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Monotone Mean L^p-Deviation Risk Measures

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Abstract: In this paper, we establish a new coherent risk measure on L^p, which we refer to as the monotone mean L^p-deviation risk measure. Then, the related properties are discussed. Furthermore, from the perspective of acceptance set, we discuss the relationship between the monotone mean L^p-deviation risk measure and the monotone Sharpe ratio risk measure. Finally, we extend the monotone mean L^p-deviation risk measure to the multivariate setting.

Keywords: coherent risk measure; monotone Sharpe ratio; mean standard deviation risk measure; portfolio

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1. Introduction

Coherent risk measures were introduced by [1], which satisfy four basic properties, monotonicity, translation invariance, positive homogeneity and subadditivity. Furthermore, convex risk measures were studied by [2,3], which relate the positive homogeneity and subadditivity to the convexity.

For more details about univariate risk measures, we refer the reader to [4,5]. At the same time, multivariate risk measures for portfolios which generalize the univariate risk measures have been extensively reported in the literature. The authors of [6] introduced multivariate coherent and convex risk measures. In [7], multivariate coherent and convex risk measures were generalized to a more general setting. For more details about multivariate risk measures, we refer the reader to [8].

The Sharpe ratio is one of the most important performance measures, which is calculated as the ratio of the expected return to its standard deviation, and is popularly used to rank financial positions. However, the Sharpe ratio lacks the property of monotonicity. Hence, it may be possible for an investment strategy to produce higher returns than another strategy, while the investment strategy has a smaller Sharpe ratio. Therefore, in order to modify the Sharpe ratio, the authors of [9] introduced the monotone Sharpe ratio, which makes the Sharpe ratio monotone, and established its connection with coherent risk measures.

It is well known that the mean standard deviation risk measure, which has a closed relationship with the Sharpe ratio, is one of the most popular risk measures due to its simplicity and tractability in practice, see [10]. Nevertheless, it is not a coherent risk measure, since it lacks the property of monotonicity. It is well known that monotonicity is one of the most basic and important properties that a risk measure is expected to have, because it represents an intuition that, for financial positions, lower profit should indicate higher riskiness. Therefore, a natural and interesting question is whether we can modify the common mean standard deviation risk measure into a coherent risk measure. Motivated by this consideration, and inspired by the idea of introducing a monotone Sharpe ratio suggested by [9], in this paper, we construct a monotone mean L^p-deviation risk measure.

It turns out that the monotone mean L^p-deviation risk measure is coherent and includes the monotone mean standard deviation risk measure as a particular case. Some basic properties of the monotone mean L^p-deviation risk measure are discussed. Furthermore, from the perspective of acceptance set, we also investigate the relationship between the monotone mean L^p-deviation risk measure and the monotone Sharpe ratio. Finally, we extend the introduced monotone mean L^p-deviation risk measure to the multivariate setting.

The definition of the mean standard deviation risk measure can be found in [4] (p. 202), which is defined by

$$\rho_c(X) := E(-X) + c \cdot \sigma(X),$$

where the random variable X represents the profit (or gain) of a financial position, $E(-X)$ means the expectation of $-X$ representing the expected loss of the financial position, $\sigma(X)$ means the standard deviation of X, and $c \geq 0$ is a constant. It is not hard to see that the mean standard deviation risk measure ρ_c does not satisfy the property of monotonicity in general when $c > 0$. The deviation measure is studied by [11]. The monotone mean L^p-deviation risk measure will modify the mean standard deviation risk measure so that it becomes monotone, and, hence, is a coherent risk measure. The L^p-deviation of a random variable was introduced by [9], Definition 1, which is defined as

$$\sigma_p(X) := \min_{c \in \mathbb{R}} ||X - c||_p \ \text{ for any } X \in L^p.$$

In particular, $\sigma_1(X)$ is the absolute deviation from the median of X, and $\sigma_2(X)$ is the standard deviation of X. Making use of this L^p-deviation, we construct the monotone mean L^p-deviation risk measure in this paper, and, thus, we provide a new coherent risk measure. The investor can evaluate the performance of the investment strategies by the monotone mean L^p-deviation risk measure. The larger the monotone mean L^p-deviation risk measure, the higher the riskiness.

The rest of this paper is organized in a straightforward manner. In Section 2, we provide necessary preliminary information, including definitions and notations. In Section 3, we introduce the main results of this paper. Finally, the conclusions are summarized in Section 4.

2. Preliminaries

Let $(\Omega, \mathscr{F}, P)$ be an atomless probability space. Let $1 \leq p \leq +\infty$, be denoted by $L^p(\Omega, \mathscr{F}, P)$ the linear space of random variables on $(\Omega, \mathscr{F}, P)$ with finite L^p norm, that is, $\|X\|_p := (E|X|^p)^{\frac{1}{p}} < \infty$ for $p < \infty$, and $\|X\|_\infty := \text{esssup}\,|X|$, the essential supremum of $|X|$, when $p = +\infty$. For simplicity, we write L^p for $L^p(\Omega, \mathscr{F}, P)$. Note that, $(L^p, \|\cdot\|_p)$ is a Banach space. Each element $X \in L^p$ represents the profit (or gain) of a financial position (or risky asset). Given a random variable $X \in L^p$, $E(X)$ and $\sigma(X)$ stand for the expectation and standard deviation of the random variable X, respectively. In general, a risk measure on L^p is defined as any mapping from L^p to the real numbers $\mathbb{R}$. For a given integer $d \geq 1$, for each $1 \leq i \leq d$, let $(\Omega_i, \mathscr{F}_i, P_i)$ be a fixed atomless probability space. Denote by $\mathcal{X}^d$ the product space of $\mathcal{X}_1, \cdots, \mathcal{X}_d$, that is, $\mathcal{X}^d := \mathcal{X}_1 \times \cdots \times \mathcal{X}_d$, where $\mathcal{X}_i := L^{p_i}(\Omega_i, \mathscr{F}_i, P_i)$. In general, a (scalar) multivariate risk measure is defined as any mapping from $\mathcal{X}^d$ to $\mathbb{R}$.

First, we recall the definition of coherent risk measures. For more details, see Definition 2.4 of [1] or Definition 2.1 of [12].

Definition 1 (Coherent risk measures on L^p)**.** *Let ρ be a risk measure on L^p. We say that ρ is a coherent risk measure, if it satisfies the following properties:*

(1) Monotonicity: $\rho(X) \geq \rho(Y)$ for any $X, Y \in L^p$ with $X \leq Y$;
(2) Translation invariance: $\rho(X + a) = \rho(X) - a$ for any $X \in L^p$ and $a \in \mathbb{R}$;
(3) Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for any $X \in L^p$ and $\lambda > 0$;
(4) Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for any $X, Y \in L^p$.

Remark 1. *Definition 1 of [2] and Definition 4 of [3] introduced the convex risk measures which relax the properties (3) and (4) for the following property of convexity:*

(5) convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ *for any* $X, Y \in L^p, \lambda \in [0,1]$.

For coherent risk measures, we have the following equivalent conditions: (For more details, we refer to [13], Lemma 2.1).

Lemma 1. *For a coherent risk measure* $\rho : L^p \to \mathbb{R}$*, the following conditions are equivalent:*

(1) The coherent risk measure ρ *is continuous;*
(2) There exists $K \in (0,\infty)$*, such that* $|\rho(X) - \rho(Y)| \leq K\|X - Y\|_p$ *for any* $X, Y \in L^p$*;*
(3) There exists $K \in (0,\infty)$*, such that* $\rho(X) \leq K\|X\|_p$ *for any* $X \in L^p$*;*
(4) There exists $K \in (0,\infty)$*, such that* $|\rho(X)| \leq K\|X\|_p$ *for any* $X \in L^p$*.*

Next, we recall the definition of an acceptance set induced by a risk measure. For more details, we refer to [4,14].

Definition 2 (Acceptance sets)**.** *Let* ρ *be a risk measure on* L^p*. The set* $\mathcal{A}_\rho := \{X \in L^p : \rho(X) \leq 0\}$ *is called the acceptance set induced by* ρ*.*

The following proposition about the acceptance set is straightforward, for instance, see [4].

Proposition 1. *If* ρ *is a coherent risk measure on* L^p*, and* $\mathcal{A}_\rho$ *is the acceptance set induced by* ρ*. Then the non-empty set* $\mathcal{A}_\rho$ *is a closed set and satisfies:*

(1)

$$\rho(X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}_\rho\}. \tag{1}$$

(2) For any $X \in \mathcal{A}_\rho$*,*

$$Y \in \mathcal{A}_\rho \ \textit{for any} \ Y \in L^p \ \textit{with} \ X \leq Y. \tag{2}$$

(3)

$$\lambda X + (1 - \lambda Y) \in \mathcal{A}_\rho \ \textit{for any} \ X, Y \in \mathcal{A}_\rho \textit{ and } \lambda \in [0,1]. \tag{3}$$

(4)

$$\lambda X \in \mathcal{A}_\rho \ \textit{for any} \ X \in \mathcal{A}_\rho \textit{ and } \lambda > 0. \tag{4}$$

Conversely, if a non-empty set $\mathcal{A} \subset L^p$ *satisfies*

$$\inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\} > -\infty \quad \textit{for all } X \in L^p, \tag{5}$$

(2)–(4)*, then the risk measure* $\rho_{\mathcal{A}}$ *defined by*

$$\rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} : \ m + X \in \mathcal{A}\}, \ \ X \in L^p$$

is a coherent risk measure, and $\mathcal{A} \subset \mathcal{A}_\rho$*. Moreover, if the set* $\mathcal{A}$ *is closed, then* $\mathcal{A} = \mathcal{A}_\rho$*.*

In practice, the Sharpe ratio is usually used to evaluate the performance of an investment. Let us recall the definition of the Sharpe ratios. For more details, see [9] (p. 4).

Definition 3 (Sharpe ratios)**.** *On* L^2*, the Sharpe ratio* $S(R)$ *of the random variable* R *is defined as*

$$S(R) := \frac{E(R)}{\sigma(R)}, \quad R \in L^2, \tag{6}$$

and $S(0) := 0$ *by convention.*

Next, we recall the definition of the mean standard deviation risk measures. For more details, see [4] (p. 202).

Definition 4 (Mean standard deviation risk measures)**.** *Let $\alpha > 0$ be a fixed constant. We call the risk measure ρ_{msd} defined by*

$$\rho_{msd}(X) := E(-X) + \alpha \cdot \sigma(X), \quad X \in L^2,$$

the mean standard deviation risk measure on L^2.

The mean standard deviation risk measure ρ_{msd} satisfies translation invariance, positive homogeneity and subadditivity, though it does not satisfy monotonicity in general. Thus, ρ_{msd} is not a coherent risk measure in general. Denote $\mathscr{X} := \{X \in L^2 : \sigma(X) \neq 0\}$. On $\mathscr{X}$, the acceptance set $\mathcal{A}_{msd}$ induced by a mean standard deviation risk measure ρ_{msd} can be described by the Sharpe ratio; that is,

$$\mathcal{A}_{msd} := \{X \in \mathscr{X} : \rho_{msd}(X) \leq 0\} = \left\{X \in \mathscr{X} : S(R) := \frac{E(X)}{\sigma(X)} \geq \alpha\right\}.$$

It is not hard to verify that the set $\mathcal{A}_{msd}$ is not monotonic, since the Sharpe ratio $S(\cdot)$ is not monotone. For more details, we refer to [9] (p. 4).

Now, we recall the definition of L^p-deviation; for instance, see [9], Definition 1.

Definition 5 (L^p-deviation)**.** *For $X \in L^p, 1 \leq p < \infty$, $\sigma_p(X) := \min_{c \in \mathbb{R}} \|X - c\|_p$ is called the L^p-deviation of X.*

Note that, when $p = 2$, the L^2-deviation of X coincides with the standard deviation of X. The following lemma is from ([9], p. 5).

Lemma 2. *The L^p-deviation $\sigma_p(\cdot)$ satisfies the following properties:*

(1) $\sigma_p(\lambda X) = |\lambda| \sigma_p(X)$ for any $\lambda \in \mathbb{R}$.
(2) For any $X \in L^p$, there is a $c_0 \in \mathbb{R}$, such that $\sigma_p(X) = \|X - c_0\|_p$.
(3) $\sigma_p(X + Y) \leq \sigma_p(X) + \sigma_p(Y)$ for any $X, Y \in L^p$.
(4) For any $X \in L^p$, $\sigma_p(X) \geq 0$. Furthermore, $\sigma_p(X) = 0$ if, and only if, X is almost surely a constant.
(5) For any $X \in L^p$, $\sigma_p(X) \leq \|X\|_p$. Thus, $\sigma_p(X)$ is uniformly continuous on L^p.

Definition 2 of [9] introduced the notion of monotone Sharpe ratios.

Definition 6 (Monotone Sharpe ratios)**.** *For any random variable $X \in L^p$, $1 \leq p < \infty$, the monotone Sharpe ratio of X is defined by*

$$\mathbb{S}_p(X) := \sup_{Y \leq X} \frac{E(Y)}{\sigma_p(Y)}, \tag{7}$$

where the supremum is taken over all $Y \in L^p$, such that $Y \leq X$ a.s.; while for $X = 0$ a.s., define $\mathbb{S}_p(0) := 0$.

By the definition, it is not hard to verify the following lemma; for instance, see Theorem 2 of [9].

Lemma 3. *The monotone Sharpe ratio $\mathbb{S}_p(\cdot)$ defined by (7) satisfies the following properties:*

(1) If $E(X) \leq 0$, then $\mathbb{S}_p(X) = 0$.
(2) If $X \geq 0$, a.s. and $P(X > 0) \neq 0$, then $\mathbb{S}_p(X) = +\infty$.
(3) If $E(X) > 0$ and $P(X < 0) \neq 0$, then $\mathbb{S}_p(X) \in (0, +\infty)$. In this situation, $\mathbb{S}_p(X)$ is continuous with respect to X.

Next, we introduce the acceptance set induced by monotone Sharpe ratios.

Definition 7 (Acceptance set induced by monotone Sharpe ratios)**.** *Let $\alpha > 0$ be a constant, and $1 \le p < \infty$. The acceptance set $\mathcal{A}_{\mathbb{S}_p}$ induced by the monotone Sharpe ratio $\mathbb{S}_p$ is defined as*

$$\mathcal{A}_{\mathbb{S}_p} := \{X \in L^p : \mathbb{S}_p(X) \ge \alpha\}. \tag{8}$$

Notice that the acceptance set $\mathcal{A}_{\mathbb{S}_p}$ induced by the monotone Sharpe ratio $\mathbb{S}_p$ satisfies (2)–(5). Hence, the risk measure, denoted by $\rho_{\mathbb{S}_p}$, induced by $\mathcal{A}_{\mathbb{S}_p}$ via (1) is a coherent risk measure.

3. Main Results

In this section, we present the main results of this paper. We construct a new coherent risk measure, which we refer to as the monotone mean L^p-deviation risk measure. Moreover, from the perspective of the acceptance set, its connection with the monotone Sharpe ratios is investigated. Finally, we also extend this univariate coherent risk measure to the multivariate setting.

3.1. Monotone Mean L^p-Deviation Risk Measures

As pointed out previously, the mean standard deviation risk measure lacks monotonicity, and, thus, it is not a coherent risk measure. Inspired by [9], in this subsection, we first construct a new kind of coherent risk measure, named the monotone mean L^p-deviation risk measure, which includes monotone mean standard deviation risk measures as a special case. Then, some basic properties of the monotone mean L^p-deviation risk measure are discussed. Furthermore, its connection with the monotone Sharpe ratios is also investigated.

Definition 8 (Monotone mean L^p-deviation risk measures)**.** *Let $\alpha > 0$ be a fixed constant. We call the risk measure defined by*

$$\rho^*(X) := \inf_{Y \le X} \{E(-Y) + \alpha \cdot \sigma_p(Y)\}, \quad X \in L^p,$$

the monotone mean L^p-deviation risk measure on L^p. When $p = 2$, the monotone mean L^p-deviation risk measure ρ^* is simply called the monotone mean standard deviation risk measure.

In the above definition, since Y represents the profit of a risky asset, the expectation $E(-Y)$ represents the expected loss of the risky asset. α represents the preference coefficient of the average profit volatility level, and L^p-deviation $\sigma_p(Y)$ indicates the average volatility level of the profit Y. For a given risky asset X, the monotone mean L^p-deviation risk measure $\rho^*(X)$ takes into account all risky assets whose profits are not greater than X, and calculates the minimum value of the possible loss of these risky assets. Similar to the monotone Sharpe ratios, the monotone mean L^p-deviation risk measure ρ^* holds the monotonicity.

Now, we are in a position to state one of the main results of this paper, which is that the mean L^p-deviation risk measure ρ^* is coherent.

Theorem 1. *The monotone mean L^p-deviation risk measure ρ^* on L^p is a coherent risk measure.*

Proof.

(1) Monotonicity: For any $X_1, X_2 \in L^p$, if $X_1 \le X_2$, then

$$\inf_{Y \le X_1} \{E(-Y) + \alpha \cdot \sigma_p(Y)\} \ge \inf_{Y \le X_2} \{E(-Y) + \alpha \cdot \sigma_p(Y)\}.$$

Thus, $\rho^*(X_1) \ge \rho^*(X_2)$.

(2) Translation invariance: For any $X \in L^p$ and $a \in \mathbb{R}$,

$$\begin{aligned}\rho^*(X+a) &= \inf_{Y \le X+a} \{E(-Y) + \alpha \cdot \sigma_p(Y)\} \\ &= \inf_{Y \le X} \{E(-Y-a) + \alpha \cdot \sigma_p(Y)\} \\ &= \inf_{Y \le X} \{E(-Y) + \alpha \cdot \sigma_p(Y)\} - a \\ &= \rho^*(X) - a.\end{aligned}$$

(3) Positive homogeneity: For any $X \in L^p$ and $\lambda > 0$,

$$\begin{aligned}\rho^*(\lambda X) &= \inf_{Y \le \lambda X} \{E(-Y) + \alpha \cdot \sigma_p(Y)\} \\ &= \inf_{Y \le X} \{E(-\lambda Y) + \alpha \cdot \sigma_p(\lambda Y)\} \\ &= \lambda \cdot \inf_{Y \le X} \{E(-Y) + \alpha \cdot \sigma_p(Y)\} \\ &= \lambda \cdot \rho^*(X).\end{aligned}$$

(4) Subadditivity: For any $X_1, X_2 \in L^p$, let $a_1 := \inf_{Y \le X_1} \{E(-Y) + \alpha \cdot \sigma_p(Y)\}$, $a_2 := \inf_{Y \le X_2} \{E(-Y) + \alpha \cdot \sigma_p(Y)\}$. Then, for any $\varepsilon_1 > 0$, there is a $Y_1 \le X_1$, such that $E(-Y_1) + \alpha \cdot \sigma_p(Y_1) \le a_1 + \varepsilon_1$. Similarly, for any $\varepsilon_2 > 0$, there is a $Y_2 \le X_2$, such that $E(-Y_2) + \alpha \cdot \sigma_p(Y_2) \le a_2 + \varepsilon_2$. Since $Y_1 + Y_2 \le X_1 + X_2$, by Lemma 2 (3), we have that

$$\begin{aligned}\inf_{Y \le X_1 + X_2} \{E(-Y) + \alpha \cdot \sigma_p(Y)\} &\le E(-Y_1 - Y_2) + \alpha \cdot \sigma_p(Y_1 + Y_2) \\ &\le E(-Y_1) + \alpha \cdot \sigma_p(Y_1) + E(-Y_2) + \alpha \cdot \sigma_p(Y_2) \\ &\le a_1 + a_2 + \varepsilon_1 + \varepsilon_2.\end{aligned}$$

Taking the limits of $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ simultaneously on both sides of the above inequality, we can get that

$$\rho^*(X_1 + X_2) \le \rho^*(X_1) + \rho^*(X_2).$$

In summary, ρ^* is a coherent risk measure. The theorem is proved. □

Next, we discuss the continuity of the monotone mean L^p-deviation risk measure ρ^* on L^p.

Proposition 2. *The monotone mean L^p-deviation risk measure ρ^* is continuous on L^p.*

Proof. By the definition of the monotone mean L^p-deviation risk measure ρ^*,

$$\begin{aligned}\rho^*(X) &= \inf_{Y \le X} \{E(-Y) + \alpha \cdot \sigma_p(Y)\} \\ &\le E(-X) + \alpha \cdot \sigma_p(X) \\ &\le \|X\|_1 + \alpha \cdot \sigma_p \|X\|_p \\ &\le \|X\|_p + \alpha \cdot \sigma_p \|X\|_p \\ &= (\alpha + 1) \cdot \|X\|_p.\end{aligned}$$

From Theorem 1, it follows that ρ^* is a coherent risk measure. Hence, by Lemma 1, ρ^* is continuous on L^p. □

Next, we discuss the properties of the acceptance set induced by the monotone mean L^p-deviation risk measure ρ^*. Recall that the acceptance set $\mathcal{A}_*$ induced by ρ^*, is defined as

$$\begin{aligned}\mathcal{A}_* &:= \{X \in L^p : \rho^*(X) \le 0\} \\ &= \{X \in L^p : \inf_{Y \le X}\{E(-Y) + \alpha \cdot \sigma_p(Y)\} \le 0\} \\ &= \{X \in L^p : \sup_{Y \le X}\{E(Y) - \alpha \cdot \sigma_p(Y)\} \ge 0\}.\end{aligned}$$

Proposition 3. *$\mathcal{A}_*$ satisfies (2)–(4); that is, $\mathcal{A}_*$ is a monotonic closed convex cone.*

Proof.

(1) Closedness: For any $X_n \in \mathcal{A}_*$, $n \ge 1$, with $X_n \to X$ in L^p norm, since $\rho^*(X)$ is continuous with respect to X, then

$$\rho^*(X) = \rho^*\left(\lim_{n\to\infty} X_n\right) = \lim_{n\to\infty} \rho^*(X_n).$$

Since $\rho^*(X_n) \le 0$, $n \ge 1$, hence, $\rho^*(X) \le 0$. Thus, $X \in \mathcal{A}_*$.

(2) Monotonicity: For any $X_1 \in \mathcal{A}_*$ and $X_2 \in L^p$ with $X_2 \ge X_1$, *a.s.*, by the monotonicity of ρ^*, we have that

$$\rho^*(X_2) \le \rho^*(X_1).$$

Since $X_1 \in \mathcal{A}_*$, hence, $\rho^*(X_2) \le 0$, and, thus, $X_2 \in \mathcal{A}_*$.

(3) Convexity: For any $X_1, X_2 \in \mathcal{A}_*, \lambda \in [0,1]$, we have that $\rho^*(X_1) \le 0$ and $\rho^*(X_2) \le 0$. From the positive homogeneity and subadditivity of ρ^*, it follows that

$$\rho^*[\lambda X_1 + (1-\lambda)X_2] \le \lambda\rho^*(X_1) + (1-\lambda)\rho^*(X_2) \le 0.$$

Hence, $\lambda X_1 + (1-\lambda)X_2 \in \mathcal{A}_*$.

(4) Cone: For any $X \in \mathcal{A}_*$, we have that $\rho^*(X) \le 0$. For any $\lambda > 0$, by the positive homogeneity of ρ^*, we know that $\rho^*(\lambda X) \le 0$. Thus, $\lambda X \in \mathcal{A}_*$.

□

Next, we turn to discuss the relationship between the acceptance sets induced by the monotone mean L^p-deviation risk measure ρ^* and the monotone Sharpe ratios $\mathbb{S}_p$. We begin by discussing the acceptance set induced by $\mathbb{S}_p$.

Recall that the acceptance set induced by the monotone Sharpe ratio $\mathbb{S}_p$ is defined as

$$\mathcal{A}_{\mathbb{S}_p} := \left\{X \in L^p : \sup_{Y \le X} \frac{E(Y)}{\sigma_p(Y)} \ge \alpha\right\} = \left\{X \in L^p : \sup_{Y \le X}\left\{\frac{E(Y) - \alpha \cdot \sigma_p(Y)}{\sigma_p(Y)}\right\} \ge 0\right\}, \quad (9)$$

where the constant α is chosen as the same as that in the definition of ρ^*.

Recall also that the acceptance set induced by the monotone mean L^p-deviation risk measure ρ^* is defined as

$$\mathcal{A}_* := \{X \in L^p : \rho^*(X) \le 0\} = \left\{X \in L^p : \sup_{Y \le X}\{E(Y) - \alpha \cdot \sigma_p(Y)\} \ge 0\right\}. \quad (10)$$

It can be seen that $\mathcal{A}_{\mathbb{S}_p}$ and $\mathcal{A}_*$ appear likely. Therefore, we discuss the possible connection between $\mathcal{A}_{\mathbb{S}_p}$ and $\mathcal{A}_*$.

Proposition 4. *The acceptance set $\mathcal{A}_{\mathbb{S}_p}$ is not closed.*

Proof. We will show the proposition by contradiction. Assume that $\mathcal{A}_{\mathbb{S}_p}$ is closed. Consider the sequence of random variables $X_n := \frac{1}{n}$, $n \ge 1$. Then, $E(X_n) = \frac{1}{n} > 0$, $X_n \ge 0$ a.s. and

$P(X_n > 0) \neq 0$. Hence, by Lemma 3, we know that $\mathbb{S}_p(X_n) = +\infty$. Thus, $X_n \in \mathcal{A}_{\mathbb{S}_p}$. On the other hand, we know that $X_n \to 0$ in L^p norm. For any $\alpha > 0$, we have that $\mathbb{S}_p(0) = 0 < \alpha$, and, hence, $0 \notin \mathcal{A}_{\mathbb{S}_p}$. Therefore, we obtain that $X_n \in \mathcal{A}_{\mathbb{S}_p}$ for any $n \geq 1$, while 0, which is the limit of the sequence $\{X_n; n \geq 1\}$, does not belong to the set $\mathcal{A}_{\mathbb{S}_p}$. This contradicts the assumption that $\mathcal{A}_{\mathbb{S}_p}$ is closed. □

Proposition 5. *$0 \notin \mathcal{A}_{\mathbb{S}_p}$, and $0 \in \mathcal{A}_*$.*

Proof. By the proof of Proposition 4, we know that $0 \notin \mathcal{A}_{\mathbb{S}_p}$. Obviously, $\rho^*(0) = 0$, which means that $0 \in \mathcal{A}_*$. □

Proposition 6. *$\overline{\mathcal{A}}_{\mathbb{S}_p} \subset \mathcal{A}_*$, that is, $\mathcal{A}_*$ contains the closure of $\mathcal{A}_{\mathbb{S}_p}$.*

Proof. For any $X \in \mathcal{A}_{\mathbb{S}_p}$, by Lemma 3, we know that, if $E(X) \leq 0$, then $\mathbb{S}_p(X) = 0 < \alpha$; thus, $X \notin \mathcal{A}_{\mathbb{S}_p}$. Therefore, without any loss of generality, we can assume that $E(X) > 0$, and we only need to consider the following two cases:

(1) Assume that $E(X) > 0$ and $X \geq 0$ a.s. Then $P(X > 0) \neq 0$. Hence, by Lemma 3, $\mathbb{S}_p(X) = +\infty$. Thus, we have that $\sup_{Y \leq X}\{E(Y) - \alpha \cdot \sigma_p(Y)\} \geq 0$. Consequently, $X \in \mathcal{A}_*$.

(2) Assume that $E(X) > 0$ and $P(X < 0) \neq 0$. Then

$$\sup_{Y \leq X}\left\{\frac{E(Y) - \alpha \cdot \sigma_p(Y)}{\sigma_p(Y)}\right\} \geq 0 \tag{11}$$

is equivalent to

$$\sup_{Y \leq X}\{E(Y) - \alpha \cdot \sigma_p(Y)\} \geq 0, \tag{12}$$

which implies that $X \in \mathcal{A}_*$.

In summary, we have shown that, if $X \in \mathcal{A}_{\mathbb{S}_p}$, then $X \in \mathcal{A}_*$. By Proposition 3, we know that $\mathcal{A}_*$ is closed, and, thus, we have that $\overline{\mathcal{A}}_{\mathbb{S}_p} \subset \mathcal{A}_*$. □

3.2. Multivariate Extension of the Monotone Mean L^p-Deviation Risk Measures

In this subsection, we extend the monotone mean L^p-deviation risk measure to the multivariate setting. We use a random vector $\overrightarrow{X} := (X_1, \cdots, X_d) \in \mathcal{X}^d$ to represent the profit vector of a portfolio consisting of d risky assets. For $1 \leq i \leq d$, the i-th component X_i stands for the profit of the i-th risky asset.

Definition 9.

(1) Order relation: For $\overrightarrow{X} = (X_1, \cdots, X_d) \in \mathcal{X}^d$, $\overrightarrow{Y} = (Y_1, \cdots, Y_d) \in \mathcal{X}^d$, $\overrightarrow{X} \leq \overrightarrow{Y}$ means $X_i \leq Y_i$ for all $1 \leq i \leq d$.

(2) Addition: For $\overrightarrow{X} = (X_1, \cdots, X_d) \in \mathcal{X}^d$, $\overrightarrow{Y} = (Y_1, \cdots, Y_d) \in \mathcal{X}^d$, define $\overrightarrow{X} + \overrightarrow{Y} := (X_1 + Y_1, \cdots, X_d + Y_d)$.

(3) Multiplication: For $\overrightarrow{X} = (X_1, \cdots, X_d) \in \mathcal{X}^d$, $\lambda \in \mathbb{R}$, define $\lambda\overrightarrow{X} := (\lambda X_1, \cdots, \lambda X_d)$.

(4) Norm: For $1 \leq i \leq d$, $1 < p_i < \infty$, define $\left\|\overrightarrow{X}\right\| := \sum_{i=1}^{d} \|X_i\|_{p_i}$ as the norm on $\mathcal{X}^d$.

In general, a multivariate risk measure ρ_d on $\mathcal{X}^d$ is defined as any mapping from $\mathcal{X}^d$ to $\mathbb{R}$. A multivariate risk measure ρ_d on $\mathcal{X}^d$ is called coherent if it satisfies the following properties:

(1) Monotonicity: For any $\overrightarrow{X}, \overrightarrow{Y} \in \mathcal{X}^d$ with $\overrightarrow{X} \leq \overrightarrow{Y}$, $\rho_d(\overrightarrow{X}) \geq \rho_d(\overrightarrow{Y})$.

(2) Translation invariance: For any $\overrightarrow{X} \in \mathcal{X}^d$ and $\overrightarrow{a} = (a_1, \cdots, a_d) \in \mathbb{R}^n$, $\rho_d(\overrightarrow{X} + \overrightarrow{a}) = \rho_d(\overrightarrow{X}) - \sum_{i=1}^{d} a_i$.

(3) Positive homogeneity: For any $\overrightarrow{X} \in \mathcal{X}^d$ and $\lambda > 0$, $\rho_d(\lambda \overrightarrow{X}) = \lambda \rho_d(\overrightarrow{X})$.
(4) Subadditivity: For any $\overrightarrow{X}, \overrightarrow{Y} \in \mathcal{X}^d$, $\rho_d(\overrightarrow{X} + \overrightarrow{Y}) \leq \rho_d(\overrightarrow{X}) + \rho_d(\overrightarrow{Y})$.

For more details about multivariate coherent risk measures, we refer the reader to [6–8].

Now, we introduce the multivariate monotone mean L^p-deviation risk measures.

Definition 10 (Multivariate monotone mean L^p-deviation risk measures)**.** *Let $\alpha_i > 0$ be a fixed constant, $1 \leq i \leq d$. The multivariate monotone mean L^p-deviation risk measure $\rho_d^* : \mathcal{X}^d \to \mathbb{R}$ is defined as*

$$\rho_d^*(\overrightarrow{X}) := \sum_{i=1}^{d} \inf_{Y_i \leq X_i} \{E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i)\}, \quad \overrightarrow{X} \in \mathcal{X}^d.$$

The next theorem is another of the main results of this paper.

Theorem 2. *The multivariate monotone mean L^p-deviation risk measure ρ_d^* is a multivariate coherent risk measure.*

Proof.

(1) Monotonicity: For any $\overrightarrow{X} = (X_1, \cdots, X_d)$, $\overrightarrow{Z} = (Z_1, \cdots, Z_d) \in \mathcal{X}^d$ with $\overrightarrow{X} \leq \overrightarrow{Z}$, then, $X_i \leq Z_i$ for all $1 \leq i \leq d$. For each $1 \leq i \leq d$, we have that

$$\inf_{Y_i \leq X_i} \{E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i)\} \geq \inf_{Y_i \leq Z_i} \{E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i)\}.$$

Thus,

$$\sum_{i=1}^{d} \inf_{Y_i \leq X_i} \{E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i)\} \geq \sum_{i=1}^{d} \inf_{Y_i \leq Z_i} \{E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i)\},$$

which means that $\rho_d^*(\overrightarrow{X}) \geq \rho_d^*(\overrightarrow{Z})$.

(2) Translation invariance: For any $\overrightarrow{X} = (X_1, \cdots, X_d) \in \mathcal{X}^d$ and $\overrightarrow{a} = (a_1, \cdots, a_d) \in \mathbb{R}^d$, we know that

$$\begin{aligned}
\rho_d^*(\overrightarrow{X} + \overrightarrow{a}) &= \sum_{i=1}^{d} \inf_{Y_i \leq X_i + a_i} \{E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i)\} \\
&= \sum_{i=1}^{d} \inf_{Y_i \leq X_i} \{E(-Y_i - a_i) + \alpha_i \cdot \sigma_{p_i}(Y_i)\} \\
&= \sum_{i=1}^{d} \inf_{Y_i \leq X_i} \{E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i)\} - \sum_{i=1}^{d} a_i \\
&= \rho_d^*\left(\overrightarrow{X}\right) - \sum_{i=1}^{d} a_i.
\end{aligned}$$

(3) Positive homogeneity: For any $\overrightarrow{X} = (X_1, \cdots, X_d) \in \mathcal{X}^d$ and $\lambda > 0$, we have that

$$
\begin{aligned}
\rho_d^*\left(\lambda \overrightarrow{X}\right) &= \sum_{i=1}^{d} \inf_{Y_i \le \lambda X_i} \left\{ E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i) \right\} \\
&= \sum_{i=1}^{d} \inf_{Y_i \le X_i} \left\{ E(-\lambda Y_i) + \alpha_i \cdot \sigma_{p_i}(\lambda Y_i) \right\} \\
&= \sum_{i=1}^{d} \lambda \cdot \inf_{Y_i \le X_i} \left\{ E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i) \right\} \\
&= \lambda \cdot \sum_{i=1}^{d} \inf_{Y_i \le X_i} \left\{ E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i) \right\} \\
&= \lambda \cdot \rho_d^*\left(\overrightarrow{X}\right).
\end{aligned}
$$

(4) Subadditivity: For any $\overrightarrow{X} = (X_1, \cdots, X_d)$, $\overrightarrow{Z} = (Z_1, \cdots, Z_d) \in \mathcal{X}^d$, we have that

$$
\begin{aligned}
\rho_d^*\left(\overrightarrow{X} + \overrightarrow{Z}\right) &= \sum_{i=1}^{d} \left\{ \inf_{Y_i \le X_i + Z_i} \left\{ E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i) \right\} \right\} \\
&\le \sum_{i=1}^{d} \left\{ \inf_{Y_i \le X_i} \left\{ E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i) \right\} + \inf_{Y_i \le Z_i} \left\{ E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i) \right\} \right\} \\
&= \sum_{i=1}^{d} \inf_{Y_i \le X_i} \left\{ E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i) \right\} + \sum_{i=1}^{d} \inf_{Y_i \le Z_i} \left\{ E(-Y_i) + \alpha_i \cdot \sigma_{p_i}(Y_i) \right\} \\
&= \rho_d^*\left(\overrightarrow{X}\right) + \rho_d^*\left(\overrightarrow{Z}\right).
\end{aligned}
$$

In summary, ρ_d^* is coherent. The theorem is proved. □

We end this subsection with the introduction of the acceptance set induced by ρ_d^*.

Definition 11. *The acceptance set induced by the multivariate monotone mean L^p-deviation risk measure ρ_d^* is defined as*

$$
\mathcal{A}_*^d := \left\{ \overrightarrow{X} \in \mathcal{X}^d : \rho_d^*(\overrightarrow{X}) \le 0 \right\}. \tag{13}
$$

By (13), we know that, for the multivariate monotone mean L^p-deviation risk measure ρ_d^* on $\mathcal{X}^d$, we consider the portfolio $\overrightarrow{X}$ as a whole, and think that the portfolio $\overrightarrow{X}$ is acceptable, as long as $\rho_d^*(\overrightarrow{X}) \le 0$. Given a portfolio $\overrightarrow{X}$ with $\rho_d^*(\overrightarrow{X}) \le 0$, if there is some $1 \le i_0 \le d$, such that $\rho^*(X_{i_0}) > 0$, then it means that the i-th risky asset is not acceptable, while the whole portfolio is acceptable for the investor. This reflects that the risk associated with one component of the portfolio can be hedged by other components.

4. Conclusions

We establish a new coherent risk measure, the monotone mean L^p-deviation risk measure. This risk measure can be considered as a sort of monotonicity-based modification of the common mean standard deviation risk measure. The properties of its acceptance set are also discussed. Moreover, its connection with the monotone Sharpe ratios is investigated. Finally, its multivariate extension is addressed.

The new coherent risk measure can be considered as a new tool to evaluate the performance of investment strategies. One could further consider its application in financial statistics, to analyze the performance of certain specific investments. Furthermore, taking into account that monotonicity is typically expected for a risk measure, this paper suggests a way of constructing monotone risk measures via non-monotone risk measures, which is of itself interesting.

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