mathematics

Article

Property (h) of Banach Lattice and Order-to-Norm Continuous Operators

Fu Zhang, Hanhan Shen * and Zili Chen

School of Mathematics, Xipu Campus, Southwest Jiaotong University, Chengdu 611756, China; zhangfu@my.swjtu.edu.cn (F.Z.); zlchen@swjtu.edu.cn (Z.C.)
* Correspondence: 20123645@my.swjtu.edu.cn

Abstract: In this paper, we introduce the property (h) on Banach lattices and present its characterization in terms of disjoint sequences. Then, an example is given to show that an order-to-norm continuous operator may not be σ-order continuous. Suppose $T : E \to F$ is an order-bounded operator from Dedekind σ-complete Banach lattice E into Dedekind complete Banach lattice F. We prove that T is σ-order-to-norm continuous if and only if T is both order weakly compact and σ-order continuous. In addition, if E can be represented as an ideal of $L_0(\mu)$, where (Ω, Σ, μ) is a σ-finite measure space, then T is σ-order-to-norm continuous if and only if T is order-to-norm continuous. As applications, we extend Wickstead's results on the order continuity of norms on E and E'.

Keywords: Banach lattices; property (h); order weakly compact operators; order-to-norm continuous operators; σ-order continuous operators

MSC: 46A40; 46B42; 47B65

check for updates

Citation: Zhang, F.; Shen, H.; Chen, Z. Property (h) of Banach Lattice and Order-to-Norm Continuous Operators. *Mathematics* **2023**, *11*, 2747. https://doi.org/10.3390/math11122747

Academic Editor: Enrique Alfonso Sánchez Pérez

Received: 23 May 2023
Revised: 13 June 2023
Accepted: 16 June 2023
Published: 17 June 2023

1. Introduction

Throughout this paper, E and F will denote Banach lattices, whereas X and Y will denote Archimedean Riesz spaces. The set of all positive vectors of X is called the positive cone of X, and is denoted by X^+. Similarly, $E^+ := \{x \in E : x \geq 0\}$. A net $\{x_\alpha\}$ in X is said to be **order convergent** to x, written as $x_\alpha \xrightarrow{o} 0$, if there exists another net $\{p_\alpha\}$ in X satisfying $p_\alpha \downarrow 0$, such that $|x_\alpha - x| \leq p_\alpha$ for all α. Put $e \in X^+$. The net $\{x_\alpha\}$ in X **converges** e**-uniformly** to x if there exists a positive real number net $\{\varepsilon_\alpha\}$ with $\varepsilon_\alpha \downarrow 0$, such that $|x_\alpha - x| \leq \varepsilon_\alpha e$ for all α. In this case, we write $x_\alpha \to x(e\text{-ru})$. And the net $\{x_\alpha\}$ in X **converges relatively uniformly** to x, denoted by $x_\alpha \xrightarrow{ru} x$, if there exists e in X^+ such that $\{x_\alpha\}$ converges e-uniformly to x. Every relatively uniformly convergent sequence is also order convergent.

Recall that E is said to

- Have **an order-continuous norm** if $x_\alpha \xrightarrow{\|\cdot\|} 0$ whenever $x_\alpha \xrightarrow{o} 0$ in E.
- Have **a** σ**-order continuous norm** if $x_n \xrightarrow{\|\cdot\|} 0$ whenever $x_n \xrightarrow{o} 0$ in E.
- Be a *KB* **space** if every increasing norm-bounded sequence in E^+ has a norm limit.

The concepts of order convergence and relative uniform convergence are identical on E if and only if E has an order-continuous norm; see [1] (Proposition 3). If H is a closed sublattice of E and $\{x_n\} \subset H$, then $x_n \xrightarrow{ru} 0$ in H if and only if $x_n \xrightarrow{ru} 0$ in E; see [2] (Proposition 2.12).

Niculescu [3] extended Lozanovskii's results on Banach lattices with σ-order continuous norms to type A operators defined on Banach lattices. In 2021, Jalili et al. continued the study of operator versions of order-continuous norm and introduced order-to-norm continuous operators (see [4]). An operator $T : E \to F$ is said to be

- **Type A** if $\{Tx_n\}$ is norm convergent whenever $0 \leq x_n \downarrow$ in E.

- **Order weakly compact** if for every $x \in E^+$, $T[0, x]$ is weakly compact in F.
- M **weakly compact** if $Tx_n \xrightarrow{\|\cdot\|} 0$ for every norm-bounded disjoint sequence $\{x_n\}$ in E.
- **Order-to-norm continuous** if $Tx_\alpha \xrightarrow{\|\cdot\|} 0$ whenever $x_\alpha \xrightarrow{o} 0$ in E.
- σ**-order-to-norm continuous** if $Tx_n \xrightarrow{\|\cdot\|} 0$ whenever $x_n \xrightarrow{o} 0$ in E.
- **order continuous** if $Tx_\alpha \xrightarrow{o} 0$ whenever $x_\alpha \xrightarrow{o} 0$ in E.
- σ**-order continuous** if $Tx_n \xrightarrow{o} 0$ whenever $x_n \xrightarrow{o} 0$ in E.

In Theorem 2, we also present that every order-bounded σ-order-to-norm continuous operator into a Banach lattice with property (h) (see Definition 1) can be defined in order relation. It is proved in [3] (Lemma 3.1) that type A operators and order weakly compact operators are equivalent. Moreover, $T : E \to F$ is order weakly compact if and only if $Tx_n \xrightarrow{\|\cdot\|} 0$ for every order bounded disjoint sequence $\{x_n\}$ in E. Meyer-Nieberg shows that every order weakly compact operator $T : E \to F$ admits a factorization through a Banach lattice G_T with an order continuous norm. These details can be found in [5] (Theorem 3.4.6) and [6] (Theorem 5.58). Now, we are in position to list some necessary notes. The natural factorization of $T : E \to F$ occurs through a quotient space. We define lattice seminorm q_T on E as $q_T(x) = \sup\{\| Ty \| : | y | \leq | x |\}$ for every $x \in E$. Let $N(T)$ be the null ideal $\{x \in E : q_T(x) = 0\}$ of q_T. And let $Q_T : E \to E/N(T)$ denote the canonical projection. Suppose that G_T is the norm completion of normed Riesz space $E/N(T)$ under the quotient norm $\| Q_T x \| = q_T(x)$. The formula $S_T(Q_T x) = Tx$ gives rise to a well-defined continuous operator $S_T : E/N(T) \to F$ with $\| S_T \| \leq 1$. Hence, S extends to all of G_T, satisfying $\| S_T \| \leq 1$. We have the factorization of T,

$$\begin{tikzcd}
E \arrow[rr, "T"] \arrow[dr, "Q_T"'] & & F \\
& G_T \arrow[ur, "S_T"'] &
\end{tikzcd}$$

Moreover, if $T : E \to F$ is order bounded and F is Dedekind complete, then the modulus $|T| : E \to F$ of T exists. Given any $0 < x \in E$, $\| |T|x \| \geq q_T(x)$, hence

$$\{x \in E : |T|(|x|) = 0\} \subset N(T).$$

To see $N(T) \subset \{x \in E : |T|(|x|) = 0\}$, put $0 < x \in N(T)$. Since $\| Ty \| \leq q_T(x) = 0$ for every $|y| \leq x$, by [6] (Theorem 1.18), we have $|T|x = \sup\{|Ty| : |y| \leq x\} = 0$. It follows that

$$\{x \in E : |T|(|x|) = 0\} = N(T).$$

In this paper, we mainly study the relative uniform order convergence of sequence on Banach lattices. At the same time, the main results of the article relate to the properties of σ-order-to-norm continuous and order-to-norm continuous operators. We refer the reader to [5–8] for unexplained terminology on Banach lattices.

2. Banach Lattices with Property (*h*)

Every relatively uniformly convergent sequence converges in norm. However, the opposite may be not true. e_n denotes the sequence of real numbers whose nth term is one and the rest are zero. $\frac{1}{n}e_n \xrightarrow{\|\cdot\|} 0$ in l_1 but not relatively uniformly convergent. Recall that E is said to be an AM-space if $\| x \vee y \| = \max\{\| x \|, \| y \|\}$ for $x, y \in E^+$. In [9] (Proposition 2), Wirth proved that E is isomorphic to an AM-space if and only if every norm convergent sequence in E is relatively uniformly convergent and if and only if every norm convergent sequence in E is order convergent. If E is a AM-space or σ-order continuous Banach lattice, then for every sequence $\{x_n\}$ in E, $x_n \xrightarrow{ru} 0$ if and only if $x_n \xrightarrow{o} 0$ and $x_n \xrightarrow{\|\cdot\|} 0$.

Definition 1. *The Banach lattice E has property (h), provided that $x_n \xrightarrow{ru} 0$ if and only if $x_n \xrightarrow{o} 0$ and $x_n \xrightarrow{\|\cdot\|} 0$ in E.*

Example 1. *Let E be the L_∞-sum of the space sequence $\{E_n : E_n = l_1, n \in \mathbb{N}\}$ i.e.,*

$$E = l_\infty(l_1) := \{a = (a_1, a_2, \dots, a_n, \dots) : a_n \in l_1 \text{ and } \parallel a \parallel := \sup_n \parallel a_n \parallel_1 < +\infty\}.$$

E is a Dedekind complete Banach lattice without order continuous norm under the pointwise ordering. In fact, E lacks property (h).

Proof. Define a sequence $\{z_m\}$ in E by

$$\begin{aligned}
&(e_1, 0, 0, \dots, 0, \dots),\\
&(0, \frac{1}{2}e_1, 0, \dots, 0, \dots), (0, \frac{1}{2}e_2, 0, \dots, 0, \dots),\\
&\dots\\
&\underbrace{(0, 0, 0, \dots, \frac{1}{k}e_1, \dots)}_{kth\ term}, \underbrace{(0, 0, 0, \dots, \frac{1}{k}e_2, \dots)}_{kth\ term}, \dots, \underbrace{(0, 0, 0, \dots, \frac{1}{k}e_k, \dots)}_{kth\ term},\\
&\dots
\end{aligned}$$

Evidently, $\{z_m\}$ is an order-bounded disjoint norm null sequence and $\parallel \sum_{j=\frac{m(m+1)}{2}+1}^{\frac{m(m+1)}{2}+m+2} z_j \parallel = 1$ for all m. We claim that $z_m \xrightarrow{ru} 0$ is not true. Otherwise, there exists $e \in E^+$ and a real number sequence $\{\varepsilon_m\}$ satisfying $\varepsilon_m \downarrow 0$, such that $0 \le z_m \le \varepsilon_m e$ for all $m \in \mathbb{N}$. It follows that for every $m \in \mathbb{N}$,

$$\sum_{j=\frac{m(m+1)}{2}+1}^{\frac{m(m+1)}{2}+m+2} z_j = \bigvee_{j=\frac{m(m+1)}{2}+1}^{\frac{m(m+1)}{2}+m+2} z_j \le \varepsilon_{\frac{m(m+1)}{2}+1} e.$$

This implies that

$$1 = \parallel \sum_{j=\frac{m(m+1)}{2}+1}^{\frac{m(m+1)}{2}+m+2} z_j \parallel \le \varepsilon_{\frac{m(m+1)}{2}+1} \parallel e \parallel$$

for every m. We obtain a contradiction. □

Recall that in a discrete Banach lattice E, $x_n \xrightarrow{o} 0$ for every order bounded norm null sequence $\{x_n\}$ in E.

Proposition 1. *Let E be a discrete Dedekind σ-complete Banach lattice. The following assertions are equivalent.*

(1) E has property (h).

(2) $x_n \xrightarrow{ru} 0$ if and only if $\{x_n\}$ is order bounded and $x_n \xrightarrow{\|\cdot\|} 0$.

Recall that Riesz space X has principal projection property if and only if every principal band is a projection band i.e., for every $u \in X$, $\bigvee_{n=1}^\infty y \wedge n|u|$ exists for each $y \in X^+$. In this case, for $u \in E^+$, the band projection from E onto $[u]$ by $P_u : E \to [u]$ is defined by $P_u x = \bigvee_{n=1}^\infty y \wedge n|u|$ for every $x \in X^+$, where $[u]$ is the principal band generated by u in E. By [6] (Theorem 1.47), every Dedekind σ-complete Riesz space has principal projection property. The following lemmas will be useful in the sequel discussions of property (h).

Lemma 1 ([5], Proposition 2.8.2). *If Riesz space X has principal projection property, then the following statements are hold.*

(1) *For any x, y in X^+, there exist disjoint elements e_1, e_2 in X, such that $x \vee y = e_1 + e_2$ and $e_1 \le x, e_2 \le y$.*

(2) *For any x, y, z in X^+ with $x \le y \vee z$, there exist disjoint elements x_1, x_2 in X, such that $x = x_1 + x_2$ and $x_1 \le y, x_2 \le z$.*

Lemma 2 ([6], Theorem 4.12). *Let X be a Riesz space and $0 \le x_n \uparrow \le x$ in X. For every $k \in \mathbb{N}$, there exist disjoint sequences $\{y_n^1\}, \{y_n^2\}, \ldots, \{y_n^k\}$ of $[0, x]$, such that for each n,*

$$y_n^1 + y_n^2 + \ldots + y_n^k \le x_{n+1} - x_n \le y_n^1 + y_n^2 + \ldots + y_n^k + \frac{2}{k+3} x.$$

Lemma 3 ([10], Theorem 105.15). *If $x_n \downarrow$ and $x_n \xrightarrow{\|\cdot\|} 0$ in E, then $x_n \xrightarrow{ru} 0$ in E.*

Next, we characterize Banach lattices with property (h) in terms of disjoint sequences.

Theorem 1. *Let E be a Dedekind σ-complete Banach lattice. The following assertions are equivalent.*

(1) *E has property (h).*

(2) *$x_n \xrightarrow{ru} 0$ for every order-bounded disjoint norm null sequence $\{x_n\}$ in E.*

Proof. (1) $\Rightarrow$ (2) is evident. (2) $\Rightarrow$ (1) Let $0 \le x_n \xrightarrow{o} 0$ and $x_n \xrightarrow{\|\cdot\|} 0$. Since E is a Dedekind σ-complete Banach lattice, according to [5] (Proposition 1.1.10), $x_n \xrightarrow{o} 0$ implies $\bigvee_{i=n}^{\infty} x_i \downarrow 0$. We assert that $\bigvee_{i=n}^{\infty} x_i \xrightarrow{\|\cdot\|} 0$. Next, we prove the assertion by contradiction. Assume that $\bigvee_{i=n}^{\infty} x_i \xrightarrow{\|\cdot\|} 0$ is not true. According to [11] (Theorem 15.4), $\{\bigvee_{i=n}^{\infty} x_i\}$ is not a Cauchy sequence since $\bigvee_{i=n}^{\infty} x_i \downarrow 0$. Thus, we can find $\varepsilon_0 > 0$ and a subsequence $\{\bigvee_{i=n_m}^{\infty} x_i\}$ of $\{\bigvee_{i=n}^{\infty} x_i\}$, such that

$$\| \bigvee_{i=n_m}^{\infty} x_i - \bigvee_{i=n_{m+1}}^{\infty} x_i \| \ge \varepsilon_0$$

for every $m \in \mathbb{N}$. Note that $0 \le \bigvee_{i=1}^{\infty} x_i - \bigvee_{i=n_m}^{\infty} x_i \uparrow \bigvee_{i=1}^{\infty} x_i$. According to Lemma 2, there are disjoint sequences $\{z_m^1\}, \{z_m^2\}, \ldots, \{z_m^k\}$, such that for every n,

$$z_m^1 + z_m^2 + \ldots + z_m^k \le \bigvee_{i=n_m}^{\infty} x_i - \bigvee_{i=n_{m+1}}^{\infty} x_i \le z_m^1 + z_m^2 + \ldots + z_m^k + \frac{2}{k+3} \bigvee_{i=1}^{\infty} x_i.$$

It follows from $\bigvee_{i=n_m}^{\infty} x_i - \bigvee_{i=n_{m+1}}^{\infty} x_i = (\bigvee_{i=n_m}^{n_{m+1}-1} x_i - \bigvee_{i=n_{m+1}}^{\infty} x_i)^+$ that $0 \le z_m^j \le \bigvee_{i=n_m}^{n_{m+1}-1} x_i$ for every $j = 1, 2, \ldots, k$ and $m \in \mathbb{N}$. By Lemma 1, for every $j = 1, 2, \ldots, k$ and $m \in \mathbb{N}$ there exist pairwise disjoint elements $w_{n_m}^j, w_{n_m+1}^j, \ldots, w_{n_{m+1}-1}^j$, such that $z_m^j = \sum_{i=n_m}^{n_{m+1}-1} w_i^j$ and $w_i^j \le x_i$ for each $i = n_m, n_m + 1, \ldots, n_{m+1} - 1$. It follows that the order-bounded disjoint sequence is $w_i^j \xrightarrow{\|\cdot\|} 0$ for every $j = 1, 2, \ldots, k$. Therefore, according to the assumption of (2), $w_i^j \xrightarrow{ru}_i 0$. We can find $e^1, e^2, \ldots, e^k \in E^+$ and $\{\varepsilon_i^1\}, \{\varepsilon_i^2\}, \ldots, \{\varepsilon_i^k\} \subset \mathbb{R}$, such that $\varepsilon_i^j \downarrow 0$ for every j and $0 \le w_i^j \le \varepsilon_i^j e^j$ for every i and j. Therefore,

$$\begin{aligned}
\varepsilon_0 &\leq \| \bigvee_{i=n_m}^{\infty} x_i - \bigvee_{i=n_{m+1}}^{\infty} x_i \| \leq \| z_m^1 \| + \| z_m^2 \| + \ldots + \| z_m^k \| + \frac{2}{k+3} \| \bigvee_{i=1}^{\infty} x_i \| \\
&= \| \sum_{i=n_m}^{n_{m+1}-1} w_i^1 \| + \| \sum_{i=n_m}^{n_{m+1}-1} w_i^2 \| + \ldots + \| \sum_{i=n_m}^{n_{m+1}-1} w_i^k \| + \frac{2}{k+3} \| \bigvee_{i=1}^{\infty} x_i \| \\
&\leq \varepsilon_{n_m}^1 \| e^1 \| + \varepsilon_{n_m}^2 \| e^2 \| + \ldots + \varepsilon_{n_m}^k \| e^k \| + \frac{2}{k+3} \| \bigvee_{i=1}^{\infty} x_i \| .
\end{aligned}$$

Letting $m \to \infty$, we determine that $\varepsilon_0 \leq \frac{2}{k+3} \| \bigvee_{i=1}^{\infty} x_i \|$ for every k. This is a contradiction. Hence, $\bigvee_{i=n}^{\infty} x_i \xrightarrow{\|\cdot\|} 0$. According to Lemma 3, $\bigvee_{i=n}^{\infty} x_i \xrightarrow{ru} 0$. This implies $x_n \xrightarrow{ru} 0$. □

Remark 1. *Let $\{y_n\}$ and $\{z_n\}$ be two sequences in Riesz space X. $\{y_n\}$ is said to be dominated by $\{z_n\}$, written as $\{y_n\} \preccurlyeq \{z_n\}$, if $|y_n| \leq |z_n|$ for every $n \in \mathbb{N}$. According to the proof of Theorem 1, for every sequence $\{x_n\}$ in Dedekind σ-complete Banach lattice, $x_n \xrightarrow{ru} 0$ if and only if the following statements hold:*

(1) $x_n \xrightarrow{o} 0$.

(2) *If $\{w_n\} \preccurlyeq \{x_n\}$, then $w_n \xrightarrow{ru} 0$.*

In general, the property (h) cannot imply AM or order-continuous property.

Example 2. *Let E be the L_1-sum of the space sequence $\{E_n : E_n = l_\infty, n \in \mathbb{N}\}$ i.e.,*

$$E = l_1(l_\infty) := \{a = (a_1, a_2, \ldots, a_n, \ldots) : a_n \in l_1 \text{ and } \| a \| := \sum_{n=1}^{\infty} \| a_n \|_1 < +\infty\}.$$

E is a Dedekind complete Banach lattice. Evidently, E is neither AM-space nor an order-continuous Banach lattice. However, E has property (h).

Proof. Suppose that $x \in E^+$ and $\{x^n\}$ is a disjoint norm null sequence in $[0, x]$, where $x = (x_1, x_2, \ldots, x_m, \ldots)$. For every $m \in \mathbb{N}$, define the projection $P_m : E \to E$ by $P_m a = a_m$ for every $a = (a_1, a_2, \ldots, a_m, \ldots) \in E$. Now, fixed m, $\{P_m x^n\}$ is a disjoint norm null sequence in $[0, P_m x] \subset P_m E = l_\infty$. Note that l_∞ is an AM-space. It follows that $P_m x^n \xrightarrow{ru} 0$ in $P_m E$ for every m. For fixed $\varepsilon > 0$, there exists $2 \leq N \in \mathbb{N}$, such that $\sum_{m=N}^{\infty} \| x_m \| \leq \varepsilon$. Therefore,

$$\| \bigvee_{m=N}^{\infty} \bigvee_{n=1}^{\infty} P_m x^n \| \leq \| \bigvee_{m=N}^{\infty} P_m x \| = \| \bigvee_{m=N}^{\infty} x_m \| \leq \varepsilon.$$

For every $n \in \mathbb{N}$, $\bigvee_{k=n}^{\infty} x^k = \bigvee_{k=n}^{\infty} \bigvee_{m=1}^{\infty} P_m x^k = \bigvee_{m=1}^{\infty} \bigvee_{k=n}^{\infty} P_m x^k = \sum_{m=1}^{N-1} \bigvee_{k=n}^{\infty} P_m x^k + \bigvee_{m=N}^{\infty} \bigvee_{k=n}^{\infty} P_m x^k$. Given any $m \in \mathbb{N}$, $\lim_{n\to\infty} \| \bigvee_{k=n}^{\infty} P_m x^k \| = 0$ since $P_m x^n \xrightarrow{ru} 0$. It follows that

$$\begin{aligned}
\limsup_{n\to\infty} \| \bigvee_{k=n}^{\infty} x^k \| &= \limsup_{n\to\infty} \| \sum_{m=1}^{N-1} \bigvee_{k=n}^{\infty} P_m x^k + \bigvee_{m=N}^{\infty} \bigvee_{k=n}^{\infty} P_m x^k \| \\
&\leq \limsup_{n\to\infty} \| \sum_{m=1}^{N-1} \bigvee_{k=n}^{\infty} P_m x^k \| + \limsup_{n\to\infty} \| \bigvee_{m=N}^{\infty} \bigvee_{k=n}^{\infty} P_m x^k \| \\
&\leq \limsup_{n\to\infty} \| \sum_{m=1}^{N-1} \bigvee_{k=n}^{\infty} P_m x^k \| + \| \bigvee_{m=N}^{\infty} x_m \| \\
&\leq \limsup_{n\to\infty} \sum_{m=1}^{N-1} \| \bigvee_{k=n}^{\infty} P_m x^k \| + \varepsilon = \varepsilon
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \| \bigvee_{k=n}^{\infty} x^k \| = 0$. According to Lemma 3, we obtain $x^n \xrightarrow{ru} 0$. □

In general, every Dedekind σ-complete Banach lattice contains a norm closed ideal which has property (h).

Definition 2. *Let E be a Banach lattice. An element $x \in E$ is said to have property (h) if every disjoint sequence $\{x_n\} \subset [0, |x|]$ with $x_n \xrightarrow{\|\cdot\|} 0$ is uniformly convergent. The collection of all elements that have property (h) is denoted by E^h.*

In Example 1, it is easy to verify that $E^h = c_0(l_1)$.

Theorem 2. *Let E be a Dedekind σ-complete Banach lattice. E^h is a norm-closed ideal of E.*

Proof.

(1) E^h is an ideal of E. Given any $x, y \in E^h$ and $\alpha, \beta \in \mathbb{R}$, if $\{z_n\}$ is a disjoint sequence such that $0 \le z_n \le |\alpha x + \beta y|$ and $z_n \xrightarrow{\|\cdot\|} 0$, then due to the Riesz decomposition property of [6] (Theorem 1.13), there exist two positive disjoint sequences, $\{z_{n,1}\}$ and $\{z_{n,2}\}$, such that $z_n = z_{n,1} + z_{n,2}$ and $z_{n,1} \le |\alpha||x|, z_{n,2} \le |\beta||y|$ for every n. Then, $z_{n,i} \xrightarrow{ru} 0$ for $i = 1, 2$. This implies that $z_n = z_{n,1} + z_{n,2} \xrightarrow{ru} 0$. We obtain that $|\alpha x + \beta y| \in E^h$ for every $x, y \in E^h$ and $\alpha, \beta \in \mathbb{R}$. It is easy to see that $x \in E^h$ if and only $|x| \in E^h$. And the condition $0 \le z \le x \in E^h$ and $z \in E$ can imply that $y \in E^h$. We have proved that E^h is an ideal of E.

(2) Assume that $x_n \xrightarrow{\|\cdot\|} x$ in E with $\{x_n\} \subset E^h$. Let $\{z_n\}$ be a disjoint sequence in $[0, |x|]$. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $\| \, |x| - |x_N| \, \| \le \varepsilon$. It follows from $x_N \in E^h$ that $z_n \wedge |x_N| \xrightarrow{ru} 0$. Thus, we can find $e \in E^+$ and a real number sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$, such that $0 \le z_n \wedge |x_N| \le \varepsilon_n e$ for every n. For every $m > n$, we have

$$|z_m \wedge |x| - z_m \wedge |x_N|| \wedge |z_n \wedge |x| - z_n \wedge |x_N|| \le (2z_m) \wedge (2z_n) = 2(z_m \wedge z_n) = 0.$$

It follows that $\{|z_n \wedge |x| - z_n \wedge |x_N||\}$ is a disjoint sequence. According to Birkhoff's inequalities (see [6], Theorem 1.9), for every $n, k \in \mathbb{N}$,

$$\begin{aligned}
&\| \sum_{i=n}^{n+k} z_i \| = \| \sum_{i=n}^{n+k} (z_i \wedge |x| - z_i \wedge |x_N| + z_i \wedge |x_N|) \| \\
&\le \| \sum_{i=n}^{n+k} |z_i \wedge |x| - z_i \wedge |x_N|| + \sum_{i=n}^{n+k} (z_i \wedge |x_N|) \| \\
&\le \| \sum_{i=n}^{n+k} |z_i \wedge |x| - z_i \wedge |x_N|| \, \| + \| \sum_{i=n}^{n+k} (z_i \wedge |x_N|) \| \\
&= \| \bigvee_{i=n}^{n+k} |z_i \wedge |x| - z_i \wedge |x_N|| \, \| + \| \bigvee_{i=n}^{n+k} (z_i \wedge |x_N|) \| \\
&\le \| \, ||x| - |x_N|| \, \| + \varepsilon_n \| e \| \le \varepsilon + \varepsilon_n \| e \| .
\end{aligned}$$

This implies that the series $\sum_{n=1}^{\infty} |z_n|$ converges in its norm. Therefore, $\sum_{i=n}^{\infty} |z_i| \xrightarrow{\|\cdot\|} 0$. According to Lemma 3, $|z_n| \le \sum_{i=n}^{\infty} |z_i| \xrightarrow{ru} 0$. This implies that $x_n \xrightarrow{ru} 0$. Based on the proceeding deduction, we conclude that $x \in E^h$. Hence, E^h is a closed subspace of E. □

3. Order-to-Norm Continuous Operators on Banach Lattices

Evidently, every σ-order-to-norm continuous operator from a Dedekind σ-complete Banach lattice to another Banach lattice is order weakly compact. The identity operator on Banach lattice without σ-order continuous norm is neither order-to-norm continuous nor order weakly compact.

Example 3. *Let $E = c_0$ and $F = l_1$. Then, there is an operator $T : E \to F$ which is order-to-norm continuous and order weakly compact but not σ-order continuous.*

Proof. According to Dvoretzky–Rogers' theorem (see [12], Theorem 2), there is an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ in l_1, such that $\| x_n \|= \frac{1}{n^{\frac{3}{4}}}$ for every n. Define $T : E \to F$ by

$$Tw = \sum_{n=1}^{\infty} w(n)x_n$$

for all $w \in E$. When using [13] (Corollary 2.5), T is not order bounded. Therefore, there exists $x \in E^+$, such that $T[0, x]$ is not order bounded, i.e., $\{\bigvee_{i=1}^{n} |Tx_i| : x_i \in [0, x], n \in \mathbb{N}\}$ is not order bounded. Note that $F = l_1$ is a KB space. $\{\bigvee_{i=1}^{n} |Tx_i| : x_i \in [0, x], n \in \mathbb{N}\}$ is not norm bounded. Therefore, there exists an increasing sequence $\{n_k\} \subset \mathbb{N}$ and a subsequence $\{x_i\}_{i=n_1}^{\infty}$ of $[0, x]$, such that $\| \bigvee_{i=n_k}^{n_{k+1}-1} |Tx_i| \|\geq k^2$ for every k. Define a sequence $\{z_i\}_{i=n_1}^{\infty}$ in $[0, x]$ by $z_i = \frac{1}{k}x_i$ for every $n_k \leq i \leq n_{k+1} - 1$ and $k \in \mathbb{N}$. Clearly, $z_i \xrightarrow{ru} 0$. We claim that T is not σ-order continuous. Otherwise, T is σ-order continuous. This implies that $Tz_i \xrightarrow{o} 0$. It follows that there is $y \in F$, such that $|Tz_i| \leq y$ for every i. Hence,

$$k \leq\| \bigvee_{i=n_k}^{n_{k+1}-1} |Tz_i| \|\leq\| y \| .$$

This is a contradiction. □

Every order-bounded σ-order-to-norm continuous operator in a Banach lattice with property (h) can be defined in order.

Proposition 2. *Let E, F be Banach lattices with F Dedekind complete. $T : E \to F$ is an order-bounded operator. The following statements hold.*

(1) If T is σ-order-to-norm continuous, then T is σ-order continuous.

(2) If F has property (h), then the following assertions are equivalent.

(2a) $Tx_n \xrightarrow{ru} 0$ for every $x_n \xrightarrow{o} 0$ in E.

(2b) T is σ-order-to-norm continuous.

Proof.

(1) Let $x_n \xrightarrow{o} 0$ in E. We have $\| \bigwedge_{k=1}^{\infty} |Tx_k| \|\leq\| Tx_n \|\to 0$. It follows that $\bigwedge_{k=1}^{\infty} |Tx_k| = 0$. According to [6] (Theorem 1.56), T is σ-order continuous.

(2) $(2a) \Rightarrow (2b)$ is evident. $(2b) \Rightarrow (2a)$ Note that $Tx_n \xrightarrow{\|\cdot\|} 0$ and $Tx_n \xrightarrow{o} 0$ for every $x_n \xrightarrow{o} 0$ in E. Since F has property (h), we have $Tx_n \xrightarrow{ru} 0$.

□

Recall that a net $\{x_\alpha\}$ in E is said to be laterally decreasing if $(x_\alpha - x_\beta) \wedge x_\beta = 0$ for all $\alpha \preceq \beta$. If E is Dedekind σ-complete, then $\{x_n\} \subset E^+$ is laterally decreasing to zero if and only if there is a disjoint sequence $\{e_n\}$ in E^+, such that $x_n = \bigvee_{i=n}^{\infty} e_i$ for every n.

Lemma 4 ([14], Proposition 0.3.5)**.** *Let E be a Dedekind σ-complete Banach lattice. The following statements hold.*

(1) If $x_\alpha \downarrow 0$ in E, then for every $\varepsilon > 0$ and index α_0, there exists a net $\{y_\alpha\}$ satisfying $y_\alpha \downarrow 0$, such that $P_{y_\alpha} x_{\alpha_0} \leq \frac{1}{\varepsilon} x_\alpha$ for all α and $x_\alpha \leq P_{y_\alpha} x_{\alpha_0} + \varepsilon x_{\alpha_0}$ for $\alpha \geq \alpha_0$. Therefore, there exists a laterally decreasing net $\{z_\alpha\} \subset E^+$ such that $z_\alpha \leq \frac{1}{\varepsilon} x_\alpha$ for all α and $x_\alpha \leq z_\alpha + \varepsilon x_{\alpha_0}$ for $\alpha \geq \alpha_0$.

(2) If $x_n \downarrow 0$ in E, then for every $\varepsilon > 0$, there exists an order-bounded disjoint sequence $\{w_n\} \subset E^+$, such that $\bigvee_{i=n}^{\infty} w_i \leq \frac{1}{\varepsilon} x_n$ and $x_n \leq \bigvee_{i=n}^{\infty} w_i + \varepsilon x_1$ for all n.

Lemma 5 ([5], Lemma 3.4.3)**.** *Let $T : E \to F$ be a norm-bounded operator between two Banach lattices. Then, $q_T(x) = \sup\{|y'T|(|x|) :\| y' \|\leq 1\}$ for every $x \in E$.*

The σ-order-to-norm continuous operators have a number of nice characterizations.

Theorem 3. *Let E be a Dedekind σ-complete Banach lattice and F Dedekind complete. If $T : E \to F$ is an order-bounded operator, then the following statements are equivalent.*

(1) T is σ-order-to-norm continuous.
(2) $T : E \to F$ admits a factorization through an order-continuous Banach lattice G_T,

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "Q_T"'] & & F \\ & G_T \arrow[ur, "S_T"'] & \end{tikzcd}$$

where the factor Q_T is a σ-order-to-norm continuous lattice homomorphism and S_T is a norm-bounded operator.

(3) $T(\bigvee_{n=1}^{\infty} w_n) = \sum_{n=1}^{\infty} Tw_n$ for every order-bounded disjoint sequence $\{w_n\}$ in E^+.
(4) $Tx_n \xrightarrow{w} 0$ for every $x_n \xrightarrow{o} 0$ in E i.e., $y'T : E \to R$ is a σ-order continuous functional for every $y' \in F'$.
(5) $Tx_n \xrightarrow{w} 0$ for every $x_n \downarrow 0$ in E.
(6) $Tx_n \xrightarrow{\|\cdot\|} 0$ for every $x_n \downarrow 0$ in E.
(7) T is order weakly compact and T is σ-order continuous.

Proof. The derivation of the proof of this theorem is shown as follows

$$(1) \Rightarrow (6) \Rightarrow (5) \Leftrightarrow (4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1) \Rightarrow (7) \Rightarrow (6).$$

(5) $\Leftrightarrow$ (4) is obtained by [6] (Theorem 1.56). (1) $\Rightarrow$ (6) and (6) $\Rightarrow$ (5) are evident.

(1) $\Rightarrow$ (7) follows directly from Proposition 2.

(1) $\Rightarrow$ (2) Let $x_n \xrightarrow{o} 0$ in E. If $q_T(x_n) \nrightarrow 0$, then passing to a subsequence, we assume that there exists $\varepsilon_0 > 0$, such that $q_T(x_n) \geq \varepsilon_0$ for every n. According to Lemma 5,

$$q_T(x_n) = \sup\{|y'T|(|x_n|) :\| y' \|\leq 1\}.$$

Therefore, for all n, we can find some $y'_n \in B_{F'}$ satisfying $|y'_n T|(|x_n|) \geq \frac{\varepsilon_0}{2}$. According to [6] (Theorem 1.18), $|y'_n T|(x_n) = \sup\{|y'_n Tz| : |z| \leq |x_n|\}$. It follows that there exists $z_n \in [-|x_n|, |x_n|]$, such that $|y'_n Tz_n| \geq \frac{\varepsilon_0}{4}$. Hence, $z_n \xrightarrow{o} 0$ and $\frac{\varepsilon_0}{4} \leq |y'_n Tz_n| \leq\| Tz_n \|$ for every $n \in \mathbb{N}$. This is a contradiction. We find that $\| Q_T x_n \|= q_T(x_n) \to 0$ for every $x_n \xrightarrow{o} 0$ in E.

(2) $\Rightarrow$ (3) Suppose that $\{w_n\}$ is an order-bounded disjoint sequence in E. Since E is Dedekind σ-complete, $\bigvee_{n=1}^{\infty} w_n$ exists in E. Therefore,

$$\sum_{i=1}^{n} Q_T e_i = \bigvee_{i=1}^{n} Q_T e_i \uparrow Q_T(\bigvee_{n=1}^{\infty} w_n).$$

Note that G_T has an order-continuous norm. $Q_T(\bigvee_{n=1}^{\infty} w_n) = \sum_{n=1}^{\infty} Q_T w_n$. This implies that

$$T(\bigvee_{n=1}^{\infty} w_n) = S_T Q_T(\bigvee_{n=1}^{\infty} w_n) = S_T(\sum_{n=1}^{\infty} Q_T w_n) = \sum_{n=1}^{\infty} Tw_n.$$

(3) $\Rightarrow$ (1) Let $\{z_n\}$ be an order-bounded disjoint sequence in E. Evidently,

$$\sum_{n=1}^{\infty}(y'T)^{+}z_n = \lim_{n\to\infty}\sum_{i=1}^{n}(y'T)^{+}z_i = \lim_{n\to\infty}(y'T)^{+}\bigvee_{i=1}^{n} z_i \leq (y'T)^{+}(\bigvee_{n=1}^{\infty} z_n)$$

for any $y' \in F'$. To see $(y'T)^{+}(\bigvee_{n=1}^{\infty} z_n) \leq \sum_{n=1}^{\infty}(y'T)^{+}z_n$, put $\varepsilon > 0$. In view of [6] (Theorem 1.18), $(y'T)^{+}x = \sup\{|y'Tz| : 0 \leq z \leq x\}$, there exists $0 \leq z \leq \bigvee_{n=1}^{\infty} z_n$, such that $(y'T)^{+}(\bigvee_{n=1}^{\infty} z_n) \leq |y'Tz| + \varepsilon$. By our assumption in (3),

$$Tz = T(\bigvee_{n=1}^{\infty} z \wedge z_n) = \sum_{n=1}^{\infty} Tz \wedge z_n.$$

It follows that

$$\begin{aligned}(y'T)^{+}(\bigvee_{n=1}^{\infty} z_n) &\leq |y'Tz| + \varepsilon = |y'T(\bigvee_{n=1}^{\infty} z \wedge z_n)| + \varepsilon \\ &= |\sum_{n=1}^{\infty} y'Tz \wedge z_n| + \varepsilon \leq \sum_{n=1}^{\infty} |y'Tz \wedge z_n| + \varepsilon \\ &\leq \sum_{n=1}^{\infty}(y'T)^{+}z \wedge z_n + \varepsilon \leq \sum_{n=1}^{\infty}(y'T)^{+}z_n + \varepsilon.\end{aligned}$$

This implies that $(y'T)^{+}(\bigvee_{n=1}^{\infty} z_n) = \sum_{n=1}^{\infty}(y'T)^{+}z_n$. Similarly,

$$(y'T)^{-}(\bigvee_{n=1}^{\infty} z_n) = \sum_{n=1}^{\infty}(y'T)^{-}z_n.$$

So $|y'T|(\bigvee_{n=1}^{\infty} z_n) = \sum_{n=1}^{\infty}|y'T|z_n$. The order continuity of norm on G_T implies the series $\sum_{n=1}^{\infty} Q_T z_n$ converges in norm. Therefore, in view of Lemma 5,

$$\begin{aligned}\| Q_T \bigvee_{k=n}^{\infty} z_k \| &= \sup_{\|y'\|\leq 1} |y'T| \bigvee_{k=n}^{\infty} z_k = \sup_{\|y'\|\leq 1} \sum_{k=n}^{\infty} |y'T|z_k \\ &= \sup_{\|y'\|\leq 1} \sup_{m} \sum_{k=n}^{m} |y'T|z_k = \sup_{m} \sup_{\|y'\|\leq 1} \sum_{k=n}^{m} |y'T|z_k \\ &= \sup_{m} \| \sum_{k=n}^{m} Q_T z_k \| \leq \| \sum_{k=n}^{\infty} Q_T z_k \| .\end{aligned}$$

Let $x_n \xrightarrow{o} 0$ in E. Suppose that $|x_n| \leq p_n \downarrow 0$. According to Lemma 4, for every $\varepsilon > 0$, there exists a positive disjoint sequence $\{w_n\}$, such that $0 \leq p_n \leq \bigvee_{i=n}^{\infty} w_i + \varepsilon p_1$ and $\bigvee_{i=n}^{\infty} w_i \leq \frac{1}{\varepsilon} p_n$ for every n. Therefore,

$$\| Q_T p_n \| \leq \| Q_T \bigvee_{i=n}^{\infty} w_i \| + \varepsilon \| Q_T p_1 \| \leq \| \sum_{i=n}^{\infty} Q_T w_i \| + \varepsilon \| Q_T p_1 \| .$$

This implies that $\limsup_{n\to\infty} \| Q_T x_n \| \leq \limsup_{n\to\infty} \| Q_T p_n \| \leq \varepsilon \| Q_T p_1 \|$ for every $\varepsilon > 0$. Therefore, $\lim_{n\to\infty} \| Q_T x_n \| = 0$, hence $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} S_T Q_T x_n = 0$.

(4) $\Rightarrow$ (1) By this assumption, T is an order weakly compact operator and $y'T$ is a σ-order continuous operator for every $y' \in F'$. Therefore, according to [5] (Theorem 3.4.4), Q_T is order weakly compact and $|y'T|$ is a σ-order continuous operator for every $y' \in F'$. We obtain that the series $\sum_{n=1}^{\infty} Q_T z_n$ converges in norm for every order-bounded disjoint

sequence $\{z_n\}$ in E. Given any an order-bounded disjoint sequence $\{z_n\}$ in E^+, we also have $|y'T|(\sum_{i=1}^{n} z_i) \uparrow |y'T|(\bigvee_{n=1}^{\infty} z_n)$. This implies that

$$\sum_{n=1}^{\infty} |y'T|z_n = \lim_{n\to\infty} |y'T|(\sum_{i=1}^{n} z_i) = |y'T|(\bigvee_{n=1}^{\infty} z_n).$$

The rest of the verification is contained in (3) $\Rightarrow$ (1).

(7) $\Rightarrow$ (6) Suppose that T is order weakly compact and σ-order continuous. Let $x_n \downarrow 0$ in E. Then, $x_1 - x_n \uparrow x_1$, according to [5] (Theorem 3.4.4), there exists $y \in F$, such that $T(x_1 - x_n) \xrightarrow{\|\cdot\|} y$. Then, the σ-order continuity of T implies $T(x_1 - x_n) \xrightarrow{o} Tx_1$. Therefore, $y = Tx_1$. We obtain that $Tx_n \xrightarrow{\|\cdot\|} 0$. □

Remark 2. *In the above theorem, since G_T has an order continuous norm, T is σ-order-to-norm continuous. It is also equivalent to Q_T is σ-order continuous.*

Let $L_{\sigma on}(E, F)$ be the collection of all order-bounded σ-order-to-norm continuous operators from E to F. We are interested in when $L_{\sigma on}(E, F)$ is an ideal in $L_b(E, F)$. Evidently, if F has an order-continuous norm, then, as demonstrated by Proposition 2 and [1] (Proposition 3), $L_{\sigma on}(E, F) = L_b(E, F)$. Recall that a Banach lattice E is said to be order-bounded AM if the series $\sum_{n=1}^{\infty} |x_n|$ converges in E for every unconditional convergent series $\sum_{n=1}^{\infty} x_n$ in E with $\{\sum_{i=1}^{n} x_i\}$ order bounded. And E has a weakly Fatou property if there exists $r > 0$ such that for every net $x_\alpha \uparrow x$ in E, it follows that $\| x_\alpha \| \leq r \sup_\alpha \| x_\alpha \|$.

Corollary 1. *Let Banach lattice E, F be Dedekind complete and $T : E \to F$ be an order-bounded operator. If one of the following statements holds*

(1) F is an AM-space with an order unit.
(2) F is order bounded AM and F has a weakly Fatou property.

Thus, T is σ-order-to-norm continuous if and only if $|T|$ is σ-order-to-norm continuous. In this case, $L_{\sigma on}(E, F)$ is an ideal in $L_b(E, F)$.

Proof.

(1) Suppose that F is an AM-space with an order unit and $T : E \to F$ is a σ-order-to-norm continuous operator. According to Theorem 3, $T : E \to F$ admits a factorization through an order-continuous Banach lattice G_T,

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "Q_T"'] & & F \\ & G_T \arrow[ur, "S_T"'] & \end{tikzcd}$$

where the factor Q_T is a σ-order continuous lattice homomorphism and S_T is a norm-bounded operator. Since F has an order unit, S_T is order bounded. Let $x_n \xrightarrow{o} 0$ in E^+. We have $Q_T x_n \xrightarrow{ru} 0$. Therefore,

$$|T|x_n = |S_T Q_T|x_n \leq |S_T|Q_T x_n \xrightarrow{ru} 0.$$

(2) For every $x \in E^+$, we write the restriction of T to I_x as T_x, where I_x is the ideal generated by x in E. Evidently, T is order weakly compact if and only if T_x is weakly compact for every $x \in E^+$. According to S. Kakutani' theorem (see [6], Theorem 4.29), for every $x \in E^+$, there is Hausdorff compact topological space K, such that I_e is isomorphic to $C(K)$. In view of [15] (Theorem 3.3), $|T_x|$ is weakly compact for every $x \in E^+$. Therefore, $|T|$ is order weakly compact. It follows from the order

continuity of T that $|T|$ is order continuous; see [6] (Theorem 1.56). By Theorem 3, $|T|$ is σ-order-to-norm continuous.

□

In general, $L_{\sigma on}(E,F)$ may not be a band in $L_b(E,F)$. For example, let P_n be the band projection from l_∞ onto $span\{e_n\}$. Then, $\sum_{k=1}^n P_k \uparrow$ and is contained in $L_{\sigma on}(l_\infty, l_\infty)$. However, $\sup_n \sum_{k=1}^n P_k \uparrow id_\infty$, the identity operator on l_∞, is not σ-order-to-norm continuous.

Theorem 4. *Let Banach lattice E be Dedekind complete. The followings are equivalent.*

(1) E has order continuous norm.
(2) Every order-bounded operator from E to l_∞ is σ-order-to-norm continuous.
(3) Every order-bounded σ-order continuous operator from E to l_∞ is σ-order-to-norm continuous.

Proof. $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are trivial.
$(3) \Rightarrow (1)$ If E is not order continuous, then there exists an order bounded sequence $\{x_n\}$ of disjoint elements such that $\lambda = \inf\limits_n \parallel x_n \parallel > 0$. As shown by [14] (Proposition 0.5.5), the sublattice

$$F = \{(o)\sum_{n=1}^{\infty} t_n x_n : (t_n) \in l_\infty\}$$

is norm closed regular and order isomorphic to l_∞, where $(o)\sum_{n=1}^{\infty} t_n x_n$ is the order limit of sequence $\{\sum_{k=1}^{n} t_k x_k\}$. There exists an interval preserving projection P from E onto F; see [14] (Proposition 0.5.5). We can find $0 < y_n' \in E'$ satisfying $y_n'(x_n) = \parallel x_n \parallel$ and $\parallel y_n' \parallel = 1$. Let P_n be the band projection onto the band $[\{x_n\}]$ generated by $\{x_n\}$ in E. Set $x_n' = \frac{1}{\|x_n\|} y_n' P_n$, then $x_n' \wedge x_m' = 0$ for $n \neq m$, $\sup\limits_n \parallel x_n' \parallel \leq \frac{1}{\lambda}$ and $x_n'(x_m) = \delta_{nm}$, where $\delta_{nm} = 1, n = m$ and $\delta_{nm} = 0, n \neq m$. The operator $P : E \to F$ is defined by

$$Pz = (o)\sum_{n=1}^{\infty} x_n'(z) x_n$$

for every $z \in E$. Clearly, P is a projection from E onto F. Now, define the lattice isomorphism $S : F \to l_\infty$ by $Su = (t_n)$ for every $u = (o)\sum_{n=1}^{\infty} t_n x_n \in F$. Suppose that $T = SP$. Clearly, T is σ-order continuous. Note that

$$Tx_n = SPx_n = S(\frac{x_n}{\parallel x_n \parallel}) = \frac{e_n}{\parallel x_n \parallel};$$

hence, $\{Tx_n\}$ does not converge to 0 in norm. This proves that the operator T is σ-order continuous but σ-order-to-norm continuous. □

Suppose that (Ω, Σ, μ) is a measure space and $L_0(\mu)$ is the Riesz space of all real valued μ-measurable functions f on Ω. Now, our aim comes down to studying the relationship between σ-order-to-norm continuous operators and order-to-norm continuous operators on the subspaces of $L_0(\mu)$. Recall that a Riesz subspace Z of X is called a regular sublattice of X if $x_\alpha \xrightarrow{o} 0$ in Z implies $x_\alpha \xrightarrow{o} 0$ in X.

Lemma 6. *Let E be a Dedekind σ-complete Banach lattice which is lattice isomorphic to a regular sublattice of $L_0(\mu)$. If (Ω, Σ, μ) is a finite measure space i.e., $\mu(\Omega) < +\infty$, then for every $x_0 \in E^+$ and every $\{y_\alpha\} \subset E^+$ satisfying $y_\alpha \downarrow$ and $P_{y_\alpha} x_0 \downarrow 0$, there exists a subsequence $\{P_{y_{\alpha_n}} x_0\}$ of $\{P_{y_\alpha} x_0\}$, such that $P_{y_{\alpha_n}} x_0 \downarrow 0$.*

Proof. Let $x_0 \in E^+$. Suppose $\{y_\alpha\}$ is a net in E^+ satisfying $y_\alpha \downarrow$ and $P_{y_\alpha} x_0 \downarrow 0$. Identifying E with its copy in $L_0(\mu)$, put $A = \{\omega \in \Omega : x_0(\omega) > 0\}$. Evidently, $x_0 = x_0 \chi_A$, where χ_A is the characteristic function of A. Fixed α, it follows that $x_0 \wedge n y_\alpha \uparrow P_{y_\alpha} x_0$ in E and $x_0 \chi_A \wedge n y_\alpha = x_0 \wedge n y_\alpha \uparrow x_0 \chi_{A_\alpha \cap A}$ in $L_0(\mu)$, where $A_\alpha = \{\omega \in \Omega : y_\alpha(\omega) > 0\}$. Since E is a regular sublattice of $L_0(\mu)$, for every α, we have $P_{y_\alpha} x_0 = x_0 \chi_{A_\alpha} = x_0 \chi_{A_\alpha \cap A}$.

Therefore, $x_0\chi_{A_\alpha\cap A} = P_{y_\alpha}x_0 \downarrow 0$ in E. Then, $\chi_{A_\alpha\cap A} \downarrow 0$ in $L_0(\mu)$; hence, $\chi_{A_\alpha\cap A} \downarrow 0$ in $L_1(\mu)$, where $L_1(\mu)$ represents real absolutely integrable functions on a measure space Ω. By the continuity of norm on $L_1(\mu)$, $\lim_\alpha \| \chi_{A_\alpha\cap A} \|_1 = \lim_\alpha \int_\Omega \chi_{A_\alpha\cap A} d\mu = 0$. Therefore, there exists an increasing subsequence $\{\alpha_n\}$ of $\{\alpha\}$, such that $\| \chi_{A_{\alpha_n}} \|_1 \downarrow 0$. It follows that $\chi_{A_{\alpha_n}\cap A} \downarrow 0$ in $L_1(\mu)$. This implies that $x_0\chi_{A_{\alpha_n}\cap A} \downarrow 0$ in $L_0(\mu)$. We obtain that $P_{y_{\alpha_n}}x_0 = x_0\chi_{A_{\alpha_n}\cap A} \downarrow 0$ in E. □

Theorem 5. *Let E be a Dedekind σ-complete Banach lattice and F a Dedekind complete Banach lattice. $T : E \to F$ is an order-bounded operator. If one of the following conditions holds:*

(H1) (Ω, Σ, μ) is a finite measure space and E is lattice isomorphic to a regular sublattice of $L_0(\mu)$;
(H2) (Ω, Σ, μ) is a σ-finite measure space and E is lattice isomorphic to an ideal of $L_0(\mu)$;

then the following statements hold.

(1) *If T is a σ-order-to-norm continuous operator, then the following assertions hold.*
   (1a) *$E/N(T)$ is an ideal in G_T.*
   (1b) *Q_T is order continuous and $N(T)$ is a band in E.*
   (1c) *T is order continuous.*
(2) *T is σ-order-to-norm continuous if and only if T is order weakly compact and order continuous.*
(3) *T is σ-order-to-norm continuous if and only if T is order-to-norm continuous.*
(4) *Every σ-order continuous functional on E is order continuous.*

Proof. (1a) If $v = Q_Tx \in E/N(T)$, then $|v| = Q_T|x| \in E/N(T)$. Put $v \in G_T$ and $0 \le v \le Q_Tx$ (where $x \ge 0$). Note that $Q_Tx_n \xrightarrow{\|\cdot\|} v$ for some sequence $\{x_n\}$ in E^+. Passing to a subsequence, we can assume that $Q_Tx_n \xrightarrow{ru} v$ in G_T. It follows that $\bigvee_{k=n}^\infty Q_Tx_k \downarrow v$ in G_T. According to Remark 2, Q_T is σ-order continuous. We have

$$\bigvee_{k=n}^{\infty}(Q_Tx \wedge Q_Tx_k) = \bigvee_{k=n}^{\infty} Q_T(x \wedge x_k) = Q_T(\bigvee_{k=n}^{\infty}(x \wedge x_k)) \downarrow Q_T(\bigwedge_{n=1}^{\infty}\bigvee_{k=n}^{\infty}(x \wedge x_k)).$$

At the same time,

$$\bigvee_{k=n}^{\infty}(Q_Tx \wedge Q_Tx_k) = Q_Tx \wedge \bigvee_{k=n}^{\infty} Q_Tx_k \downarrow Q_Tx \wedge v = v.$$

This implies that $v = Q_T(\bigwedge_{n=1}^\infty \bigvee_{k=n}^\infty (x \wedge x_k)) \in E/N(T)$.

(1b) Firstly, in two cases, we will prove that Q_T is order continuous. To see this, it is sufficient to prove that $\bigwedge_\alpha |Q_Tx_\alpha| = 0$ for every $x_\alpha \downarrow 0$ in E. Let $x_\alpha \downarrow 0$ in E. For a fixed α_0 and given any $\varepsilon > 0$, by Lemma 4, there exists a decreasing net $\overline{x_\alpha}$, such that $P_{\overline{x_\alpha}}x_{\alpha_0} \le \frac{1}{\varepsilon}x_\alpha$ and $0 \le x_\alpha \le P_{\overline{x_\alpha}}x_{\alpha_0} + \varepsilon x_{\alpha_0}$ for every $\alpha \ge \alpha_0$.

Case 1. *The assertion (H2) is true.*

According to Lemma 6, there exists a subsequence $\{\alpha_n\}$ of $\{\alpha\}$, such that $P_{\overline{x_{\alpha_n}}}x_{\alpha_0} \downarrow 0$. Therefore, based on the statements (2) and (3) of Theorem 3, $Q_TP_{\overline{x_{\alpha_n}}}x_{\alpha_0} \xrightarrow{\|\cdot\|} 0$. For every $\varepsilon > 0$,

$$\limsup_{n\to\infty} \| Q_Tx_{\alpha_n} \| \le \limsup_{n\to\infty} \| Q_TP_{\overline{x_{\alpha_n}}}x_{\alpha_0} + \varepsilon Q_Tx_{\alpha_0} \| \le \varepsilon \| Q_Tx_{\alpha_0} \|$$

It follows that $\lim_{n\to\infty} \| Q_Tx_{\alpha_n} \| = 0$. Hence,

$$\| \bigwedge_\alpha |Q_Tx_\alpha| \| \le \| \bigwedge_{n=1}^{\infty} |Q_Tx_{\alpha_n}| \| \le \| Q_Tx_{\alpha_n} \| \to 0.$$

We have $\bigwedge_\alpha |Q_T x_\alpha| = 0$. This implies that $\bigwedge_\alpha |Q_T x_\alpha| = 0$ for every $x_\alpha \downarrow 0$ in E. According to [6] (Theorem 1.56), Q_T is order continuous.

Case 2. *The assertion $(H1)$ is true.*

Since (Ω, Σ, μ) is a σ-finite measure space, there is a disjoint sequence $\{B_m\}$ of subsets of Ω, such that $\bigcup_{m=1}^{\infty} B_m = \Omega$ and $\mu(B_m) < +\infty$ for all m. Identifying E with its copy in $L_0(\mu)$, since E is an ideal in $L_0(\mu)$, we have $x_{\alpha_0}\chi_{B_m} \in E$ for every m. Then, $x_{\alpha_0} = \bigvee_{m=1}^{\infty} x_{\alpha_0}\chi_{B_m}$. For every α, put $A_\alpha = \{\omega \in \Omega : \overline{x_\alpha}(\omega) > 0\}$. Hence,

$$P_{\overline{x_\alpha}} x_{\alpha_0} = x_{\alpha_0}\chi_{A_\alpha} = \bigvee_{m=1}^{\infty} x_{\alpha_0}\chi_{A_\alpha \cap B_m}$$

for every α. Fixed $m \in \mathbb{N}$, $x_{\alpha_0}\chi_{A_\alpha \cap B_m} \downarrow 0$ in $L_0(B_m, \mu)$ since $x_{\alpha_0}\chi_{A_\alpha \cap B_m} \le P_{\overline{x_\alpha}} x_{\alpha_0} \downarrow 0$ in $L_0(\mu)$ for every m. Note that the band generated $[x_{\alpha_0}\chi_{B_m}]$ by $x_{\alpha_0}\chi_{B_m}$ in E is contained in $L_0(B_m, \mu)$. Then, $x_{\alpha_0}\chi_{A_\alpha \cap B_m} \downarrow 0$ in $[x_{\alpha_0}\chi_{B_m}]$. In the view of Case 1, $Q_T x_{\alpha_0}\chi_{A_\alpha \cap B_m} \downarrow 0$. Let $v = \bigwedge_\alpha \bigvee_{m=1}^{\infty} Q_T x_{\alpha_0}\chi_{A_\alpha \cap B_m}$. Without loss of generality, assume that $\alpha \ge \alpha_0$ for every α.

$$v \wedge Q_T x_{\alpha_0}\chi_{A_{\alpha_0} \cap B_m} \le Q_T x_{\alpha_0}\chi_{A_{\alpha_0} \cap B_m} \wedge \bigvee_{m=1}^{\infty} Q_T x_{\alpha_0}\chi_{A_\alpha \cap B_m} = Q_T x_{\alpha_0}\chi_{A_\alpha \cap B_m} \downarrow 0.$$

We have $v \wedge Q_T x_{\alpha_0}\chi_{A_{\alpha_0} \cap B_m} = 0$ for all m hence $v = \bigvee_{m=1}^{\infty} v \wedge Q_T x_{\alpha_0}\chi_{A_{\alpha_0} \cap B_m} = 0$ i.e., $\bigvee_{m=1}^{\infty} Q_T x_{\alpha_0}\chi_{A_\alpha \cap B_m} \downarrow 0$. Note that Q_T is σ-order continuous. We obtain

$$Q_T P_{\overline{x_\alpha}} x_{\alpha_0} = Q_T \bigvee_{m=1}^{\infty} x_{\alpha_0}\chi_{A_\alpha \cap B_m} = \bigvee_{m=1}^{\infty} Q_T x_{\alpha_0}\chi_{A_\alpha \cap B_m} \downarrow 0.$$

By the order continuity of norm on G_T, $\lim_\alpha \| Q_T P_{\overline{x_\alpha}} x_{\alpha_0} \| = 0$. For every $\varepsilon > 0$,

$$\limsup_\alpha \| Q_T x_\alpha \| \le \limsup_\alpha \| Q_T P_{\overline{x_\alpha}} x_{\alpha_0} + \varepsilon Q_T x_{\alpha_0} \| \le \varepsilon \| Q_T x_{\alpha_0} \| .$$

Hence, $\lim_\alpha \| Q_T x_\alpha \| = 0$. And $\| \bigwedge_\alpha |Q_T x_\alpha| \| \le \| Q_T x_\alpha \|$ implies that $\bigwedge_\alpha |Q_T x_\alpha| = 0$. We also obtain $\bigwedge_\alpha |Q_T x_\alpha| = 0$ for every $x_\alpha \downarrow 0$ in E. According to [6] (Theorem 1.56), Q_T is order continuous.

Next, we prove that $N(T)$ is a band in E. Since $q_T(\cdot)$ is a Riesz seminorm on E, $N(T)$ is an ideal in E. If $0 \le z_\alpha \uparrow z$ in E with $z_\alpha \in N(T)$, then $Q_T z_\alpha \uparrow Q_T z$. Note that G_T has an order continuous norm. $0 = q_T(x_\alpha) = \| Q_T z_\alpha \| \to \| Q_T z \| = q_T(z)$. We have $q_T(z) = 0$ hence $z \in N_T$. This implies taht $N(T)$ is a band in E.

(1c) According to the proof of (1b), Q_T is order continuous; hence, it is also order-to-norm continuous. If $x_\alpha \downarrow 0$ in E, then $Q_T x_\alpha \xrightarrow{\|\cdot\|} 0$ in G_T. We have $\| \bigwedge_\alpha |T x_\alpha| \| \le \| T x_\alpha \| = \| S_T Q_T x_\alpha \| \to 0$. Hence, $\bigwedge_\alpha |T x_\alpha| = 0$. According to [6] (Theorem 1.56), T is order continuous.

(2) and (3) are obtained by (1) and Theorem 3. (4) is a simple indirect argument of (3). $\square$

Recall that a net $\{x_\alpha\}$ in E is said to be **unbounded order convergent** to 0 (or *uo*-converges to 0), written as $x_\alpha \xrightarrow{uo} 0$, if $|x_\alpha| \wedge u \xrightarrow{o} 0$ for every $u \in E^+$. The details on unbounded order convergence can be found in [16,17]. As an application of σ-order-to-norm continuous operators, we give a characterization of σ-order continuous M-weakly compact operators.

Theorem 6. *Let E be Dedekind σ-complete Banach lattice and F Dedekind complete. If $T : E \to F$ is an order-bounded operator, then the following statements are equivalent.*

(1) $T x_n \xrightarrow{\|\cdot\|} 0$ *for every norm-bounded* $x_n \xrightarrow{uo} 0$ *in* E.

(2) *T is M-weakly compact and T is σ-order continuous.*

Proof. $(1) \Rightarrow (2)$ is obvious in Proposition 2.

$(2) \Rightarrow (1)$ This is similar to the proof of $(1) \Rightarrow (2)$ in Theorem 3 (replace "$0 \leq x_n \xrightarrow{o} 0$ in E" with "disjoint $\{x_n\}$ in B_E") in which Q_T is M-weakly compact. Note that every M-weakly compact operator is order weakly compact. In view of Theorem 3 and Theorem 5, Q_T is order-to-norm continuous. According to [6] (Theorem 5.60), for every $\varepsilon > 0$, there exists $z \in E^+$ such that $\| Q_T(|x| - z)^+ \| < \varepsilon$ for every $x \in B_E$. Given any $\{x_n\} \subset B_E$ with $x_n \xrightarrow{uo} 0$, we have $|x_n| \wedge z \xrightarrow{o} 0$; hence,

$$\begin{aligned} \limsup_{n\to\infty} \| Q_T x_n \| &= \limsup_{n\to\infty} \| Q_T((|x_n| - z)^+ + |x_n| \wedge z) \| \\ &\leq \varepsilon + \limsup_{n\to\infty} \| Q_T(|x_n| \wedge z) \| = \varepsilon \end{aligned}$$

for every $\varepsilon > 0$. This implies that $Q_T x_n \xrightarrow{\|\cdot\|} 0$. We obtain that $Tx_n = S_T Q_T x_n \xrightarrow{\|\cdot\|} 0$. The proof is complete. □

Recall that $T : E \to F$ is said to be L-weakly compact if every disjoint sequence in the solid hull $sol(TB_E)$ of TB_E converges to zero. Meyer-Nieberg prove in [6] (Theorem 5.64) that the notions of L- and M-weakly compact operators are in duality to each other. By [6] (Theorem 1.73), every order-bounded operator has an order-continuous adjoint. Note that the norm dual of a Banach lattice is a Dedekind complete Banach lattice. We are now in the position to give a new characterization of L-weakly compact operators.

Corollary 2. *Let E, F be Banach lattices and $T : E \to F$ be an order-bounded operator. The following statements are equivalent.*

(1) *$T'x'_n \xrightarrow{\|\cdot\|} 0$ for every norm-bounded $x'_n \xrightarrow{uo} 0$ in E'.*
(2) *T is L-weakly compact.*

In [17] (Theorem 5), A.W. Wickstead gave a characterization of the order continuity of norms on E and E' in terms of unbounded order convergent nets. To be precise, E and E' have order continuous norms if and only if every norm-bounded net in E which uo-converges to zero must converge weakly to zero. It does not suffice to consider only sequences in this equivalence, as is shown by the space of all continuous real-valued functions on the one point compactification of an uncountable discrete space. A norm-bounded uo-convergent sequence there must converge in norm, so certainly weakly ([18], Example 33.1).

Theorem 7. *Let E be a Banach lattice. The following statements are equivalent.*

(1) *E is Dedekind σ-complete and $x_n \xrightarrow{w} 0$ for every norm-bounded $x_n \xrightarrow{uo} 0$ in E.*
(2) *E and E' have order-continuous norms.*

Proof. $(2) \Rightarrow (1)$ is by [17] (Theorem 5). $(1) \Rightarrow (2)$ Based on the assumption of (2), for every $x' \in E'$, $x'(x_n) \to 0$ for every norm-bounded $x_n \xrightarrow{uo} 0$. According to Theorem 6, for every $x' \in E'$, x' is M-weakly compact and σ-order continuous. In the view of [11] (Theorem 39.5), E has a σ-order continuous norm if and only if for each $x' \in (E')^+$, $x'(x_n) \downarrow 0$ whenever $x_n \downarrow 0$ in E. We find that E has a σ-order continuous norm. Note that E is Dedekind σ-complete. According to [14] (Theorem 1.1), E has an order-continuous norm. It is proved in [10] (Theorem 116.1) that E' has an order-continuous norm if and only if $x_n \xrightarrow{w} 0$ for every norm-bounded disjoint sequence $\{x_n\}$ in E. Note that every disjoint sequence must uo-converge to zero. Again, based on the assumption of (2), E' has an order-continuous norm. □

4. Conclusions and Recommendation

This paper under review aims to further explore σ-order-to-norm continuous and order-to-norm continuous operators. To prove that the σ-order-to-norm continuous operator can be defined using the order relation, we introduce a basic property of Banach lattices, namely the property (h). Theorem 3 plays an important role in this paper, in which we present a number of nice characterizations for a σ-order-to-norm continuous operator. This result is also the basis for our extension on Wickstead's results on the order continuity of norms on E and E'.

This research can be categorized under the topic "Operators Acting on Banach Lattices and Related Applications". Our future work is to extend these results to more general spaces (e.g., topological Riesz spaces) as well as applying them to the field of non-linear analysis. In [19], Çevik and Altun introduce and investigate a class of spaces equipped with vetor metrics, mainly vector metric spaces. One of the research components could be the application of relative uniform convergence to the fixed point theory on vector metric spaces. Based on [20–22], we could consider replacing order convergence, in the definition of vectorial convergence (see [19], Definition 2.4), with relative uniform convergence to obtain more of the properties of vectorially continuous functions (see [20], Definition 3).

Author Contributions: Methodology, F.Z. and Z.C.; validation, F.Z. and H.S.; investigation, F.Z. and H.S.; data curation, F.Z. and H.S.; project administration, H.S. and Z.C.; writing—original draft, F.Z. and Z.C.; writing—review and editing, Z.C. and H.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bedingfield, S.E.; Wirth, A. Norm and order properties of Banach lattices. *J. Aust. Math. Soc.* **1980**, *29*, 331–336. [CrossRef]
2. Taylor, M.A.; Troitsky, V.G. Bibasic sequences in Banach lattices. *J. Funct. Anal.* **2020**, *278*, 108448. [CrossRef]
3. Niculescu, C. Order σ-continuous operators on Banach lattices. Banach Space Theory and its Applications. In Proceedings of the First Romanian-GDR Seminar, Bucharest, Romania, 31 August–6 September 1981; Springer: Berlin/Heidelberg, Germany, 2006; pp. 188–201.
4. Jalili, S.A.R.; Azar, K.H.; Moghimi, M.B.F. Order-to-topology continuous operators. *Positivity* **2021**, *25*, 1313–1322. [CrossRef]
5. Meyer-Nieberg, P. *Banach Lattices*; Springer: Berlin/Heidelberg, Germany; NewYork, NY, USA, 1991.
6. Aliprantis, C.D.; Burkinshaw, O. *Positive Operators*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2006.
7. Aliprantis, C.D.; Burkinshaw, O. *Locally Solid Riesz Spaces with Applications to Economics*; American Mathematical Society: Providence, RI, USA, 2003.
8. Luxemburg, W.A.J. *Riesz Spaces*; Elsevier: Amsterdam, The Netherlands, 2000.
9. Wirth, A. Relatively uniform Banach lattices. *Proc. Am. Math. Soc.* **1975**, *52*, 178–180. [CrossRef]
10. Zaanen, A.C. *Riesz Spaces II*; Elsevier: Amsterdam, The Netherlands, 1983.
11. Zaanen, A.C. *Introduction to Operator Theory in Riesz Spaces*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2012.
12. Dvoretzky, A.; Rogers, C. A. Absolute and unconditional convergence in normed linear spaces. *Proc. Natl. Acad. Sci. USA* **1950**, *36*, 192–197. [CrossRef] [PubMed]
13. Xiong, H.Y. On whether or not $L(E,F) = L_r(E,F)$ for some classical Banach lattices E and F. *Indag. Math.* **1984**, *87*, 267–282. [CrossRef]
14. Wnuk, W. *Banach Lattices with Order Continuous Norms*; Polish Scientific Publishers PWN: Warszawa, Poland, 1999.
15. Groenewegen, G.; van Rooij, A. The modulus of a weakly compact operator. *Math. Z.* **1987**, *195*, 473–480. [CrossRef]
16. Gao N.; Troitsky, V.G.; Xanthos, F. Uo-convergence and its applications to Cesro means in Banach lattices. *Isr. J. Math.* **2017**, *220*, 649–689. [CrossRef]
17. Wickstead, A.W. Weak and unbounded order convergence in Banach lattices. *J. Aust. Math. Soc.* **1977**, *24*, 312–319. [CrossRef]
18. Luxemburg, W.A.J.; Zaanen, A.C. Notes on Banach function spaces. XI. *Indag. Math.* **1964**, *26*, 507–518. [CrossRef]
19. Çevik, C.; Altun, I. Vector metric space and some properties. *Topol. Met. Nonlinear Anal.* **2009**, *34*, 375–382. [CrossRef]
20. Çevik, C. On continuity functions between vector metric spaces. *J. Funct. Spaces* **2014**, *2014*, 753969. [CrossRef]

21. Rahimi, H.; Abbas, M.; Soleimani Rad, G. Common fixed point results for four mappings on ordered vector metric spaces. *Filomat* **2015**, *29*, 865–878. [CrossRef]
22. Sahin, H. Best proximity point theory on vetor metric spaces. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2021**, *70*, 130–142. [CrossRef]