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Sandwich-Type Theorems for a Family of Non-Bazilevič Functions Involving a q-Analog Integral Operator

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Abstract: This article presents a new q-analog integral operator, which generalizes the q-Srivastava–Attiya operator. Using this q-analog operator, we define a family of analytic non-Bazilevič functions, denoted as $\mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta,\lambda,\mathcal{M},\mathcal{N})$. Furthermore, we investigate the differential subordination properties of univalent functions using q-calculus, which includes the best dominance, best subordination, and sandwich-type properties. Our results are proven using specialized techniques in differential subordination theory.

Keywords: univalent functions; q-calculus; non-Bazilevič function; best dominant; best subordinate; sandwich-type theorems

MSC: 33E12; 30C45

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1. Introduction to Differential Subordination

The set of all functions that are holomorphic in $\mathbb{U}$ is defined as the family $\mathcal{A}(\mathbb{U})$, and $\mathcal{H}[a,l]$ is the subfamily of $f(z) \in \mathcal{A}(\mathbb{U})$ defined by

$$\mathcal{H}[a,l] = \left\{ f : f(z) = a + a_l z^l + a_{l+1} z^{l+1} + \dots \right\}, \quad (l \in \mathbb{N}, a \in \mathbb{C}),$$

where

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

is the open unit disc.

In addition, let $\mathcal{A}(l)$ be the subfamily of $f \in \mathcal{A}(\mathbb{U})$ with the normalized form

$$f(z) = z + \sum_{j=l+1}^{\infty} a_j z^j. \tag{1}$$

It is obvious that $\mathcal{A}(1) = \mathcal{A}$.

The convolution $(*)$ of any holomorphic functions $f, \hbar \in \mathcal{A}(l)$ is given as

$$(f * \hbar)(z) = z + \sum_{j=l+1}^{\infty} a_j d_j z^j = (\hbar * f)(z), \tag{2}$$

where $\hbar(z) = z + \sum_{j=l+1}^{\infty} d_j z^j$.

For any analytic functions $f, h \in \mathcal{A}(l)$, we say f is subordinate to h, which is denoted by ($f \prec h$), if we have a Schwarz function Φ with

$$\Phi(0) = 0 \text{ and } |\Phi(z)| < 1,$$

such that

$$f(z) = h(\Phi(z)).$$

Moreover, the functions f and h are subordinated $(\prec)$ if they satisfy the condition:

$$f(z) \prec h(z) \Leftrightarrow f(0) = h(0) \text{ and } f(\mathbb{U}) \subset h(\mathbb{U}).$$

The theory of differential subordination, along with its dual counterpart, the theory of differential superordination, was introduced by Miller and Mocanu [1]. These theories are based on reinterpreting certain inequalities that originally apply to real-valued functions and extending them to the case of complex-valued functions. The study of subordination and superordination properties using various types of operators is still a widely-used technique, and some studies have resulted in the discovery of sandwich-type theorems, as is the case in the present paper.

Assume that γ and $\hbar$ in $\mathbb{U}$ are two analytic functions, and let

$$\Lambda(\mathrm{r}, e, \ell; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}.$$

(i) Let γ satisfy the second-order subordination in $\mathbb{U}$,

$$\Lambda\Big(\gamma(z), z\gamma'(z), z^2\gamma''(z); z\Big) \prec \hbar(z). \tag{3}$$

If γ satisfies the differential subordination (3), it is considered a solution. A univalent function κ is considered a dominant in the differential subordination in Equation (3) if every function γ satisfying (3) is subordinated to κ, i.e., $\gamma(z) \prec \kappa(z)$. If $\varpi(z) \prec \kappa(z)$ for all functions γ, a dominant ϖ is said to be the best dominant of Equation (3).

(ii) Let γ satisfy the second-order superordination in $\mathbb{U}$,

$$\hbar(z) \prec \Lambda\Big(\gamma(z), z\gamma'(z), z^2\gamma''(z); z\Big). \tag{4}$$

If γ satisfies the differential superordination (4), it is considered a solution. A univalent function κ is considered a subordinate of the differential superordination in Equation (4) if it dominates every function γ satisfying (4), i.e., $\kappa(z) \prec \gamma(z)$. If $\kappa(z) \prec \varpi(z)$ for all the subordinates ϖ, a subordinate ϖ is said to be the best subordinate of Equation (4).

Miller and Mocanu [1] established conditions that were sufficient to draw the following conclusion regarding the functions γ, $\hbar$, and Λ:

$$\hbar(z) \prec \Lambda\Big(\gamma(z), z\gamma'(z), z^2\gamma''(z); z\Big) \Rightarrow \kappa(z) \prec \gamma(z). \tag{5}$$

Using the same techniques as before, Bulboacă ([2,3]) introduced broad families of first-order differential subordinations and integral operators, such as the Alexander operator, Libera operator, and Bernardi–Libera–Livingston operator, that preserved superordination. Furthermore, Ali et al. [4] explored sufficient conditions for functions f to satisfy the next subordination based on Bulboacă's [3] results:

$$\kappa_1(z) \prec \frac{zf'(z)}{f(z)} \prec \kappa_2(z),$$

where the univalent functions κ_1 and κ_2 are in $\mathbb{U}$ with $\kappa_1(0) = 1$ and $\kappa_2(0) = 1$.

Moreover, Tuneski [5] discovered a sufficient condition for the starlikeness of the functions f, which is given by $\frac{f''(z)f(z)}{f'(z)^2}$.

Shanmugam et al. [6] provided another sufficient condition for a normalized holomorphic function

$$\kappa_1(z) \prec \frac{f(z)}{zf'(z)} \prec \kappa_2(z)$$

and

$$\kappa_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec \kappa_2(z).$$

In recent years, there has been a significant focus on investigating the principles of differential subordination and superordination in various studies (for example, see [7–16]).

2. Main Concepts of Quantum Calculus

The concept of quantum calculus, also known as q-calculus, has played a significant role in the advancement of Geometric Function Theory (GFT) and its extensive application in diverse fields, including mathematical science and quantum physics. The q-calculus idea, which includes the q-derivative and q-integral, was first proposed by Jackson ([17,18]). As the study of q-calculus has expanded, various related topics have been investigated, including the q-Gamma function, the q-Beta function, and the q-Mittag–Leffler function (for further information, refer to [19–22]). In the field of GFT, the application of q-calculus has been effective in studying classes of functions, including $\mathcal{S}$ and $\mathcal{S}(p)$. The introduction of q-calculus by Ismail et al. [23] has led to the development of several Ma and Minda classes of analytic functions in the unit disk $\mathbb{U}$, which are closely related to the subordination concept. In addition, q-calculus operators, such as fractional "q-integral and q-derivative" operators, have been used to establish various analytic functions. Furthermore, numerous studies have explored specific classes of analytic functions in $\mathbb{U}$ using q-calculus (for example, see [24–27]).

Recently, intriguing findings have been published regarding the use of an integral operator, as evidenced in [28]. Building on these findings, and inspired by previous results obtained by applying q-calculus to various derivative and integral operators (as seen in [29–31]), we decided to extend our study to the q-Srivastava–Attiya operator in this paper.

Definition 1 ([18])**.** *The expression for the q-derivative $\mathfrak{D}_q$ is*

$$\mathfrak{D}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \ \ (z \neq 0).$$

Subsequently,

$$\mathfrak{D}_q f(z) := \mathfrak{D}_q \left\{ z + \sum_{j=l+1}^{\infty} a_j z^j \right\} = 1 + \sum_{j=l+1}^{\infty} [j]_q a_j z^{j-1},$$

where the q-number $[\nu]_q$ represents the following:

$$[\nu]_q := \begin{cases} \sum\limits_{\nu=0}^{\mathrm{m}-1} q^\nu, & (\nu = \mathrm{m} \in \mathbb{N}), \\ \dfrac{1-q^\nu}{1-q}, & (\nu \in \mathbb{C}), \\ 0, & \nu = 0. \end{cases}$$

Obviously,

$$f'(z) := lim_{q\to 1-}\mathfrak{D}_q\left\{1+\sum_{j=l+1}^{\infty}[j]_q a_j z^j\right\} = 1+\sum_{j=l+1}^{\infty} j a_j z^{j-1}.$$

The definition of the q-factorial, denoted by $[l]_q!$, is as follows:

$$[l]_q! = \begin{cases} [l]_q[l-1]_q\dots[2]_q[1]_q, & l=1,2,3,\dots, \\ 1, & l=0. \end{cases}$$

The q-generalized Pochhammer is expressed by

$$[\mathfrak{s};j]_q = [\mathfrak{s}]_q[\mathfrak{s}+1]_q[\mathfrak{s}+2]_q\dots\dots[\mathfrak{s}+j-1]_q.$$

The following is the concept of the q-Gamma function:

$$\Gamma_q(\mathfrak{s}+1) = [\mathfrak{s}]_q\Gamma_q(\mathfrak{s}) \text{ and } \Gamma_q(1) = 1.$$

The focus of q-calculus is primarily on the q-analog, which is motivated by the symmetric nature of quantum calculus. A new operator is introduced in this paper that connects the q-Srivastava–Attyia operator and the q-analog Ruscheweyh operator using the definitions already known in q-calculus. This operator is used to define the class $\mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta,\lambda,\mathcal{M},\mathcal{N})$ of non-Bazilevič functions. By employing this class, we implement the methods of the theory of differential subordination and differential superordination to obtain interesting new differential subordination and superordination, respectively, for which the best dominant and subordinate functions are found.

Definition 2. *Using the q-derivative, we define the q-analog Ruscheweyh operator $\mathcal{F}_q^{\mu+1}f(z)$: $\mathcal{A}(l) \to \mathcal{A}(l)$ using*

$$\mathcal{F}_q^{\mu+1}(z) = z+\sum_{j=l+1}^{\infty}\frac{[\mu+1;j]_q}{[j-1]_q!}z^j, \quad (\mu > -1, l\in\mathbb{N}, z\in\mathbb{U}). \tag{6}$$

Given the importance of studying q-calculus, Shah and Noor [29] introduced and investigated the q-Srivastava–Attiya operator, which is studied using the q-Hurwitz Lerch Zeta function $\Phi_q(u,\tau;z)$ provided in [26] and is given as

$$\mathcal{J}^u_{q,\tau}f(z) = z+\sum_{j=l+1}^{\infty}\left(\frac{[1+u]_q}{[j+u]_q}\right)^{\tau}a_j z^j, \quad (l\in\mathbb{N}), \tag{7}$$

where $u\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $\tau\in\mathbb{C}$ when $|z|<1$, and $\mathcal{R}(\tau)>1$ when $|z|=1$.

In recent years, numerous studies have highlighted the concept of the q-Srivastava–Attiya operator (for example, see [32,33]).

Now, we introduce the q-derivative of the q-Srivastava–Attiya operator $\mathcal{J}^u_{q,\tau}f(z)$ for $0<q<1$ as follows:

$$\mathfrak{D}_q\mathcal{J}^u_{q,\tau}f(z) := \frac{\mathcal{J}^u_{q,\tau}f(qz)-\mathcal{J}^u_{q,\tau}f(z)}{(q-1)z}, \ (z\in\mathbb{U}, z\neq 0).$$

Hence

$$\mathfrak{D}_q\mathcal{J}^u_{q,\tau}f(z) = 1+\sum_{j=l+1}^{\infty}\left(\frac{[1+u]_q}{[j+u]_q}\right)^{\tau}[j]_q\, a_j z^{j-1}. \tag{8}$$

We conclude from (8) that

$$z\mathfrak{D}_q \mathcal{J}^u_{q,\tau} f(z) = z + \sum_{j=l+1}^{\infty} \left(\frac{[1+u]_q}{[j+u]_q} \right)^{\tau} [j]_q \, a_j z^j.$$

For $\mu > -1$, we define a q-analog integral operator $\mathcal{JF}^{\mu}_{q,\tau,u} f(z) : \mathcal{A}(l) \to \mathcal{A}(l)$ as follows:

$$\mathcal{JF}^{\mu}_{q,\tau,u} f(z) * \mathcal{F}^{\mu+1}_q(z) = z\mathfrak{D}_q \mathcal{J}^u_{q,\tau} f(z),$$

where $\mathcal{F}^{\mu+1}_q(z)$ is defined in (2).

From the above operator, we conclude that

$$\begin{aligned} \mathcal{JF}^{\mu}_{q,\tau,u} f(z) &= z + \sum_{j=l+1}^{\infty} \left(\frac{[1+u]_q}{[j+u]_q} \right)^{\tau} \frac{[j]_q [j-1]_q!}{[\mu+1; j]_q} a_j z^j \\ &= z + \sum_{j=l+1}^{\infty} \left(\frac{[1+u]_q}{[j+u]_q} \right)^{\tau} \frac{[j]_q!}{[\mu+1; j]_q} a_j z^j, \end{aligned} \tag{9}$$

$$(u \in \mathbb{C}\backslash\mathbb{Z}_0^-, \mu > -1, \tau \in \mathbb{C}, \text{ when } |z| < 1, \text{ and } \mathcal{R}(\tau) > 1 \text{ when } |z| = 1).$$

We note that

$$lim_{q\to 1-} \mathcal{JF}^{\mu}_{q,\tau,u} f(z) := \mathcal{IF}^{\mu}_{\tau,u} f(z) = z + \sum_{j=l+1}^{\infty} \left(\frac{[1+u]}{[j+u]} \right)^{\tau} \frac{j!}{[\mu+1; j]} a_j z^j. \tag{10}$$

It follows from (9) that

$$z\mathfrak{D}_q(\mathcal{JF}^{\mu}_{q,\tau+1,u} f(z)) = \left(1 + \frac{[u]_q}{q^u}\right) \mathcal{JF}^{\mu}_{q,\tau,u} f(z) - \frac{[u]_q}{q^u} \mathcal{JF}^{\mu}_{q,\tau+1,u} f(z), \tag{11}$$

and

$$z\mathfrak{D}_q\left(\mathcal{JF}^{\mu}_{q,\tau,u} f(z)\right) = \left(1 + \frac{[\mu]_q}{q^{\mu}}\right) \mathcal{JF}^{\mu+1}_{q,\tau,u} f(z) - \frac{[\mu]_q}{q^{\mu}} \mathcal{JF}^{\mu}_{q,\tau,u} f(z). \tag{12}$$

In complex analysis, the q-analog integral operator is a generalization of the classical integral operator. It has several advantages over the classical operator. The q-analog integral operator has applications in physics, particularly in quantum field theory and statistical mechanics research. The q-analog integral operator can be used to analyze non-commutative geometry, a type of generalizable classical geometry that includes non-commutative algebraic structures. This is advantageous for the study of quantum groups and non-commutative differential geometry. Subsequently, by utilizing the q-analog integral operator $\mathcal{JF}^{\mu}_{q,\tau,u} f(z)$, we provide the following class of non-Bazilevič functions.

Definition 3. *The class $\mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta, \lambda, \mathcal{M}, \mathcal{N})$ contains a function $f(z)$ if the following condition holds:*

$$(1+\vartheta)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u} f(z)}\right)^{\lambda} - \vartheta \frac{\mathcal{JF}^{\mu}_{q,\tau,u} f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u} f(z)} \left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u} f(z)}\right)^{\lambda} \prec \frac{1+\mathcal{M}z}{1+\mathcal{N}z}, \tag{13}$$

$$(\vartheta \in \mathbb{C}, 0 < \lambda < 1, -1 \leq \mathcal{N} < \mathcal{M} \leq 1).$$

Furthermore, it contains a function $f(z) \in \mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta, \lambda, \varpi)$ if the following inequality holds:

$$\mathcal{R}\{(1+\vartheta)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \vartheta\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\} > \varpi, \ (0 \leq \varpi < 1).$$

Remark 1. *We obtain the following subclasses by specifying the parameters $q, \tau, \mu, \vartheta, \mathcal{M}$, and $\mathcal{N}$:*

1. *If $q \to 1-$, the class reduces to the subclass $\mathcal{K}^{\mu}_{\tau,u}(\vartheta, \lambda, \mathcal{M}, \mathcal{N})$ given by*

$$\begin{aligned}\mathcal{K}^{\mu}_{\tau,u}(\vartheta, \lambda, \mathcal{M}, \mathcal{N}) := \{&(1+\vartheta)\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda} \\ &- \vartheta\frac{\mathcal{IF}^{\mu}_{\tau,u}f(z)}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda} \prec \frac{1+\mathcal{M}z}{1+\mathcal{N}z}\}.\end{aligned} \tag{14}$$

2. *For $q \to 1-, \tau = 0$, and $\mu = 1$, we obtain the class, as introduced by Wang et al. [34].*
3. *Taking $q \to 1-, \tau = 0, \mu = 1, \vartheta = -1, j = 1, \mathcal{M} = 1$, and $\mathcal{N} = -1$, we have non-Bazilevič functions, as defined by Obradovic [35].*
4. *For $q \to 1-, \tau = 0, \mu = 1, \vartheta = -1, j = 1, \mathcal{M} = 1 - 2\varpi$, and $\mathcal{N} = -1$, we obtain non-Bazilevič functions, as provided by Tuneski and Darus [36].*

There are many previous studies related to these classes (for instance, see [37,38]).

3. Main Lemmas

The Lemmas listed below are part of the classical methods used to obtain original results related to operators. The most interesting aspect is the form that the results take due to the operator used. By combining the results obtained using both theories, we establish a sandwich-type theorem, which generates two corollaries for the particular functions involved.

Definition 4 ([1])**.** *Let Q denote the family of all analytic and injective functions f on $\overline{\mathbb{U}} \backslash E(f)$, where*

$$E(f) = \left\{\eta \in \partial\mathbb{U} : \lim_{z\to\eta} f(z) = \infty\right\},$$

such that $f'(\eta) \neq 0$ for $\eta \in \partial\mathbb{U}\backslash E(f)$.

Lemma 1 ([9])**.** *In the disc $\mathbb{U}$, let $\aleph(z)$ be a convex and analytic function, with $\aleph(0) = 1$. In addition, let $\gamma(z)$ be a function defined by*

$$\gamma(z) = 1 + s_l z^l + s_{l+1} z^{l+1} + \cdots \in \mathcal{H}[a, l], \tag{15}$$

that is an analytic function in $\mathbb{U}$ if

$$\gamma(z) + \frac{z\mathfrak{D}_q\gamma(z)}{\omega} \prec \aleph(z) \ \ \{\mathcal{R}(\omega) \geq 0, \omega \neq 0\}. \tag{16}$$

Then,

$$\gamma(z) \prec \psi(z) := \frac{\omega}{l} z^{-\left(\frac{\omega}{l}\right)} \int_0^z t^{\left(\frac{\omega}{l}\right)-1}\aleph(t)dt \prec \aleph(z), \tag{17}$$

where $\psi(z) \in \mathcal{H}[a, l]$ is the convex function and the best dominant.

Lemma 2 ([26])**.** *Let $\kappa(z)$ be a univalent function that is convex in $\mathbb{U}$ and satisfies the condition $\kappa(0) = 1$, with*

$$\mathcal{R}(\rho\kappa(z) + \sigma) > 0, \quad (\rho, \sigma \in \mathbb{C}).$$

If $\gamma(z)$ is analytic in $\mathbb{U}$ and $\gamma(0) = 1$, then

$$\gamma(z) + \frac{z\mathfrak{D}_q(\gamma(z))}{\rho\gamma(z) + \sigma} \prec \kappa(z),$$

implies that

$$\gamma(z) \prec \kappa(z).$$

Lemma 3 ([26])**.** *Suppose that $\kappa(z)$ is a univalent and convex function in $\mathbb{U}$. Let ζ and ρ be complex numbers such that $\rho \in \mathbb{C}\backslash\{0\}$, $\zeta \in \mathbb{C}$ and*

$$\mathcal{R}\left(1 + \frac{z\kappa''(z)}{\kappa'(z)}\right) > max\left\{0; -\mathcal{R}\left(\frac{\zeta}{\rho}\right)\right\}.$$

If γ is analytic in $\mathbb{U}$ with

$$\zeta\gamma(z) + \rho z\mathfrak{D}_q(\gamma(z)) \prec \zeta\kappa(z) + \rho z\mathfrak{D}_q(\kappa(z)), \tag{18}$$

then, $\gamma(z) \prec \kappa(z)$ and κ is the best dominant of (18)*.*

Lemma 4 ([26])**.** *Suppose that $\kappa(z)$ is a convex function in $\mathbb{U}$, $\rho \in \mathbb{C}$ with $\mathcal{R}(\rho) > 0$. Let $\gamma \in \mathcal{H}[\kappa(0), 1] \cap Q$ and $\gamma(z) + \rho z\mathfrak{D}_q(\gamma(z))$ be univalent in $\mathbb{U}$. Then, we obtain:*

$$\kappa(z) + \rho z\mathfrak{D}_q(\kappa(z)) \prec \gamma(z) + \rho z\mathfrak{D}_q(\gamma(z)), \tag{19}$$

then, $\kappa \prec \gamma$.

Lemma 5 ([39])**.** *In the unit disc $\mathbb{U}$, let $F(z)$ be a convex analytic function. If $f, h \in \mathcal{A}$, with $f \prec F$ and $h \prec F$, then*

$$\varsigma_1 f(z) + (1 - \varsigma_1)h(z) \prec F(z), \ \ (\varsigma_1 \in [0, 1]).$$

4. Differential Subordination and Sandwich-Type Results

Using Lemma 1, we start by introducing the first subordination property.

Theorem 1. *Let $\vartheta \in \mathbb{C}, \mathcal{R}(\vartheta) > 0, 0 < \lambda < 1$, $-1 \leq \mathcal{N} < \mathcal{M} \leq 1$, and $f(z) \in \mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta, \lambda, \mathcal{M}, \mathcal{N})$. Then,*

$$\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u} f(z)}\right)^{\lambda} \prec \aleph(z) = \frac{\lambda}{\vartheta\varkappa_q l} \int_0^1 v^{\left(\frac{\lambda}{\vartheta\varkappa_q l}\right)-1} \frac{1 + \mathcal{M}zv}{1 + \mathcal{N}zv} dv \prec \frac{1 + \mathcal{M}z}{1 + \mathcal{N}z},$$

where $\varkappa_q = \frac{[u]_q}{q^u}$ and $\aleph(z)$ is the best dominant.

Proof. Suppose that the function $\gamma(z)$ is defined by

$$\gamma(z) = \left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u} f(z)}\right)^{\lambda}. \tag{20}$$

Then, $\gamma(z)$ in (15) is analytic in $\mathbb{U}$, with $\gamma(0) = 1$.

We can derive the following expression by applying logarithmic differentiation with respect to z to Equation (20)

$$(1+\vartheta)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}-\vartheta\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} = \gamma(z)+\frac{\vartheta\varkappa_q}{\lambda}z\mathfrak{D}_q(\gamma(z)), \tag{21}$$

where $\varkappa_q=\frac{[u]_q}{q^u}$.

Since $f(z)\in\mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta,\lambda,\mathcal{M},\mathcal{N})$, we obtain

$$\gamma(z)+\frac{\vartheta\varkappa_q}{\lambda}z\mathfrak{D}_q(\gamma(z))\prec\frac{1+\mathcal{M}z}{1+\mathcal{N}z}.$$

By applying Lemma 1 to (21) with $\omega=\frac{\lambda}{\vartheta\varkappa_q}$ and $\mathcal{R}(\frac{\lambda}{\vartheta\varkappa_q})\geq 0$, we have

$$\left(\frac{z}{\mathcal{J}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\prec\aleph(z)=\frac{\lambda}{\vartheta\varkappa_q l}z^{-\left(\frac{\lambda}{\vartheta\varkappa_q l}\right)}\int_z^1 t^{\left(\frac{\lambda}{\vartheta\varkappa_q l}\right)-1}\frac{1+\mathcal{M}t}{1+\mathcal{N}t}dt$$

$$=\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 v^{\left(\frac{\lambda}{\vartheta\varkappa_q l}\right)-1}\frac{1+\mathcal{M}zv}{1+\mathcal{N}zv}dv\prec\frac{1+\mathcal{M}z}{1+\mathcal{N}z}$$

and $\aleph(z)$ is the best dominant. □

Corollary 1. *Let $\vartheta\in\mathbb{C},0<\lambda<1,-1\leq\mathcal{N}<\mathcal{M}\leq 1$, and $f(z)\in\mathcal{K}^{\mu}_{\tau,u}(\vartheta,\lambda,\mathcal{M},\mathcal{N})$ with $\mathcal{R}(\vartheta)>0$. Then,*

$$\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda}\prec\aleph(z)=\frac{\lambda}{\vartheta ul}\int_0^1 v^{\left(\frac{\lambda}{\vartheta ul}\right)-1}\frac{1+\mathcal{M}zv}{1+\mathcal{N}zv}dv\prec\frac{1+\mathcal{M}z}{1+\mathcal{N}z}$$

and $\aleph(z)$ is the best dominant.

Theorem 2. *Let $0\leq\vartheta_1\leq\vartheta_2,\mathcal{R}(\vartheta_1)>0,\mathcal{R}(\vartheta_2)>0,0<\lambda<1$, and $-1\leq\mathcal{N}_1\leq\mathcal{N}_2<\mathcal{M}_2\leq\mathcal{M}_1\leq 1$. Then,*

$$\mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta_2,\lambda,\mathcal{M}_2,\mathcal{N}_2)\subset\mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta_1,\lambda,\mathcal{M}_1,\mathcal{N}_1). \tag{22}$$

Proof. Let $f\in\mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta_2,\lambda,\mathcal{M}_2,\mathcal{N}_2)$. Then, it follows that

$$(1+\vartheta_2)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}-\vartheta_2\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\prec\frac{1+\mathcal{M}_2z}{1+\mathcal{N}_2z}.$$

Since $-1\leq\mathcal{N}_1\leq\mathcal{N}_2<\mathcal{M}_2\leq\mathcal{M}_1\leq 1$, we obtain

$$(1+\vartheta_2)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}-\vartheta_2\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\prec\frac{1+\mathcal{M}_2z}{1+\mathcal{N}_2z}\prec\frac{1+\mathcal{M}_1z}{1+\mathcal{N}_1z}. \tag{23}$$

Hence, $f(z)\in\mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta_1,\lambda,\mathcal{M}_1,\mathcal{N}_1)$. Then, Theorem 2 is satisfied when $0\leq\vartheta_1=\vartheta_2$.

When $0 \leq \vartheta_1 < \vartheta_2$, by Theorem 1, we find that $f(z) \in \mathcal{T}^{\mu}_{q,\tau+1,u}(0, \lambda, \mathcal{M}_1, \mathcal{N}_1)$, that is,

$$\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} \prec \frac{1+\mathcal{M}_1 z}{1+\mathcal{N}_1 z}. \tag{24}$$

Simultaneously, we have

$$\begin{aligned}(1+\vartheta_1)\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \vartheta_1 \frac{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau,u}f(z)}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} \\ = (1-\frac{\vartheta_1}{\vartheta_2})\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} \\ +\frac{\vartheta_1}{\vartheta_2}\left[(1+\vartheta_2)\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \vartheta_2 \frac{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau,u}f(z)}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\right].\end{aligned} \tag{25}$$

Furthermore, because $0 \leq \frac{\vartheta_1}{\vartheta_2} < 1$ and the function $\frac{1+\mathcal{M}_1 z}{1+\mathcal{N}_1 z}$ are analytic and convex in $\mathbb{U}$. Using (23)–(25) and Lemma 5, we deduce that

$$(1+\vartheta_1)\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \vartheta_1 \frac{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau,u}f(z)}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} \prec \frac{1+\mathcal{M}_1 z}{1+\mathcal{N}_1 z}.$$

That is, $f \in \mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta_1, \lambda, \mathcal{M}_1, \mathcal{N}_1)$ implies that the assertion in (22) is true. □

Corollary 2. *Let* $0 \leq \vartheta_1 \leq \vartheta_2, 0 < \lambda < 1$, *and* $-1 \leq \mathcal{N}_1 \leq \mathcal{N}_2 < \mathcal{M}_2 \leq \mathcal{M}_1 \leq 1$. *Then,*

$$\mathcal{K}^{\mu}_{\tau,u}(\vartheta_2, \lambda, \mathcal{M}_2, \mathcal{N}_2) \subset \mathcal{K}^{\mu}_{\tau,u}(\vartheta_1, \lambda, \mathcal{M}_1, \mathcal{N}_1). \tag{26}$$

Theorem 3. *In the unit disc* $\mathbb{U}$, *let* $\kappa(z)$ *be a univalent function that satisfies the following condition*

$$\mathcal{R}\left(1+\frac{z\kappa''(z)}{\kappa'(z)}\right) > max\left\{0; -\mathcal{R}\left(\frac{\lambda}{\vartheta\varkappa_q}\right)\right\}, \quad (\mathcal{R}(\vartheta) > 0). \tag{27}$$

If $f(z) \in \mathcal{A}(l)$ *satisfies the condition below*

$$\begin{aligned}(1+\vartheta)\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \vartheta \frac{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau,u}f(z)}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} \\ \prec \kappa(z) + \frac{\vartheta\varkappa_q}{\lambda} z\mathfrak{D}_q(\kappa(z)),\end{aligned} \tag{28}$$

then,

$$\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} \prec \kappa(z)$$

and κ *is the best dominant of* (28).

Proof. Let $\gamma(z)$ be as determined in (20). In addition, note that (28) is correct. By combining (21) and (28), we obtain

$$\gamma(z) + \frac{\vartheta\varkappa_q}{\lambda} z\mathfrak{D}_q(\gamma(z)) \prec \kappa(z) + \frac{\vartheta\varkappa_q}{\lambda} z\mathfrak{D}_q(\kappa(z)). \tag{29}$$

So, we can readily derive the statement of Theorem 3 using Lemma 3 and (29). □

For $q \to 1-$, we obtain Corollary 3 below:

Corollary 3. *In the unit disc $\mathbb{U}$, let $\kappa(z)$ be a univalent function that satisfies the following condition*

$$\mathcal{R}\left(1+\frac{z\kappa''(z)}{\kappa'(z)}\right) > max\left\{0; -\mathcal{R}\left(\frac{\lambda}{\vartheta u}\right)\right\}, \quad (\mathcal{R}(\vartheta) > 0). \tag{30}$$

If $f(z) \in \mathcal{A}(l)$ satisfies the condition below

$$(1+\vartheta)\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda} - \vartheta\frac{\mathcal{IF}^{\mu}_{\tau,u}f(z)}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda} \prec \kappa(z) + \frac{\vartheta u}{\lambda}z\kappa'(z), \tag{31}$$

then,

$$\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda} \prec \kappa(z)$$

and κ is the best dominant of (31).

In Theorem 3, if we take $\kappa(z) = \frac{1+\mathcal{M}z}{1+\mathcal{N}z}$, we obtain the following result:

Corollary 4. *Let $\vartheta \in \mathbb{C}, \mathcal{R}(\vartheta) > 0, 0 < \lambda < 1,$ and $-1 \le \mathcal{N} < \mathcal{M} \le 1$. Suppose that*

$$\mathcal{R}\left(\frac{1-\mathcal{N}z}{1+\mathcal{N}z}\right) > max\left\{0; -\mathcal{R}\left(\frac{\lambda}{\vartheta\varkappa_q}\right)\right\}.$$

If $f(z) \in \mathcal{A}(l)$ satisfies the following condition

$$\begin{aligned}(1+\vartheta)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \vartheta\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} \\ \prec \frac{1+\mathcal{M}z}{1+\mathcal{N}z} + \frac{\vartheta\varkappa_q}{\lambda}\frac{(\mathcal{M}-\mathcal{N})z}{(1+\mathcal{N}z)^2},\end{aligned} \tag{32}$$

then,

$$\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} \prec \frac{1+\mathcal{M}z}{1+\mathcal{N}z}$$

and $\frac{1+\mathcal{M}z}{1+\mathcal{N}z}$ is the best dominant of (32).

Next, we investigate the superordination properties for the class $\mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta, \lambda, \mathcal{M}, \mathcal{N})$.

Theorem 4. *In the unit disc $\mathbb{U}$, let $\kappa(z)$ be a univalent function that satisfies the following condition*

$$\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} \in \mathcal{H}[\kappa(0), 1] \cap Q.$$

If

$$\kappa(z) + \frac{\vartheta\varkappa_q}{\lambda}z\mathfrak{D}_q(\kappa(z)) \prec (1+\vartheta)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \vartheta\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda},$$

such that $(1+\vartheta)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \vartheta\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}$ *be a univalent in* $\mathbb{U}$, *then,*

$$\kappa(z) \prec \left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}$$

and κ *is the best subordinate.*

Proof. If we define $\gamma(z)$, as given in (20), we can derive that

$$\begin{aligned}\kappa(z) + \frac{\vartheta\varkappa_q}{\lambda}z\mathfrak{D}_q(\kappa(z)) &\prec (1+\vartheta)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \vartheta\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\\ &= \gamma(z) + \frac{\vartheta\varkappa_q}{\lambda}z\mathfrak{D}_q(\gamma(z)).\end{aligned}$$

Theorem 4 is asserted as a result of applying Lemma 4. □

If $q \to 1-$, we deduce the following corollary:

Corollary 5. *In the unit disc* $\mathbb{U}$, *let* $\kappa(z)$ *be a univalent function that satisfies the following condition*

$$\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda} \in \mathcal{H}[\kappa(0),1] \cap Q.$$

If

$$\kappa(z) + \frac{\vartheta u}{\lambda}z\kappa'(z) \prec (1+\vartheta)\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda} - \vartheta\frac{\mathcal{IF}^{\mu}_{\tau,u}f(z)}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda},$$

such that $(1+\vartheta)\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda} - \vartheta\frac{\mathcal{IF}^{\mu}_{\tau,u}f(z)}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda}$ *be a univalent in* $\mathbb{U}$, *then,*

$$\kappa(z) \prec \left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda}$$

and κ *is the best subordinate.*

In Theorem 4, if we take $\kappa(z) = \frac{1+\mathcal{M}z}{1+\mathcal{N}z}$ $(-1 \le \mathcal{N} < \mathcal{M} \le 1)$, we obtain the result as follows:

Corollary 6. *Let* $\vartheta \in \mathbb{C}$ *and* $0 < \lambda < 1$. *Suppose that*

$$\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} \in \mathcal{H}[\kappa(0),1] \cap Q.$$

If

$$\begin{aligned}\frac{1+\mathcal{M}z}{1+\mathcal{N}z} + \frac{\vartheta\varkappa_q}{\lambda}\frac{(\mathcal{M}-\mathcal{N})z}{(1+\mathcal{N}z)^2} &\prec (1+\vartheta)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\\ &- \vartheta\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda},\end{aligned}$$

such that $(1+\vartheta)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}-\vartheta\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}$ *is a univalent in* $\mathbb{U}$, *then,*

$$\frac{1+\mathcal{M}z}{1+\mathcal{N}z}\prec\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}$$

and $\frac{1+\mathcal{M}z}{1+\mathcal{N}z}$ *is the best subordinate.*

Next, we investigate the sandwich-type results for the class $\mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta,\lambda,\mathcal{M},\mathcal{N})$.

Theorem 5. *Let* $\kappa_1(z)$ *and* $\kappa_2(z)$ *be convex functions in* $\mathbb{U}$, *with* $\kappa_1(0)=\kappa_2(0)=1$, *and let* $\vartheta\in\mathbb{C}$ *with* $\mathcal{R}(\vartheta)>0$. *Let* $\kappa_2(z)$ *satisfy the conditions in* (27), (28) *and*

$$\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\in\mathcal{H}[1,1]\cap Q$$

and

$$\begin{aligned}\kappa_1(z)+\frac{\vartheta\varkappa_q}{\lambda}z\mathfrak{D}_q(\kappa_1(z))\prec\Omega(z)&=(1+\vartheta)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\\&-\vartheta\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\prec\kappa_2(z)+\frac{\vartheta\varkappa_q}{\lambda}z\mathfrak{D}_q(\kappa_2(z)).\end{aligned}$$

Then,

$$\kappa_1(z)\prec\phi(z):=\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\prec\kappa_2(z),$$

where $\kappa_1(z)$ *and* $\kappa_2(z)$ *are the best subordinate and dominant, respectively.*

Corollary 7. *Let* $\kappa_1(z)$ *and* $\kappa_2(z)$ *be convex functions in* $\mathbb{U}$, *with* $\kappa_1(0)=\kappa_2(0)=1$, *and let* $\vartheta\in\mathbb{C}$ *with* $\mathcal{R}(\vartheta)>0$. *Let* $\kappa_2(z)$ *satisfy the conditions in* (27), (28) *and*

$$\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\in\mathcal{H}[1,1]\cap Q$$

and

$$\begin{aligned}\kappa_1(z)+\frac{\vartheta u}{\lambda}z\kappa_1'(z)\prec\bar{\Omega}(z)&=(1+\vartheta)\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda}\\&-\vartheta\frac{\mathcal{IF}^{\mu}_{\tau,u}f(z)}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda}\prec\kappa_2(z)+\frac{\vartheta u}{\lambda}z\kappa_2'(z).\end{aligned}$$

Then,

$$\kappa_1(z)\prec\bar{\phi}(z):=\left(\frac{z}{\mathcal{IF}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda}\prec\kappa_2(z),$$

where $\kappa_1(z)$ *and* $\kappa_2(z)$ *are the best subordinate and dominant, respectively.*

Example 1. *Taking* $\kappa_m=e^{\mathrm{r}_m z}$ $(m=1,2)$, *such that* $0<\mathrm{r}_1<\mathrm{r}_2\leq 1$, *the following examples are derived from Theorem* 5 *and Corollary* 7*:*

1. *If $f \in \mathcal{A}(l)$ and the subordination conditions in Theorem 4 hold, then*

$$\left(1+\frac{\vartheta\varkappa_q}{\lambda}z\right)e^{\mathrm{r}_1 z} \prec \Omega(z) \prec \left(1+\frac{\vartheta\varkappa_q}{\lambda}z\right)e^{\mathrm{r}_2 z} \Rightarrow e^{\mathrm{r}_1 z} \prec \phi(z) \prec e^{\mathrm{r}_2 z},$$

where $\Omega(z)$ and $\phi(z)$ are defined in Theorem 5, and $e^{\mathrm{r}_1 z}$ and $e^{\mathrm{r}_2 z}$ are the best "subordinate and dominant", respectively.

2. *If $f \in \mathcal{A}(l)$ and the subordination conditions in Corollary 5 hold, and if the operator $\mathcal{JF}^{\mu}_{q,\tau,u}f(z)$ is replaced with $\mathcal{IF}^{\mu}_{\tau,u}f(z)$, then*

$$\left(1+\frac{\vartheta u}{\lambda}z\right)e^{\mathrm{r}_1 z} \prec \bar{\Omega}(z) \prec \left(1+\frac{\vartheta u}{\lambda}z\right)e^{\mathrm{r}_2 z} \Rightarrow e^{\mathrm{r}_1 z} \prec \bar{\phi}(z) \prec e^{\mathrm{r}_2 z},$$

where $\bar{\Omega}(z)$ and $\bar{\phi}(z)$ are defined in Corollary 7. Thus, $e^{\mathrm{r}_1 z}$ and $e^{\mathrm{r}_2 z}$ are the best "subordinate and dominant", respectively.

Theorem 6. *If $f(z) \in \mathcal{T}^{\mu}_{q,\tau+1,u}(0,\lambda,\varpi)$, with $0 \le \varpi < 1$, then $f(z) \in \mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta,\lambda,\varpi)$ for $|z| < \boldsymbol{R}$, where*

$$\boldsymbol{R} = \left(\sqrt{\frac{|\vartheta|^2\varkappa_q^2 l^2}{\lambda^2}+1} - \frac{|\vartheta|\varkappa_q l}{\lambda}\right)^{\frac{1}{l}}, \quad 0<\lambda<1 \tag{33}$$

and the bound $\boldsymbol{R}$ is the best possible.

Proof. For $f(z) \in \mathcal{T}^{\mu}_{q,\tau+1,u}(0,\lambda,\varpi)$, suppose that the function $\mathcal{G}(z)$ is given by

$$\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} = (1-\varpi)\mathcal{G}(z)+\varpi. \tag{34}$$

Hence, $\mathcal{G}(z)$ is an analytic function in $\mathbb{U}$ and $\mathcal{G}(0)=1$. Taking the q-derivative of the function (34), we find that

$$\begin{aligned}&\frac{1}{1-\varpi}\{(1+\vartheta)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \vartheta\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \varpi\}\\ &\quad= \mathcal{G}(z)+\frac{\vartheta\varkappa_q}{\lambda}z\mathfrak{D}_q(\mathcal{G}(z)).\end{aligned}$$

Hence, we find that

$$\begin{aligned}&\mathcal{R}\{\frac{1}{1-\varpi}[(1+\vartheta)\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \vartheta\frac{\mathcal{JF}^{\mu}_{q,\tau,u}f(z)}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} - \varpi]\}\\ &\quad\ge \mathcal{R}\{\mathcal{G}(z)\}[1-\frac{2|\vartheta|\varkappa_q l r^l}{\lambda(1-r^{2l})}],\ (|z|=r<1).\end{aligned} \tag{35}$$

The right-hand side of inequality (35) is positive such that $\mathrm{r} < \boldsymbol{R}$, where $\boldsymbol{R}$ is defined by (33), and using the relation in (34), we obtain the following estimate (see [40])

$$\frac{|z\mathfrak{D}_q(\mathcal{G}(z))|}{\mathcal{R}\{\mathcal{G}(z)\}} \ge \frac{2l\mathrm{r}^l}{1-\mathrm{r}^{2l}}.$$

In order to prove that the bound $\boldsymbol{R}$ is the best possible, we let $f(z) \in \mathcal{S}(l)$, which is given by

$$\left(\frac{z}{\mathcal{JF}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda} = (1-\varpi)\frac{1+z^l}{1-z^l}+\varpi.$$

Note that

$$\frac{1}{1-\varpi}\{(1+\vartheta)\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}-\vartheta\frac{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau,u}f(z)}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}-\varpi\}$$
$$=\frac{1+z^l}{1-z^l}+\frac{2|\vartheta|\varkappa_q l \mathrm{r}^l}{\lambda(1-\mathrm{r}^l)^2}=0, \tag{36}$$

for $|z|=\boldsymbol{R}$, so we deduce that $\boldsymbol{R}$ is the best possible. □

Theorem 7. *Let the function* $f(z)\in\mathcal{T}^{\mu}_{q,\tau+1,u}(\vartheta,\lambda,\mathcal{M},\mathcal{N})$ *with* $\mathcal{R}(\vartheta)>0$ *and* $-1\leq\mathcal{N}<\mathcal{M}\leq 1$. *Then,*

$$\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}\frac{1-\mathcal{M}v}{1-\mathcal{N}v}dv<\mathcal{R}\left\{\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\right\}<\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}\frac{1+\mathcal{M}v}{1+\mathcal{N}v}dv. \tag{37}$$

In addition, for $|z|=\mathrm{r}<1$, *we obtain*

$$\mathrm{r}\left(\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}\frac{1+\mathcal{M}v\mathrm{r}}{1+\mathcal{N}v\mathrm{r}}dv\right)^{-\frac{1}{\lambda}}\leq\left|\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)\right|$$
$$\leq\mathrm{r}\left(\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}\frac{1-\mathcal{M}v\mathrm{r}}{1-\mathcal{N}v\mathrm{r}}dv\right)^{-\frac{1}{\lambda}}.$$

These are the best possible inequalities.

Proof. Using the hypotheses of Theorem 1, it follows that

$$\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\prec\aleph(z)=\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}\frac{1+\mathcal{M}zv}{1+\mathcal{N}zv}dv.$$

Since $\aleph(z)$ is a convex function and $z\in\mathbb{U}$, then

$$\mathcal{R}\left\{\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\right\}<\sup\mathcal{R}\left\{\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}\frac{1+\mathcal{M}zv}{1+\mathcal{N}zv}dv\right\}$$
$$\leq\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1\sup\mathcal{R}\left(\frac{1+\mathcal{M}zv}{1+\mathcal{N}zv}\right)v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}dv<\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1\frac{1+\mathcal{M}zv}{1+\mathcal{N}zv}v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}dv$$

and

$$\mathcal{R}\left\{\left(\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right)^{\lambda}\right\}>inf\mathcal{R}\left\{\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}\frac{1-\mathcal{M}zv}{1-\mathcal{N}zv}dv\right\}$$
$$\geq\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 inf\mathcal{R}\left(\frac{1-\mathcal{M}zv}{1-\mathcal{N}zv}\right)v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}dv>\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1\frac{1-\mathcal{M}zv}{1-\mathcal{N}zv}v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}dv.$$

In addition, since

$$\left|\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right|^{\lambda}\leq\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1\left|\frac{1+\mathcal{M}zv}{1+\mathcal{N}zv}\right|v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}dv$$
$$\leq\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1\frac{1+\mathcal{M}|z|v}{1+\mathcal{N}|z|v}v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}dv=\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1\frac{1+\mathcal{M}\mathrm{r}v}{1+\mathcal{N}\mathrm{r}v}v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}dv,\ |z|=\mathrm{r}<1,$$

we obtain
$$\left|\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)\right| \geq \mathrm{r}(\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 \frac{1+\mathcal{M}\mathrm{r}v}{1+\mathcal{N}\mathrm{r}v}v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}dv)^{\frac{-1}{\lambda}}$$
and
$$\left|\frac{z}{\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)}\right|^{\lambda} \geq \frac{\lambda}{\vartheta\varkappa_q l}\int_0^1\left|\frac{1-\mathcal{M}zv}{1-\mathcal{N}zv}\right|v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}dv$$
$$\geq \frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 \frac{1-\mathcal{M}|z|v}{1-\mathcal{N}|z|v}v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}dv = \frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 \frac{1-\mathcal{M}\mathrm{r}v}{1-\mathcal{N}\mathrm{r}v}v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}dv,$$
and so
$$\left|\mathcal{J}\mathcal{F}^{\mu}_{q,\tau+1,u}f(z)\right| \leq \mathrm{r}(\frac{\lambda}{\vartheta\varkappa_q l}\int_0^1 \frac{1-\mathcal{M}\mathrm{r}v}{1-\mathcal{N}\mathrm{r}v}v^{\frac{\lambda}{\vartheta\varkappa_q l}-1}dv)^{\frac{-1}{\lambda}}.$$

These inequalities are the best subordinations. □

Corollary 8. *Let the function $f(z) \in \mathcal{K}^{\mu}_{\tau,u}(\vartheta,\lambda,\mathcal{M},\mathcal{N})$ with $\mathcal{R}(\vartheta) > 0$ and $-1 \leq \mathcal{N} < \mathcal{M} \leq 1$. Then,*
$$\frac{\lambda}{\vartheta ul}\int_0^1 v^{\frac{\lambda}{\vartheta ul}-1}\frac{1-\mathcal{M}v}{1-\mathcal{N}v}dv < \mathcal{R}\left\{\left(\frac{z}{\mathcal{I}\mathcal{F}^{\mu}_{\tau+1,u}f(z)}\right)^{\lambda}\right\} < \frac{\lambda}{\vartheta ul}\int_0^1 v^{\frac{\lambda}{\vartheta ul}-1}\frac{1+\mathcal{M}v}{1+\mathcal{N}v}dv. \quad (38)$$

In addition, for $|z| = \mathrm{r} < 1$, we obtain
$$\mathrm{r}(\frac{\lambda}{\vartheta ul}\int_0^1 v^{\frac{\lambda}{\vartheta ul}-1}\frac{1+\mathcal{M}v\mathrm{r}}{1+\mathcal{N}v\mathrm{r}}dv)^{-\frac{1}{\lambda}} \leq \left|\mathcal{I}\mathcal{F}^{\mu}_{\tau+1,u}f(z)\right| \leq \mathrm{r}(\frac{\lambda}{\vartheta ul}\int_0^1 v^{\frac{\lambda}{\vartheta ul}-1}\frac{1-\mathcal{M}v\mathrm{r}}{1-\mathcal{N}v\mathrm{r}}dv)^{-\frac{1}{\lambda}}.$$

These are the best possible inequalities.

If we take $q \to 1-, \tau = 0$, and $\mu = 1$ in Theorem 7 and Corollary 8, we obtain the following results, provided by [34]:

Corollary 9 ([34]). *Let the function $f(z) \in \mathcal{T}^{1}_{0,u}(\vartheta,\lambda,\mathcal{M},\mathcal{N})$ and $\vartheta \in \mathbb{C}$ such that $\mathcal{R}(\vartheta) > 0$ and $-1 \leq \mathcal{N} < \mathcal{M} \leq 1$. Then,*
$$\frac{\lambda}{\vartheta l}\int_0^1 v^{\frac{\lambda}{\vartheta l}-1}\frac{1-\mathcal{M}v}{1-\mathcal{N}v}dv < \mathcal{R}\left(\frac{z}{f(z)}\right)^{\lambda} < \frac{\lambda}{\vartheta l}\int_0^1 v^{\frac{\lambda}{\vartheta l}-1}\frac{1+\mathcal{M}v}{1+\mathcal{N}v}dv.$$

In addition, for $|z| = \mathrm{r} < 1$, we obtain
$$\mathrm{r}(\frac{\lambda}{\vartheta l}\int_0^1 v^{\frac{\lambda}{\vartheta l}-1}\frac{1+\mathcal{M}v\mathrm{r}}{1+\mathcal{N}v\mathrm{r}}dv)^{-\frac{1}{\lambda}} \leq |f(z)| \leq \mathrm{r}(\frac{\lambda}{\vartheta l}\int_0^1 v^{\frac{\lambda}{\vartheta l}-1}\frac{1-\mathcal{M}v\mathrm{r}}{1-\mathcal{N}v\mathrm{r}}dv)^{-\frac{1}{\lambda}}.$$

Example 2. *By putting $\vartheta = \lambda = l = 1, \mathcal{M} = 1 - 2\alpha$ $(0 \leq \alpha < 1)$, and $\mathcal{N} = -1$, we obtain*

1. *The following inequality satisfies*
$$\mathcal{R}(\frac{z}{f(z)}) > 2\alpha - 1 + 2(1-\alpha)ln2.$$

2. *Let $\mathrm{r} = |z| = 0.9$, and we obtain*
$$\frac{0.9}{1+3.11685\alpha} \leq |f(z)| \leq \frac{0.9}{1-0.57365\alpha}.$$

5. Concluding Remarks

We have utilized q-calculus to introduce new results on the differential subordination and sandwich-type properties for a class of analytic non-Bazilevič functions defined by the q-analog integral operator $\mathcal{JF}^{\mu}_{q,\tau,u}f(z)$. The present study has the potential to inspire the use of other operators. Additionally, the best subordinates of the differential subordinations given can provide the basis for investigating conditions for the univalence of the operator introduced in this paper. Further research could involve the introduction of new classes of non-Bazilevič functions using the operator $\mathcal{JF}^{\mu}_{q,\tau,u}f(z)$ defined in Equation (9). As the classes obtained using this operator are likely to be distinct and interesting compared to previously obtained classes using other operators, relations to other known classes could be explored and coefficient estimates could be established. Additional works on the properties of non-Bazilevič functions, such as neighborhoods, subordinate sequences, the Fekete–Szegö inequality, and the Hankel determinant, could be investigated. This may also shed light on new concepts in geometric function theory.

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