mathematics

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Weak Nearly Sasakian and Weak Nearly Cosymplectic Manifolds

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Abstract: Weak contact metric structures on a smooth manifold, introduced by V. Rovenski and R. Wolak in 2022, have provided new insight into the theory of classical structures. In this paper, we define new structures of this kind (called weak nearly Sasakian and weak nearly cosymplectic and nearly Kähler structures), study their geometry and give applications to Killing vector fields. We introduce weak nearly Kähler manifolds (generalizing nearly Kähler manifolds), characterize weak nearly Sasakian and weak nearly cosymplectic hypersurfaces in such Riemannian manifolds and prove that a weak nearly cosymplectic manifold with parallel Reeb vector field is locally the Riemannian product of a real line and a weak nearly Kähler manifold.

Keywords: weak nearly Sasakian manifold; weak nearly cosymplectic manifold; Killing vector field; hypersurface; weak nearly Kähler manifold

MSC: 53C15; 53C25; 53D15

1. Introduction

Nearly Kähler manifolds (M, J, g) are defined by the condition that only the symmetric part of ∇J vanishes, in contrast to the Kähler case where $\nabla J = 0$. Nearly Sasakian and nearly cosymplectic manifolds $M(\varphi, \xi, \eta, g)$ are defined (see [1,2]) using a similar condition—by a constraint only on the symmetric part of φ—starting from Sasakian and cosymplectic manifolds, respectively:

$$(\nabla_X \varphi)X = \begin{cases} g(X,X)\,\xi - \eta(X)X\,, & \text{nearly Sasakian.} \\ 0\,, & \text{nearly cosymplectic.} \end{cases} \tag{1}$$

Here, φ is a $(1,1)$-tensor, ξ is a vector field (called Reeb vector field) and η is a 1-form, satisfying $\varphi^2 = -\mathrm{id}_{TM} + \eta \otimes \xi$ and $\eta(\xi) = 1$.

These two classes of odd-dimensional counterparts of nearly Kähler manifolds play a key role in the classification of almost-contact metric manifolds, see [3]. They also appeared in the study of harmonic almost-contact structures: a nearly cosymplectic structure, identified with a section of a twistor bundle, defines a harmonic map, see [4]. The Reeb vector field ξ of a nearly Sasakian and a nearly cosymplectic structure is a unit Killing vector field. The influence of constant-length Killing vector fields on Riemannian geometry has been studied by many authors, e.g., [5,6].

In dimensions greater than 5, every nearly Sasakian manifold is Sasakian, see [7], and a nearly cosymplectic manifold M^{2n+1} is locally the Riemannian product $\mathbb{R} \times F^{2n}$ or $B^5 \times F^{2n-4}$, where F is a nearly Kähler manifold and B is a nearly cosymplectic manifold, see [8]. Moreover, in dimension 5, any nearly cosymplectic manifold is Einsteinian with positive scalar curvature, see [8]. There are integrable distributions with totally geodesic leaves in a nearly Sasakian manifold, which are either Sasakian or five-dimensional nearly Sasakian manifolds, see [8,9].

In [10–12], we introduced and studied metric structures on a smooth manifold that generalize the almost-contact, Sasakian, cosymplectic, etc., metric structures. Such so-called "weak" structures (the complex structure on the contact distribution is replaced by a

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Citation: Rovenski, V. Weak Nearly Sasakian and Weak Nearly Cosymplectic Manifolds. *Mathematics* **2023**, *11*, 4377. https://doi.org/10.3390/math11204377

Academic Editor: Marian Ioan Munteanu

Received: 14 September 2023
Revised: 18 October 2023
Accepted: 20 October 2023
Published: 21 October 2023

nonsingular skew-symmetric tensor) made it possible to take a new look at the theory of classical structures and find new applications.

In this paper, we define new structures of this kind and study their geometry. In Section 2, following the introductory Section 1, we recall some results regarding weak almost-contact manifolds. In Section 3, we define weak nearly Sasakian and weak nearly cosymplectic structures and study their geometry. In Section 4, we define weak nearly Kähler manifolds (generalizing nearly Kähler manifolds), characterize weak nearly Sasakian and weak nearly cosymplectic hypersurfaces in such Riemannian spaces and prove that a weak nearly cosymplectic manifold with parallel Reeb vector field is locally the Riemannian product of a real line and a weak nearly Kähler manifold.

The proofs use the properties of new tensors, as well as classical constructions.

2. Preliminaries

A weak almost-contact structure on a smooth manifold M^{2n+1} $(n \geq 1)$ is a set (φ, Q, ξ, η), where φ is a $(1,1)$-tensor, Q is a nonsingular $(1,1)$-tensor, ξ is a vector field (called Reeb vector field) and η is a 1-form on TM, satisfying the following, see [10,11],

$$\varphi^2 = -Q + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad Q\,\xi = \xi. \tag{2}$$

According to (2), $\ker \eta$ is a $2n$-dimensional distribution. Assume that $\ker \eta$ is φ-invariant:

$$\varphi X \in \ker \eta, \quad \forall\, X \in \ker \eta, \tag{3}$$

as in the classical theory [13], where $Q = \mathrm{id}_{TM}$. By the first equation in (2) and (3), $\ker \eta$ is invariant for Q, $Q(\ker \eta) = \ker \eta$, and the following is true:

$$\varphi\,\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta \circ Q = \eta, \quad [Q,\, \varphi] := Q \circ \varphi - \varphi \circ Q = 0.$$

The "small" (1,1)-tensor $\widetilde{Q} = Q - \mathrm{id}_{TM}$ is a measure of the difference between a weakly contact structure and a contact one. Note that

$$[\widetilde{Q}, \varphi] := \widetilde{Q} \circ \varphi - \varphi \circ \widetilde{Q} = 0, \qquad \eta \circ \widetilde{Q} = 0, \qquad \widetilde{Q}\,\xi = 0.$$

The weak almost-contact structure (φ, Q, ξ, η) on a manifold M will be called *normal* if the following tensor $N^{(1)}$ is identically zero:

$$N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) + 2\, d\eta(X, Y)\, \xi, \quad X, Y \in \mathfrak{X}_M.$$

Here, $d\eta(X, Y) = \frac{1}{2}\, \{X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])\}$ is the exterior derivative, and the Nijenhuis torsion $[\varphi, \varphi]$ of φ is given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \quad X, Y \in \mathfrak{X}_M. \tag{4}$$

If there exists a Riemannian metric g on M such that

$$g(\varphi X, \varphi Y) = g(X, Q\,Y) - \eta(X)\,\eta(Y), \quad X, Y \in \mathfrak{X}_M, \tag{5}$$

then $(\varphi, Q, \xi, \eta, g)$ is called a *weak almost-contact metric structure* on M.

A weak almost-contact manifold $M(\varphi, Q, \xi, \eta)$ endowed with a compatible Riemannian metric is said to be a *weak almost-contact metric manifold* and is denoted by $M(\varphi, Q, \xi, \eta, g)$. Setting $Y = \xi$ in (5), we obtain, as in the classical theory, $\eta(X) = g(X, \xi)$. By (5), we obtain $g(X, Q\,X) = g(\varphi X, \varphi X) > 0$ for any nonzero vector $X \in \ker \eta$; thus, Q is positive-definite. Using the Levi-Civita connection ∇ of g, (4) can be written as

$$[\varphi, \varphi](X, Y) = (\varphi \nabla_Y \varphi - \nabla_{\varphi Y} \varphi)X - (\varphi \nabla_X \varphi - \nabla_{\varphi X} \varphi)Y. \tag{6}$$

The weak contact metric structure is defined in [10] as a weak almost-contact metric structure satisfying the equality $d\eta = \Phi$, where $\Phi(X,Y) = g(X, \varphi Y)$ $(X, Y \in \mathfrak{X}_M)$ is called the fundamental 2-form. A normal weak contact metric manifold is called a *weak Sasakian manifold*. A weak almost-contact metric structure is said to be *weak almost-cosymplectic* if both Φ and η are closed. If a weak almost-cosymplectic structure is normal, then it is called *weak cosymplectic*.

Remark 1. *Recall [13] that if an almost-contact metric structure is a normal and contact metric, then it is called Sasakian; equivalently,*

$$(\nabla_X \varphi)Y = -g(X,Y)\xi + \eta(Y)X. \tag{7}$$

A weak almost-contact manifold is weak Sasakian if and only if it is Sasakian, see Theorem 4.1 in [10]. For any weak almost-cosymplectic manifold, the ξ-curves are geodesics, see Corollary 1 in [10], and if $\nabla\varphi = 0$, then the manifold is weak cosymplectic, see Theorem 5.2 in [10].

Three tensors $N^{(2)}, N^{(3)}$ and $N^{(4)}$ are well known in the classical theory, see [13]:

$$\begin{aligned} N^{(2)}(X,Y) &= (\pounds_{\varphi X}\,\eta)(Y) - (\pounds_{\varphi Y}\,\eta)(X),\\ N^{(3)}(X) &= (\pounds_{\xi}\,\varphi)X = [\xi, \varphi X] - \varphi[\xi, X],\\ N^{(4)}(X) &= (\pounds_{\xi}\,\eta)(X) = 2\,d\eta(\xi, X). \end{aligned}$$

Remark 2 (See [12])**.** *Let $M(\varphi, Q, \xi, \eta)$ be a weak almost-contact manifold. Consider the product manifold $\bar M = M \times \mathbb{R}$, where $\mathbb{R}$ has the Euclidean basis ∂_t, and define a (1,1)-tensor field $\bar\varphi$ on $\bar M$ by putting*

$$\bar\varphi(U,\, a\,\partial_t) = (\varphi U - a\,\xi,\; \eta(U)\,\partial_t),$$

where $a \in C^\infty(M)$. Thus, $\bar\varphi(U,0) = (\varphi U, 0)$ for $U \perp \ker\varphi$, $\bar\varphi(\xi,0) = (0,\partial_t)$ and $\bar\varphi(0,\partial_t) = (-\xi, 0)$. The tensors $N^{(i)}$ $(i = 1,\ 2,\ 3,\ 4)$ appear when we derive the integrability condition $[\bar\varphi, \bar\varphi] = 0$ (i.e., vanishing of the Nijenhuis torsion of $\bar\varphi$) and express the normality condition $N^{(1)} = 0$ of (φ, Q, ξ, η) on M.

For a weak contact metric structure $(\varphi, Q, \xi, \eta, g)$, the tensors $N^{(2)}$ and $N^{(4)}$ vanish, and $N^{(3)}$ vanishes if and only if ξ is a Killing vector field, see Theorem 2.2 in [10]. Moreover, on a weak Sasakian manifold, ξ is a Killing vector field, see Proposition 4.1 in [10].

3. Main Results

Definition 1. *A weak almost-contact metric structure is called weak nearly Sasakian if*

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 2\,g(X,Y)\xi - \eta(Y)X - \eta(X)Y. \tag{8}$$

The weak almost-contact metric structure is called weak nearly cosymplectic if φ is Killing,

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0, \tag{9}$$

or, equivalently, (1) is satisfied.

Example 1. *Let a Riemannian manifold (M^{2n+1}, g) admit two nearly Sasakian structures (or, nearly cosymplectic structures) with common Reeb vector field ξ and one-form $\eta = g(\xi,\,\cdot)$. Suppose that $\varphi_1 \neq \varphi_2$ are such that $\psi := \varphi_1\,\varphi_2 + \varphi_2\,\varphi_1 \neq 0$. Then, $\varphi := (\cos t)\,\varphi_1 + (\sin t)\,\varphi_2$ for small $t > 0$ satisfies (8) (and (9), respectively) and $\varphi^2 = -\mathrm{id} + (\sin t \cos t)\,\psi + \eta\otimes\xi$. Thus, $(\varphi, Q, \xi, \eta, g)$ is a weak nearly Sasakian (and weak nearly symplectic, respectively) structure on M with $Q = \mathrm{id} - (\sin t\cos t)\,\psi$.*

The following result extends Proposition 3.1 in [2] and gives new applications to Killing vector fields.

Proposition 1. *Both on weak nearly Sasakian and weak nearly cosymplectic manifolds the vector field ξ is geodesic; moreover, if the condition (trivial when $Q = \mathrm{id}_{TM}$)*

$$(\nabla_X Q)Y = 0 \quad (X, Y \in TM,\ Y \perp \xi) \tag{10}$$

is valid, then the vector field ξ is Killing.

Proof. Putting $X = Y = \xi$ in (8) or (9), we find $(\nabla_\xi \varphi)\xi = 0$; hence, $\varphi\nabla_\xi \xi = 0$. Applying φ to this and using (2) and $\eta(\nabla_\xi \xi) = 0$, we obtain

$$0 = \varphi^2\nabla_\xi \xi = -Q\nabla_\xi \xi + \eta(\nabla_\xi \xi)\,\xi = -Q\nabla_\xi \xi.$$

Since the (1,1)-tensor Q is nonsingular, we obtain that ξ is a geodesic vector field: $\nabla_\xi \xi = 0$. Then, we calculate

$$(\nabla_\xi \eta)X = \xi(g(\xi, X)) - g(\xi, \nabla_\xi X) = g(\nabla_\xi \xi, X) = 0.$$

Thus, $\nabla_\xi \eta = 0$ is true. Since $\nabla_\xi \xi = 0$, we obtain $(\pounds_\xi g)(\xi, \cdot) = 0$. Applying the ξ-derivative to (5) and using (10) and $\nabla_\xi \eta = 0$, we find (for $Y \perp \xi$)

$$\begin{aligned} &g((\nabla_\xi \varphi)X, \varphi Y) + g(\varphi X, (\nabla_\xi \varphi)Y) = \nabla_\xi\, g(\varphi X, \varphi Y) \\ &= g(X, (\nabla_\xi Q)Y) + (\nabla_\xi \eta)(X)\,\eta(Y) + \eta(X)(\nabla_\xi \eta)(Y) = 0. \end{aligned}$$

For a weak nearly Sasakian manifold, using (8) and $\eta \circ \widetilde{Q} = 0$ yields

$$\begin{aligned} &g((\nabla_\xi \varphi)X, \varphi Y) + g(\varphi X, (\nabla_\xi \varphi)Y) \\ &= -g((\nabla_X \varphi)\xi, \varphi Y) - g(\varphi X, (\nabla_Y \varphi)\xi) \\ &\quad + g(2\,\eta(X)\,\xi - X - \xi, \varphi Y) + g(2\,\eta(Y)\,\xi - Y - \xi, \varphi X) \\ &= -g(\nabla_X \xi, \varphi^2 Y) - g(\varphi^2 X, \nabla_Y \xi) - g(X, \varphi Y) - g(Y, \varphi X) \\ &= g(\nabla_X \xi, QY) + g(QX, \nabla_Y \xi) \\ &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + g(\nabla_X \xi, \widetilde{Q}Y) + g(\widetilde{Q}X, \nabla_Y \xi) \\ &= (\pounds_\xi g)(X, Y) - g(\xi, (\nabla_X \widetilde{Q})Y) - g((\nabla_Y \widetilde{Q})X, \xi). \end{aligned}$$

Similarly, for a weak nearly cosymplectic manifold, using (9) yields

$$(\pounds_\xi g)(X, Y) - g(\xi, (\nabla_X \widetilde{Q})Y) - g((\nabla_Y \widetilde{Q})X, \xi) = 0.$$

From the above, using (10), for both cases we obtain $\pounds_\xi g = 0$, that is, ξ is Killing. □

Proposition 2. *Let $M(\varphi, Q, \xi, \eta)$ be a weak almost-contact manifold satisfying (10) and*

$$Q\,|_{\ker \eta} = \lambda\, \mathrm{id}|_{\ker \eta}$$

for a positive function $\lambda \in C^\infty(M)$. Then, $\lambda = const$ and the following is valid:

(i) *$(\tilde\varphi, \xi, \eta)$ is an almost-contact structure on M, where $\tilde\varphi$ is given by*

$$\varphi = \sqrt{\lambda}\,\tilde\varphi. \tag{11}$$

(ii) *If $(\varphi, Q, \xi, \eta, g)$ is a weak nearly Sasakian or weak nearly cosymplectic structure on M with the conditions (11) and*

$$g|_{\ker \eta} = \lambda^{-\frac{1}{2}}\,\tilde g|_{\ker \eta}, \quad g(\xi, \cdot) = \tilde g(\xi, \cdot), \tag{12}$$

then $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ is a nearly Sasakian or nearly cosymplectic structure, respectively.

Proof. (i) We obtain $(\nabla_X Q)Y = X(\lambda)\,Y$ for any $X, Y \in TM$ and $Y \perp \xi$. Thus, using these conditions, $\lambda = const$. The rest is proved in Proposition 1.2(i) in [10] with $\nu = 1$.
(ii) This follows from calculations $(\tilde{\nabla}_X\,\tilde{\varphi})X$ and the Levi-Civita connection formula. □

Example 2. *Let $M(\varphi, Q, \xi, \eta, g)$ be a three-dimensional weak almost-contact metric manifold. The tensor Q has on the plane field $\ker\eta$ in the form $\lambda\,\mathrm{id}_{\ker\eta}$ for some positive function $\lambda \in C^\infty(M)$. Suppose that the condition (10) is true, then $\lambda = const$ and this structure reduces to the almost-contact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ satisfying (11) and (12).*

Let either (8) or (9) hold for $M(\varphi, Q, \xi, \eta, g)$. By Proposition 2(ii), $M(\tilde{\varphi}, \xi, \eta, \tilde{g})$ is nearly Sasakian or nearly cosymplectic, respectively. Since $\dim M = 3$, we obtain Sasakian (Theorem 5.1 in [14]) or cosymplectic (see [15]) structures $(\tilde{\varphi}, \xi, \eta, \tilde{g})$, respectively.

We will generalize Theorem 5.2 in [1].

Theorem 1. *There are no weak nearly cosymplectic structures with the condition (10), which are weak contact metric structures.*

Proof. Suppose that our weak nearly cosymplectic manifold is a weak contact metric. Since also ξ is Killing (see Proposition 1), then M is a weak K-contact. Recall [12] that a weak K-contact manifold is defined as a weak contact metric manifold, whose Reeb vector field ξ is Killing. By Theorem 2 in [12], the following holds: $\nabla\xi = -\varphi$. Also, by Corollary 2 in [12], the ξ-sectional curvature is positive, i.e., $K(\xi, X) > 0$ $(X \perp \xi)$. Thus, if $X \neq 0$ is a vector orthogonal to ξ, then

$$
\begin{aligned}
0 < K(\xi, X) &= g(\nabla_\xi\nabla_X\xi - \nabla_X\nabla_\xi\xi - \nabla_{[\xi,X]}\xi, X) \\
&= g(-(\nabla_\xi\varphi)(X) + \varphi^2 X, X) = g((\nabla_X\varphi)\xi, X) - g(\varphi X, \varphi X) \\
&= -g(\varphi(\nabla_X\xi), X) + g(\varphi^2 X, X) = 2\,g(\varphi^2 X, X).
\end{aligned}
$$

This contradicts the following equality: $g(\varphi^2 X, X) = -g(\varphi X, \varphi X) \leq 0$. □

We generalize from Theorem 3.2 in [2] that a normal nearly Sasakian structure is Sasakian.

Theorem 2. *For a weak nearly Sasakian structure with the condition (10), normality $(N^{(1)} = 0)$ is equivalent to a weak contact metric $(d\eta = \Phi)$.*

Proof. First, we will show that a weak nearly Sasakian structure with conditions (10) and $\eta \circ N^{(1)} = 0$ is a weak contact metric structure. Applying the ξ-derivative to (2) and using (10), $\nabla_\xi\,\eta = 0$ and $\nabla_\xi\,\xi = 0$, we find (for $X \perp \xi$)

$$
(\nabla_\xi\,\varphi)\,\varphi X + \varphi(\nabla_\xi\,\varphi)X = (\nabla_\xi\,\varphi^2)X = -(\nabla_\xi\,Q)X + \nabla_\xi(\eta(X)\,\xi) = 0. \tag{13}
$$

We calculate, using (6), (8) and (13) and $\eta \circ \varphi = 0$,

$$
\begin{aligned}
\eta([\varphi, \varphi](X, Y)) &\overset{(6)}{=} \eta((\nabla_{\varphi X}\,\varphi)Y - (\nabla_{\varphi Y}\,\varphi)X) \\
&\overset{(8)}{=} \eta((\nabla_X\,\varphi)\,\varphi Y - (\nabla_Y\,\varphi)\,\varphi X) - 4\,g(X, \varphi Y) \\
&\overset{(13)}{=} \eta(\varphi\,(\nabla_Y\,\varphi)X - \varphi\,(\nabla_X\,\varphi)\,Y) - 4\,g(X, \varphi Y) = -4\,g(X, \varphi Y).
\end{aligned}
$$

Thus, if $\eta(N^{(1)}(X, Y)) = 0$, then $d\,\eta(X, Y) = 2\,g(X, \varphi\,Y)$.

Conversely, if a weak nearly Sasakian structure with the condition (10) is also a weak contact metric structure, then $\Phi = d\,\eta$; hence $d\,\Phi = 0$, where

$$d\Phi(X,Y,Z) = \frac{1}{3}\,\{X\,\Phi(Y,Z) + Y\,\Phi(Z,X) + Z\,\Phi(X,Y) \\ - \Phi([X,Y],Z) - \Phi([Z,X],Y) - \Phi([Y,Z],X)\}.$$

It is easy to calculate

$$\begin{aligned} 3\,d\,\Phi(X,Y,Z) &= -g((\nabla_X\,\varphi)Y,Z) + g((\nabla_Y\,\varphi)X,Z) - g((\nabla_Z\,\varphi)X,Y) \\ &= -g((\nabla_X\,\varphi)Y,Z) + g(-(\nabla_X\,\varphi)Y - 2\,g(X,Y)\,\xi + \eta(X)Y + \eta(Y)X,Z) \\ &\quad -g(-(\nabla_X\,\varphi)Z - 2\,g(X,Z)\,\xi + \eta(X)Z + \eta(Z)X,Y) \\ &= -3\,g((\nabla_X\,\varphi)Y,Z) - 3\,g(X,Y)\,\eta(Z) + 3\,g(X,Z)\,\eta(Y). \end{aligned}$$

Thus (7) holds. Using (7) in (6) gives us a normal structure: $[\varphi,\varphi] = -\,2\,d\eta\otimes\xi$. □

As a consequence of Theorem 2, we obtain a rigidity result for Sasakian manifolds.

Corollary 1. *A normal weak nearly Sasakian structure with the condition* (10) *is Sasakian.*

Proof. By Theorem 2, a weak nearly Sasakian structure with conditions (10) and $N^{(1)} = 0$ is weak Sasakian (see Section 2). Using Theorem 4.1 in [10] completes the proof. □

4. Hypersurfaces and Weak Nearly Kähler Manifolds

Here, we define weak nearly Kähler manifolds (generalizing nearly Kähler manifolds) and study weak nearly Sasakian and weak nearly cosymplectic hypersurfaces in such Riemannian spaces.

Definition 2. A Riemannian manifold $(\bar M,\bar g)$ of even dimension equipped with a skew-symmetric (1,1)-tensor $\bar\varphi$ such that the tensor $\bar\varphi^{\,2}$ is negative-definite will be called a *weak Hermitian manifold*. Such $(\bar M,\bar\varphi,\bar g)$ will be called a *weak nearly Kähler manifold*, if $(\overline\nabla_X\,\bar\varphi)X = 0$ $(X\in T\bar M)$, where $\overline\nabla$ is the Levi-Civita connection of $\bar g$, or, equivalently,

$$(\overline\nabla_X\,\bar\varphi)Y + (\overline\nabla_Y\,\bar\varphi)X = 0 \quad (X,Y\in T\bar M). \tag{14}$$

A weak Hermitian manifold will be called a *weak Kähler manifold* if $\overline\nabla\,\bar\varphi = 0$.

Remark 3. *Several authors studied the problem of finding parallel skew-symmetric 2-tensors (different from almost-complex structures) on a Riemannian manifold and classified such tensors (e.g., [16]) or proved that some spaces do not admit them (e.g., [17]).*

Example 3. *Let $(\bar M,\bar\varphi,\bar g)$ be a weak nearly Kähler manifold. To construct a weak nearly cosymplectic structure (φ,Q,ξ,η,g) on the Riemannian product $M = \bar M\times\mathbb{R}$ of $(\bar M,\bar g)$ and a Euclidean line $(\mathbb{R},\partial_t)$, we take any point (x,t) of M and set*

$$\xi = (0,\partial_t),\quad \eta = (0,dt),\quad \varphi(X,\partial_t) = (\bar\varphi X,0),\quad Q(X,\partial_t) = (-\bar\varphi^{\,2}X,\partial_t),$$

where $X\in T_x\bar M$. Note that if $\overline\nabla_X\,\bar\varphi^2 = 0$ $(X\in T\bar M)$, then (10) *holds.*

The scalar second fundamental form h of an orientable hypersurface $M\subset(\bar M,\bar g)$ with a unit normal N is related with $\overline\nabla$ and the Levi-Civita connection ∇ induced on the M metric g via the Gauss equation

$$\overline\nabla_X Y = \nabla_X Y + h(X,Y)\,N \quad (X,Y\in TM). \tag{15}$$

The Weingarten operator $A_N : X \mapsto -\overline{\nabla}_X N$ is related with h via the following equality:

$$\bar{g}(h(X,Y),N) = g(A_N(X),Y) \quad (X,Y \in TM).$$

A hypersurface is *totally geodesic* if $h = 0$. A hypersurface is called *quasi-umbilical* if

$$h(X,Y) = c_1\, g(X,Y) + c_2\, \mu(X)\, \mu(Y),$$

where c_1, c_2 are smooth functions on M and μ is a nonvanishing one-form.

Theorem 3. *For a weak nearly cosymplectic manifold, $\nabla\xi = 0$ if and only if the manifold is locally isometric to the Riemannian product of a real line and a weak nearly Kähler manifold.*

Proof. For all vector fields X, Y orthogonal to ξ, we have

$$2\, d\eta(X,Y) = g(\nabla_X\, \xi, Y) - g(\nabla_Y\, \xi, X). \tag{16}$$

Thus, by the condition $\nabla\xi = 0$, the distribution $\ker\eta$ is integrable. Any integral submanifold of $\ker\eta$ is a totally geodesic hypersurface. Indeed, for every $X, Y \perp \xi$, we have

$$g(\nabla_X\, Y, \xi) = -g(Y, \nabla_X\, \xi) = 0.$$

Since $\nabla_\xi\, \xi = 0$, by de Rham decomposition theorem (e.g., [18]), the manifold is locally the Riemannian product $\bar{M} \times \mathbb{R}$. The weak almost-contact metric structure induces on $\bar{M}$ a weak almost-Hermitian structure, which, by these conditions, is weak nearly Kähler.

Conversely, if a weak nearly cosymplectic manifold is locally the Riemannian product $\bar{M} \times \mathbb{R}$, where $\bar{M}$ is a weak nearly Kähler manifold and $\xi = (0, \partial_t)$ (see also Example 3), then $d\eta(X,Y) = 0$ $(X, Y \perp \xi)$. By (16) and $\nabla_\xi\, \xi = 0$, we obtain $\nabla\xi = 0$. □

Lemma 1. *A hypersurface M with a unit normal N and induced metric g in a weak Hermitian manifold $(\bar{M}, \bar{\varphi}, \bar{g})$ inherits a weak almost-contact structure $(\varphi, Q, \xi, \eta, g)$ given by*

$$\xi = \bar{\varphi}\, N, \quad \eta = \bar{g}(\bar{\varphi}\, N, \cdot), \quad \varphi = \bar{\varphi} + \bar{g}(\bar{\varphi}\, N, \cdot\,)\, N, \quad Q = -\bar{\varphi}^{\,2} + \bar{g}(\bar{\varphi}^{\,2} N, \cdot\,)\, N.$$

Proof. Using the skew-symmetry of $\bar{\varphi}$ (e.g., $\bar{g}(\bar{\varphi}N, N) = 0$), we verify (2) for $X \in TM$:

$$\begin{aligned}
\varphi^2 X &= \varphi(\bar{\varphi}X - \bar{g}(\bar{\varphi}X, N)\, N) \\
&= \bar{\varphi}(\bar{\varphi}X - \bar{g}(\bar{\varphi}X, N)\, N) - \bar{g}(\bar{\varphi}(\bar{\varphi}X - \bar{g}(\bar{\varphi}X, N)\, N), N)\, N \\
&= \bar{\varphi}^{\,2} X - \bar{g}(\bar{\varphi}^{\,2} N, X) + \bar{g}(\bar{\varphi}\, N, X)\, \bar{\varphi}\, N + \bar{g}(\bar{\varphi}X, N)\, \bar{g}(\bar{\varphi}\, N, N)\, N \\
&= -QX + \eta(X)\, \xi.
\end{aligned}$$

Since $\bar{\varphi}^{\,2}$ is negative-definite,

$$g(QX, X) = \bar{g}(-\bar{\varphi}^{\,2} X + \bar{g}(\bar{\varphi}^{\,2} N, X)\, N,\ X) = -\bar{g}(\bar{\varphi}^{\,2} X, X) > 0$$

for $X \in TM$, i.e., the tensor Q is positive-definite. □

The following theorem generalizes the fact (see [1,2]) that a hypersurface of a nearly Kähler manifold is nearly Sasakian or nearly cosymplectic if and only if it is quasi-umbilical with respect to the (almost) contact form.

Theorem 4. *Let M^{2n+1} be a hypersurface with a unit normal N of a weak nearly Kähler manifold $(\bar{M}^{2n+2}, \bar{\varphi}, \bar{g})$. Then, the induced structure $(\varphi, Q, \xi, \eta, g)$ on M is*

(i) *Weak nearly Sasakian;*

(ii) *Weak nearly cosymplectic.*

This is true if and only if M is quasi-umbilical with the following scalar second fundamental form:

$$(i)\; h(X,Y) = g(X,Y) + (h(\xi,\xi) - 1)\,\eta(X)\,\eta(Y), \quad (ii)\; h(X,Y) = h(\xi,\xi)\,\eta(X)\,\eta(Y). \tag{17}$$

In both cases, $A_N\varphi + \varphi A_N = 2\varphi$ *is true, and if*

$$((\overline{\nabla}_X\,\bar{\varphi}^2)Y)^\top = 0 \quad (X,Y \in TM,\ Y \perp \xi),$$

then (10) *holds on M.*

Proof. Substituting $\bar{\varphi}\,Y = \varphi\,Y - \bar{g}(\bar{\varphi}\,N,Y)\,N$ in $(\overline{\nabla}_X\bar{\varphi})Y$, and using (15) and Lemma 1, we obtain

$$\begin{aligned}
&(\overline{\nabla}_X\bar{\varphi})Y = \overline{\nabla}_X(\bar{\varphi}\,Y) - \bar{\varphi}(\overline{\nabla}_X Y)\\
&= (\nabla_X\varphi)Y + \eta(Y)A_N(X) - h(X,Y)\,\xi + [X(\eta(Y)) - \eta(\nabla_X Y) + h(X,\varphi Y)]\,N.
\end{aligned}$$

Thus, the TM-component of the weak nearly Kähler condition (14) takes the form

$$\begin{aligned}
&((\overline{\nabla}_X\bar{\varphi})Y + (\overline{\nabla}_Y\bar{\varphi})X)^\top\\
&= (\nabla_X\varphi)Y + (\nabla_Y\varphi)X - 2\,h(X,Y)\,\xi + \eta(X)A_N(Y) + \eta(Y)A_N(X) = 0.
\end{aligned} \tag{18}$$

Then we calculate $(\nabla_X Q)Y$ for $X,Y \in TM$, $Y \perp \xi$, using Lemma 1, (15) and $\bar{\varphi}^2 N = -N$,

$$\begin{aligned}
&(\nabla_X\,Q)Y = \nabla_X(QY) - Q(\nabla_X Y)\\
&= (\overline{\nabla}_X(-\bar{\varphi}^2 Y + g(\bar{\varphi}^2 N,Y)N) - h(X,QY)N + \bar{\varphi}^2(\overline{\nabla}_X Y - h(X,Y)N)\\
&- g(\bar{\varphi}^2 N, \overline{\nabla}_X Y - h(X,Y)N)N)^\top\\
&= (-(\overline{\nabla}_X(\bar{\varphi}^2 Y)) + \bar{\varphi}^2(\overline{\nabla}_X Y))^\top = -((\overline{\nabla}_X\,\bar{\varphi}^2)Y)^\top,
\end{aligned}$$

where $^\top$ is the TM-component of a vector.

(i) If the structure is weak nearly Sasakian, see (8), then, from (18), we obtain

$$2\,g(X,Y)\,\xi - \eta(Y)X - \eta(X)Y - 2\,h(X,Y)\,\xi + \eta(X)A_N(Y) + \eta(Y)A_N(X) = 0,$$

from which, taking the scalar product with ξ, we obtain

$$2\,g(X,Y) - 2\,\eta(Y)\,\eta(X) - 2\,h(X,Y) + \eta(X)\,h(Y,\xi) + \eta(Y)\,h(X,\xi) = 0. \tag{19}$$

Setting $Y = \xi$ and taking the scalar product with ξ, we obtain

$$h(X,\xi) = h(\xi,\xi)\,\eta(X). \tag{20}$$

Using this in (19), we obtain (17)(i).

Conversely, if (17)(i) is valid, then substituting $Y = \xi$ yields (20). Using (17)(i), we express the Weingarten operator as

$$A_N(X) = X + (h(\xi,\xi) - 1)\,\eta(X)\,\xi.$$

Substituting the above expressions of $h(X,Y), h(X,\xi)$ and A_N in (18) gives (8); thus, the structure is weak nearly Sasakian.

(ii) If the structure is weak nearly cosymplectic, see (9), then, from (18), we obtain

$$2\,h(X,Y)\,\xi = \eta(X)A_N(Y) + \eta(Y)A_N(X),$$

From which, taking the scalar product with ξ, we obtain

$$2\,h(X,Y) = \eta(X)\,h(Y,\xi) + \eta(Y)\,h(X,\xi). \tag{21}$$

Setting $Y = \xi$ and taking the scalar product with ξ, we obtain (20). Using this in (21), we obtain (17)(ii).

Conversely, if (17)(ii) is valid, then substituting $Y = \xi$ yields (20). Using (17)(ii), we express the Weingarten operator as $A_N(X) = h(\xi,\xi)\,\eta(X)\,\xi$. Substituting the expressions of h and A_N in (18) gives (9); thus, the structure is weak nearly cosymplectic. □

5. Conclusions

We have shown that weak nearly Sasakian and weak nearly cosymplectic structures are useful tools for studying almost-contact metric structures and Killing vector fields. Some classical results have been extended in this paper to weak nearly Sasakian and weak nearly cosymplectic structures. Based on the numerous applications of nearly Sasakian and nearly cosymplectic structures, we expect that certain weak structures will also be useful for geometry and physics, e.g., in QFT.

The idea of considering the entire bundle of almost-complex structures compatible with a given metric led to the twistor construction and then to twistor string theory. Thus, it may be interesting to consider the entire bundle of weak Hermitian or weak nearly Kähler structures (see Definition 2) that are compatible with a given metric.

In conclusion, we ask the following questions for dimensions greater than three: find conditions under which

(i) A weak nearly Sasakian manifold is Sasakian;

(ii) A weak nearly cosymplectic manifold with $\nabla\xi \not\equiv 0$ is a Riemannian product.

It would be interesting to study the geometry of five-dimensional weak almost-Sasakian and weak almost-cosymplectic manifolds. We also consider the following (inspired by Corollary 6.4 in [8]): whether a hypersurface in a weak nearly Kähler six-dimensional manifold has Sasaki–Einstein structure.

These questions can be answered by generalizing some deep results on nearly Sasakian and nearly cosymplectic manifolds (e.g., [3,7,8,14]) to their weak analogues.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflicts of interest.

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