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# Music through Curve Insights

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**Abstract:** This manuscript endeavors to establish a framework for the mapping of music onto a three-dimensional structure. Our objective is to transform the guitar choruses of Beatles songs into curves, with each chorus corresponding to its respective curve. We aim to investigate and characterize the intricacy of each song by employing mathematical techniques derived from differential geometry, specifically focusing on the total curvature of the chorus curve. Given that a single song may possess varying chord progressions in different verses, the performer can determine the geometric representation they aim to convey through the number of loops and the direction of the curve. The overarching objective of our study is to enable viewers to identify specific songs or motives by visually examining an object and exploring its geometric properties. Furthermore, we posit that these ideas can provide composers with a fresh perspective on their own musical compositions while also granting non-professional audiences a glimpse into the intricacies involved in the process of composing.

**Keywords:** differential geometry; three-dimensional design; curvature; index of a curve

**MSC:** 51-08; 53Z99; 00A65

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## 1. Introduction

The inception of our research was initiated through an informal dialogue among amateur musicians, centering on the phenomenon of vibrational occurrences within the low-frequency spectrum of music, thereby endowing it with a palpable, corporeal quality. This discourse prompted us to embark on an exploration of the prospect of characterizing music as a tangible entity. This contemplation, in turn, provoked our interest in examining music as an object of study, with the aim of unraveling its intricacies and assessing its resemblance to other specific compositions. Figure 1 is one of our results; three Beatles songs (with the help of a designer) have been produced as physical objects by our mathematical model. This visualization holds particular significance for an audience that lacks familiarity with the fundamental principles of music theory, such as deciphering musical notation on a sheet. By examining the object derived from the visualization, viewers can gain valuable insights into the complexity of the music itself. This approach provides a unique opportunity for individuals without a formal background in music to develop a rudimentary understanding of its intricacies and appreciate the nuances involved.

To attain our objective, we will utilize initial chord data presented in the Western chromatic scale. The objects and explorations will be specific to the guitar choruses of Beatles songs, as referenced in the previous work by the authors [1].

Our proposed method involves the mapping of a given song, comprising triads conforming to classical Western harmony, onto three-dimensional curves that can be mathematically explored.

The western chromatic scale is based on 12 notes, without loss of generality, starting with the note $A$:

$$\{A, A\#, B, C, C\#, D, D\#, E, F, F\#, G, G\#\},$$

where each note in this list can be mapped to a number in the range of 0 through 11 ($A \to 0$ and $G\# \to 11$). Chords are defined as a set of notes played together. A triad is a common chord that is defined by a set of specific three different notes, as $F_{maj} = (F, A, C) = (8, 0, 3)$. By this definition, if we transform the notes to natural numbers, then the notes can be represented as elements of $\mathbb{Z}_{12}$ (modulo 12) and a given triad $(x, y, z) \in \mathbb{Z}_{12}^3$. Note that this identification does not distinguish different voicings of a given chord. An example of popular chords are major and minor. The major triad with root $r$, which is the relative shift in terms of semi-tones, is composed of three tones, $(r, r+4, r+7)$. As $r$ represents the same tone as $r+12$, the respective intervals within the triad are $(4, 3, 5)$, with the third entry, 5, being the distance from $r+7$ to $r+12$. Similarly, the minor triad with root $r$ is composed of the three tones $(r, r+3, r+7)$, inducing the intervals $(3, 4, 5)$. In the other direction, each of the 24 pairs $(t, (a, b, c))$, where $t \in \mathbb{Z}_{12}$ and $(a, b, c)$ is either $(4, 3, 5)$ or $(3, 4, 5)$, corresponds to a triad $(t, t+a, t+a+b)$.

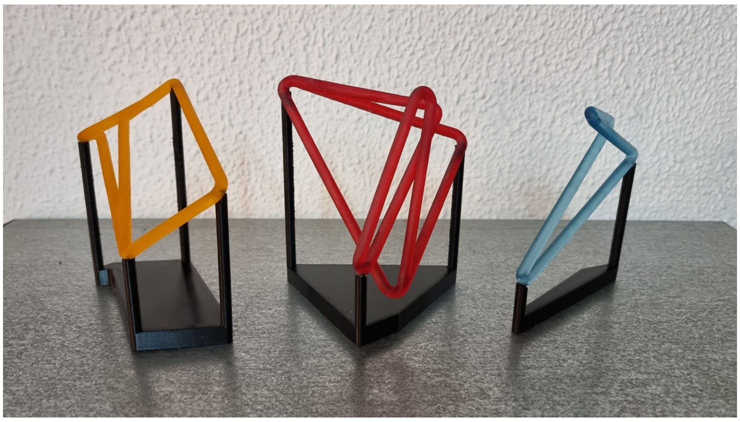


**Figure 1.** Three Beatles songs which we transformed into 3*D* physical objects, from left to right: Hello Goodbye; All You Need Is Love; Like Dreamers Do. 3*D* printing Pla/Sla, by Lior Bar.

This formulation naturally lends itself to mathematical concepts rooted in group theory, as explored in previous works such as [2,3].

We are intrigued by the possibility of providing a three-dimensional geometrical representation for a set of triads, which could serve as a gateway to exploring music through various mathematical tools. We contemplate whether a well-defined mathematical representation, based on the geometrical properties of the triads, can offer insights into the intended message of the composer. This approach could potentially benefit not only individuals without a background in reading musical notation but also those with a mathematical aptitude, allowing them to engage with music in a novel and analytical manner.

Geometrical ideas for representing music have already been considered before. In [4,5], a geometrical representation of harmony has been introduced with defining distances in a polygon, which leads to better visualization of music. In [6], scale-theoretical ideas which relate to geometry from the standpoint of voice have been presented. In [7], a measure for determining the similarity between two melodies with multiple-note change has been discussed based on pitch distance, the effects of primacy, and two-note changes. In [8], a geometric model of psychoacoustics has been defined to express roughness and harmonicity with the help of a height function.

Also, [9,10] propose a natural geometrical scheme for investigating and generating varieties of pitch simultaneity and sound intensity using Eulerian flows with $n$-tuple Hopf singularity for a sufficiently large $n$.

Even topological methods have also been considered. In [11] homotopy theory has been applied to transformational theory, where two-dimensional diagrams capturing voice leadings have been presented.

Affected by these ideas, we wonder how to transform songs into geometric objects and evaluate or explore their complexity with a proper measure.

The intuitive geometric approach involves defining a suitable distance metric, denoted as $d$, between pairs of chords, represented as $x$ and $y$, to establish a notion of proximity in the geometric representation. This distance metric, denoted as $d(x,y)$, provides a local perspective by quantifying the dissimilarity or similarity between individual chords. However, a limitation of this approach is that it primarily captures local relationships and fails to provide a comprehensive understanding of the song's global complexity. To overcome this limitation, it becomes necessary to explore additional measures or techniques that can offer a broader, more holistic perspective on the overall structure and complexity of the song.

A chord is defined by three notes or more, but the first three notes: root, third, and fifth, are the most important ones for harmonic sound. We mainly focused on choruses defined by triads (we will not include in our model single notes) such that we reduce the musical elements to the first three notes, as is the case in [2].

Each triad can be represented as a point in three-dimensional space. The song's progression is captured by establishing directed edges between every two successive triads, forming a connected curve. This curve serves as a representation of the song or chorus and encapsulates its complexity. By analyzing the geometric properties of this curve, we can compare and evaluate different songs or choruses, enabling a quantitative assessment of their relative complexities.

Following an extensive brainstorming session, we have reached a consensus to investigate this geometric object using differential geometry tools, specifically employing the concept of total curvature. The theoretical foundation and initial findings of our approach have already been presented at the Bridges conference; for more details see [12]. By leveraging total curvature as a measure, we aim to compare different songs and provide a straightforward means for non-professional audiences to gain insights into the relative complexities of the music, without requiring a prerequisite understanding of music theory. Moreover, this method offers professional players and composers a novel tool to assess the complexity and overall dynamics of a song.

To exemplify our findings, we select choruses from the Beatles' repertoire, as documented in references such as [1,13]. To ensure simplicity in our demonstration, we establish an initial condition. While we acknowledge that some musicians may express reservations regarding this seemingly "dry" formulation, we maintain that the geometric properties inherent in the analysis can still provide valuable insights into the measure of complexity exhibited by the musical compositions.

## 2. Preliminaries

Given a sequence of chords, we aim to give a visual representation with a three-dimensional geometrical structure and explore its mathematical properties.

In differential geometry, the curvature of a given point on the curve is the amount by which a curve deviates from being a straight line; i.e., it measures the change in the angle of the tangent along the curve at every point. In the plane, if a curve $\gamma$ is twice differentiable in the plane represented by $\gamma(t) = ((x(t), y(t)))$, then the curvature $\kappa$ of a given point can be calculated by

$$\kappa = \frac{\det(\gamma', \gamma'')}{||\gamma'||^3}\ .$$

If the curve is given by an arc length parametrization, then the absolute value of the curvature can be defined by $\kappa(s) = ||\gamma''(s)||$ and can be generalized to any number of dimensions.

In a two-dimensional plane, the sign of curvature (positive or negative) is determined by the direction of the curve: clockwise rotation corresponds to negative curvature, while counterclockwise rotation corresponds to positive curvature. However, the concept of clockwise or counterclockwise rotation is not applicable in higher dimensions, as it is inherently tied to the two-dimensional plane. Therefore, we adopt the convention of considering the absolute value of curvature in higher dimensions, as the sign of the curvature is not defined in those cases. The curvature of a line is zero, and the curvature of a circle with radius $R$ is constant $\kappa = \pm\frac{1}{R}$ since it deformed uniformly. It turns out that if the derivative never vanishes, the curvature can give a local approximation of a circle, which is centered at $E(t) = \gamma(t) + \kappa(t)$. In the case of two segments (vectors), which are attached as $\overline{CD}$ and $\overline{DE}$, as in Figure 2a, the curvature is defined by the exterior angle between the segments (vectors), i.e., $\alpha_3$ in the respective change.

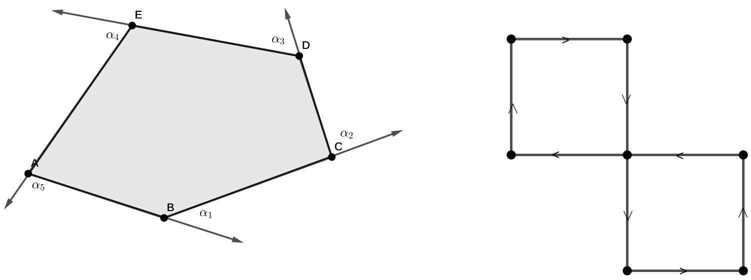
This reasoning leads us to the definition of total curvature for a given curve, as outlined in [14]. The total curvature is computed by summing the curvatures at all points along the curve. For instance, in the case of a triangle, as all points on a straight line possess zero curvature, only the vertices of the triangle contribute to the curvature, resulting in three defined points of curvature.

Surprisingly, in the two-dimensional plane, if a closed curve is simple (non-self-intersecting), the total curvature is always either $\pm 2\pi$, as described in [14]. In the case of a triangle, the total curvature is precisely $2\pi$ when the curve is defined in a counterclockwise direction or $-2\pi$ when defined in a clockwise direction, as depicted in Figure 2a.

A simple closed curve can define various domains, each with distinct properties. One such domain is referred to as a convex domain, which satisfies the condition that for any two points, $A$ and $B$, within the domain, the entire line segment connecting them, denoted as $\overline{AB}$, lies entirely within the domain. This can be visualized as the gray domain in Figure 2a.

Another type of domain is a star domain, which possesses a unique central point, denoted as $A$, such that for every point $B$ within the domain, the line segment connecting $A$ and $B$, represented as $\overline{AB}$, is entirely contained within the domain.

It turns out that in two-dimensional planes, the complexity of the shape defined by a closed simple curve, such as whether it is convex, non-convex, or exhibits other intricate characteristics, does not affect the total curvature. Irrespective of the shape's complexity, the total curvature of a closed simple curve in the plane remains constant at $\pm 2\pi$. This intriguing property suggests that the total curvature is solely determined by the topological properties of the curve (number of laps), disregarding its specific geometric configuration.




(**a**) (**b**)

**Figure 2.** Different polygonal curves in the plane. (**a**) Total curvature $2\pi$. (**b**) Total curvature zero.

When dealing with a closed but non-simple curve, the approach is to calculate the total curvature by decomposing the curve into a collection of simply closed curves, each with a clockwise or counterclockwise orientation. Each of these simply closed curves individually possesses a total curvature of $\pm 2\pi$. By summing up the curvatures of these closed curves, we can determine the total curvature of the initial closed curve.

Various techniques exist for calculating the total curvature, as have been given in [14]. These techniques offer valuable insights into quantifying the total curvature of curves with different complexities and configurations.

Figure 2a illustrates a closed convex polygon defined by the ordered vertices $\{A, B, C, D, E\}$. In this case, non-zero curvature occurs only at the external angles $\{\alpha_1 \ldots \alpha_5\}$, which correspond to changes in the vectors of the polygon's edges. Since the curve is both simple and closed in a counterclockwise orientation, the total curvature of the curve is determined to be $2\pi$.

In Figure 2b, the domain is formed by two polygonal closed curves: one is defined in a counterclockwise direction (with a respective curvature of $2\pi$), while the other is defined in a clockwise direction (with a respective curvature of $-2\pi$). As a result, the sum of the curvatures of these two curves is zero.

These examples highlight how the orientation and arrangement of the curves can affect their respective curvatures and, in turn, influence the total curvature of the closed domain.

This leads us to the conclusion that the number of laps are the topological property. If the lap is counterclockwise, then the index of the curve is $+1$, and if the curve is clockwise, then the index is $-1$ (for more details, see [14]).

## 3. Our Results

### 3.1. Curve Interpretation

Our objective is to establish a framework in which a given song, represented in three-dimensional space, can be analyzed using the "total curvature" approach. This approach enables us to evaluate the complexity of the song based on its geometric properties. By understanding and incorporating these concepts, composers and musicians can enhance the process of composition and performance. They can consider factors such as the number of laps, direction, and other related elements to determine the appropriate chord progressions and musical movements that align with the desired artistic expression. In the subsequent discussion, we will delve further into these aspects.

To facilitate our geometrical representation, we will focus on guitar chords consisting of triads, which are composed of three notes. In the case where a chord consists of more than three notes, we will consider only the first three notes, following the approach outlined in [2].

To represent these triads in a geometric context, we will map them to points in the three-dimensional space $\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$. This space allows for the incorporation of the twelve possible pitches or tones in Western music theory. Each component of the point represents a note within the chord, and the combination of these three components defines the specific triad in our representation. By utilizing this mapping, we can explore and analyze the geometric properties of the triads within a three-dimensional framework.

By considering each chord as a vertex, the sequence of chords can be represented as an oriented curve. This curve connects the vertices in the order of their appearance in the song, providing a visual depiction of the chord progression.

It is important to note that multiple curves can pass through the given set of vertices, as there are various ways to connect them while preserving the sequence. However, our mathematical exploration aims to find consistent results regardless of the specific curve chosen. The analysis of the geometric properties and the application of mathematical tools will ensure that the chosen curve accurately represents the underlying musical structure and allows for meaningful comparisons and evaluations.

Notice that, with our geometrical approach, the sequence $(A, G, B, A)$ and $(A, B, G, A)$ define the same curve but in opposite directions.

To facilitate the conversion of a computer plot into a physical object, we initially developed a "working model" or technical model, as depicted in Figure 3. In this model, we identified two key visual elements that we aimed to explore: (1) the curve representing the chord progression and (2) the coordinate system.

The wire in the model represents the edges defined by the vertices, forming a polygonal curve within three-dimensional space. This model served as a valuable tool from both a mathematical and design perspective. Mathematically, it provided a natural representation of a polygonal curve, which is one of the possible curve families that can pass through a given sequence of vertices. From a design standpoint, the physical aspect of interacting with the wire stimulated discussions regarding the use of different materials to convey distinct sensory experiences associated with the transitions between successive chords.

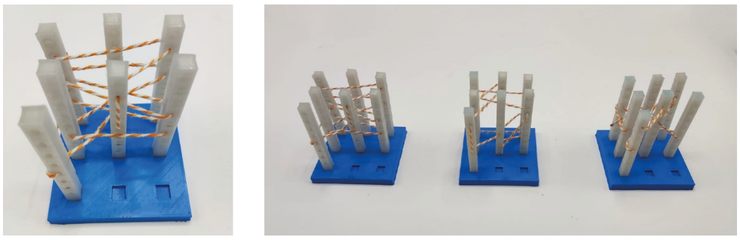
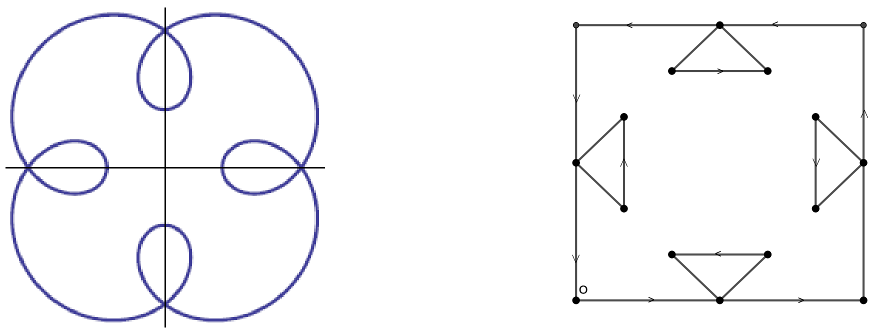


**Figure 3.** Our first experiment to define a sequence of triads into three-dimensional physical objects. The chords as coordinates/vertices embedded in three-dimensional space. The wire moves along the coordinates and represents chord progression, which leads to a polygonal curve.

Overall, this working model played a crucial role in bridging mathematical concepts and design considerations, allowing us to further explore the geometric representation of the music and consider the tangible aspects of the chord progression.

This polygonal curve will be our geometrical approach to expressing music. Figure 4, shows how a smooth curve (the derivative is continuous as well) can be transformed to a polygonal curve which preserves the total curvature properties. By these ideas, any simple curve that goes through these vertices can be selected, so we choose the closed polygonal curve.



(**a**) (**b**)

**Figure 4.** Take both curves counterclockwise. The index of the curves is five and the total curvature is $10\pi$. (**a**) A curve with four laps. (**b**) Approximation as a polygonal curve.

The following proposition will define exactly the set of songs/curves we will deal with.

**Assertion 1.** *To streamline our analysis, we specifically focused on the choruses of songs where the musical composition starts and ends with the same triad. This characteristic ensures that the chord progression forms a closed curve, simplifying our geometric representation.*

*In instances where the chords or vertices give rise to two-sided directional edges, we define the curvature as zero. This choice allows us to effectively handle cases where the curve transitions between chords without introducing curvature variations.*

*Lastly, if the same chord repeats in a successive manner, from the geometrical point of view there is no change in the chorus structure, so we will write this chord only once; as an example, see the sequence in Equation (1), which defines the chorus of "Across the Universe".*

By imposing these simplifications, we aim to establish a clear and manageable framework for studying the geometric properties of choruses in music. This focused approach enables us to delve deeper into the analysis and interpretation of the musical structure within a three-dimensional context.

The special case of choruses of two chords is trivial and will not be discussed in our exploration.

Indeed, the characteristic of choruses in Western music often starting and ending with the same chord is quite prevalent. A preliminary examination of Beatles songs, as documented in [1], reveals a considerable number of instances where this pattern is observed. Songs such as “Across the Universe”, “All Together Now”, “Ask Me Why”, “Baby in Black”, “Cry Baby Cry”, “Golden Slumbers”, “I’m So Tired”, “Julia”, and many others follow this structure.

Having developed our mathematical formulation, we are now prepared to showcase our construction and present the results, specifically focusing on the Beatles’ choruses. By applying the total curvature approach and exploring the geometric properties of the chord progressions, we aim to offer valuable insights into the complexity and structure of these iconic songs. This analysis provides a unique perspective that is accessible to both professional musicians and non-professional audiences with an interest in mathematics and music. Through this endeavor, we hope to enrich the understanding and appreciation of the Beatles’ music and contribute to the broader exploration of music through a mathematical lens.

The chorus of the Beatles’ song “Across the Universe” is defined by a sequence of three chords. In this particular case, the chord progression follows the sequence

$$\{(5,9,0),(0,4,7),(0,4,7),(10,2,5),(5,9,0),(0,4,7),(10,2,5),(5,9,0)\}\,, \tag{1}$$

which is a closed curve; see Figure 5. This visualization emphasizes the simplicity of the chorus for the non-professional audience with the three-dimensional triangle (respective to the three different chords), while from the mathematical point of view, the curve wraps the triangle twice, which can be considered as an additional insight into the song structure. In our model, we decided to include this number of laps (wrapping).

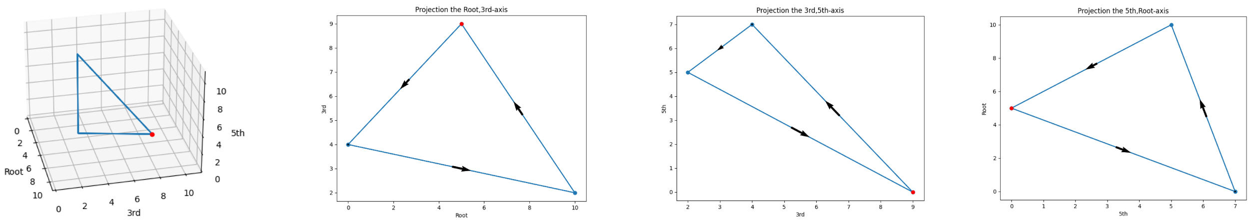



**Figure 5.** The Beatles: Across the Universe. The red dot represents the first vertex. The arrows represent the curve direction. From left to right, $3D$ representation, projection $(x,y)$ respective to (root, third), projection $(y,z)$ respective to (third, fifth), projection $(z,x)$ respective to (third, root).

The chorus of “Ask Me Why”, is obtained by

$$\{(7,11,2),(9,0,4),(9,2,6),(0,4,7),(9,2,6),(0,4,7),(7,11,2)\}\,.$$

While the number of chords is similar to “Across the Universe”, it visualization is different. First, as seen in Figure 6, while the three-dimensional curve is a simple curve, the projection of the root and third leads to a point where the curve intersects itself. In addition, in each of the projections, there are no laps. So the behavior of these two choruses that are represented by curves is different (by a proper measure).

Note that in three-dimensions, only the absolute value of the curvature is well defined, but since there is no curve direction (as clockwise or counterclockwise), the total curvature is meaningless. This leads us to project the three-dimensional curve onto the plane in three different ways by omitting one of the axes in each of the projections, where the total curvature or index is well defined. This projection onto three planes $(x,y),(y,z),(z,x)$ leads to three total curvatures, which will be the respective characterizations of the three-dimensional closed curve; i.e., each chorus has a respective vector of curvature which is obtained by the projections. As we already discussed, this calculation of total curvature or index for a close curve is determined by the index that holds the curve and not by local property, so we can choose the edges as straight lines, a polygonal curve, and the curvature will be obtained only in the vertices.

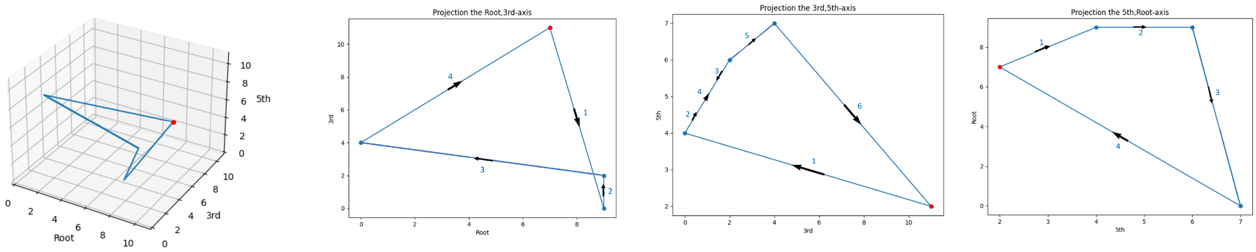



**Figure 6.** The Beatles: Ask Me Why. From left to right, a 3*D* representation and the respective projections. The numbers on the edges represent the direction of movement.

**Remark 1.** *These projections show 'the contribution' of each note in the triad. If the plot is a thin domain (as an acute triangle), this hints that the respective notes are located 'nearby' and vice versa, as in some of the projections in the Beatles songs below.*

**Remark 2.** *Notice that curvature zero can be obtained in different cases, where the number of clockwise laps is equal to the number of counterclockwise laps. This complexity, for a chorus with more than three chords, led us to think of this case as an indicator function, i.e., if there exists such a 'turn' (curvature zero) or not.*

### 3.2. Curvature as a Tool to Estimate Musical Complexity

Now, we are ready to apply our model. Figure 7 describes the process from the chorus of "Get Back" to the respective curves. In a similar way to "Across the Universe", a 3*D* triangle is obtained. The $(x, y)$ projection (green triangle) is a closed counterclockwise curve with a respective total curvature of $2\pi$. The second projection leads to the red triangle, with a respective total curvature of $-2\pi$ (closed clockwise curve). The third projection leads to the blue triangle, with a total curvature $-2\pi$. So, using our visualization, all choruses (geometric objects) which are defined by $(2\pi, -2\pi, -2\pi)$ in the same equivalent class, i.e., have the same total maneuver which leans on the total curvature vector of the curve.

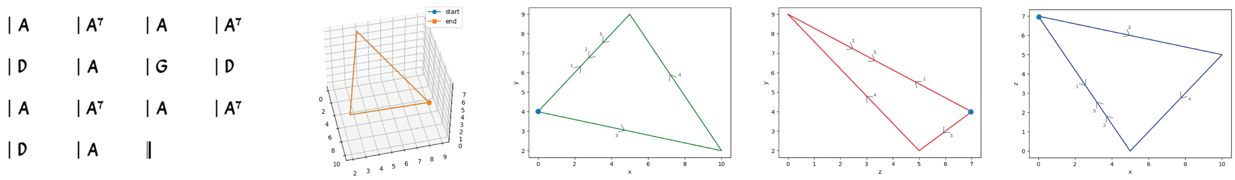



**Figure 7.** The Beatles: Get Back. From left to right: the chorus, the three-dimensional representation, and the three projections. The dot in every curve is the first triad in the chorus. The total curvature of the respective projection in the plane is $(2\pi, -2\pi, -2\pi)$.

Figure 8 represents the Beatles "Like Dreamers Do" chorus; as can be seen, despite the projections defining a non-convex domain, the $(x, y)$ and $(z, x)$ do not affect the total curvature (or index). In the $(y, z)$ projection, a kind of knot, which leads to curvature zero, is obtained that can indicate a more complex move between chords compared to "Get Back" and may indicate the guiding hand of the composer trying to transmit. Finally, this chorus is defined by the total curvature vector $(2\pi, 0, -2\pi)$.

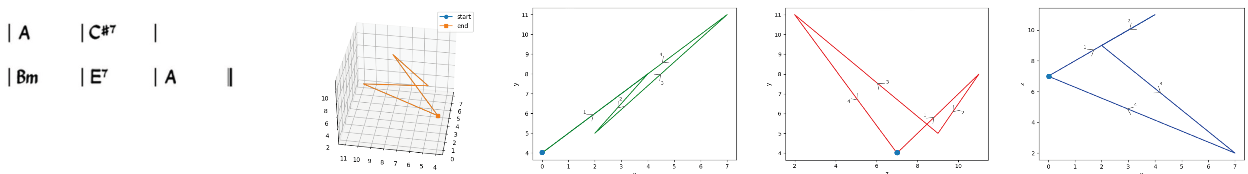



**Figure 8.** The Beatles: Like Dreamers Do. The respective projection's total curvature is $(2\pi, 0, -2\pi)$.

Figure 9 represents the Beatles "Hello, Goodbye" chorus. Notice that in the second and third projections, some of the chords are on the same edge, which are repetitions, and the numbering explains the direction of movement and gives the progression's feel along the curve. Notice that in the $(y, z)$ projection the curve is wrapped twice—two laps. In the first projection $(x, y)$, the total curvature is zero and leads to a knot, 'feel'. The respective vector in this case is $(0, -4\pi, 4\pi)$.

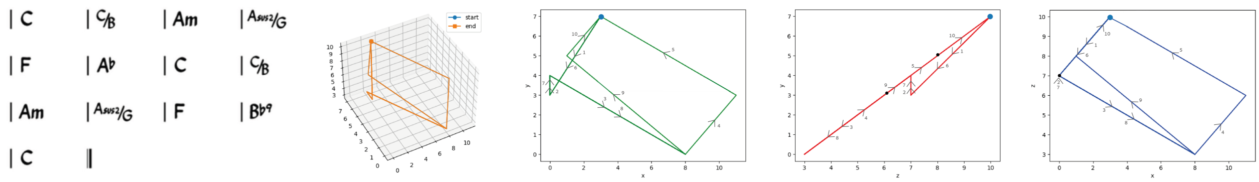



**Figure 9.** The Beatles: Hello, Goodbye. The respective projection's total curvature is $(0, -4\pi, 4\pi)$.

Figure 10 leads to the vector $(-4\pi, 2\pi, 6\pi)$ and can give a glance to the audience with no music theory background on how complex music can go with our geometrical representation, where a few laps (clockwise or counterclockwise) can give a "dizzy feel", which may indicate that there exists at least one point in the projection, in which the curve defined its index (for more details, see Winding numbers in [14]).

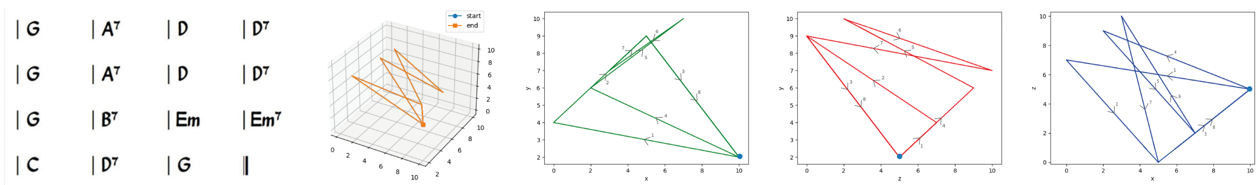



**Figure 10.** The Beatles: All You Need Is Love. The curve reveals the complexity of this song. The respective projection's total curvature is $(-4\pi, 2\pi, 6\pi)$.

**Remark 3.** *Since a major triad with root $r$ is defined by $(r, r+4, r+7)$ and a minor triad by $(r, r+3, r+7)$, if the chorus is not defined by a translation of a single chord, then the polygonal curve (where the number of chords $\geq 3$) cannot be embedded in the two-dimensional space; i.e., there exists a polygonal projection for each of the two-dimensional axes $(x, y)$, $(y, z)$ and $(z, x)$.*

This exploration by a vector of curvature or index (by omitting $2\pi$) can give us a new dimension for exploring songs; it does not give attention to the number of chords which define the song or the distance between them but examines the sequence of triads that constructs the chorus as a curve. The direction of progression along them leads to the respective behavior of the chorus, which is defined by laps, knots, and more, and all together can indicate the global structure which has been determined by the composer.

Indeed, the theoretical approach of exploring songs as geometric objects based on the progression of chords can establish connections between different songs that may not share the exact same chord progression but exhibit similar global progressions along their choruses. This highlights the importance of the overall pattern and movement of the curve rather than the specific chords themselves.

Furthermore, considering that a song can have multiple versions with different chord arrangements (such as live performances or recorded variations), the player or composer has the flexibility to choose the desired geometrical properties that best represent the intended musical motive. They can make decisions such as determining the number of geometrical loops present in the curve, whether the curve changes direction, and more. By manipulating the geometrical structure of the curve, the user can convey their artistic interpretation and enhance the musical experience for themselves and the audience.

This approach provides a creative and intuitive way for players and composers to engage with and shape the musical composition, allowing them to express their artistic vision through the chosen geometric properties of the curve. It opens up possibilities for exploration, interpretation, and customization, ultimately enriching the musical experience and offering new avenues for creativity.

One of the intriguing aspects of the method is its potential to sort choruses into equivalent classes based on their total curvature. By assigning a measure of complexity through the total curvature, choruses with similar geometric structures can be grouped together, regardless of the specific chord progressions or musical styles they belong to. This allows for a novel way of categorizing and comparing different songs, transcending traditional genre boundaries.

Furthermore, the geometric structure of a chorus can serve as a common ground to relate and connect various musical styles. Different musical styles may have distinct chord progressions and harmonic patterns, but their underlying geometric structures can reveal shared characteristics or similarities. This offers a fresh perspective on exploring and understanding the relationships between different musical genres and styles, potentially bridging the gap between diverse musical traditions.

By employing a geometrical framework, our method enables a unified approach to analyze and interpret choruses across musical styles, fostering a deeper understanding of the underlying structures that shape musical compositions. It opens up possibilities for cross-genre collaborations, creative inspiration, and the development of new musical ideas by drawing upon the geometric connections and relationships between different styles.

In Appendix A additional exploration of Beatles choruses, where all explorations together give the reader an immediate chosen song comparison

## 4. Future Work

Approaching this subject matter from a mathematical standpoint, our primary objective is to delineate mathematical curves that encapsulate the entirety of a given song, encompassing its various sections, such as the chorus, verse, and other distinct parts. Each of these mathematical curves will be uniquely characterized by a different color, and when combined, they will collectively serve as a graphical representation of the entire musical composition, which may even be defined by different knots.

In addition, our ambition extends to the generalization of our findings into the realm of surfaces, akin to the approach outlined in [15], where the Gauss–Bonnet theorem, as expounded in [14], will play a pivotal role. This theorem provides a means of classifying topological surfaces based on their total Gaussian curvature and geodesic curvature, affording us a broader perspective on the mathematical representation of music within a geometric context.

Lastly, from a design perspective, our current endeavor revolves around the development of models that afford viewers the opportunity to physically engage with and tactilely experience the progression of a musical composition, as illustrated in Figure 1. The designer, through the judicious selection of materials, including those possessing varying textures such as roughness and softness is poised to convey the nuanced interplay between harmonious and cacophonous elements within the musical structure, particularly evident when transitioning between two chords.

## 5. Summary and Conclusions

In this study, we present a methodology for representing the chorus of a song as a curve and propose an approach to explore its characteristics using geometric principles. By employing geometric ideas, we aim to convey the intricacy of a song to a non-professional musical audience, providing them with a fresh perspective grounded in geometry. Furthermore, this framework offers musicians a means to characterize and analyze their own compositions through a geometric lens. We assert that our model has the potential to establish connections between compositions originating from diverse musical styles, employing geometric objects and properties, such as total curvature, as unifying elements. Overall, our findings suggest that incorporating geometry into the analysis of musical pieces enables a deeper understanding and appreciation of their complexity and facilitates novel avenues for artistic exploration.

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## Appendix A. Additional Explorations

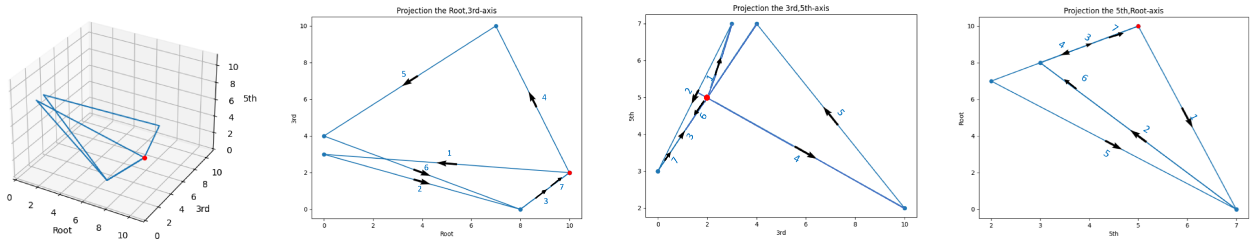


**Figure A1.** The Beatles: Cry Baby Cry. $(4\pi, 4\pi, 0)$.

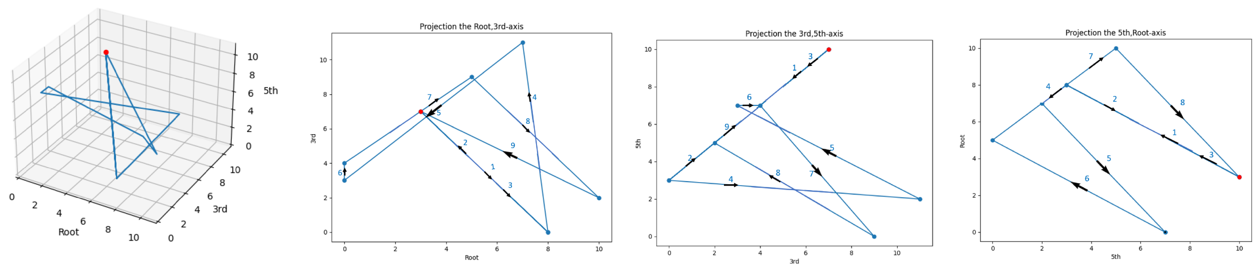


**Figure A2.** The Beatles: Golden Slumbers. $(-2\pi, -2\pi, -4\pi)$.

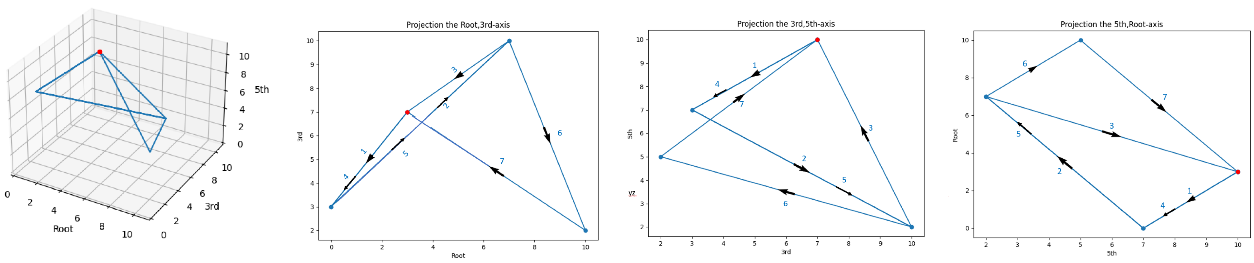


**Figure A3.** The Beatles: Julia. $(2\pi, 2\pi, -4\pi)$.

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