equation (2) as follows

$$Q(x,t) = \pm\left[\frac{-2}{3(c_1-4\gamma k^2)}\left(b\omega k - ak^2 - \omega - \sqrt{\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{m^4+14m^2+1}} \times\Big\{(m^2+6m+1) - 3(\,m\,\operatorname{sn}(\Omega[x-vt]) + \operatorname{ns}(\Omega[x-vt]))^2\Big\}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{39}$$

where $\Omega = \left[\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{(a-bv)^2(m^4+14m^2+1)}\right]^{\frac{1}{4}}$. As $m \to 1$, solution (39) results in singular soliton solutions as follows

$$Q(x,t) = \pm\Bigg[\frac{-2}{3(c_1-4\gamma k^2)}\Bigg(b\omega k - ak^2 - \omega - \frac{1}{4}\sqrt{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4} \times\Big\{8 - 3\,(\,\tanh(\Omega[x-vt]) + \coth(\Omega[x-vt])\,)^2\Big\}\Bigg)\Bigg]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{40}$$

where $\Omega = \left[\frac{(b\omega k - ak^2-\omega)^2 - 3(c_1-4\gamma k^2)L_4}{16(a-bv)^2}\right]^{\frac{1}{4}}$.

- Set 2

$$a_0 = \frac{-3L_2}{8L_3}, a_1 = \pm\frac{1}{8L_3}\sqrt{\frac{6h_4(32L_1L_3 - 9L_2^2)}{h_2}}, a_2 = 0, \quad \Omega = \sqrt{\frac{9L_2^2 - 32L_1L_3}{8L_3h_2L_0}},\ \ L_4 = \frac{48L_1L_2L_3 - 9L_2^3}{128L_3^2}. \tag{41}$$

From (41), the general form of JEF solutions of Equation (20) can be expressed as

$$P(\xi) = \frac{1}{8L_3}\left(-3L_2 \pm \sqrt{\frac{6h_4(32L_1L_3 - 9L_2^2)}{h_2}}\, F(\Omega\xi)\right), \tag{42}$$

provided that $h_2 \neq 0$ and $L_3 \neq 0$. By virtue of relations (3) and (19), the general solution of Equation (2) is written as

$$\begin{aligned} Q(x,t) = {} & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1-4\gamma k^2) \pm \sqrt{\frac{6h_4[32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2]}{h_2}}\right.\right. \\ & \left.\left.\times F(\Omega[x-vt])\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \end{aligned} \tag{43}$$

where $\Omega = \sqrt{\frac{-32(b\omega k - ak^2-\omega)(c_2-\delta)+9(c_1-4\gamma k^2)^2}{8h_2(c_2-\delta)(a-bv)}}$ provided that $c_2 \neq \delta$ and $a \neq bv$. Thus, implementing some of the Jacobi elliptic functions generates distinct types of solutions displayed as follows.

Case 1. If $h_0 = 1,\ h_2 = -(1+m^2),\ h_4 = m^2$, then $F(\xi) = \mathrm{sn}(\xi)$. Hence, we secure JEF solutions of Equation (2) as

$$\begin{aligned} Q(x,t) = {} & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1-4\gamma k^2) \pm \sqrt{\frac{-6m^2[32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2]}{m^2+1}}\right.\right. \\ & \left.\left.\times\, \mathrm{sn}(\Omega[x-vt])\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \end{aligned} \tag{44}$$

where $\Omega = \sqrt{\frac{32(b\omega k - ak^2-\omega)(c_2-\delta)-9(c_1-4\gamma k^2)^2}{8(c_2-\delta)(m^2+1)(a-bv)}}$. As $m \to 1$, solution (44) degenerates to soliton solutions as

$$\begin{aligned} Q(x,t) = {} & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1-4\gamma k^2) \pm \sqrt{-3\,[32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2]}\right.\right. \\ & \left.\left.\times \tanh(\Omega[x-vt])\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \end{aligned} \tag{45}$$

where $\Omega = \sqrt{\frac{32(b\omega k - ak^2-\omega)(c_2-\delta)-9(c_1-4\gamma k^2)^2}{16(c_2-\delta)(a-bv)}}$.

Case 2. If $h_0 = 1-m^2,\ h_2 = 2m^2-1,\ h_4 = -m^2$, then $F(\xi) = \mathrm{cn}(\xi)$. Therefore, we arrive at JEF solutions of Equation (2) given by

$$\begin{aligned} Q(x,t) = {} & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1-4\gamma k^2) \pm \sqrt{\frac{-6m^2[32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2]}{2m^2-1}}\right.\right. \\ & \left.\left.\times\, \mathrm{cn}(\Omega[x-vt])\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \end{aligned} \tag{46}$$

where $\Omega = \sqrt{\frac{-32(b\omega k - ak^2-\omega)(c_2-\delta)+9(c_1-4\gamma k^2)^2}{8(c_2-\delta)(2m^2-1)(a-bv)}}$ provided that $m \neq \frac{1}{\sqrt{2}}$. As $m \to 1$, solution (46) is converted to soliton solutions in the form

$$\begin{aligned} Q(x,t) = {} & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1-4\gamma k^2) \pm \sqrt{-6[32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2]}\right.\right. \\ & \left.\left.\times\, \mathrm{sech}(\Omega[x-vt])\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \end{aligned} \tag{47}$$

where $\Omega = \sqrt{\frac{-32(b\omega k - ak^2-\omega)(c_2-\delta)+9(c_1-4\gamma k^2)^2}{8\,(c_2-\delta)(a-bv)}}$.

Case 3. If $h_0 = h_4 = \frac{1}{4}$, $h_2 = \frac{1-2m^2}{2}$, then $F(\xi) = \frac{\mathrm{sn}(\xi)}{1+\mathrm{cn}(\xi)}$. As a consequence, we obtain JEF solutions of Equation (2) as follows

$$
\begin{aligned}
Q(x,t) = \; & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1-4\gamma k^2) \pm \sqrt{\frac{3[32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2]}{-2m^2+1}}\right.\right. \\
& \left.\left.\times \frac{\mathrm{sn}(\Omega[x-vt])}{1+\mathrm{cn}(\Omega[x-vt])}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)},
\end{aligned} \tag{48}
$$

where $\Omega = \sqrt{\frac{32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2}{4(c_2-\delta)(2m^2-1)(a-bv)}}$ provided that $m \neq \frac{1}{\sqrt{2}}$. As $m \to 1$, solution (48) reduces to the soliton solutions as

$$
\begin{aligned}
Q(x,t) = \; & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1-4\gamma k^2) \pm \sqrt{-3[32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2]}\right.\right. \\
& \left.\left.\times \frac{\tanh(\Omega[x-vt])}{1+\mathrm{sech}(\Omega[x-vt])}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)},
\end{aligned} \tag{49}
$$

where $\Omega = \sqrt{\frac{32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2}{4(c_2-\delta)(a-bv)}}$.

Case 4. If $h_0 = 1-m^2$, $h_2 = 2-m^2$, $h_4 = 1$, then $F(\xi) = \mathrm{cs}(\xi)$. Subsequently, we obtain JEF solutions of Equation (2) with the form

$$
\begin{aligned}
Q(x,t) = \; & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1-4\gamma k^2) \pm \sqrt{\frac{6[32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2]}{2-m^2}}\right.\right. \\
& \left.\left.\times\, \mathrm{cs}(\Omega[x-vt])\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)},
\end{aligned} \tag{50}
$$

where $\Omega = \sqrt{\frac{-32(b\omega k - ak^2-\omega)(c_2-\delta) + 9(c_1-4\gamma k^2)^2}{8(c_2-\delta)(2-m^2)(a-bv)}}$. As $m \to 1$, solution (50) gives rise to singular soliton solutions as

$$
\begin{aligned}
Q(x,t) = \; & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1-4\gamma k^2) \pm \sqrt{6[32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2]}\right.\right. \\
& \left.\left.\times\, \mathrm{csch}(\Omega[x-vt])\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)},
\end{aligned} \tag{51}
$$

where $\Omega = \sqrt{\frac{-32(b\omega k - ak^2-\omega)(c_2-\delta) + 9(c_1-4\gamma k^2)^2}{8(c_2-\delta)(a-bv)}}$.

Case 5. If $h_0 = m^2$, $h_2 = -(1+m^2)$, $h_4 = 1$, then $F(\xi) = \mathrm{ns}(\xi)$. From this, we obtain JEF solutions of Equation (2) as

$$
\begin{aligned}
Q(x,t) = \; & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1-4\gamma k^2) \pm \sqrt{\frac{-6[32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2]}{m^2+1}}\right.\right. \\
& \left.\left.\times\, \mathrm{ns}(\Omega[x-vt])\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)},
\end{aligned} \tag{52}
$$

where $\Omega = \sqrt{\frac{32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2}{8(c_2-\delta)(m^2+1)(a-bv)}}$. As $m \to 1$, solution (52) changes into singular soliton solutions

$$
\begin{aligned}
Q(x,t) = \; & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1-4\gamma k^2) \pm \sqrt{-3[32(b\omega k - ak^2-\omega)(c_2-\delta) - 9(c_1-4\gamma k^2)^2]}\right.\right. \\
& \left.\left.\times\, \coth(\Omega[x-vt])\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)},
\end{aligned} \tag{53}
$$

where $\Omega = \sqrt{\dfrac{32(b\omega k - ak^2 - \omega)(c_2 - \delta) - 9(c_1 - 4\gamma k^2)^2}{16\,(c_2 - \delta)(a - bv)}}$.

Case 6. If $h_0 = 4m(1+m)^2,\ h_2 = -(4m + (1+m)^2),\ h_4 = 1$, then $F(\xi) = m\,\mathrm{sn}(\xi) + \mathrm{ns}(\xi)$. As a result, we obtain JEF solutions of Equation (2) as

$$
\begin{aligned}
Q(x,t) = \quad & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1 - 4\gamma k^2) \pm \sqrt{\frac{6[32(b\omega k - ak^2 - \omega)(c_2-\delta) - 9(c_1 - 4\gamma k^2)^2]}{-m^2 - 6m - 1}}\right.\right. \\
& \left.\left.\times\{m\,\mathrm{sn}(\Omega[x - vt]) + \mathrm{ns}(\Omega[x - vt])\}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)},
\end{aligned}
\tag{54}
$$

where $\Omega = \sqrt{\dfrac{32(b\omega k - ak^2 - \omega)(c_2 - \delta) - 9(c_1 - 4\gamma k^2)^2}{8(c_2 - \delta)(m^2 + 6m + 1)(a - bv)}}$. As $m \to 1$, solution (54) degenerates to singular soliton solutions in the form

$$
\begin{aligned}
Q(x,t) = \quad & \pm\left[\frac{1}{8(c_2-\delta)}\left(-3(c_1 - 4\gamma k^2) \pm \tfrac{1}{2}\sqrt{-3[32(b\omega k - ak^2 - \omega)(c_2-\delta) - 9(c_1 - 4\gamma k^2)^2]}\right.\right. \\
& \left.\left.\times\{\tanh(\Omega[x - vt]) + \coth(\Omega[x - vt])\}\right)\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)},
\end{aligned}
\tag{55}
$$

where $\Omega = \sqrt{\dfrac{32(b\omega k - ak^2 - \omega)(c_2 - \delta) - 9(c_1 - 4\gamma k^2)^2}{64(c_2 - \delta)(a - bv)}}$.

3.2. Bernoulli Sub-ODE Method

Now, we aim to procure soliton solutions to LPD Equation (2) through applying a specific form of the Bernoulli sub-ODE method. Thus, we consider that the general solution of Equation (22) is given as

$$
W(\zeta) = b_0 + b_1\,G(\zeta) + b_2\,G^2(\zeta)\,,
\tag{56}
$$

where b_0, b_1 and b_2 are constants to be identified. The function $G(\zeta)$ satisfies the type of Bernoulli equation given by

$$
G'(\zeta) = \eta^2\,G^2(\zeta) - \eta\,G(\zeta)\,,
\tag{57}
$$

which has the solution of the form

$$
G(\zeta) = \frac{1}{\eta + \rho\,e^{\eta\zeta}},
\tag{58}
$$

where η and ρ are arbitrary constants.

Substituting (56) and (57) into Equation (22) gives a polynomial in $G^i(\zeta)$, $i = 0, 1, \ldots, 6$. Equating the coefficients of various powers of G to zero, we arrive at the following system of algebraic equations.

$$
\begin{aligned}
G^0 &: \quad 4L_1 b_0 + 3L_2 b_0^2 + \frac{8}{3}L_3 b_0^3 + 4L_4 = 0, \\
G^1 &: \quad \Omega^2\eta^2 b_1 L_0 + 8b_0^2 b_1 L_3 + 6b_0 b_1 L_2 + 4b_1 L_1 = 0, \\
G^2 &: \quad -3\Omega^2\eta^3 b_1 L_0 + 4\Omega^2\eta^2 b_2 L_0 + 8b_0^2 b_2 L_3 + 8b_0 b_1^2 L_3 + 6b_0 b_2 L_2 + 3b_1^2 L_2 + 4b_2 L_1 = 0, \\
G^3 &: \quad 6l_2 b_1 b_2 + \frac{8}{3}l_3 b_1^3 + 2\Omega^2 L_0\eta^4 b_1 - 10\Omega^2 L_0\eta^3 b_2 + 16L_3 b_0 b_1 b_2 = 0, \\
G^4 &: \quad 6\Omega^2\eta^4 b_2 L_0 + 8b_0 b_2^2 L_3 + 8b_1^2 b_2 L_3 + 3b_2^2 L_2 = 0, \\
G^5 &: \quad 8L_3 b_1 b_2^2 = 0, \\
G^6 &: \quad \frac{8}{3}L_3 b_2^3 = 0.
\end{aligned}
\tag{59}
$$

Solving the above system yields several sets of solutions for the constants b_0, b_1 and b_2 which are presented below.

- Set 1

$$
\begin{aligned}
&b_0 = \frac{2\left(L_1 \pm \sqrt{L_1^2 - 3L_2L_4}\right)}{-3L_2},\ b_1 = \pm\frac{8\eta\sqrt{L_1^2 - 3l_2L_4}}{L_2},\ b_2 = \mp\frac{8\eta^2\sqrt{L_1^2 - 3L_2L_4}}{L_2},\\
&\Omega = \pm\frac{2}{\eta}\left(\frac{L_1^2 - 3L_2L_4}{L_0^2}\right)^{\frac{1}{4}},\ L_3 = 0.
\end{aligned} \tag{60}
$$

From (60) and using (58) together with (21), the general solution (56) to Equation (22) collapses to

$$
P(\xi) = \frac{-2}{3L_2}\left(L_1 \pm \sqrt{L_1^2 - 3L_2L_4}\left\{1 - \frac{12\eta}{\eta + \rho\, e^{\eta\Omega\xi}} + \frac{12\,\eta^2}{(\eta + \rho\, e^{\eta\Omega\xi})^2}\right\}\right), \tag{61}
$$

provided that $L_1^2 \neq 3L_2L_4$ and $L_2 \neq 0$. Hence, the soliton solution of Equation (2) can be written as

$$
\begin{aligned}
Q(x,t) = \pm\ &\left[\frac{-2}{3(c_1 - 4\gamma k^2)}\left(b\omega k - ak^2 - \omega \pm \sqrt{(b\omega k - ak^2 - \omega)^2 - 3(c_1 - 4\gamma k^2)L_4}\right.\right.\\
&\left.\left.\left\{1 - \frac{12\eta}{\eta + \rho\, e^{\eta\Omega\xi}} + \frac{12\,\eta^2}{(\eta + \rho\, e^{\eta\Omega\xi})^2}\right\}\right)\right]^{\frac{1}{2}} e^{i(-kx + \omega t + \theta)},
\end{aligned} \tag{62}
$$

where $\Omega = \pm\frac{2}{\eta}\left(\frac{(b\omega k - ak^2 - \omega)^2 - 3(c_1 - 4\gamma k^2)L_4}{(a - bv)^2}\right)^{\frac{1}{4}}$ provided that $c_1 \neq 4\lambda k^2$, $a \neq bv$ and $c_2 = \delta$.

- Set 2

$$
\begin{aligned}
&b_0 = \frac{3L_2 \pm \sqrt{3(9L_2^2 - 32L_1L_3)}}{-8L_3},\ b_1 = \pm\frac{\eta\sqrt{3(9L_2^2 - 32L_3L_1)}}{4L_3},\ b_2 = 0,\\
&\Omega = \pm\frac{1}{2\,\eta}\left(\frac{32L_1L_3 - 9L_2^2}{L_0L_3}\right)^{\frac{1}{2}},\ L_4 = \frac{48L_1L_2L_3 - 9L_2^3}{128L_3^2}.
\end{aligned} \tag{63}
$$

From (63) and using (58) along with (21), the general solution (56) to Equation (22) becomes

$$
P(\xi) = \frac{-1}{8L_3}\left(3L_2 \pm \sqrt{3(9L_2^2 - 32L_1L_3)}\left\{1 - \frac{2\eta}{\eta + \rho\, e^{\eta\Omega\xi}}\right\}\right), \tag{64}
$$

provided that $9L_2^2 \neq 32L_1L_3$ and $L_3 \neq 0$. Therefore, the soliton solution of Equation (2) is introduced as

$$
\begin{aligned}
Q(x,t) = \pm\ &\left[\frac{-1}{8(c_2 - \delta)}\left(3(c_1 - 4\gamma k^2) \pm \sqrt{3(9(c_1 - 4\gamma k^2)^2 - 32(b\omega k - ak^2 - \omega)(c_2 - \delta))}\right.\right.\\
&\left.\left.\left\{1 - \frac{2\eta}{\eta + \rho\, e^{\eta\Omega\xi}}\right\}\right)\right]^{\frac{1}{2}} e^{i(-kx + \omega t + \theta)},
\end{aligned} \tag{65}
$$

where

$$
\Omega = \pm\frac{1}{2\,\eta}\left(\frac{32(b\omega k - ak^2 - \omega)(c_2 - \delta) - 9(c_1 - 4\gamma k^2)^2}{(a - bv)(c_2 - \delta)}\right)^{\frac{1}{2}}, \tag{66}
$$

$$
L_4 = \frac{48(b\omega k - ak^2 - \omega)(c_1 - 4\gamma k^2)(c_2 - \delta) - 9(c_1 - 4\gamma k^2)^3}{128(c_2 - \delta)^2}, \tag{67}
$$

provided that $a \neq bv$ and $c_2 \neq \delta$.

- Set 3

$$b_0 = \frac{-4L_1}{3L_2},\ b_1 = \frac{8\eta L_1}{L_2},\ b_2 = \frac{-8\eta^2 L_1}{L_2}, \Omega = \pm\frac{2}{\eta}\sqrt{\frac{L_1}{L_0}},\ L_3 = L_4 = 0\,. \tag{68}$$

From these findings and using (58) in conjunction with (21), the general solution (56) to Equation (22) reduces to

$$P(\xi) = \frac{-4L_1}{3L_2}\left\{1 - \frac{6\eta}{\eta + \rho\, e^{\eta\Omega\xi}} + \frac{6\eta^2}{(\eta + \rho\, e^{\eta\Omega\xi})^2}\right\}, \tag{69}$$

provided that $L_1 \neq 0$ and $L_2 \neq 0$. As a result, the soliton solution of Equation (2) becomes

$$Q(x,t) = \pm\left[\frac{-4(b\omega k - ak^2 - \omega)}{3(c_1 - 4\gamma k^2)}\left\{1 - \frac{6\eta}{\eta + \rho\, e^{\eta\Omega\xi}} + \frac{6\eta^2}{(\eta + \rho\, e^{\eta\Omega\xi})^2}\right\}\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{70}$$

where $\Omega = \pm\frac{2}{\eta}\sqrt{\frac{b\omega k - ak^2 - \omega}{a - bv}}$ provided that $c_1 \neq 4\lambda k^2$, $a \neq bv$ and $c_2 = \delta$.

- Set 4

$$b_0 = 0,\ b_1 = \frac{-8\eta L_1}{L_2},\ b_2 = \frac{8\eta^2 L_1}{L_2}, \Omega = \pm\frac{2}{\eta}\sqrt{\frac{-L_1}{L_0}},\ L_3 = L_4 = 0. \tag{71}$$

From (71) and using (58) in conjunction with (21), the general solution (56) to Equation (22) changes into the form

$$P(\xi) = \frac{-8\eta L_1}{L_2}\left\{\frac{1}{\eta + \rho\, e^{\eta\Omega\xi}} - \frac{\eta}{\left(\eta + \rho\, e^{\eta\Omega\xi}\right)^2}\right\}, \tag{72}$$

provided that $L_1 \neq 0$ and $L_2 \neq 0$. For this reason, the soliton solution of Equation (2) is presented as

$$Q(x,t) = \pm\left[\frac{-8\eta\,(b\omega k - ak^2 - \omega)}{c_1 - 4\gamma k^2}\left\{\frac{1}{\eta + \rho\, e^{\eta\Omega\xi}} - \frac{\eta}{\left(\eta + \rho\, e^{\eta\Omega\xi}\right)^2}\right\}\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{73}$$

where $\Omega = \pm\frac{2}{\eta}\sqrt{\frac{-\,(b\omega k - ak^2 - \omega)}{a - bv}}$ provided that $c_1 \neq 4\lambda k^2$, $a \neq bv$ and $c_2 = \delta$.

- Set 5

$$b_0 = \frac{-4L_1}{L_2},\ b_1 = \frac{4\eta L_1}{L_2},\ b_2 = 0, \Omega = \pm\frac{2}{\eta}\sqrt{\frac{-L_1}{L_0}}, L_3 = \frac{3L_2^2}{16L_1}, L_4 = 0. \tag{74}$$

From these results and using (58) along with (21), the general solution (56) to Equation (22) has the form

$$P(\xi) = \frac{-4L_1}{L_2}\left\{1 - \frac{\eta}{\eta + \rho\, e^{\eta\Omega\xi}}\right\}, \tag{75}$$

provided that $L_1 \neq 0$ and $L_2 \neq 0$. Thus, the soliton solution of Equation (2) is addressed as

$$Q(x,t) = \pm\left[\frac{-4(b\omega k - ak^2 - \omega)}{c_1 - 4\gamma k^2}\left\{1 - \frac{\eta}{\eta + \rho\, e^{\eta\Omega\xi}}\right\}\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{76}$$

where $\Omega = \pm\dfrac{2}{\eta}\sqrt{\dfrac{-(b\omega k - ak^2 - \omega)}{a - bv}}$ and $16(c_2 - \delta)(b\omega k - ak^2 - \omega) = 3(c_1 - 4\gamma k^2)^2$ provided that $c_1 \neq 4\lambda k^2$ and $a \neq bv$.

- Set 6

$$b_0 = 0,\; b_1 = \frac{-4\eta L_1}{L_2},\; b_2 = 0, \Omega = \pm\frac{2}{\eta}\sqrt{\frac{-L_1}{L_0}}, L_3 = \frac{3L_2^2}{16L_1}, L_4 = 0. \tag{77}$$

From (77) and using (58) together with (21), the general solution (56) to Equation (22) has the form

$$P(\xi) = \frac{-4\,\eta L_1}{L_2\left(\,\eta + \rho\, e^{\eta\Omega\xi}\,\right)}, \tag{78}$$

provided that $L_1 \neq 0$ and $L_2 \neq 0$. Therefore, the soliton solution of Equation (2) is given by

$$Q(x,t) = \pm\left[\frac{-4\,\eta\,(b\omega k - ak^2 - \omega)}{(c_1 - 4\gamma k^2)\left(\,\eta + \rho\, e^{\eta\Omega\xi}\,\right)}\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{79}$$

where $\Omega = \pm\dfrac{2}{\eta}\sqrt{\dfrac{-\,(b\omega k - ak^2 - \omega)}{a - bv}}$ and $16(c_2 - \delta)(b\omega k - ak^2 - \omega) = 3(c_1 - 4\gamma k^2)^2$ provided that $c_1 \neq 4\lambda k^2$ and $a \neq bv$.

- Set 7

$$b_0 = \mp\frac{1}{2}\sqrt{\frac{-6L_1}{L_3}},\; b_1 = \pm\eta\sqrt{\frac{-6L_1}{L_3}},\; b_2 = 0,\; \Omega = \pm\frac{2}{\eta}\sqrt{\frac{2L_1}{L_0}},\; L_2 = L_4 = 0. \tag{80}$$

Based on (80) and using (58) in addition to (21), the general solution (56) to Equation (22) reads as

$$P(\xi) = \mp\frac{1}{2}\sqrt{\frac{-6L_1}{L_3}}\left\{1 - \frac{2\,\eta}{\eta + \rho\, e^{\eta\Omega\xi}}\right\}, \tag{81}$$

provided that $L_1 \neq 0$ and $L_3 \neq 0$. Hence, the soliton solution of Equation (2) is

$$Q(x,t) = \pm\left[\mp\frac{1}{2}\sqrt{\frac{-6(b\omega k - ak^2 - \omega)}{c_2 - \delta}}\left\{1 - \frac{2\eta}{\eta + \rho e^{\eta\Omega\xi}}\right\}\right]^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \tag{82}$$

where $\Omega = \pm\dfrac{2}{\eta}\sqrt{\dfrac{2\,(b\omega k - ak^2 - \omega)}{a - bv}}$ provided that $c_2 \neq \delta$, $a \neq bv$ and $c_1 = 4\gamma k^2$.

4. Modulation Instability Analysis

Our purpose now is to discuss the modulation instability of the LPD model (2) with the help of the standard linear stability analysis. In order to achieve this target, the steady-state solution of the LPD model (2) is assumed to be

$$Q(x,t) = \sqrt{P}\, e^{i\Phi t}, \tag{83}$$

where $\Phi = c_1P + (c_2 - \delta)P^2$, and P is the normalized optical power. According to the standard linear stability analysis, we include the perturbation term in (83) to obtain

$$Q(x,t) = \left[\sqrt{P} + \Upsilon(x,t)\right] e^{i\Phi t}, \tag{84}$$

where $\Upsilon(x,t)$ is a small perturbation such that $\Upsilon(x,t) \ll \sqrt{P}$. The perturbation $\Upsilon(x,t)$ is examined by utilizing linear stability analysis through inserting (84) into Equation (2) and collecting the linearized terms to obtain

$$\begin{aligned} &i\frac{\partial Q}{\partial t} + (a-\gamma P)\frac{\partial^2 Q}{\partial x^2} + b\frac{\partial^2 Q}{\partial x \partial t} - \sigma\frac{\partial^4 Q}{\partial x^4} - \lambda P\,\frac{\partial^2 Q^*}{\partial x^2} \\ &+P[c_1 + 2P(c_2-\delta)](Q+Q^*) + ibP[c_1 + (c_2-\delta)P]\frac{\partial Q}{\partial x} = 0, \end{aligned} \tag{85}$$

where $*$ denotes the conjugate of the complex function $Q(x,t)$. Then, suppose that the solution of Equation (85) is expressed as

$$Q(x,t) = \beta_1 e^{i(\kappa x - \varpi t)} + \beta_2 e^{-i(\kappa x - \varpi t)}, \tag{86}$$

where κ is the normalized wave number, and ϖ is the frequency of perturbation. Substituting solution (86) into Equation (85) and separating the coefficients of $e^{i(\kappa x - \varpi t)}$ and $e^{-i(\kappa x - \varpi t)}$ results in two equations in β_1 and β_2 given as

$$\begin{aligned} &\left(P\kappa^2\lambda - 2P^2\delta + 2P^2c_2 + Pc_1\right)\beta_1 + \left(-P^2b\delta\kappa + P^2b\kappa c_2 - \kappa^4\sigma\right. \\ &\left.+Pb\kappa c_1 + P\gamma\kappa^2 + \varpi b\kappa - 2P^2\delta + 2P^2c_2 - a\kappa^2 + Pc_1 - \varpi\right)\beta_2 = 0, \\ &\left(P^2b\delta\kappa - P^2b\kappa c_2 - \kappa^4\sigma - Pb\kappa c_1 + P\gamma\kappa^2 + \varpi b\kappa - 2P^2\delta + 2P^2c_2\right. \\ &\left.-a\kappa^2 + Pc_1 + \varpi\right)\beta_1 + \left(P\kappa^2\lambda - 2P^2\delta + 2P^2c_2 + Pc_1\right)\beta_2 = 0. \end{aligned} \tag{87}$$

From the coupled Equations (87), one can construct the coefficient matrix of β_1 and β_2. The determinant of this matrix has to vanish to secure nontrivial solution. Accordingly, the dispersion relation is obtained as

$$\begin{aligned} &\left(b^2\kappa^2 - 1\right)\varpi^2 - 2b\left\{\sigma\kappa^5 + (a-\gamma P)\kappa^3 - P[3(c_2-\delta)P + 2c_1]\kappa\right\}\varpi + \sigma^2\kappa^8 \\ &+ 2\sigma(a-\gamma P)\kappa^6 - \left\{\left[4\sigma(c_2-\delta) + \lambda^2 - \gamma^2\right]P^2 + 2(a\gamma + c_1\sigma)P - a^2\right\}\kappa^4 \\ &- \left\{b^2(c_2-\delta)^2P^4 + [2b^2c_1 + 4(\lambda-\gamma)](c_2-\delta)P^3\right. \\ &\left.+[b^2c_1^2 + 2(\lambda-\gamma)c_1 + 4a(c_2-\delta)]P^2 + 2ac_1P\right\}\kappa^2 = 0, \end{aligned} \tag{88}$$

and its solution has the form

$$\begin{aligned} \varpi = \frac{\kappa}{b^2\kappa^2 - 1}\Big[&b\sigma\kappa^4 + b(a-\gamma P)\kappa^2 - 3b(c_2-\delta)P^2 - 2bc_1P \\ &\pm\sqrt{\sigma^2\kappa^6 + \chi_4\kappa^4 + \chi_2\kappa^2 + \chi_0}\Big], \end{aligned} \tag{89}$$

where $b^2\kappa^2 \neq 1$. The parameters χ_0, χ_2 and χ_4 are defined as

$$\chi_4 = \left[\lambda^2 - 2\sigma(c_2-\delta)\right]b^2P^2 - 2(b^2c_1 + \gamma)\sigma P + 2\sigma a, \tag{90}$$

$$\begin{aligned} \chi_2 =\,&b^4(c_2-\delta)^2P^4 + 2b^2(c_2-\delta)(b^2c_1 + \gamma + 2\lambda)P^3 \\ &+ \left\{c_1^2b^4 + 2[c_1(\gamma+\lambda) - a(c_2-\delta)]b^2 - 4\sigma(c_2-\delta) + \gamma^2 - \lambda^2\right\}P^2 \\ &- 2(ab^2c_1 + a\gamma + c_1\sigma)P + a^2, \end{aligned} \tag{91}$$

$$\chi_0 = \left\{4b^2(c_2-\delta)P^2 + (3b^2c_1 + 2\gamma - 2\lambda)P - 2a\right\}[2(c_2-\delta)P^2 + c_1P]. \tag{92}$$

If $\sigma^2\kappa^6 + \chi_4\kappa^4 + \chi_2\kappa^2 + \chi_0 \geq 0$, the frequency ϖ is real for all real values of $a, b, c_1, c_2, \sigma, \gamma, \lambda, \delta, \kappa$, and P. Hence, the steady state is stable against small perturbations. Conversely, if $\sigma^2\kappa^6 + \chi_4\kappa^4 + \chi_2\kappa^2 + \chi_0 < 0$, then ϖ is imaginary, which means that the perturbation grows exponentially. Hence, the steady-state solution becomes unstable.

Ultimately, the growth rate of modulation instability gain spectrum $G(\kappa)$ can be expressed as follows:

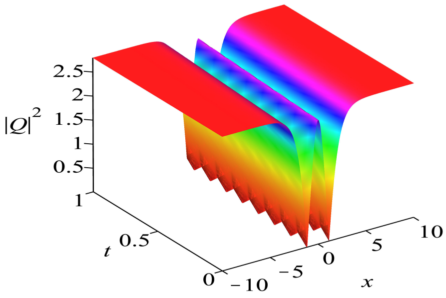
$$
\begin{aligned}
G(\kappa) = \quad & 2Im(\varpi) = 2Im\bigg(\frac{\kappa}{b^2\kappa^2 - 1}\Big[b\sigma\kappa^4 + b(a - \gamma P)\kappa^2 - 3b(c_2 - \delta)P^2 - 2bc_1P \\
& \pm\sqrt{\sigma^2\kappa^6 + \chi_4\kappa^4 + \chi_2\kappa^2 + \chi_0}\Big] \bigg).
\end{aligned}
$$

5. Results and Discussion

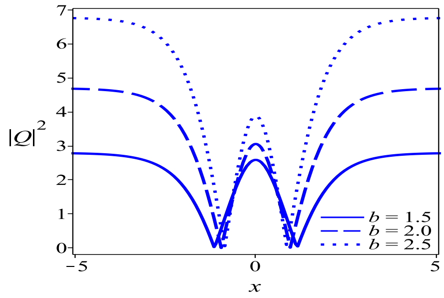
The analytic processes presented above demonstrate that the two proposed integration approaches are highly efficient at providing many structures of optical soliton solutions for the LPD model (2). To add physical understanding to our mathematical analysis, the 3D-plot of the intensity profiles of optical solitons are exhibited with suitable values of parameters. Additionally, the effect of spatiotemporal dispersion on the wave propagation is reported through depicting the 2D-plot of solutions for three distinct values of the coefficient of spatiotemporal dispersion, b.

Since the auxiliary equation method yielded abundant exact solutions, we display the behaviors of some of the extracted solutions. In Figure 1, the 3D-plots given in (Figure 1a,c) show the propagation of W-shaped solitons which describe solutions (32) and (34) with the values of parameters $a = 0.05, b = 1.5, k = v = \omega = 1, \gamma = c_1 = 0.1$, and $L_4 = 0.5$. The 2D-plots in (Figure 1b,d) present a noteworthy influence of spatiotemporal dispersion on the amplitude of W-shaped solitons which is enhanced by increasing the value of b. Further to this, the evolution of soliton solution (40) illustrates a singular-type wave as delineated in Figure 2a for the same values of parameters as in Figure 1. It is clear that the amplitude of singular soliton is stretched by increasing the value of b. In Figure 3, the graph presents a kink-dark soliton for solution (45) with the same values of the parameters as those in Figure 1 besides $c_2 = 0.1$ and $\delta = 0.5$. As shown in the plot in (Figure 3b), the spatiotemporal dispersion amplifies the amplitude of kink-dark soliton. It can be clearly seen that the plot in Figure 4 represents the profile of bright soliton pulse characterizing solution (47) with the same values of parameters as those in Figure 1 except $a = c_2 = 0.5$, $b = -0.5, \delta = 0.1$. The spatiotemporal dispersion can be seen to adversely affect the amplitude of bright soliton which undergoes a continuous decline once b increases, as shown in Figure 4b.

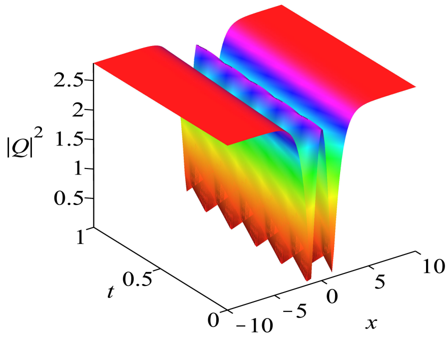
Similarly, some of soliton solutions created by the Bernoulli sub-ODE method are graphically represented to recognize the physical characteristics of solitons. In Figure 5, the graph describes the W-shaped soliton pulse for solution (70) with the values of parameters $a = 0.05, b = \gamma = \eta = 0.5, k = v = \omega = \rho = 1$ and $c_1 = 0.3$. The increase in the value of spatiotemporal dispersion reduces the amplitude of soliton wave as given in Figure 5b. Moreover, we can observe that the plot in Figure 6 represents a bright soliton wave for solution (73) for the same values of parameters as those in Figure 5 except $a = 0.5, b = 0.1, \eta = \rho = 0.3, \gamma = 1$. The amplitude of bright soliton experiences a gradual decrease with an increase in the value of b. One can clearly see that the plots in Figure 7a,c show the structures of kink- and antikink-type waves for solutions (76) and (79), respectively, for the same values of parameters as those in Figure 6 except $\eta = 0.5, \rho = 1$ in both graphs in addition to $\gamma = -1$ in Figure 7c. Both amplitudes of kink and antikink waves suffer reductions when the value of b increases as presented in Figure 7b,d. Finally, the evolution of soliton solution (82) is depicted in Figure 8 with the same values of parameters as those in Figure 5, where the plot characterizes the profile of dark soliton. It is easily noticed in Figure 8b that the growth in the value of b leads to a collapse in the amplitude of dark soliton.



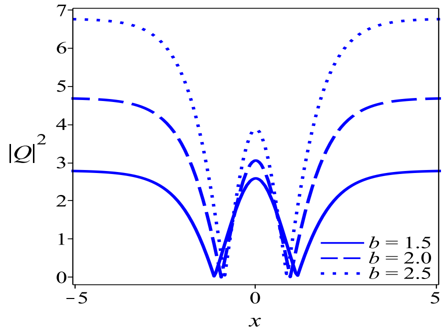
(**a**) 3D-plot of W-shaped soliton



(**b**) Effect of STD on W-shaped soliton

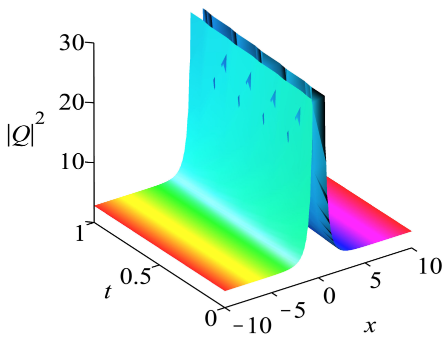


(**c**) 3D-plot of W-shaped soliton

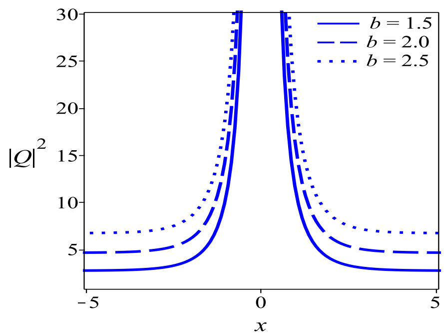


(**d**) Effect of STD on W-shaped soliton

Figure 1. The behaviors of soliton solutions (32) and (34) with the values of parameters $a = 0.05$, $b = 1.5$, $k = v = \omega = 1$, $\gamma = c_1 = 0.1$ and $L_4 = 0.5$.

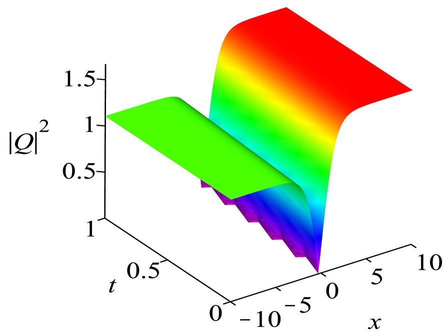


(**a**) 3D-plot of singular soliton

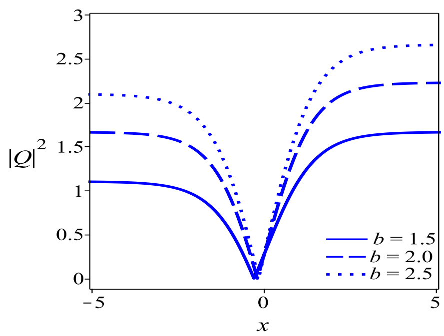


(**b**) Effect of STD on singular soliton

Figure 2. The behavior of soliton solution (40) for the same values of parameters as in Figure 1.

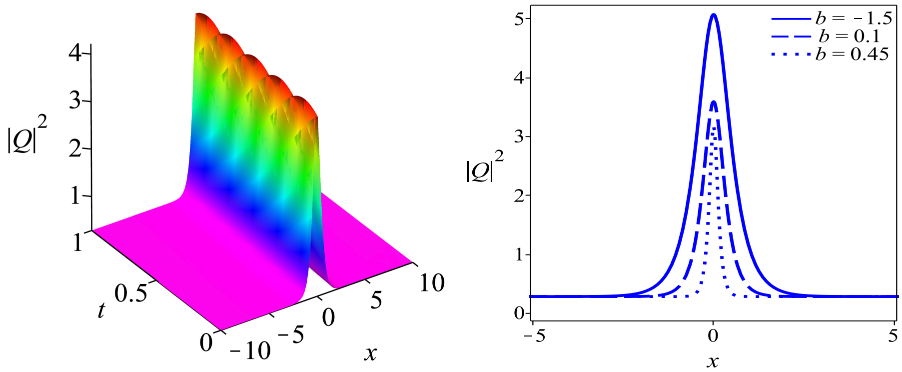


(**a**) 3D-plot of kink-dark soliton



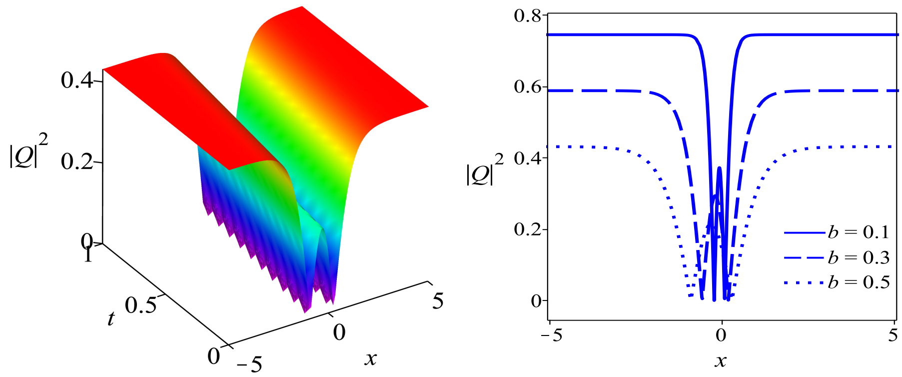
(**b**) Effect of STD on kink-dark soliton

Figure 3. The behavior of soliton solution (45) with the same values of parameters as in Figure 1 besides $c_2 = 0.1$ and $\delta = 0.5$.



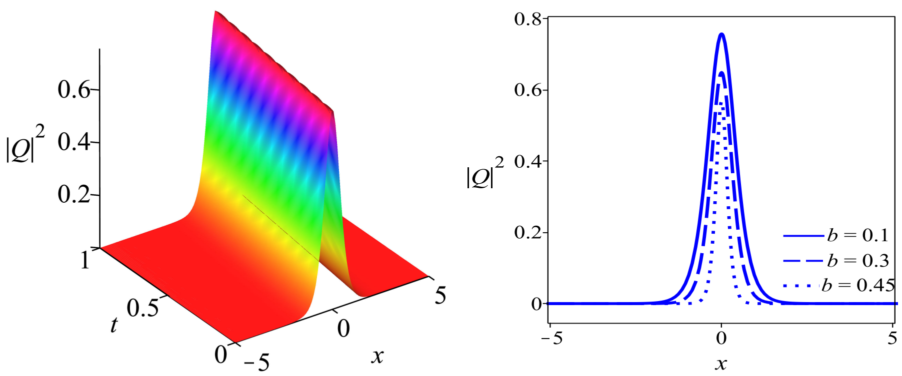
(**a**) 3D-plot of bright soliton (**b**) Effect of STD on bright soliton

Figure 4. The behavior of soliton solution (47) with the same values of parameters as those in Figure 1 except $a = c_2 = 0.5, b = -0.5, \delta = 0.1$.



(**a**) 3D-plot of W-shaped soliton (**b**) Effect of STD on W-shaped soliton

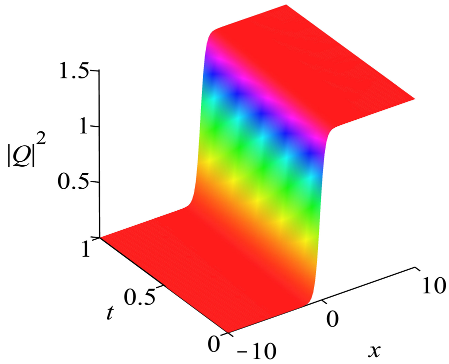
Figure 5. The behavior of soliton solution (70) with the values of parameters $a = 0.05$, $b = \gamma = \eta = 0.5$, $k = v = \omega = \rho = 1$ and $c_1 = 0.3$.



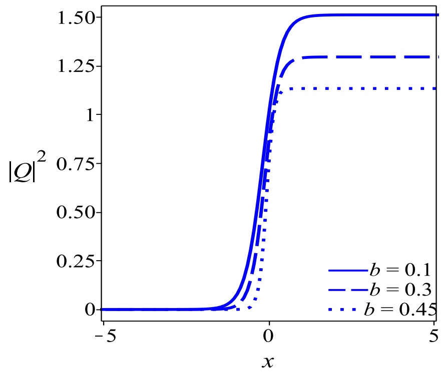
(**a**) 3D-plot of bright soliton (**b**) Effect of STD on bright soliton

Figure 6. The behavior of soliton solution (73) for the same values of parameters as in Figure 5 except $a = 0.5, b = 0.1, \eta = \rho = 0.3, \gamma = 1$.

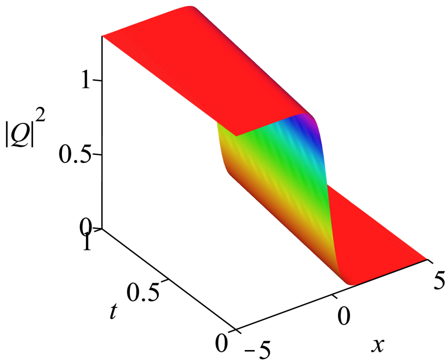
From the above illustrated graphs, one can see that the obtained analytic solutions demonstrate various types of soliton profiles which are dominated by the model parameters. Furthermore, it can be obviously deduced that the spatiotemporal dispersion causes an impressive evolution to the amplitude of pulses. This intensive impact of the spatiotemporal dispersion can be exploited to manipulate the crisis of internet bottleneck. In comparison with mathematical approaches used in the previous studies [24,36], the applied integral schemes in this study have given rise to an abundance of entirely new, exact solutions that describe different wave structures including the W-shaped, bright, dark, kink-dark, singular, kink, and anti-kink type solitons. Further to this, the effect of spatiotemporal dispersion on the soliton propagation is discussed more thoroughly as compared to the studies that have previously addressed the LPD model in the past.



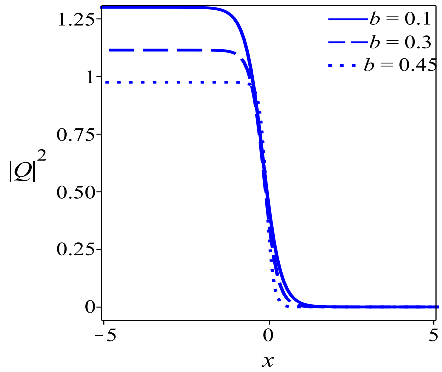
(**a**) 3D-plot of kink soliton



(**b**) Effect of STD on kink soliton

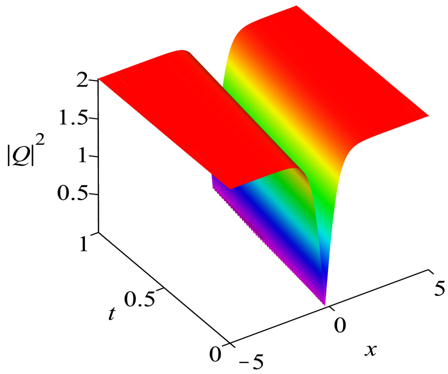


(**c**) 3D-plot of anti-kink soliton

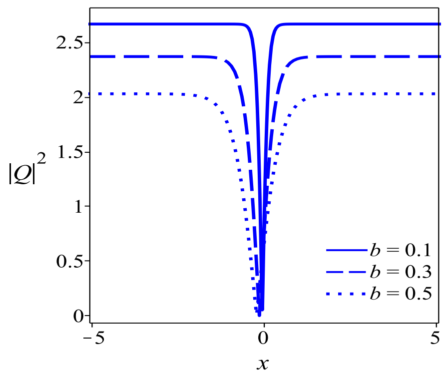


(**d**) Effect of STD on anti-kink soliton

Figure 7. The behaviors of soliton solutions (76) and (79) for the same values of parameters as in Figure 6 except $\eta = 0.5, \rho = 1$, and in (**c**) $\gamma = -1$.



(**a**) 3D-plot of dark soliton



(**b**) Effect of STD on dark soliton

Figure 8. The behavior of soliton solution (82) with the same values of parameters as those in Figure 5.

6. Conclusions

This work discusses the optical soliton solutions of the LPD equation with the parabolic law of nonlinearity which describes the propagation of optical pulses through optical fibers. The spatiotemporal dispersion is included in this model because of its effective role in handling the internet bottleneck problem. Two powerful integration schemes, the auxiliary equation method and Bernoulli sub-ODE method, are employed to explore the soliton solutions analytically. Consequently, slow-light optical solitons of different profiles such as W-shaped, bright, dark, kink-dark, singular, kink, and anti-kink solitons were revealed under specific restrictions. The outcomes indicate that the spatiotemporal dispersion causes a significant variation to the wave dynamics. Hence, these types of pulses can be employed to control the internet bottleneck issue and to allow smooth internet traffic flow. Some of obtained solutions have been represented graphically to give a clear insight into the optical soliton behaviors. In addition, the modulation instability (MI) of the LPD model was examined in conjunction with the MI gain formula. The results of this work can be exploited for possible applications in the engineering and physics of nonlinear optics.

Author Contributions: Conceptualization, K.K.A.-K. and K.S.A.-G.; methodology, K.S.A.-G., E.V.K. and A.B.; formal analysis, K.K.A.-K. and K.S.A.-G.; writing—original draft preparation, K.K.A.-K.; writing—review and editing, K.S.A.-G., E.V.K. and A.B.; supervision, K.S.A.-G. and E.V.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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