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General Fractional Noether Theorem and Non-Holonomic Action Principle

Vasily E. Tarasov [1,2]

[1] Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia; tarasov@theory.sinp.msu.ru
[2] Department of Physics, 915, Moscow Aviation Institute (National Research University), Moscow 125993, Russia

Abstract: Using general fractional calculus (GFC) of the Luchko form and non-holonomic variational equations of Sedov type, generalizations of the standard action principle and first Noether theorem are proposed and proved for non-local (general fractional) non-Lagrangian field theory. The use of the GFC allows us to take into account a wide class of nonlocalities in space and time compared to the usual fractional calculus. The use of non-holonomic variation equations allows us to consider field equations and equations of motion for a wide class of irreversible processes, dissipative and open systems, non-Lagrangian and non-Hamiltonian field theories and systems. In addition, the proposed GF action principle and the GF Noether theorem are generalized to equations containing general fractional integrals (GFI) in addition to general fractional derivatives (GFD). Examples of field equations with GFDs and GFIs are suggested. The energy–momentum tensor, orbital angular-momentum tensor and spin angular-momentum tensor are given for general fractional non-Lagrangian field theories. Examples of application of generalized first Noether's theorem are suggested for scalar end vector fields of non-Lagrangian field theory.

Keywords: action principle; noether theorem; non-Lagrangian field theory; dissipative systems; variational equation; non-holonomic functional

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1. Introduction

In physics and mechanics, principles of action stationarity make it possible to obtain field equations and equations of motion, which have great importance. The Noether theorems are used to derive conservation laws in physics. The action principle and the Noether theorem are actively applied in different branches of mechanics and physics. For example, classical field theories are usually based on the Lagrangian and Hamiltonian formalisms, the standard action principle and the Noether theorems [1–8]. The Noether theorem is used in classical mechanics [9–11], continuum mechanics [12,13] optical systems [14], statistical mechanics [15], fluid dynamics [16,17], classical field theory [1,7,18], and in quantum field theory [1–3,8].

The standard action principle is usually described by a holonomic variational equation, which has the form of the variation of some holonomic functional being equal to zero. However, not all equations of motion and field equations can be derived from the holonomic variational equation and be written in the form of the Euler–Lagrange equations for Lagrangian function or Lagrangian density. Theories, whose equations of motion and/or field equations cannot be derived from the standard action principle, are called non-Lagrangian or non-Hamiltonian theories. Non-Lagrangian and non-Hamiltonian theories make it possible to describe a wide class of irreversible processes and dissipative and open systems. Therefore, variational equations can be important for such processes and systems.

The standard Noether theorem is formulated for the Lagrangian field theories with holonomic variational equations and expresses the invariance of the Lagrangian with respect to some continuous group of transformations. The Noether theorem formulates a sufficient condition for the existence of conservation laws [19–22]. the Noether theorem states that any continuous symmetry of a physical system corresponds to some conservation law. Since the standard Noether theorem is also based on a holonomic variational equation, it inherits the same shortcomings as the standard action principle.

Generalizations of the standard action principle and the standard Noether theorems to a wider class of physical systems and processes are being actively studied. One of the main goals of such generalizations is to extend the formalism to dissipative and non-conservative systems. Let us note two main approaches to generalizing the action principle and the Noether theorem:

(I) The first approach is based on non-holonomic variational equations. The standard action principles and the Noether theorems use only holonomic variation equations. Therefore, there are significant restrictions on the classes of field equations, equations of motions and on the construction of the theories, in which they are used. For a correct, self-consistent description of the widest class of processes and systems with irreversibility and dissipation one should use the non-holonomic variational equations. An important generalization of the standard action principle was proposed by Sedov [23] in works [24–28]. The Sedov non-holonomic variational equations for electromagnetic, gravitational, hydrodynamic and thermodynamic fields are used to construct different models of continuous media with irreversibility and dissipation in papers [24–31] and in books [31–35]. The variational principles for non-potential operators are described in the review of Filippov, Savchin, and Shorokhov [36].

(II) The second approach is based on fractional calculus. Fractional calculus of differential and integral operators of non-integer order [37–43] is used to describe nonlocal systems and processes in physics (for example, see handbooks [44,45] and books [46–54]). The well-known fractional derivatives and integrals can be used to generalize the standard action principle and the standard Noether theorems. These generalizations allow us to describe a wide range of systems and fields that are non-Lagrangian and non-Hamiltonian in a standard sense. These types of systems and fields are fractional Lagrangian and fractional Hamiltonian systems and fields. An important area of research is the generalization of the Lagrangian and Hamiltonian mechanics by including derivatives of non-integer order [55–58]. In this case, Lagrangians and Hamiltonians with fractional derivatives lead directly to equations of motion with non-conservative and dissipative forces.

Let us consider in more detail the second approach and describe the basic results in this direction. The main results obtained in the use of fractional calculus to generalize the action principle and Noether theorems are summarized to the following achievements.

(A) The mathematical basis of the action principles and the Noether theorems is variational calculus. The standard variational calculus considers the holonomic functionals that are represented by definite integrals of integer orders involving functions and their derivatives of integer orders. An important concept of variational calculus is the concept of a functional derivative (variation derivative). One of the directions of generalizations of variational calculus is related to fractional calculus of integrals and derivatives of non-integer orders.

   (A1) The holonomic functionals can be considered as definite integer-order integrals involving functions and their fractional derivatives of non-integer orders. This type of generalization is called “Fractional calculus of variations” (FCofV) or “Fractional variational calculus” [59–68]. Note that in paper [61], the fractional integrals are considered in addition to fractional derivatives in FCofV.

   (A2) There are other approaches to the generalization of the calculus of variations. For example, one can define a fractional generalization of functional (variational) derivatives on non-integer orders, or the functional itself can be defined as a fractional integral of non-integer orders. However, these approaches are less developed. Attempts to formulate such generalizations are discussed in the papers [69–74].

(B) The standard action principle and Noether's theorem can be generalized by using the fractional calculus in the framework of the FCofV-approach. Lagrangians with fractional derivatives lead directly to equations of motion with non-conservative forces such as resistance forces, friction, and dissipative forces. Using Lagrangian functions, which depend on coordinates and its fractional derivatives with respect to time, once can obtain fractional differential equations of motion as fractional Euler–Lagrange equations. Generalizations of the standard action principle for systems that are described by equation with fractional derivatives of non-integer orders are proposed in works [51,75–91].

- (B1) A fractional action principle and fractional Euler–Lagrange equations are proposed in [51,75–91].
- (B2) A fractional action principle for fractional field theories is considered in [92–99].
- (B3) Noether's theory for classical non-conservative mechanics is discussed in [55–57].
- (B4) Generalizations of the Noether theorem for fractional Lagrangian and Hamiltonian systems is considered in [51,86,100–118].
- (B5) The Noether theorem for fractional Birkhoffian systems is considered in [119–123].

(C) As an important extension of the FCofV, one can use operator kernels belonging to a wide class of functions, while retaining the fractional analogs of the fundamental theorems. Attempts to make such generalizations are considered in works [124–127].

To describe a wider type of non-localities, we can use the general fractional calculus (GFC), which is based on Sonin's ideas [128,129]. The most convenient form of GFC is the Luchko's form of GFC [130–138]. The Luchko approach is also developed and appied to various sciences in [139–149]. Other forms of GFC are described and applied in [150–163]. The general fractional integrals (GFIs) and general fractional derivatives (GFDs) form the GFC, in which generalizations of the fundamental theorems of standard calculus are satisfied.

In this proposed paper, the generalizations are built within the framework of the ideology (approach) of the general fractional dynamics (GFDynamics) suggested in [164]. This means that the non-local properties of dynamical systems are studied by using of GFC, equations with GFIs and GFDs. The approach implies research and results for general form of nonlocality, which can be described by general-form operator kernels, and not thier particular form.

The novelty of the proposed work in comparison with other papers devoted the generalizations of action principle and Noether theorems, as well as the main differences of the proposed GF action principle and GF Noether theorems, are the following.

(1) $[FC \longrightarrow GFC]$. Firstly, in the proposed generalization of the standard action principle and standard Noether theorems, we use general fractional calculus (GFC) instead of the usual fractional calculus, which mainly uses power-type operator kernels. The GFC allows us to consider different types of non-locality in space and time, in contrast to the usual fractional calculus. In the proposed paper, the Luchko GFC is used to generalize standard and fractional action principles and Noether theorems. The GFC is a tool that allows us to consider different types of nonlocalities in space and time.

(2) $[Holonomic \longrightarrow GFNon-Holonomic]$. Secondly, the use of a non-holonomic variation equation is proposed, in contrast to the use of a holonomic variation equation in the standard action principle. This allows us to derive field equations and equations of motion for non-Lagrangian field theories and systems. It is known that the equations, which are derived from a Lagrangian with non-integer (fractional) order derivatives, are non-Lagrangian systems in the standard sense. The proposed approach makes it possible to obtain equations of motion (and field equations) not only for non-Lagrangian field theory, but even for generalized non-Lagrangian systems in the sense of fractional dynamics. In the proposed paper, we consider the non-holonomic functionals that are definite integer-order

integrals involving functions and their GFDs and GFIs. A generalization of the Sedov non-holonomic variational equation for the case fractional dynamics and general fractional dynamics is proposed.

(3) $[FDs \longrightarrow GFDs + GFIs]$. Thirdly, the proposed principle allows us to obtain not only fractional differential equations, but also equations containing both general fractional derivatives and general fractional integrals. Moreover, the GF integral equations can be not only fractional integral equations, but also belong to a wide class of equations with general fractional integrals. In a non-holonomic variational equation, we consider variations of the GFIs in addition to the variations of the GFDs. In a particular case, this gives a generalization of the standard Sedov variational equation by taking into account changes in the integrals of fields, if the system depends on them. In proposed paper, a generalization of the action principle and Noether theorem for fractional field equations is proposed, in contrast to the fractional action principle, which is usually suggested to derive equations of motion of mechanical systems. By virtue of this, the principle of action for fractional dynamic systems is a special case of the proposed principle.

The GF action principle and the GF Noether theorems, which are proposed in this paper, contain the standard and fractional forms of action principle and Noether's theorems as special cases.

In Section 2, a short introduction to the GFC on $[a_\mu, b_\mu]$ and new notations for partial GFDs and GFIs are described. In Section 3, non-holonomic variational equation as a generalization of action principle is proposed and proved. The GF Euler–Lagrange equation with GFDs and GFIs are derived. In Section 4, GF Noether theorem is proved. In Section 5, an example of the application of the GF action principle and the GF Noether theorem to fractional field equations is described. GF energy–momentum tensor, GF orbital and spin angular-momentum tensors are derived. Example of field equations for real scalar field and real vector field are proposed.

2. Preliminary: GFC on $[a_\mu, b_\mu]$ and Notations

Let us consider n-dimensional space-time. The point x of this space-time is labeled by coordinates x_μ, where $\mu = 1, \ldots, n$ and x_μ are independent variables.

The GFC is formulated for the Luchko set $L_1(\mathbb{R}^n_+)$ of kernel pairs $\{(M_\mu(x_\mu),\, K_\mu(x_\mu),\, \mu = 1, \ldots n)\}$ for $x \in \mathbb{R}_+$. The kernels of this set satisfy two conditions:

$$\int_0^{x_\mu} M_\mu(x_\mu - z_\mu)\, K_\mu(z_\mu)\, dz_\mu = \{1\} = \begin{cases} 1 & \text{if } x_\mu \in (0, b_\mu - a_\mu] \\ 0 & \text{if } x_\mu \notin (0, b_\mu - a_\mu] \end{cases}, \tag{1}$$

$$M_\mu(x_\mu),\, K_\mu(x_\mu) \in C_{-1}(0, b_\mu - a_\mu], \tag{2}$$

where $f(x_\mu) \in C_{-1}(0, b_\mu - a_\mu]$ if there is such a function $g(x_\mu) \in C_{-1}(0, b_\mu - a_\mu]$ that $f(x_\mu) = (x_\mu - a_\mu)^{p_\mu}\, g(x_\mu)$ with $p > -1$.

Nonlocality in space and time is characterized and described by kernel pairs of integral and differential operators. Let us give some examples of kernel pairs $(M_\mu(x_\mu),\, K_\mu(x_\mu))$ that belong to the Luchko set (see Table 1 of [147] (pp. 5–7), Table 1 of [148] (p. 15), [149] (p. 11), [144] (pp. 21–22), [145] (p. 10)). Note that we can also consider the kernel pairs $(M_{\mu,new} = \lambda^{-1} K_\mu(x_\mu), K_{\mu,new} = \lambda M_\mu(x_\mu))$, where $(M_\mu(x_\mu), K_\mu(x_\mu))$ are pairs of this list of examples.

- An example of kernel pairs describing the power-law type of nonlocality.

$$M_\mu(x_\mu) = h_{\alpha_\mu}(\lambda x_\mu) = \frac{(\lambda\, x_\mu)^{\alpha_\mu - 1}}{\Gamma(\alpha_\mu)},$$

$$K_\mu(x_\mu) = \lambda\, h_{1-\alpha_\mu}(\lambda x_\mu) = \frac{\lambda\, (\lambda\, x_\mu)^{-\alpha_\mu}}{\Gamma(1 - \alpha_\mu)}. \tag{3}$$

- An example of kernel pairs describing gamma-distributed nonlocality.

$$M_\mu(x_\mu) \;=\; h_{\alpha_\mu,\lambda}(\lambda\, x_\mu) \;=\; \frac{(\lambda\, x_\mu)^{\alpha_\mu-1}}{\Gamma(\alpha_\mu)} e^{-\lambda\, x_\mu},$$

$$K_\mu(x_\mu) \;=\; \lambda\, h_{1-\alpha_\mu,\lambda}(\lambda x_\mu) \;+\; \frac{\lambda}{\Gamma(1-\alpha_\mu)}\, \gamma(1-\alpha_\mu, \lambda x_\mu). \tag{4}$$

- An example of kernel pairs describing the Mittag–Leffler type of nonlocality.

$$M_\mu(x_\mu) \;=\; (\lambda\, x_\mu)^{\beta_\mu-1}\, E_{\alpha_\mu,\beta_\mu}[-(\lambda\, x_\mu)^{\alpha_\mu}],$$

$$K_\mu(x_\mu) \;=\; \frac{\lambda\, (\lambda\, x_\mu)^{\alpha_\mu-\beta_\mu}}{\Gamma(\alpha_\mu-\beta_\mu+1)} \;+\; \frac{\lambda\, (\lambda\, x_\mu)^{-\beta_\mu}}{\Gamma(1-\beta_\mu+1)}. \tag{5}$$

- An example of kernel pairs describing the Bessel type of nonlocality.

$$M_\mu(x_\mu) \;=\; (\sqrt{\lambda\, x_\mu})^{\alpha_\mu-1}\, J_{\alpha_\mu-1}(2\sqrt{\lambda\, x_\mu}),$$

$$K_\mu(x_\mu) \;=\; \lambda\, (\sqrt{\lambda\, x_\mu})^{-\alpha_\mu}\, I_{-\alpha_\mu}(2\sqrt{\lambda\, x_\mu}). \tag{6}$$

- An example of kernel pairs describing the hypergeometric type of nonlocality.

$$M_\mu(x_\mu) \;=\; (\lambda x_\mu)^{\alpha_\mu-1} \Phi(\beta_\mu, \alpha_\mu; -\lambda\, x_\mu),$$

$$K_\mu(x_\mu) \;=\; \frac{\lambda\, \sin(\pi\alpha_\mu)}{\pi}\, (\lambda\, x_\mu)^{-\alpha_\mu}\, \Phi(-\beta_\mu, 1-\alpha_\mu; -\lambda\, x_\mu). \tag{7}$$

- An example of kernel pairs describing the cosine type of nonlocality.

$$M_\mu(x_\mu) \;=\; \frac{\cos(2\sqrt{\lambda\, x_\mu})}{\sqrt{\pi\, \lambda\, x_\mu}}, \quad K_\mu(x_\mu) \;=\; \frac{\lambda\, \cosh(2\sqrt{\lambda\, x_\mu})}{\sqrt{\pi\, \lambda\, x_\mu}}. \tag{8}$$

Here, $\gamma(\beta, x)$ is the incomplete gamma function; $E_{\alpha,\beta}[x]$ is the two-parameters Mittag–Leffler function; $J_\nu(x)$ is the Bessel function; $I_\nu(x)$ is the modified Bessel function; $\Phi(\beta, \alpha; x)$ is the confluent hypergeometric Kummer function. The physical dimensions of the kernels are $[M_\mu(x_\mu)] = [1]$ and $[K_\mu(x_\mu)] = [x_\mu]^{-1}$, where $\lambda > 0$, $[\lambda] = [x_\mu]^{-1}$, $0 < \alpha_\mu \le \beta_\mu < 1$, and $x_\mu > 0$.

The left-sided GFDs of the Riemann–Liouville (RL) and Caputo (C) types are defined as

$$(\partial^{(K)}_{\mu,a+} f)(x) \;=\; \frac{\partial}{\partial x_\mu} \int_{a_\mu}^{x_\mu} dz_\mu\, K_\mu(x_\mu \,-\, z_\mu)\, f(z), \tag{9}$$

$$({}_*\partial^{(K)}_{\mu,a+} f)(x) \;=\; \int_{a_\mu}^{x_\mu} dz_\mu\, K_\mu(x_\mu \,-\, z_\mu)\, \frac{\partial f(z)}{\partial z_\mu}, \tag{10}$$

where $a_\mu < x_\mu \le b_\mu$, $\mu = 1, \ldots n$, and $f(x) \in C^1_{-1}(W_+)$. The condition $f(x) \in C^1_{-1}(W_+)$ means that first-order partial derivatives of function $f(x)$ can be represented as

$$\frac{\partial f(x)}{\partial x_\mu} \;=\; (x_\mu \,-\, a_\mu)^{p_\mu}\, g(x)$$

for all $\mu = 1, \ldots n$, where $g(x) \in C(W_+)$, $p_\mu > -1$ and

$$W_+ \;=\; \{x : \quad a_\mu \,<\, x_\mu \,\le\, b_\mu, \quad \mu \,=\, 1, \ldots, n\}.$$

The right-sided GFDs of RL and C types are defined as

$$(\partial^{(K)}_{\mu,b-} f)(x) = \frac{\partial}{\partial x_\mu} \int_{x_\mu}^{b_\mu} dz_\mu\, K_\mu(z_\mu - x_\mu)\, f(z), \tag{11}$$

$$({}_*\partial^{(K)}_{\mu,b-} f)(x) = \int_{x_\mu}^{b_\mu} dz_\mu\, K_\mu(z_\mu - x_\mu)\, \frac{\partial f(z)}{\partial x_\mu}, \tag{12}$$

where $a_\mu \le x_\mu < b_\mu$, $\mu = 1, \dots n$, and $f(x) \in C^1_{-1}(W_-)$. The condition $f(x) \in C^1_{-1}(W_-)$ means that first-order partial derivatives of function $f(x)$ can be represented as

$$\frac{\partial f(x)}{\partial x_\mu} = (b_\mu - x_\mu)^{p_\mu}\, g(x)$$

for all $\mu = 1, \dots n$, where $g(x) \in C(W_-)$, $p_\mu > -1$ and

$$W_- = \{x : \quad a_\mu \le x_\mu < b_\mu, \quad \mu = 1, \dots, n\}.$$

The left-sided and right-sided GFIs are defined as

$$(\mathcal{I}^{(M),\mu}_{a+} f)(x) = \int_{a_\mu}^{x_\mu} dz_\mu\, M_\mu(x_\mu - z_\mu)\, f(z) \tag{13}$$

for $f(x) \in C_{-1}(W_+)$ and

$$(\mathcal{I}^{(M),\mu}_{b-} f)(x) = \int_{x_\mu}^{b_\mu} dz_\mu\, M_\mu(z_\mu - x_\mu)\, f(z) \tag{14}$$

for $f(x) \in C_{-1}(W_-)$. Function $f(x)$ belongs to the set $C_{-1}(W_+)$, if it can be represented as

$$f(x_1, \dots, x_\mu, \dots, x_n) = (x_\mu - a_\mu)^{p_\mu}\, g(x)$$

for all $\mu = 1, \dots n$, where $g(x) \in C(W_+)$, and $p_\mu > -1$. Function $f(x)$ belongs to the set $C_{-1}(W_-)$, if it can be represented as

$$f(x_1, \dots, x_\mu, \dots, x_n) = (b_\mu - x_\mu)^{p_\mu}\, g(x)$$

for all $\mu = 1, \dots n$, where $g(x) \in C(W_-)$, and $p_\mu > -1$.

Note some basic properties of GFDs and DFIs [165]:

(a) The GFIs satisfy the semi-group and commutativity properties, which are proved as Proposition 3 in [165] (pp. 5–6).

(b) Equations expressing GFDs of the Riemann–Liouville type in terms of GFDs of the Caputo type are proved as Proposition 4 in [165] (pp. 8–9).

(c) The GFDs satisfy rules of fractional integration by parts, which are proved as Proposition 5 in [165] (p. 8).

(d) The GFIs satisfy rules of fractional integration by parts, which are proved as Proposition 2 in [165] (p. 5).

(e) The fundamental theorems of GFC are proved as Theorems 1 and 2 in [165] (pp. 9–11).

For the derivation of field equations and equations of motions from the action principle and from non-holonomic variational equation, and also for the proof of the general fractional Noether theorem it is important the rules of integration by parts for GFDs and GFIs. Let us write down these rules without proofs in the new notation. Equation of the integration by parts for the left-sided GFDs of RL type is

$$\int_{a_\mu}^{b_\mu} dx_\mu\, f(x)\, (\partial^{(K)}_{\mu,a+} g)(x) = \int_{a_\mu}^{b_\mu} dx_\mu\, g(x)\, ({}_*\partial^{(K)}_{\mu,b-} f)(x) + \int_{a_\mu}^{b_\mu} \frac{\partial}{\partial x_\mu} \Big(f(x)\, (\mathcal{I}^{(K)}_{\mu,a+} g)(x) \Big)\, dx_\mu, \tag{15}$$

where $f(x) \in C^1_{-1}[a_\mu, b_\mu)$ and $g(x) \in C^1_{-1}(a_\mu, b_\mu]$. The inverse to Equation (15) with redesignation of f and g is the equation of the integration by parts for the right-sided GFDs of Caputo type

$$\int_{a_\mu}^{b_\mu} dx_\mu\, f(x)\, ({}_*\partial^{(K)}_{\mu,b-} g)(x) \;=\; \int_{a_\mu}^{b_\mu} dx_\mu\, g(x)\, (\partial^{(K)}_{\mu,a+} f)(x) \;-\; \int_{a_\mu}^{b_\mu} \frac{\partial}{\partial x_\mu} \Big(g(x)\, (\mathcal{I}^{(K)}_{\mu,a+} f)(x) \Big)\, dx_\mu, \tag{16}$$

where $g(x) \in C^1_{-1}[a_\mu, b_\mu)$ and $f(x) \in C^1_{-1}(a_\mu, b_\mu]$. Equations (15) and (16) are proved as Equations (28) and (32) in Proposition 5 of [165]. Equation of the integration by parts for the left-sided GFDs of Caputo type is

$$\int_{a_\mu}^{b_\mu} dx_\mu\, f(x)\, ({}_*\partial^{(K)}_{\mu,a+} g)(x) \;=\; \int_{a_\mu}^{b_\mu} dx_\mu\, g(x)\, (\partial^{(K)}_{\mu,b-} f)(x) \;+\; \int_{a_\mu}^{b_\mu} \frac{\partial}{\partial x_\mu} \Big(g(x)\, (\mathcal{I}^{(K)}_{\mu,b-} f)(x) \Big)\, dx_\mu, \tag{17}$$

where $f(x) \in C^1_{-1}[a_\mu, b_\mu)$ and $g(x) \in C^1_{-1}(a_\mu, b_\mu]$. The inverse to Equation (17) with redesignation of f and g is the equation of the integration by parts for the right-sided GFDs of RL type

$$\int_{a_\mu}^{b_\mu} dx_\mu\, f(x)\, (\partial^{(K)}_{\mu,b-} g)(x) \;=\; \int_{a_\mu}^{b_\mu} dx_\mu\, g(x)\, ({}_*\partial^{(K)}_{\mu,a+} f)(x) \;-\; \int_{a_\mu}^{b_\mu} \frac{\partial}{\partial x_\mu} \Big(f(x)\, (\mathcal{I}^{(K)}_{\mu,b-} g)(x) \Big)\, dx_\mu, \tag{18}$$

where $g(x) \in C^1_{-1}[a_\mu, b_\mu)$ and $f(x) \in C^1_{-1}(a_\mu, b_\mu]$. Equations (17) and (18) are proved as Equations (29) and (30) in Proposition 5 of [165].

For further consideration, it is convenient to use the sum (or linear combination) of Equations (15)–(18). For example,

$$\begin{gathered}
\int_{a_\mu}^{b_\mu} dx_\mu \Big(F_1\, (\partial^{(K)}_{\mu,a+} G_1) \;+\; F_2\, (\partial^{(K)}_{\mu,b-} G_2) \;+\; F_3\, ({}_*\partial^{(K)}_{\mu,a+} G_3) \;+\; F_4\, ({}_*\partial^{(K)}_{\mu,b-} G_4) \Big) \;= \\
\int_{a_\mu}^{b_\mu} dx_\mu \Big(G_1\, ({}_*\partial^{(K)}_{\mu,b-} F_1) \;+\; G_2\, ({}_*\partial^{(K)}_{\mu,a+} F_2) \;+\; G_3\, (\partial^{(K)}_{\mu,b-} F_3) \;+\; G_4\, (\partial^{(K)}_{\mu,a+} F_4) \Big) \;+ \\
\int_{a_\mu}^{b_\mu} \frac{\partial}{\partial x_\mu} \Big(F_1\, (\mathcal{I}^{(K)}_{\mu,a+} G_1) \;-\; F_2\, (\mathcal{I}^{(K)}_{\mu,b-} G_2) \;+\; G_3\, (\mathcal{I}^{(K)}_{\mu,b-} F_3) \;-\; G_4\, (\mathcal{I}^{(K)}_{\mu,a+} F_4) \Big)\, dx_\mu.
\end{gathered} \tag{19}$$

Expressions of form (19) are very bulky, especially if we consider all possible sequential action of two GFD on the field functions, for example, such as $\partial^{(K)}_{\mu,a+}\, \partial^{(K)}_{\nu,a+}$, $\partial^{(K)}_{\mu,a+}\, \partial^{(K)}_{\nu,b-}$, and so on. To simplify equations, we use the notations

$$\widehat{\partial} \;:=\; \{\widehat{\partial}_{(\mu,q)}, \quad q \;=\; 1, 2, 3, 4\} \;=\; \{\partial^{(K)}_{\mu,a+}, \partial^{(K)}_{\mu,b-}, {}_*\partial^{(K)}_{\mu,a+}, {}_*\partial^{(K)}_{\mu,b-}\}. \tag{20}$$

The notation $\widehat{\partial}_{(\mu,q)}$ means that $q = 1$ gives the left-sided GFD of RL type, $q = 2$ gives the right-sided GFD of RL type, $q = 3$ gives the left-sided GFD of C type, $q = 4$ gives the right-sided GFD of C type with respect to coordinate x_μ. Let us also define the operator notation with the reverse sequence

$$\widehat{\partial}^{\dagger} \;:=\; \{\widehat{\partial}^{\dagger}_{(\mu,q)}, \quad q \;=\; 1, 2, 3, 4\} \;=\; \{{}_*\partial^{(K)}_{\mu,b-}, {}_*\partial^{(K)}_{\mu,a+}, \partial^{(K)}_{\mu,b-}, \partial^{(K)}_{\mu,a+}.\}. \tag{21}$$

Note that

$$\widehat{\partial}^{\dagger}_{(\mu,q)} \;=\; \widehat{\partial}^{\dagger}_{(\mu,5-q)}, \quad (q \;=\; 1, 2, 3, 4).$$

It should be noted the non-commutativity of the GFDs in the general case

$$\widehat{\partial}_{(\mu,q)}\, \widehat{\partial}_{(\nu,p)}\, f(x) \;\neq\; \widehat{\partial}_{(\nu,p)}\, \widehat{\partial}_{(\mu,q)}\, f(x). \tag{22}$$

For GFIs, we can use the notations

$$\widehat{\mathcal{I}} := \{\widehat{\mathcal{I}}_{(\mu,q)}, \quad q = 1, 2\} = \{\mathcal{I}^{(K)}_{\mu,a+}, \mathcal{I}^{(K)}_{\mu,b-}\}, \tag{23}$$

$$\widehat{\mathcal{I}}^{\dagger} := \{\widehat{\mathcal{I}}^{\dagger}_{(\mu,q)}, \quad q = 1, 2\} = \{\mathcal{I}^{(K)}_{\mu,b-}, \mathcal{I}^{(K)}_{\mu,a+}\}. \tag{24}$$

Note that

$$\widehat{\mathcal{I}}^{\dagger}_{(\mu,q)} = \widehat{\mathcal{I}}_{(\mu,3-q)}, \quad (q = 1, 2).$$

Using the proposed notations, Equation (19) takes the form

$$\int_{a_\mu}^{b_\mu} \Big(\sum_{q=1}^{4} F_q \, \widehat{\partial}_{(\mu,q)} \, G_q \Big) \, dx_\mu = \int_{a_\mu}^{b_\mu} \Big(\sum_{q=1}^{4} G_q \, \widehat{\partial}^{\dagger}_{(\mu,q)} \, F_q \Big) \, dx_\mu +$$

$$\int_{a_\mu}^{b_\mu} \frac{\partial}{\partial x_\mu} \Big(\sum_{q=1}^{2} (-1)^{q+1} \Big(F_q \, \widehat{\mathcal{I}}_{(\mu,q)} \, G_q + G_{q+2} \, \widehat{\mathcal{I}}^{\dagger}_{(\mu,q)} \, F_{q+2} \Big) \Big) \, dx_\mu, \tag{25}$$

if $F_q(x) \in C^1_{-1}(W_q)$ and $G_q(x) \in C^1_{-1}(W^{\dagger}_q)$, where $W_q = W_-$ for $q = 1, 3$ and $W_q = W_+$ for $q = 2, 4$, $W^{\dagger}_{\pm} = W^{\dagger}_{\mp}$, i.e., $W^{\dagger}_q = W_+$ for $q = 1, 3$ and $W^{\dagger}_q = W_-$ for $q = 2, 4$.

It should be emphasized that GFIs in (25) contain kernels of GFDs.

In general, the Lagrangian density and the density of the non-holonomic functional can depend not only on the GFDs, but also on the GFIs of fields. In this case, the rules of integration by parts for GFIs (Proposition 2 in [165] (p. 5)) should be used by the equation

$$\int_{a_\mu}^{b_\mu} dx_\mu \, f(x) \, (\mathcal{I}^{(M)}_{\mu,a+} g)(x) = \int_{a_\mu}^{b_\mu} dx_\mu \, g(x) \, (\mathcal{I}^{(M)}_{\mu,b-} f)(x), \tag{26}$$

if $f(x) \in C_{-1}[a_\mu, b_\mu)$ and $g(x) \in C_{-1}(a_\mu, b_\mu]$. The inverse to Equation (26) with redesignation of f and g is

$$\int_{a_\mu}^{b_\mu} dx_\mu \, f(x) \, (\mathcal{I}^{(M)}_{\mu,b-} g)(x) = \int_{a_\mu}^{b_\mu} dx_\mu \, g(x) \, (\mathcal{I}^{(M)}_{\mu,a+} f)(x). \tag{27}$$

if $f(x) \in C_{-1}(a_\mu, b_\mu]$ and $g(x) \in C_{-1}[a_\mu, b_\mu)$. For sum (or linear combinations) of the left-sided and right-sided GFIs, we have the equation

$$\int_{a_\mu}^{b_\mu} \Big(\sum_{q=1}^{2} F_q \, \widehat{\mathcal{I}}_{(\mu,q)} \, G_q \Big) \, dx_\mu = \int_{a_\mu}^{b_\mu} \Big(\sum_{q=1}^{2} G_q \, \widehat{\mathcal{I}}^{\dagger}_{(\mu,q)} \, F_q \Big) \, dx_\mu, \tag{28}$$

if $F_q(x) \in C_{-1}(W_q)$ and $G_q(x) \in C_{-1}(W^{\dagger}_q)$, Equation can also be used for theories, in which the Lagrangian density and the density of the non-holonomic functional depend on the GFIs of fields.

3. Non-Holonomic Action Principle

The standard action principle is actively used in physics and mechanics and field theory. This principle is represented by holonomic variational equations. It is well-known that there are a lot of equations of motion and field equations that cannot be derived from the standard action principle. Therefore it is important to have generalization of the action principle.

In this section, we proposed to use the Sedov non-holonomic variational equations as generalizations of the standard action principle. The proposed non-holonomic variational equation is a fractional generalization, or more precisely a general fractional generalization, of Sedov variational equations for general fractional field theory.

Let a point x of space–time has coordinates x^{μ}, where $\mu = 1, \ldots, n$ and x^{μ} are independent variables. Fractional generalizations of the action principle and Noether's theorem are considered for tensor field $U = U(x)$, where $U_j(x)$ are components of this field and x is a point of n-dimensional space. As a field $U_j(x)$, one can consider, for example, a scalar field $U_j(x) = \Phi(x)$, a vector field $U_j(x) = V_{\mu}(x)$, a tensor field $U_j(x) = F_{\mu\nu}$.

For simplicity, the generalization of the action principle and the Noether theorem will be considered for the region R in the form of a multi-dimensional box R in the Cartesian coordinate system

$$R = \{x: \quad a_{\mu} \leq x_{\mu} \leq b_{\mu}, \quad \mu = 1, \ldots, n\}. \tag{29}$$

To consider a wider class of areas, one should use orthogonal curvilinear coordinates (OCC) and the methods described in work [140].

3.1. Variations of Fields and Coordinates

To derive GF field equations (or GF equation of motion) from the holonomic and non-holonomic variational equations, the infinitesimal transformations of fields, its GFDs and GFIs should be used

$$U_j(x) \quad \longrightarrow \quad {U'}_j(x) = U_j(x) + \overline{\delta}\, U_j(x), \tag{30}$$

$$\widehat{\partial}_{(\mu,q)} U_j(x) \quad \longrightarrow \quad \widehat{\partial}_{(\mu,q)} {U'}_j(x) = \widehat{\partial}_{(\mu,q)} U_j(x) + \overline{\delta}\, \widehat{\partial}_{(\mu,q)} U_j(x), \tag{31}$$

$$\widehat{\Im}_{(\mu,q)} U_j(x) \quad \longrightarrow \quad \widehat{\Im}_{(\mu,q)} {U'}_j(x) = \widehat{\Im}_{(\mu,q)} U_j(x) + \overline{\delta}\, \widehat{\Im}_{(\mu,q)} U_j(x). \tag{32}$$

To prove the Noether theorem, consider the infinitesimal transformations of coordinates, fields and its GFDs and GFIs

$$x^{\mu} \quad \longrightarrow \quad {x'}^{\mu} = x^{\mu} + \delta\, x^{\mu}, \tag{33}$$

$$U_j(x) \quad \longrightarrow \quad {U'}_j(x') = U_j(x) + \delta\, U_j(x), \tag{34}$$

$$\widehat{\partial}_{(\mu,q)} U_j(x) \quad \longrightarrow \quad \widehat{\partial}_{(\mu,q)} {U'}_j(x') = \widehat{\partial}_{(\mu,q)} U_j(x) + \delta\, \widehat{\partial}_{(\mu,q)} U_j(x), \tag{35}$$

$$\widehat{\Im}_{(\mu,q)} U_j(x) \quad \longrightarrow \quad \widehat{\Im}_{(\mu,q)} {U'}_j(x') = \widehat{\Im}_{(\mu,q)} U_j(x) + \delta\, \widehat{\Im}_{(\mu,q)} U_j(x), \tag{36}$$

where the variations of the GFDs $\widehat{\partial}_{(\mu,q)} U_j(x)$ and the GFIs $\widehat{\Im}_{(\mu,q)} U_j(x)$ cannot be represented as the GFDs and GFIs of the variations $\delta\, U_j(x)$, i.e., we have the inequalities

$$\delta\, \widehat{\partial}_{(\mu,q)}\, U_j(x) \neq \widehat{\partial}_{(\mu,q)}\, \delta\, U_j(x), \quad \delta\, \widehat{\Im}_{(\mu,q)}\, U_j(x) \neq \widehat{\Im}_{(\mu,q)}\, \delta\, U_j(x), \tag{37}$$

that means the non-commutativity of $\widehat{\partial}_{(\mu,q)}$, $\widehat{\Im}_{(\mu,q)}$, and δ.

Note that $\delta\, U_j(x)$ is the variation of the field functions $U_j(x)$ due to both the change in its form and the change in its argument. The variation due to the change in the form of the field

$$\overline{\delta}\, U_j(x) = {U'}_j(x) - U_j(x) \tag{38}$$

which commutates with GFDs $\widehat{\partial}_{(\mu,q)}$ and GFIs $\widehat{\Im}_{(\mu,q)}$ by definition

$$\overline{\delta}\, \widehat{\partial}_{(\mu,q)} U_j(x) = \widehat{\partial}_{(\mu,q)}\, \overline{\delta}\, U_j(x), \tag{39}$$

$$\overline{\delta}\, \widehat{\Im}_{(\mu,q)} U_j(x) = \widehat{\Im}_{(\mu,q)}\, \overline{\delta}\, U_j(x). \tag{40}$$

The variation

$$\delta\, U_j(x) \,=\, U'_j(x') \,-\, U_j(x)$$

can be represented as

$$\delta\, U_j(x) \,=\, \overline{\delta}\, U_j(x') \,+\, U_j(x') \,-\, U_j(x) \,=\, \overline{\delta}\, U_j(x') \,+\, \partial_\mu U_j(x)\, \delta x^\mu, \tag{41}$$

where the first-order derivative ∂_μ is used, since δ is standard variation of the first-order.

In the derivation of the field equations, the variations of the holonomic and non-holonomic functionals are realized due to a change in the forms of the fields and their GFDs and GFIs. These variations are denoted as $\overline{\delta}\, U_j(x)$ and $\overline{\delta}\, U_j(x)$. Since the coordinates x^μ do not participate in the variation, the integration limits remained fixed. Therefore, the variations of the the holonomic functional (action) is completely determined by the variation of the Lagrangian density. Since the field functions on the boundaries of the integration regions remain fixed, the integral over the surface, which bounds the region of integration, vanishes.

In the proof of the Noether's theorem, the coordinates are involved in the variation in addition to the variation of field functions. Such variations of fields U_j are denoted as $\delta\, U_j$. The variation of coordinates leads to the fact that the limits of integration are changed. Therefore, the result of the variation takes a different form.

3.2. Field Equations from Non-Holonomic Variational Equation

In the general case, to derive the field equation one can use the Sedov non-holonomic variational equation

$$\overline{\delta}\, S \,+\, \overline{\delta}\, W^* \,=\, 0, \tag{42}$$

where $\overline{\delta}\, S$ is a variation of the holonomic functional

$$\overline{\delta}\, S \,=\, \overline{\delta} \int_R \Lambda(x,\, U,\, \widehat{\partial} U,\, \widehat{\mathcal{I}} U)\, d^n x \,=\, \int_R \overline{\delta}\, \Lambda(x,\, U,\, \widehat{\partial} U,\, \widehat{\mathcal{I}} U)\, d^n x \tag{43}$$

with Lagrangian density $\Lambda(x,\, U,\, \widehat{\partial} U,\, \widehat{\mathcal{I}} U) \,=\, \Lambda(x,\, \{U_j(x)\},\, \{\widehat{\partial}_{(\mu,q)} U_j(x)\},\, \{\widehat{\mathcal{I}}_{(\mu,q)} U_j(x)\})$, and $\overline{\delta}\, W^*$ is the non-holonomic functional

$$\overline{\delta}\, W^* \,=\, \int_R \overline{\delta}\, \mathcal{W}^*\, d^n x \,=\, \int_R \Big(B^j(x,\, U,\, \widehat{\partial} U,\, \widehat{\mathcal{I}} U)\, \overline{\delta}\, U_j(x) +$$

$$\sum_{j=1}^{4} C_q^{j\,\mu}(x,\, U,\, \widehat{\partial} U,\, \widehat{\mathcal{I}} U)\, \overline{\delta}\, \widehat{\partial}_{(\mu,q)} U_j(x) \,+\, \sum_{j=1}^{2} A_q^{j\,\mu}(x,\, U,\, \widehat{\partial} U,\, \widehat{\mathcal{I}} U)\, \overline{\delta}\, \widehat{\mathcal{I}}_{(\mu,q)} U_j(x) \Big)\, d^n x \tag{44}$$

with density $\overline{\delta}\, \mathcal{W}^*$. Here, the following notations are used

$$\widehat{\partial} U \,:=\, \{\widehat{\partial}_{(\mu,q)}\, U_j\} \,=\, \{\partial^{(K)}_{\mu,a+}\, U_j,\, \partial^{(K)}_{\mu,b-}\, U_j,\, {}_*\partial^{(K)}_{\mu,a+}\, U_j,\, {}_*\partial^{(K)}_{\mu,b-}\, U_j\}, \tag{45}$$

$$\widehat{\mathcal{I}} U \,:=\, \{\mathcal{I}^{(M)}_{\mu,a+}\, U_j,\, \mathcal{I}^{(M)}_{\mu,b-}\, U_j\}. \tag{46}$$

and

$$\widehat{\partial}^2\, U \,:=\, \{\widehat{\partial}_{(\nu,p)}\, \widehat{\partial}_{(\mu,q)}\, U_j\}, \quad \widehat{\mathcal{I}}^2\, U \,:=\, \{\widehat{\partial}_{(\nu,p)}\, \widehat{\partial}_{(\mu,q)}\, \widehat{\partial}_{(\nu,p)}\, U_j\}. \tag{47}$$

In general, $B^j\, \overline{\delta}\, U_j \,=\, g^{ij}\, B_i\, \overline{\delta}\, U_j$ and $C_q^{j\,\mu}\, \overline{\delta}\, \widehat{\partial}_{(\mu,q)} U_j(x) \,=\, g^{ij}\, C^{mu}_{q\;,j}\, \overline{\delta}\, \widehat{\partial}_{(\mu,q)} U_j(x)$, where $g^{ij} \,=\, g^{ij}(x)$ is a metric tensor. Note that summations are implied over repeated indices. The symbol $\sum_{q=1}$ for the sum over indices q is given explicitly, since the upper limit of summation is different for GFDs and GFIs.

The non-holonomic functional contains the functions $B^j = B^j(x, U, \widehat{\partial}U, \widehat{\Im}U)$, $C_q^{j\mu} = C_q^{j\mu}(x, U, \widehat{\partial}U, \widehat{\Im}U)$ and $A_q^{j\mu} = A_q^{j\mu}(x, U, \widehat{\partial}U, \widehat{\Im}U)$ that cannot be represented as

$$B^j = \frac{\partial F}{\partial U_j}, \quad C_q^{j\mu} = \frac{\partial F}{\partial \widehat{\partial}_{(\mu,q)} U_j}, \quad A_q^{j\mu} = \frac{\partial F}{\partial \widehat{\Im}_{(\mu,q)} U_j} \tag{48}$$

with a function $F = F(x, U, \widehat{\partial}U, \widehat{\Im}U)$. If all conditions (48) are satisfied, then $\bar{\delta} W^*$ is holonomic functional.

Theorem 1. *Let the GF action functional be defined by Equation (43) and the GF non-holonomic functional be defined as (44), where* $\Lambda(x, U, \widehat{\partial}U, \widehat{\Im}U)$, $B^j(x, U, \widehat{\partial}U, \widehat{\Im}U)$, $C_q^{j\mu}(x, U, \widehat{\partial}U, \widehat{\Im}U)$, $A_q^{j\mu}(x, U, \widehat{\partial}U, \widehat{\Im}U)$ *are continuously differentiable function with respect to its arguments* $U_j(x)$, $\widehat{\partial}_{(\mu,q)} U_j(x)$, $\widehat{\Im}_{(\mu,q)} U_j(x)$, *and*

$$\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} \in C^1_{-1}(W_q), \quad C_q^{j\mu} \in C^1_{-1}(W_q), \tag{49}$$

$$\frac{\partial \Lambda}{\partial \widehat{\Im}_{(\mu,q)} U_j} \in C_{-1}(W_q), \quad A_q^{j\mu} \in C_{-1}(W_q). \tag{50}$$

If the variation of the fields $\bar{\delta} U_j(x)$ *and* $\bar{\delta}(\widehat{\Im}_{(\mu,q)} U_j)(x)$ *are equal to zero on the boundaries of the region R, then the field equations (equations of motion) have the form*

$$\left(\frac{\partial \Lambda}{\partial U_j} + B^j\right) + \sum_{q=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} + C_q^{j\mu}\right) + \sum_{q=1}^{2} \widehat{\Im}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\Im}_{(\mu,q)} U_j} + A_q^{j\mu}\right) = 0, \tag{51}$$

where $j = 1, \ldots, m$, *and the summation over the index* μ *from 1 to n is implied.*

Proof. The variation of Λ is represented as

$$\begin{aligned} &\bar{\delta} \Lambda(x, U, \widehat{\partial}U, \widehat{\Im}U) = \\ &\frac{\partial \Lambda}{\partial U_j} \bar{\delta} U_j(x) + \sum_{q=1}^{4} \frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} \bar{\delta} \widehat{\partial}_{(\mu,q)} U_j(x) + \sum_{q=1}^{2} \frac{\partial \Lambda}{\partial \widehat{\Im}_{(\mu,q)} U_j} \bar{\delta} \widehat{\Im}_{(\mu,q)} U_j(x) = \\ &\frac{\partial \Lambda}{\partial U_j} \bar{\delta} U_j(x) + \sum_{q=1}^{4} \frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} \widehat{\partial}_{(\mu,q)} \bar{\delta} U_j(x) + \sum_{q=1}^{2} \frac{\partial \Lambda}{\partial \widehat{\Im}_{(\mu,q)} U_j} \widehat{\Im}_{(\mu,q)} \bar{\delta} U_j(x), \end{aligned} \tag{52}$$

where the properties $\bar{\delta} \widehat{\partial}_{(\mu,q)} U_j = \widehat{\partial}_{(\mu,q)} \bar{\delta} U_j$ and $\bar{\delta} \widehat{\Im}_{(\mu,q)} U_j = \widehat{\Im}_{(\mu,q)} \bar{\delta} U_j(x)$ are used. Equation (52) describes the variation of Λ due to variations in the forms of $U_j(x)$, $\widehat{\partial}_{(\mu,q)} U_j(x)$ and $\widehat{\Im}_{(\mu,q)} U_j(x)$.

Similarly, we obtain the expression

$$\begin{aligned} \bar{\delta} W &= B^j \bar{\delta} U_j(x) + \sum_{q=1}^{4} C_q^{j\mu} \bar{\delta} \widehat{\partial}_{(\mu,q)} U_j(x) + \sum_{q=1}^{2} A_q^{j\mu} \bar{\delta} \widehat{\Im}_{(\mu,q)} U_j(x) = \\ &B^j \bar{\delta} U_j(x) + \sum_{q=1}^{4} C_q^{j\mu} \widehat{\partial}_{(\mu,q)} \bar{\delta} U_j(x) + \sum_{q=1}^{2} A_q^{j\mu} \widehat{\Im}_{(\mu,q)} \bar{\delta} U_j(x). \end{aligned} \tag{53}$$

Note that the GFIs in (53) contain the kernel $M_\mu(x_\mu)$.

The rules of integration by parts for GFDs (25) give

$$\int_R \sum_{q=1}^{4} \frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} \widehat{\partial}_{(\mu,q)} \bar{\delta} U_j(x)\, d^n x = \sum_{q=1}^{4} \int_R \widehat{\partial}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} \right) \bar{\delta} U_j(x)\, d^n x +$$

$$\int_R \frac{\partial}{\partial x_\mu} \sum_{q=1}^{2} (-1)^{q+1} \left(\left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} \right) (\widehat{\mathfrak{I}}_{(\mu,q)} \bar{\delta} U_j)(x) + \bar{\delta} U_j(x)\, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q+2)} U_j} \right) \right) d^n x. \tag{54}$$

Similarly, we obtain the expression

$$\int_R \sum_{q=1}^{4} C^{j\,\mu}_q\, (\widehat{\partial}_{(\mu,q)} \bar{\delta} U_j)(x)\, d^n x = \sum_{q=1}^{4} \int_R (\widehat{\partial}^{\dagger}_{(\mu,q)} C^{j\,\mu}_q)\, \bar{\delta} U_j(x)\, d^n x +$$

$$\int_R \frac{\partial}{\partial x_\mu} \sum_{q=1}^{2} (-1)^{q+1} \left(C^{j\,\mu}_q\, (\widehat{\mathfrak{I}}_{(\mu,q)} \bar{\delta} U_j)(x) + (\widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} C^{j,\mu}_{q+2})\, \bar{\delta} U_j(x) \right) d^n x. \tag{55}$$

Note that in Equations (54) and (55), the GFIs contain the GFD kernel $K_\mu(x_\mu)$. The rules of integration by parts GFIs (28) give

$$\sum_{q=1}^{2} \int_R \frac{\partial \Lambda}{\partial \widehat{\mathfrak{I}}_{(\mu,q)} U_j} \widehat{\mathfrak{I}}_{(\mu,q)} \bar{\delta} U_j(x)\, d^n x = \sum_{q=1}^{2} \int_R \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\mathfrak{I}}_{(\mu,q)} U_j} \right) \bar{\delta} U_j(x)\, d^n x. \tag{56}$$

Similarly, we obtain

$$\sum_{q=1}^{2} \int_R A^{j\,\mu}_q\, (\widehat{\mathfrak{I}}_{(\mu,q)} \bar{\delta} U_j)(x)\, d^n x = \sum_{q=1}^{2} \int_R (\widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} A^{j\,\mu}_q)\, \bar{\delta} U_j(x)\, d^n x. \tag{57}$$

In Equations (56) and (57), the GFIs contain the GFI kernel $M_\mu(x_\mu)$.

As a result, the left-hand side of the non-holonomic variational Equation (42) takes the form

$$\bar{\delta} S + \bar{\delta} W^* = \int_R \left(\frac{\partial \Lambda}{\partial U_j} + B^j \right) \bar{\delta}\, U_j(x)\, d^n x +$$

$$\int_R \left(\sum_{q=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} + C^{j\,\mu}_q \right) + \sum_{q=1}^{2} \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\mathfrak{I}}_{(\mu,q)} U_j} + A^{j\,\mu}_q \right) \right) \bar{\delta}\, U_j(x)\, d^n x +$$

$$\int_R \frac{\partial}{\partial x_\mu} \sum_{q=1}^{2} (-1)^{q+1} \left(\left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} + C^{j\,\mu}_q \right) (\widehat{\mathfrak{I}}_{(\mu,q)} \bar{\delta} U_j)(x) + \right.$$

$$\left. \bar{\delta} U_j(x)\, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q+2)} U_j} + C^{j\,\mu}_{q+2} \right) \right) d^n x. \tag{58}$$

The standard Gauss theorem can be used to transform the divergence in Equation (58) and obtain

$$\int_R \frac{\partial}{\partial x_\mu} \sum_{q=1}^{2} (-1)^{q+1} \left(\left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} + C^{j\,\mu}_q \right) (\widehat{\mathfrak{I}}_{(\mu,q)} \bar{\delta} U_j)(x) + \right.$$

$$\left. \bar{\delta} U_j(x)\, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q+2)} U_j} + C^{j\,\mu}_{q+2} \right) \right) d^n x =$$

$$\int_{\partial R} \sum_{q=1}^{2} (-1)^{q+1} \left(\left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} + C^{j\,\mu}_q \right) (\widehat{\mathfrak{I}}_{(\mu,q)} \bar{\delta} U_j)(x) + \right.$$

$$\bar{\delta} U_j(x)\, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial\, \widehat{\partial}_{(\mu,q+2)} U_j} + C^{j\,\mu}_{q+2} \right) \Big) \, d^{n-1}x, \tag{59}$$

where ∂R is the boundary of the region R.

To derive field equations, we can use assumption that the variation of the fields $\bar{\delta}\, U_j(x)$ are equal to zero on the boundaries of the region R, one can obtain the field equations.

$$\bar{\delta}\, U_j(x) \;=\; 0 \quad \text{for } x \in \partial R. \tag{60}$$

Let us also assume the conditions

$$\bar{\delta}(\widehat{\mathfrak{I}}_{(\mu,q)}\, U_j)(x) \;=\; (\widehat{\mathfrak{I}}_{(\mu,q)}\, \bar{\delta} U_j)(x) \;=\; 0 \quad \text{for } x \in \partial R,\, q \;=\; 1 \text{ and } q \;=\; 2 \tag{61}$$

are satisfied.

As a result, Equations (42) and (58) give the field equations

$$\left(\frac{\partial \Lambda}{\partial U_j} + B^j \right) + \sum_{q=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial\, \widehat{\partial}_{(\mu,q)} U_j} + C^{j\,\mu}_{q} \right) + \sum_{q=1}^{2} \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial\, \widehat{\mathfrak{I}}_{(\mu,q)} U_j} + A^{j\,\mu}_{q} \right) \;=\; 0, \tag{62}$$

where the functions $B^j \;=\; B^j(x,\, U,\, \widehat{\partial} U,\, \widehat{\mathfrak{I}} U)$, $C^{j\,\mu}_q \;=\; C^{j\,\mu}_q(x,\, U,\, \widehat{\partial} U,\, \widehat{\mathfrak{I}} U)$ and $A^{j\,\mu}_q \;=\; A^{j\,\mu}_q(x,$ $U,\, \widehat{\partial} U,\, \widehat{\mathfrak{I}} U)$ cannot be represented in form (48). In Equation (62), the summations are implied over repeated indices $\mu \;=\; 1, \ldots, n$. $\square$

Remark 1. *There is a problem for term* (54) *with* $(\widehat{\mathfrak{I}}_{(\mu,q)}\, \bar{\delta} U_j)(x)$ *since the applicability of the conditions* (61) *are not obvious, in the general case. In the case of constructing general fractional field models, in which condition* (61) *cannot be satisfied on the boundaries of the region R, we should consider only models with* $C^{j\,\mu}_q \;=\; 0$ *for* $q \;=\; 1\,2$ *and the Lagrangian density* Λ *independent of the GFDs* $\widehat{\partial}_{(\mu,q)} U_j$ *for* $q \;=\; 1\,2$*. If conditions* (60) *and* (61) *are satisfied, then expression* (59) *is equal to zero.*

3.3. Remark about Functionals with $\widehat{\partial}^2 U_j$*,* $\widehat{\mathfrak{I}}^2 U_j$

Let us make a remark about the dependence of the functional and the function on $\widehat{\partial}^2 U_j$, $\widehat{\mathfrak{I}}^2 U_j$. One can consider theories, in which the Lagrangian density Λ depends on $\widehat{\partial}_{(\nu,p)}\, \widehat{\partial}_{(\mu,q)}\, U_j(x)$, $\widehat{\mathfrak{I}}_{(\nu,p)}\, \widehat{\mathfrak{I}}_{(\mu,q)}\, U_j(x)$ and the non-holonomic functional depends on the variation on $\bar{\delta}\, \widehat{\partial}_{(\nu,p)}\, \widehat{\partial}_{(\mu,q)}\, U_j(x)$, $\bar{\delta}\, \widehat{\mathfrak{I}}_{(\nu,p)}\, \widehat{\mathfrak{I}}_{(\mu,q)}\, U_j(x)$.

Using equation of (55) of the integration of GFDs by parts twice, we obtain the equation

$$\int_R \sum_{q,p=1}^{4} C^{j\,\mu\nu}_{q,p}\, \bar{\delta}\, (\widehat{\partial}_{(\nu,p)}\, \widehat{\partial}_{(\mu,q)}\, U_j)(x)\, d^n x \;=\; \int_R \sum_{q,p=1}^{4} C^{j\,\mu\nu}_{q,p}\, (\widehat{\partial}_{(\nu,p)}\, \widehat{\partial}_{(\mu,q)}\, \bar{\delta} U_j)(x)\, d^n x \;=$$

$$\sum_{q,p=1}^{4} \int_R (\widehat{\partial}^{\dagger}_{(\nu,p)}\, C^{j\,\mu\nu}_{q,p})\, \widehat{\partial}_{(\mu,q)}\, \bar{\delta} U_j(x)\, d^n x +$$

$$\int_R \frac{\partial}{\partial x_\nu} \sum_{q=1}^{4} \sum_{p=1}^{2} (-1)^{p+1} \left(C^{j\,\mu\nu}_{q,p}\, (\widehat{\mathfrak{I}}_{(\nu,p)}\, \widehat{\partial}_{(\mu,q)}\, \bar{\delta} U_j)(x) \;+\; (\widehat{\mathfrak{I}}^{\dagger}_{(\nu,p)}\, C^{j\,\mu\nu}_{q+2,p})\, (\widehat{\partial}_{(\mu,q)} \bar{\delta} U_j)(x) \right) d^n x, \tag{63}$$

where

$$\sum_{q,p=1}^{4} \int_R (\widehat{\partial}^{\dagger}_{(\nu,p)}\, C^{j\,\mu\nu}_{q,p})\, \widehat{\partial}_{(\mu,q)}\, \bar{\delta} U_j(x)\, d^n x \;=$$

$$\sum_{q,p=1}^{4} \int_R (\widehat{\partial}^{\dagger}_{(\mu,q)}\, \widehat{\partial}^{\dagger}_{(\nu,p)}\, C^{j\,\mu\nu}_{q,p})\, \bar{\delta} U_j(x)\, d^n x +$$

$$\int_R \frac{\partial}{\partial x_\mu} \sum_{p=1}^{4} \sum_{q=1}^{2} (-1)^{q+1} \Big((\widehat{\partial}^{\dagger}_{(\nu,p)} C^{j\,\mu\nu}_{q,p}) \, (\widehat{\mathfrak{I}}_{(\mu,q)} \, \overline{\delta} U_j)(x) \, + \, (\widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \, \widehat{\partial}^{\dagger}_{(\nu,p)} \, C^{j\,\mu\nu}_{q+2,p}) \, \overline{\delta} U_j(x) \Big) \, d^n x. \tag{64}$$

As a result, we have

$$\int_R \sum_{q=1}^{4} C^{j\,\mu\nu}_{q,p} \, \overline{\delta} \, (\widehat{\partial}_{(\nu,p)} \, \widehat{\partial}_{(\mu,q)} \, U_j)(x) \, d^n x \, = \, \sum_{q=1}^{4} \int_R (\widehat{\partial}^{\dagger}_{(\mu,q)} \, \widehat{\partial}^{\dagger}_{(\nu,p)} \, C^{j\,\mu\nu}_{q,p}) \, \overline{\delta} U_j(x) \, d^n x \, +$$

$$\int_R \frac{\partial}{\partial x_\mu} \sum_{q=1}^{2} \sum_{p=1}^{4} (-1)^{q+1} \Big((\widehat{\partial}_{(\mu,q)} C^{j\,\mu\nu}_{q,p}) \, (\widehat{\mathfrak{I}}_{(\nu,p)} \, \overline{\delta} U_j)(x) \, + \, (\widehat{\mathfrak{I}}^{\dagger}_{(\nu,p)} \, \widehat{\partial}_{(\mu,q)} \, C^{j\,\mu\nu}_{q+2,p}) \, \overline{\delta} U_j(x) \Big) \, d^n x \, +$$

$$\int_R \frac{\partial}{\partial x_\mu} \sum_{q=1}^{4} \sum_{p=1}^{2} (-1)^{p+1} \Big(C^{j\,\nu\mu}_{q,p} \, (\widehat{\mathfrak{I}}_{(\mu,p)} \, \widehat{\partial}_{(\nu,q)} \, \overline{\delta} U_j)(x) \, + \, (\widehat{\mathfrak{I}}^{\dagger}_{(\mu,p)} \, C^{j\,\nu\mu}_{q+2,p}) \, (\widehat{\partial}_{(\nu,q)} \overline{\delta} U_j)(x) \Big) \, d^n x. \tag{65}$$

Equation (65) is valid for wide class of functions $C^{j\,\mu\nu}_{q,p} = C^{j\,\mu\nu}_{q,p}(x, U, \widehat{\partial} U, \widehat{\partial}^2 U)$ as well as for the derivative of the Lagrangian density. Therefore, Equation (65) holds at the replacement

$$C^{j\,\mu\nu}_{q,p} \longrightarrow \frac{\partial \Lambda}{\partial \, \widehat{\partial}_{(\nu,p)} \, \widehat{\partial}_{(\mu,q)} U_j}. \tag{66}$$

Using equation of (57) of the integration of GFDs by parts twice, we obtain the equation

$$\sum_{q=1}^{2} \int_R A^{j\,\mu\nu}_{q,p} \, (\widehat{\mathfrak{I}}_{(\nu,p)} \, \widehat{\mathfrak{I}}_{(\mu,q)} \, \overline{\delta} U_j)(x) \, d^n x \, = \, \sum_{q=1}^{2} \int_R (\widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \, \widehat{\mathfrak{I}}^{\dagger}_{(\nu,p)} \, A^{j\,\mu\nu}_{q,p}) \, \overline{\delta} U_j(x) \, d^n x. \tag{67}$$

Equation (67) is satisfied for wide class of functions $A^{j\,\mu\nu}_{q,p} = A^{j\,\mu\nu}_{q,p}(x, U, \widehat{\partial} U, \widehat{\partial}^2 U)$ as well as for the derivative of the Lagrangian density. Therefore, Equation (67) holds at the replacement

$$A^{j\,\mu\nu}_{q,p} \longrightarrow \frac{\partial \Lambda}{\partial \, \widehat{\mathfrak{I}}_{(\nu,p)} \, \widehat{\mathfrak{I}}_{(\mu,q)} U_j}. \tag{68}$$

To derive field equations in this case, one should assume that the variation of the fields $\overline{\delta} \, U_j(x)$ and its GFDs and GFIs with kernels $K_\mu(x)$ are equal to zero on the region boundaries ∂R , one can obtain the field equations.

$$\overline{\delta} \, U_j(x) \, = \, 0, \quad (\widehat{\partial}_{(\nu,q)} \overline{\delta} U_j)(x) \, = \, 0 \quad \text{for } x \, \in \, \partial R, \tag{69}$$

$$(\widehat{\mathfrak{I}}_{(\nu,p)} \, \overline{\delta} U_j)(x) \, = \, 0, \quad (\widehat{\mathfrak{I}}_{(\mu,p)} \, \widehat{\partial}_{(\nu,q)} \, \overline{\delta} U_j)(x) \, = \, 0 \quad \text{for } x \, \in \, \partial R. \tag{70}$$

As a result, the non-holonomic variational equation gives the field equations

$$\left(\frac{\partial \Lambda}{\partial U_j} + B^j \right) + \sum_{q=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \, \widehat{\partial}_{(\mu,q)} U_j} + C^{j\,\mu}_{q} \right) + \sum_{q=1}^{2} \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \, \widehat{\mathfrak{I}}_{(\mu,q)} U_j} + A^{j\,\mu}_{q} \right) +$$

$$\sum_{q,p=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} \, \widehat{\partial}^{\dagger}_{(\nu,p)} \left(\frac{\partial \Lambda}{\partial \, \widehat{\partial}_{(\nu,p)} \, \widehat{\partial}_{(\mu,q)} U_j} + C^{j\,\mu\nu}_{q,p} \right) + \sum_{q,p=1}^{2} \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \, \widehat{\mathfrak{I}}^{\dagger}_{(\nu,p)} \left(\frac{\partial \Lambda}{\partial \, \widehat{\mathfrak{I}}_{(\nu,p)} \, \widehat{\mathfrak{I}}_{(\mu,q)} U_j} + A^{j\,\mu\nu}_{q,p} \right) \, = \, 0. \tag{71}$$

One can consider general fractional field theories with the dependencies of the functional and the density functions on the pairs of GFIs and GFDs: $\widehat{\partial}_{(\mu,q)} \, \widehat{\mathfrak{I}}_{(\nu,p)} \, U_j$ and $\widehat{\mathfrak{I}}_{(\nu,p)} \widehat{\partial}_{(\mu,q)} \, U_j$.

4. General Fractional Noether Theorem

The standard first Noether theorem states that to every finite-parameter (depending on s constant parameters) continuous transformation of fields and coordinates, which ensures that the variation of the action S is zero, there correspond s dynamic invariants, i.e., combinations of fields and their derivatives that are conserved in time [1] (pp. 16–17).

Using non-holonomic variational equations and general fractional calculus, a generalization of the standard and fractional Noether theorems is proposed and proved.

4.1. General Fractional Noether Theorem and Its Proof

The general fractional Noether theorem states the following. *For every finite-parameter (depending on constant parameters) continuous transformation of fields and coordinates, which ensures that the non-holonomic variational equation holds, there correspond dynamic invariants.*

Similarly to the classical work of Bogoliubov and Shirkov [1] (pp. 15–24), we will carry out the proof of Noether's theorem in a general form, without specifying the type of transformations under which the non-holonomic variational equation is satisfied for the fields under consideration. This general consideration has the advantage that it allows one to obtain not only a mathematical formulation of the conservation laws that follow from the space–time symmetry, but also, if necessary, to use the general results of the theorem for cases not related to space-time transformations. For example, using this approach to Noether's theorem, one can relate the conservation law of the so-called isotopic spin to the symmetry of some conditional isotopic space. This law, in particular, can be used to derive the law of conservation of charge [1].

In the proposed general formulation of Noether's theorem, the condition that the sum of the variations of the action and the non-holonomic functional vanishes means the vanishing of the sum of small changes in the action and the non-holonomic functional due to infinitesimal transformations of coordinates and functions, i.e., the fulfillment of the non-holonomic variational equation with these transformations. The generality of Noether's theorem is that the theorem is valid for all continuous transformations for which the non-holonomic variational equation is satisfied when considering any systems and fields.

We will consider a group of continuous transformations with a finite number of parameters ω_m, $m = 1, \dots, s$. The infinitesimal transformations of coordinates and fields

$$x^{\mu} \quad \longrightarrow \quad x'^{\mu} \, = \, x^{\mu} \, + \, \delta \, x^{\mu}, \tag{72}$$

$$U_j(x) \quad \longrightarrow \quad U'_j(x') \, = \, U_j(x) \, + \, \delta \, U_j(x). \tag{73}$$

The variations $\delta \, x^{\mu}$ and $\delta \, U_j(x)$ can be expressed in terms of the infinitesimal linearly independent transformation parameters ω_m, $m = 1, \dots, s$, by the equations

$$\delta \, x^{\mu} \, = \, \sum_{m=1}^{s} X^{\mu}_{(m)}(x) \, \delta \, \omega_m, \tag{74}$$

$$\delta \, U_j(x) \, = \, \sum_{m=1}^{s} Y_{j\,(m)}(x) \, \delta \, \omega_m. \tag{75}$$

The indices j and (m) of the fields $U_j(x)$ and the transformation parameters ω_m can have a simple tensorial one. The repeated indices mean the summation. Substitution of Equations (74) and (75) into Equations (72) and (73) gives

$$x^{\mu} \quad \longrightarrow \quad x'^{\mu} \, = \, x^{\mu} + \sum_{m=1}^{s} X^{\mu}_{(m)}(x) \, \delta \, \omega_m, \tag{76}$$

$$U_j(x) \quad \longrightarrow \quad U'_j(x') \, = \, U_j(x) + \sum_{m=1}^{s} Y_{j\,(m)}(x) \, \delta \, \omega_m. \tag{77}$$

The variations δ and $\overline{\delta}$ are connected as

$$\delta\, U_j(x) \,=\, \overline{\delta}\, U_j(x) \,+\, \partial_\nu U_j\, \delta x^\nu , \tag{78}$$

$$\delta\, \widehat{\partial}_{(\mu,q)} U_j(x) \,=\, \overline{\delta}\, (\widehat{\partial}_{(\mu,q)} U_j)(x) \,+\, \partial_\nu\, (\widehat{\partial}_{(\mu,q)} U_j)(x)\, \delta x^\nu , \tag{79}$$

$$\delta\, \widehat{\Im}_{(\mu,q)} U_j(x) \,=\, \overline{\delta}\, (\widehat{\Im}_{(\mu,q)} U_j)(x) \,+\, \partial_\nu\, (\widehat{\Im}_{(\mu,q)} U_j)(x)\, \delta x^\nu , \tag{80}$$

where $\partial_\nu \,=\, \partial / \partial x^\nu$ is the standard first-order derivative with respect to x_ν.

To prove the general fractional Noether theorem, let us use the Sedov non-holonomic variational equation

$$\delta\, S \,+\, \delta\, W^* \,=\, 0, \tag{81}$$

where the variation of the action functional is

$$\delta\, S \,=\, \delta \int_R \Lambda(x, U_j(x), \widehat{\partial}_{(\mu,q)} U_j(x), \widehat{\Im}_{(\mu,q)} U_j(x))\, d^n x \,:=$$

$$\int_R \Lambda(x', U'_j(x'), \widehat{\partial}'_{(\mu,q)} U'_j(x'), \widehat{\Im}'_{(\mu,q)} U_j(x'))\, d^n x' \,-\, \int_R \Lambda(x, U_j(x), \widehat{\partial}_{(\mu,q)} U_j(x), \widehat{\Im}_{(\mu,q)} U_j(x))\, d^n x, \tag{82}$$

and the variation of non-holonomic functional $\delta\, W^*$ is

$$\delta\, W^* \,=\, \int_R \Big(B^j(x,\, U,\, \widehat{\partial} U,\, \widehat{\Im} U)\, \delta\, U_j(x) \,+\, \sum_{q=1}^{4} C_q^{j\,\mu}(x,\, U,\, \widehat{\partial} U,\, \widehat{\Im} U)\, \delta\, \widehat{\partial}_{(\mu,q)} U_j(x) +$$

$$\sum_{q=1}^{2} A_q^{j\,\mu}(x,\, U,\, \widehat{\partial} U,\, \widehat{\Im} U)\, \delta\, \widehat{\Im}_{(\mu,q)} U_j(x) \,+\, E_\mu(x,\, U,\, \widehat{\partial} U,\, \widehat{\Im} U, \dots)\, \delta\, x^\mu \Big)\, d^n x. \tag{83}$$

Note that $\delta\, S \,\neq\, \overline{\delta}\, S$, and $\delta\, W^* \,\neq\, \overline{\delta}\, W^*$, in the general case.

In Equation (83), the functions $B^j \,=\, B^j(x,\, U,\, \widehat{\partial} U,\, \widehat{\Im} U)$, $C_q^{j\,\mu} \,=\, C_q^{j\,\mu}(x,\, U,\, \widehat{\partial} U,\, \widehat{\Im} U)$ and $A_q^{j\,\mu} \,=\, A_q^{j\,\mu}(x,\, U,\, \widehat{\partial} U,\, \widehat{\Im} U)$ cannot be represented as

$$B^j \,=\, \frac{\partial\, F}{\partial U_j}, \quad C_q^{j\,\mu} \,=\, \frac{\partial\, F}{\partial\, \widehat{\partial}_{(\mu,q)} U_j}, \quad A_q^{j\,\mu} \,=\, \frac{\partial\, F}{\partial\, \widehat{\Im}_{(\mu,q)} U_j} \tag{84}$$

with a function $F \,=\, F(x,\, U,\, \widehat{\partial} U,\, \widehat{\Im} U)$.

Note that the function $E_\mu \,=\, E_\mu(x,\, U,\, \widehat{\partial} U,\, \widehat{\Im} U, \dots)$ is not used in the non-holonomic variational equation of GF action principle to drive field equations, and the term with E_μ is absent in the field equations. Therefore, the function E_μ generates some additional arbitrariness in the definition of the non-holonomic functional $\delta\, W^*$. This arbitrariness can be used to simplify the consideration. Using Equations (78)–(80), which express variations δ through variations $\overline{\delta}$, Equation (83) takes the form

$$\delta\, W^* \,=\, \overline{\delta}\, W^* \,+\, \int_R \Big(B^j\, \partial_\nu U_j(x) \,+\, \sum_{q=1}^{4} C_q^{j\,\mu}\, \partial_\nu\, (\widehat{\partial}_{(\mu,q)} U_j)(x) +$$

$$\sum_{q=1}^{2} A_q^{j\,\mu}\, \partial_\nu\, (\widehat{\Im}_{(\mu,q)} U_j)(x) \,+\, E_\nu \Big)\, \delta\, x^\nu\, d^n x, \tag{85}$$

where $\overline{\delta}\, W^*$ is defined by Equation (44). Equation (85) allows us to simplify the consideration by assuming that

$$E_\nu \;=\; -\, B^j\, \partial_\nu U_j(x) \;-\; \sum_{q=1}^{4} C_q^{j\,\mu}\, \partial_\nu\, (\widehat{\partial}_{(\mu,q)} U_j)(x) \;-\; \sum_{q=1}^{2} A_q^{j\,\mu}\, \partial_\nu\, (\widehat{\mathfrak{I}}_{(\mu,q)} U_j)(x). \tag{86}$$

Assumption (86) leads to

$$\delta\, W^* \;=\; \overline{\delta}\, W^* \;=\; \int_R \Big(B^j\, \overline{\delta}\, U_j(x) \;+\; \sum_{q=1}^{4} C_q^{j\,\mu}\, \overline{\delta}\, (\widehat{\partial}_{(\mu,q)} U_j)(x) \;+\; \sum_{q=1}^{2} A_q^{j\,\mu}\, \overline{\delta}\, (\widehat{\mathfrak{I}}_{(\mu,q)} U_j)(x) \Big)\, d^n x, \tag{87}$$

where the summation over the index μ from 1 to n is implied.

The general fractional Noether theorem can be presented in the form.

Theorem 2. *Let the variation of GF action functional be defined by Equation (82) and the variation of the GF non-holonomic functional be defined as (83) with condition (86), where $\Lambda(x,\, U,\, \widehat{\partial} U,\, \widehat{\mathfrak{I}} U)$, $B^j(x,\, U,\, \widehat{\partial} U,\, \widehat{\mathfrak{I}} U)$, $C_q^{j\,\mu}(x,\, U,\, \widehat{\partial} U,\, \widehat{\mathfrak{I}} U)$, $A_q^{j\,\mu}(x,\, U,\, \widehat{\partial} U,\, \widehat{\mathfrak{I}} U)$ are continuously differentiable function with respect to its argument, and*

$$\frac{\partial \Lambda}{\partial\, \widehat{\partial}_{(\mu,q)} U_j} \;\in\; C^1_{-1}(W_q), \qquad \frac{\partial \Lambda}{\partial\, \widehat{\mathfrak{I}}_{(\mu,q)} U_j} \;\in\; C_{-1}(W_q), \tag{88}$$

$$C_q^{j\,\mu} \;\in\; C^1_{-1}(W_q), \quad A_q^{j\,\mu} \;\in\; C_{-1}(W_q). \tag{89}$$

Let field equations (equations of motion) (51) be satisfied also.

For every finite-parameter (depending on constant parameters) continuous transformation of fields and coordinates in forms (76) and (77), which ensures that the non-holonomic variational Equation (81) holds, there correspond dynamic invariants

$$C_{(m)}(t) \;=\; \int_{\partial R} \Theta^\mu_{(m)}(x)\, d^{n-1} x \quad (m \;=\; 1 \ldots,\, s) \tag{90}$$

are independent of time $x^n \;=\; t$, where

$$\Theta^\mu_{(m)}(x) \;=\; -\sum_j \sum_{q=1}^{2} (-1)^{q+1} \left(\frac{\partial \Lambda}{\partial\, \widehat{\partial}_{(\mu,q)} U_j} \;+\; C_q^{j\,\mu} \right) \widehat{\mathfrak{I}}_{(\mu,q)} \left(Y_{j\,(m)} \;-\; (\partial_\alpha\, U_j)\, X^\alpha_{(m)} \right) -$$

$$\sum_j \sum_{q=1}^{2} (-1)^{q+1}\, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial\, \widehat{\partial}_{(\mu,q+2)} U_j} \;+\; C_{q+2}^{j\,\mu} \right) \left(Y_{j\,(m)} \;-\; (\partial_\alpha U_j)\, X^\alpha_{(m)} \right) \;-\; \Lambda\, X^\mu_{(m)}. \tag{91}$$

Proof. The sum of variation of the action functional (82) and non-holonomic functional (87) can be written as

$$\delta\, S \;+\; \delta\, W^* \;=\; \mathbb{A}_1 \;+\; \mathbb{A}_2 \;+\; \mathbb{A}_3, \tag{92}$$

where

$$\mathbb{A}_1 \;=\; \int_R \delta\, \Lambda(x,\, U_j(x),\, \widehat{\partial}_{(\mu,q)} U_j(x),\, \widehat{\mathfrak{I}}_{(\mu,q)} U_j(x))\, d^n x, \tag{93}$$

$$\mathbb{A}_2 \;=\; \int_R \Lambda(x,\, U_j(x),\, \widehat{\partial}_{(\mu,q)} U_j(x),\, \widehat{\mathfrak{I}}_{(\mu,q)} U_j(x))\, d^n x' \;-$$

$$\int_R \Lambda(x,\, U_j(x),\, \widehat{\partial}_{(\mu,q)} U_j(x),\, \widehat{\mathfrak{I}}_{(\mu,q)} U_j(x))\, d^n x, \tag{94}$$

$$\mathbb{A}_3 \;=\; \int_R \Big(B^j\, \overline{\delta}\, U_j(x) \;+\; \sum_{q=1}^{4} C_q^{j\,\mu}\, \overline{\delta}\, (\widehat{\partial}_{(\mu,q)} U_j)(x) \;+\; \sum_{q=1}^{2} A_q^{j\,\mu}\, \overline{\delta}\, (\widehat{\mathfrak{I}}_{(\mu,q)} U_j)(x) \Big)\, d^n x. \tag{95}$$

STEP 1. Let us consider the term $\mathbb{A}_1$. Using Equations (78)–(80), the variation of the Lagrangian density is

$$\delta\,\Lambda(x,\,U_j(x),\,\widehat{\partial}_{(\mu,q)}U_j(x),\,\widehat{\Im}_{(\mu,q)}U_j(x)) \;=$$

$$\frac{\partial\Lambda}{\partial x^\mu}\,\delta\,x^\mu \,+\, \frac{\partial\Lambda}{\partial U_j}\,\delta\,U_j \,+\, \frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q)}U_j}\,\delta\,\widehat{\partial}_{(\mu,q)}U_j \,+\, \frac{\partial\Lambda}{\partial\,\widehat{\Im}_{(\mu,q)}U_j}\,\delta\,\widehat{\Im}_{(\mu,q)}U_j \;=$$

$$\frac{\partial\Lambda}{\partial x^\mu}\,\delta\,x^\mu \,+\, \frac{\partial\Lambda}{\partial U_j}\,(\bar{\delta}\,U_j(x) \,+\, \partial_\mu U_j(x)\,\delta x^\mu) \,+$$

$$\frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q)}U_j}\,\Big(\bar{\delta}\,\widehat{\partial}_{(\mu,q)}U_j(x) \,+\, \partial_\nu\,\widehat{\partial}_{(\mu,q)}U_j(x)\,\delta x^\nu\Big) \,+$$

$$\frac{\partial\Lambda}{\partial\,\widehat{\Im}_{(\mu,q)}U_j}\,\Big(\bar{\delta}\,\widehat{\Im}_{(\mu,q)}U_j(x) \,+\, \partial_\nu\,\widehat{\Im}_{(\mu,q)}U_j(x)\,\delta x^\nu\Big). \tag{96}$$

As a result, the term $\mathbb{A}_1$ can be written as

$$\mathbb{A}_1 \;=\; \int_R \delta\,\Lambda(x,\,U,\,\widehat{\partial}U,\,\widehat{\Im}U)\,d^n x \;=$$

$$\int_R \Big(\bar{\delta}\,\Lambda(x,\,U,\,\widehat{\partial}U,\,\widehat{\Im}U) \,+\, \frac{d}{dx^\mu}\,\Lambda(x,\,U,\,\widehat{\partial}U,\,\widehat{\Im}U)\,\delta x^\mu\Big)\,d^n x, \tag{97}$$

where $\bar{\delta}\,\Lambda$ is the variation of Λ due to variations in the field forms of $U_j(x)$, $\widehat{\partial}_{(\mu,q)}U_j(x)$ and $\widehat{\Im}_{(\mu,q)}U_j(x)$ such that

$$\bar{\delta}\,\Lambda(x,\,U,\,\widehat{\partial}U\,\widehat{\Im}U) \;=\; \frac{\partial\Lambda}{\partial U_j}\,\bar{\delta}\,U_j(x) \,+\, \frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q)}U_j}\,\bar{\delta}\,\widehat{\partial}_{(\mu,q)}U_j(x) \,+\, \frac{\partial\Lambda}{\partial\,\widehat{\Im}_{(\mu,q)}U_j}\,\bar{\delta}\,\widehat{\Im}_{(\mu,q)}U_j(x), \tag{98}$$

and the term $d\Lambda/dx^\mu$ is the total variation due to variations in the coordinates:

$$\frac{d}{dx^\mu}\,\Lambda(x,\,U(x),\,\widehat{\partial}U(x),\,\widehat{\Im}U(x)) \;=$$

$$\frac{\partial\Lambda}{\partial x^\mu} \,+\, \frac{\partial\Lambda}{\partial U_j}\,\frac{\partial U_j}{\partial x^\mu} \,+\, \frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\nu,p)}U_j}\,\frac{\partial\,\widehat{\partial}_{(\nu,p)}U_j}{\partial x^\mu} \,+\, \frac{\partial\Lambda}{\partial\,\widehat{\Im}_{(\nu,p)}U_j}\,\frac{\partial\,\widehat{\Im}_{(\nu,p)}U_j}{\partial x^\mu}. \tag{99}$$

STEP 2. Let us consider the term $\mathbb{A}_2$. Using the equation

$$\det\,\|\frac{\partial\,x'^{\,\mu}}{\partial\,x^\nu}\| \;=\; \exp\Big(Sp\,\ln\Big(\|\frac{\partial\,x'^{\,\mu}}{\partial\,x^\nu}\|\Big)\Big),$$

and the linear approximation

$$\|\frac{\partial\,x'^{\,\mu}}{\partial\,x^\nu}\| \;\approx\; \|\delta^\mu_\nu \,+\, \frac{\partial\delta x^\mu}{\partial x^\nu}\|,$$

we get

$$d^n x' \;=\; dx'_0\,dx'_1\,dx'_2\,dx'_3 \;=\; \det\|\frac{\partial\,x'^{\,\mu}}{\partial\,x^\nu}\|\,d^n x \approx \Big(1 \,+\, \frac{\partial\delta x^\mu}{\partial x^\mu}\Big)\,d^n x. \tag{100}$$

As a result, the term $\mathbb{A}_2$, which describes the difference between the two terms that describes the variation in the region of integration, can be written as

$$\mathbb{A}_2 \;=\; \int_R \Lambda(x,\,U,\,\widehat{\partial}U,\,\widehat{\Im}U)\,\frac{\partial\delta x^\mu}{\partial x^\mu}\,d^n x. \tag{101}$$

Then, the variation of the action functional $\delta S = \mathbb{A}_1 + \mathbb{A}_2$ can be written as

$$\mathbb{A}_1 + \mathbb{A}_2 =$$

$$\int_R \left(\bar{\delta}\,\Lambda(x, U, \widehat{\partial} U, \widehat{\Im} U) + \frac{d\,\Lambda(x, U, \widehat{\partial} U, \widehat{\Im} U)}{dx^\mu}\,\delta x^\mu + \Lambda(x, U, \widehat{\partial} U, \widehat{\Im} U)\,\frac{\partial \delta x^\mu}{\partial x^\mu}\right) d^n x =$$

$$\int_R \left(\bar{\delta}\,\Lambda(x, U, \widehat{\partial} U, \widehat{\Im} U) + \frac{d}{dx^\mu}\left(\Lambda(x, U, \widehat{\partial} U, \widehat{\Im} U)\,\delta x^\mu\right)\right) d^n x. \tag{102}$$

STEP 3. Let us consider the first term of Equation (102). Using the properties $\bar{\delta}\,\widehat{\partial}_{(\mu,q)}\,U_j = \widehat{\partial}_{(\mu,q)}\,\bar{\delta}\,U_j$ and $\bar{\delta}\,\widehat{\Im}_{(\mu,q)}\,U_j = \widehat{\Im}_{(\mu,q)}\,\bar{\delta} U_j(x)$, the first term of Equation (102) is the variation of Λ gives

$$\int_R \left(\bar{\delta}\,\Lambda(x, U, \widehat{\partial} U, \widehat{\Im} U)\right) d^n x =$$

$$\int_R \left(\frac{\partial \Lambda}{\partial U_j}\,\bar{\delta}\,U_j(x) + \sum_{q=1}^{4} \frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q)} U_j}\,\bar{\delta}\,\widehat{\partial}_{(\mu,q)} U_j(x) + \sum_{q=1}^{2} \frac{\partial \Lambda}{\partial\,\widehat{\Im}_{(\mu,q)} U_j}\,\bar{\delta}\,\widehat{\Im}_{(\mu,q)} U_j(x)\right) d^n x =$$

$$\int_R \left(\frac{\partial \Lambda}{\partial U_j}\,\bar{\delta}\,U_j(x) + \sum_{q=1}^{4} \frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q)} U_j}\,\widehat{\partial}_{(\mu,q)}\,\bar{\delta} U_j(x) + \sum_{q=1}^{2} \frac{\partial \Lambda}{\partial\,\widehat{\Im}_{(\mu,q)} U_j}\,\widehat{\Im}_{(\mu,q)}\,\bar{\delta} U_j(x)\right) d^n x. \tag{103}$$

Equation (103) describes the variation of Λ due to variations in the forms of $U_j(x)$, $\widehat{\partial}_{(\mu,q)} U_j(x)$ and $\widehat{\Im}_{(\mu,q)} U_j(x)$.

Using the rules of integration by parts for GFDs (25), the second term of (103) is

$$\int_R \sum_{q=1}^{4} \frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q)} U_j}\,\widehat{\partial}_{(\mu,q)}\,\bar{\delta} U_j(x)\,d^n x = \sum_{q=1}^{4} \int_R \widehat{\partial}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q)} U_j}\right) \bar{\delta} U_j(x)\,d^n x +$$

$$\int_R \frac{\partial}{\partial x_\mu} \sum_{q=1}^{2} (-1)^{q+1} \left(\left(\frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q)} U_j}\right) (\widehat{\Im}_{(\mu,q)}\,\bar{\delta} U_j)(x) + \bar{\delta} U_j(x)\,\widehat{\Im}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q+2)} U_j}\right)\right) d^n x. \tag{104}$$

Using the rules of integration by parts for GFIs (28), the third term of (103) is

$$\sum_{q=1}^{2} \int_R \frac{\partial \Lambda}{\partial\,\widehat{\Im}_{(\mu,q)} U_j}\,\widehat{\Im}_{(\mu,q)}\,\bar{\delta} U_j(x)\,d^n x = \sum_{q=1}^{2} \int_R \widehat{\Im}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial\,\widehat{\Im}_{(\mu,q)} U_j}\right) \bar{\delta} U_j(x)\,d^n x. \tag{105}$$

As a result, we obtain

$$\mathbb{A}_1 + \mathbb{A}_2 =$$

$$\int_R \left(\left(\frac{\partial \Lambda}{\partial U_j}\right) + \sum_{q=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q)} U_j}\right) + \sum_{q=1}^{2} \widehat{\Im}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial\,\widehat{\Im}_{(\mu,q)} U_j}\right)\right) \bar{\delta}\,U_j(x)\,d^n x +$$

$$\int_R \frac{\partial}{\partial x_\mu} \left(\sum_{q=1}^{2} (-1)^{q+1} \left(\frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q)} U_j}\right) (\widehat{\Im}_{(\mu,q)}\,\bar{\delta} U_j)(x) +\right.$$

$$\left.\sum_{q=1}^{2} (-1)^{q+1}\,\widehat{\Im}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q+2)} U_j}\right) \bar{\delta} U_j(x) + \Lambda\,\delta x_\mu\right) d^n x. \tag{106}$$

STEP 4. Using the properties $\overline{\delta}\,\widehat{\partial}_{(\mu,q)}\,U_j = \widehat{\partial}_{(\mu,q)}\,\overline{\delta}\,U_j$ and $\overline{\delta}\,\widehat{\mathfrak{I}}_{(\mu,q)}\,U_j = \widehat{\mathfrak{I}}_{(\mu,q)}\,\overline{\delta}U_j(x)$, we obtain the expression

$$\mathbb{A}_3 = \int_R \Big(B^j\,\overline{\delta}\,U_j(x) + \sum_{q=1}^{4} C_q^{j\,\mu}\,\overline{\delta}\,(\widehat{\partial}_{(\mu,q)}U_j)(x) + \sum_{q=1}^{2} A_q^{j\,\mu}\,\overline{\delta}\,(\widehat{\mathfrak{I}}_{(\mu,q)}U_j)(x) \Big)\, d^n x =$$

$$\int_R \Big(B^j\,\overline{\delta}\,U_j(x) + \sum_{q=1}^{4} C_q^{j\,\mu}\,\widehat{\partial}_{(\mu,q)}\,\overline{\delta}U_j(x) + \sum_{q=1}^{2} A_q^{j\,\mu}\,\widehat{\mathfrak{I}}_{(\mu,q)}\,\overline{\delta}U_j(x) \Big)\, d^n x. \tag{107}$$

Using the rules of integration by parts for GFDs (25), the second term of Equation (107) is

$$\int_R \sum_{q=1}^{4} C_q^{j\,\mu}\,\widehat{\partial}_{(\mu,q)}\,\overline{\delta}U_j(x)\, d^n x = \sum_{q=1}^{4} \int_R \widehat{\partial}^{\dagger}_{(\mu,q)}\,C_q^{j\,\mu}\,\overline{\delta}U_j(x)\, d^n x +$$

$$\int_R \frac{\partial}{\partial x_\mu} \sum_{q=1}^{2} (-1)^{q+1} \Big(C_q^{j\,\mu}\,(\widehat{\mathfrak{I}}_{(\mu,q)}\,\overline{\delta}U_j)(x) + (\widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)}\,C_{q+2}^{j,\mu})\,\overline{\delta}U_j(x) \Big)\, d^n x. \tag{108}$$

Using the rules of integration by parts for GFIs (28), the third term of Equation (107) is

$$\sum_{q=1}^{2} \int_R A_q^{j\,\mu}\,\widehat{\mathfrak{I}}_{(\mu,q)}\,\overline{\delta}U_j(x)\, d^n x = \sum_{q=1}^{2} \int_R \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)}\,A_q^{j\,\mu}\,\overline{\delta}U_j(x)\, d^n x. \tag{109}$$

As a result, we obtain

$$\mathbb{A}_3 = \int_R \Big(B^j + \sum_{q=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} C_q^{j\,\mu} + \sum_{q=1}^{2} \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} A_q^{j\,\mu} \Big)\,\overline{\delta}\,U_j(x)\, d^n x +$$

$$\int_R \frac{\partial}{\partial x_\mu} \Big(\sum_{q=1}^{2} (-1)^{q+1}\, C_q^{j\,\mu}\,(\widehat{\mathfrak{I}}_{(\mu,q)}\,\overline{\delta}U_j)(x) + \sum_{q=1}^{2} (-1)^{q+1}\,\widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)}\,C_{q+2}^{j\,\mu}\,\overline{\delta}U_j(x) + \Lambda\,\delta x_\mu \Big)\, d^n x. \tag{110}$$

STEP 5. Using Equations (106) and (110), the left hand side of the non-holonomic variational Equation (81) takes the form

$$\delta\,S + \delta\,W^* = \mathbb{A}_1 + \mathbb{A}_2 + \mathbb{A}_3 = \int_R \Bigg(\Big(\frac{\partial \Lambda}{\partial U_j} + B^j \Big) +$$

$$\sum_{q=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} \Big(\frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q)} U_j} + C_q^{j\,\mu} \Big) + \sum_{q=1}^{2} \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \Big(\frac{\partial \Lambda}{\partial\,\widehat{\mathfrak{I}}_{(\mu,q)} U_j} + A_q^{j\,\mu} \Big) \Bigg)\,\overline{\delta}\,U_j(x)\, d^n x +$$

$$\int_R \frac{\partial}{\partial x_\mu} \Bigg(\sum_{q=1}^{2} (-1)^{q+1} \Big(\frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q)} U_j} + C_q^{j\,\mu} \Big)\,(\widehat{\mathfrak{I}}_{(\mu,q)}\,\overline{\delta}U_j)(x) +$$

$$\sum_{q=1}^{2} (-1)^{q+1}\,\widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \Big(\frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q+2)} U_j} + C_{q+2}^{j\,\mu} \Big)\,\overline{\delta}U_j(x) + \Lambda\,\delta x_\mu \Bigg)\, d^n x. \tag{111}$$

Using the field equations

$$\Big(\frac{\partial \Lambda}{\partial U_j} + B^j \Big) + \sum_{q=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} \Big(\frac{\partial \Lambda}{\partial\,\widehat{\partial}_{(\mu,q)} U_j} + C_q^{j\,\mu} \Big) + \sum_{q=1}^{2} \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \Big(\frac{\partial \Lambda}{\partial\,\widehat{\mathfrak{I}}_{(\mu,q)} U_j} + A_q^{j\,\mu} \Big) = 0,$$

expression (111) takes the form

$$\delta S + \delta W^* = \int_R \frac{\partial}{\partial x_\mu} \Big(\sum_{q=1}^{2} (-1)^{q+1} \left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} + C_q^{j\,\mu} \right) (\widehat{\mathfrak{I}}_{(\mu,q)} \, \bar{\delta} U_j)(x) +$$

$$\sum_{q=1}^{2} (-1)^{q+1} \, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q+2)} U_j} + C_{q+2}^{j\,\mu} \right) \bar{\delta} U_j(x) + \Lambda \, \delta x_\mu \Big) \, d^n x. \tag{112}$$

STEP 6. Substitution of Equations (74) and (75) into Equations (78) and (80) gives

$$\bar{\delta} \, U_j(x) = \delta \, U_j(x) - \partial_\nu U_j(x) \, \delta x^\nu = \sum_{m=1}^{s} \Big(Y_{j\,(m)}(x) - \partial_\nu U_j(x) \, X^\nu_{(m)}(x) \Big) \, \delta \, \omega_m, \tag{113}$$

and

$$\bar{\delta} \, \widehat{\mathfrak{I}}_{(\mu,q)} \, U_j(x) = \widehat{\mathfrak{I}}_{(\mu,q)} \, \bar{\delta} \, U_j(x) = \widehat{\mathfrak{I}}_{(\mu,q)} \Big(\delta \, U_j(x) - \partial_\nu \, U_j(x) \, \delta x^\nu \Big) =$$

$$\sum_{m=1}^{s} \omega_m \, \widehat{\mathfrak{I}}_{(\mu,q)} \Big(Y_{j\,(m)}(x) - \partial_\nu \, U_j(x) \, X^\mu_{(m)}(x) \Big), \tag{114}$$

where the property

$$(\widehat{\mathfrak{I}}_{(\mu,q)} \, \bar{\delta} U_j)(x) = \bar{\delta} (\widehat{\mathfrak{I}}_{(\mu,q)} \, U_j)(x)$$

is used.

As a result, using transformations (74) and (75), and Equations (113) and (114) one can obtain

$$\delta \, S + \bar{\delta} \, W^* = - \sum_{m=1}^{s} \int_R \left(\frac{\partial}{\partial x^\mu} \, \Theta^\mu_{(m)}(x) \right) d^n x \, \delta \omega_m, \tag{115}$$

where

$$\Theta^\mu_{(m)}(x) = - \sum_j \sum_{q=1}^{2} (-1)^{q+1} \left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q)} U_j} + C_q^{j\,\mu} \right) \widehat{\mathfrak{I}}_{(\mu,q)} \Big(Y_{j\,(m)} - (\partial_\alpha \, U_j) \, X^\alpha_{(m)} \Big) -$$

$$\sum_{j=1}^{m} \sum_{q=1}^{2} (-1)^{q+1} \, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q+2)} U_j} + C_{q+2}^{j\,\mu} \right) \Big(Y_{j\,(m)} - (\partial_\alpha U_j) \, X^\alpha_{(m)} \Big) - \Lambda \, X^\mu_{(m)}, \tag{116}$$

and $\Lambda = \Lambda(x, U, \widehat{\partial} U, \widehat{\mathfrak{I}} U)$, $C^{j\,\mu} = C^{j\,\mu}(x, U, \widehat{\partial} U, \widehat{\mathfrak{I}} U)$. In Equation (116), summation over j is highlighted in order to emphasize the absence of summation over μ.

STEP 7. Using the non-holonomic variational Equation (81), and the independence of the transformation parameters, we obtain

$$\int_R \left(\frac{\partial}{\partial x^\mu} \, \Theta^\mu_{(m)}(x) \right) d^n x = 0. \tag{117}$$

Using that the region R of integration is arbitrary, Equation (117) gives the conserved current equation

$$\frac{\partial}{\partial x^\mu} \, \Theta^\mu_{(m)}(x) = 0 \tag{118}$$

for all $x \in R$, where summation over a repeating index μ is assumed.

STEP 8. The standard Gauss theorem can be used to transform the right-hand side of Equation (117) and to obtain the conservation laws for the corresponding surface integrals. Let us assume that the integral in (117) is valuated over a region that expands without limit in space-like directions, but is bounded in time-like directions by space-like $(n-1)$-

dimensional surfaces σ_1, σ_2, and let the field be equal to zero on the boundaries of the spatial region. Then, one can obtain

$$\int_{\sigma_1} \Theta^{\mu}_{(m)}(x)\, d\sigma_\mu \,-\, \int_{\sigma_2} \Theta^{\mu}_{(m)}(x)\, d\sigma_\mu \,=\, 0, \tag{119}$$

where $d\sigma_\mu$ is the projection of the surface area element σ onto the $(n-1)$-dimensional plane perpendicular to the x^μ-axis. Equation (119) means that the surface integrals

$$C_{(m)}(\sigma) \,=\, \int_{\sigma} \Theta^{\mu}_{(m)}(x)\, d\sigma_\mu \tag{120}$$

are independent of the surface σ. In the special case, where the surfaces are the $(n-1)$-dimensional planes $x^n = t =$ const, the integral is calculated over the $(n-1)$-dimensional configuration space, and the integrals

$$C_{(m)}(t) \,=\, \int_{\partial R} \Theta^{\mu}_{(m)}(x)\, d^{n-1}x \tag{121}$$

are independent of time $x^n = t$. $\square$

As a result, it was proved that to each continuous s-parameter transformation of coordinates (76) and fields (77), there correspond time-independent invariants $C_m = C_{(m)}(t)$, $m = 1, \ldots, s$ given by Equations (121).

4.2. Remark about General Form of Gf Non-Holonomic Functional

The GF non-holonomic functional δW^*, which is given by Equation (85), can be written as

$$\delta W^* \,=\, \bar{\delta} W^* \,+\, \int_R J_\alpha(x)\, \delta x^\alpha\, d^n x, \tag{122}$$

where $\bar{\delta} W^*$ is defined by Equation (44) and

$$J_\alpha(x) \,=\, E_\alpha \,+\, B^j\, \partial_\alpha U_j(x) \,+\, \sum_{q=1}^{4} C_q^{j\,\mu}\, \partial_\alpha\, (\widehat{\partial}_{(\mu,q)} U_j)(x) \,+\, \sum_{q=1}^{2} A_q^{j\,\mu}\, \partial_\alpha\, (\widehat{\mathfrak{I}}_{(\mu,q)} U_j)(x). \tag{123}$$

Here $J_\alpha(x) \,=\, J_\alpha(x, U, \widehat{\partial} U, \widehat{\mathfrak{I}} U, \partial\widehat{\partial} U, \partial\widehat{\mathfrak{I}} U)$.

Using (76), we obtain

$$\delta W^* \,=\, \bar{\delta} W^* \,+\, \sum_{m=1}^{s} \int_R J_{(m)}(x)\, d^n x\, \delta\omega_m, \tag{124}$$

where

$$J_{(m)}(x) \,=\, J_\alpha(x)\, X^\alpha_{(m)}(x) \,=\, B^j\, \partial_\alpha U_j(x)\, X^\alpha_{(m)}(x) \,+\, \sum_{q=1}^{4} C_q^{j\,\mu}\, \partial_\alpha\, (\widehat{\partial}_{(\mu,q)} U_j)(x)\, X^\alpha_{(m)}(x) \,+$$

$$\sum_{q=1}^{2} A_q^{j\,\mu}\, \partial_\alpha\, (\widehat{\mathfrak{I}}_{(\mu,q)} U_j)(x)\, X^\alpha_{(m)}(x) \,+\, E_\alpha\, X^\alpha_{(m)}(x). \tag{125}$$

As a result, using transformations (74) and (75), and Equations (113) and (114) one can obtain

$$\delta S \,+\, \bar{\delta} W^* \,=\, -\, \sum_{m=1}^{s} \int_R \left(\frac{\partial}{\partial x^\mu}\, \Theta^{\mu}_{(m)}(x) \,-\, J_{(m)}(x) \right) d^n x\, \delta\omega_m, \tag{126}$$

where $\Theta^{\mu}_{(m)}(x)$ is defined by Equation (91).

Similar to the proof of the GF Noether's theorem (Theorem 2), non-holonomic variational Equation (81), and the independence of the transformation parameters give

$$\int_R \left(\frac{\partial}{\partial x^\mu} \Theta^\mu_{(m)}(x) \,-\, J_{(m)}(x) \right) d^n x \,=\, 0. \tag{127}$$

Using that the region R of integration is arbitrary, Equation (117) gives the law

$$\frac{\partial}{\partial x^\mu} \Theta^\mu_{(m)}(x) \,=\, J_{(m)}(x). \tag{128}$$

In the simplest case $C^{j\,\mu}_q \,=\, 0$, $A^{j\,\mu}_q \,=\, 0$, $E_\mu \,=\, 0$, we have

$$J_{(m)}(x) \,=\, B^j \,\partial_\alpha U_j(x)\, X^\alpha_{(m)}(x), \tag{129}$$

where $m \,=\, 1\ldots, s$, and summation over indices j and α is meant. Equation (128) describe generalized conservation laws (sometimes called the laws of change or balance equations).

The assumption (86) means that

$$J_{(m)}(x) \,=\, 0 \quad \text{for all } x \in R. \tag{130}$$

This case is similar to the case of the existence of dissipative structures [166], when the loss of the system energy is compensated by its influx from the outside. The field equations with condition $J_{(m)}(x) \,=\, 0$ are similar to the case of steady states of open classical and quantum systems, which coincide with stationary states of closed classical and quantum systems that are described in [167–169] and in Chapter 21 of [170] (pp. 453–462) (see also [171–173]). A dissipative structure is a dynamical state that can be interpreted as an reproducible steady state. This is a state that occurs in a nonequilibrium environment, provided that the dissipation of energy occurs, which comes from the outside.Dissipative structures are formed only in open nonequilibrium systems that exchange energy, substance, or both with the environment. A dissipative structure is a dissipative system that develops in an environment with which it is exchanged by energy or substance. In other words, this is an open dissipative system, which is characterized by the balance of its exchanges (energy and substance exchange). In such systems, a violation of spatial symmetry (anisotropy) may occur, and chaotic complex structures may appear.

5. Example of Application to Fractional Field Equations

5.1. General Energy–Momentum Tensor and Energy–Momentum Vector

For infinitesimal space-time translations

$$x^\mu \quad\longrightarrow\quad x'^\mu \,=\, x^\mu \,+\, \delta\, x^\mu, \tag{131}$$

one can consider $\delta\, x^\mu$ as the transformation parameters $\delta\,\omega_m \,=\, \delta\,\omega^\mu$. Using the transformation law

$$U_\mu(x) \quad\longrightarrow\quad U'_\mu(x') \,=\, U_\mu(x). \tag{132}$$

one can see

$$X^\mu_{(m)} \,=\, X^\mu_\nu \,=\, \delta^\mu_\nu, \quad Y_{\mu(m)} \,=\, 0, \tag{133}$$

where $\mu,\, \nu \,=\, 1, \ldots, n$. In this case, Equation (116) gives

$$\begin{aligned} \Theta^\mu_{(m)}(x) \,=\, T^\mu_\nu(x) \,=\, \sum_{\alpha=1}^{n} \sum_{q=1}^{2} (-1)^{q+1} \left(\frac{\partial \Lambda}{\partial\, \widehat{\partial}_{(\mu,q)} U_\alpha} \,+\, C^{\alpha\,\mu}_q \right) \widehat{\Im}_{(\mu,q)}\, \partial_\nu\, U_\alpha(x) \,+ \\ \sum_{\alpha=1}^{n} \sum_{q=1}^{2} (-1)^{q+1}\, \widehat{\Im}^\dagger_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial\, \widehat{\partial}_{(\mu,q+2)} U_\alpha} \,+\, C^{\alpha\,\mu}_{q+2} \right) \partial_\nu U_\alpha(x) \,-\, \Lambda\, \delta^\mu_\nu. \end{aligned} \tag{134}$$

As a result, we have

$$T^{\mu\nu}(x) \;=\; \sum_{\alpha=1}^{n}\sum_{q=1}^{2}(-1)^{q+1}\left(\frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q)}U_\alpha} \;+\; C_q^{\alpha\,\mu}\right)\widehat{\mathfrak{I}}_{(\mu,q)}\,g^{\nu\beta}\,\partial_\beta\,U_\alpha(x)\,+$$
$$\sum_{\alpha=1}^{n}\sum_{q=1}^{2}(-1)^{q+1}\,\widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)}\left(\frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q+2)}U_\alpha} \;+\; C_{q+2}^{\alpha\,\mu}\right)g^{\nu\beta}\,\partial_\beta U_\alpha(x) \;-\; \Lambda\,g^{\mu\nu}. \tag{135}$$

Equation of the conservation law (118) has the form

$$\frac{\partial}{\partial x^\mu}\,T^{\mu\nu}(x) \;=\; 0. \tag{136}$$

Integrals of $T^{\mu\nu}(x)$ that is written by Equation (121) is the time-conserved vector

$$C_{(m)}(t) \;=\; P^\mu \;=\; \int_{\partial R} T^{\mu n}(x)\,d^{n-1}\,x. \tag{137}$$

The component of vector (137) with $\mu = n$ is the Hamilton function, which is interpreted as the energy. The vector P^μ is called the general fractional energy–momentum vector, and the tensor $T^{\mu\nu}(x)$ is interpreted as the general fractional energy–momentum tensor.

5.2. *General Orbital and Spin Angular-Momentum Tensors*

Let us consider the infinitesimal rotations

$$x^\mu \quad\longrightarrow\quad x'^{\mu} \;=\; x^\mu \;+\; x_\nu\,\delta\,\Omega^{\mu\nu}, \tag{138}$$

where the indices μ and ν define the plane, in which rotation with the parameter $\Omega^{\mu\nu}$ takes place. Using that $\Omega^{\mu\nu} = -\,\Omega^{\nu\mu}$, one can consider $\delta\,\Omega^{\mu\nu}$ with $\mu < \nu$ as the linear independent transformation parameters

$$\delta\,\omega_m \;=\; \delta\,\omega^{\mu\nu} \;=\; \delta\,\Omega^{\mu\nu},$$

where $\mu < \nu$.

The variations $\delta\,x^\mu$ and $\delta\,U_\mu(x)$ can be expressed in terms of the infinitesimal linearly independent transformation parameters $\omega^{\mu\nu}$, [1], pp. 20–22. The variations $\delta\,x^\mu$ and $\delta\,U_\mu(x)$ can be expressed by the equations

$$\delta\,x^\mu \;=\; \sum_{\alpha<\beta} X^\mu_{\ \alpha\beta}(x)\,\delta\,\omega^{\alpha\beta}, \tag{139}$$

$$X^\mu_{\ \alpha\beta}(x) \;=\; x_\beta\,\delta^\mu_\alpha \;-\; x_\alpha\,\delta^\mu_\beta, \tag{140}$$

where $\alpha < \beta$.

The variations $\delta\,U_\mu(x)$ of field $U_\mu(x)$ can be expressed by the equations

$$\delta\,U_\mu(x) \;=\; \sum_{\alpha<\beta} Y_{\mu\,\alpha\beta}(x)\,\delta\,\omega^{\alpha\beta}, \tag{141}$$

where

$$Y_{\mu\alpha\beta}(x) \;=\; \sum_{\nu<\beta} A^\nu_{\mu\alpha\beta}\,U_\nu(x). \tag{142}$$

For scalar fields $A^\nu_{\mu\alpha\beta} = 0$. For the vector fields

$$A^\nu_{\mu\alpha\beta} \;=\; g_{\mu\alpha}\,\delta^\nu_\beta \;-\; g_{\mu\beta}\,\delta^\nu_\alpha, \tag{143}$$

where $\alpha < \beta$. For the vector fields, one can write

$$Y_{\mu\alpha\beta}(x) = g_{\mu\alpha}\, U_{\beta}(x) - g_{\mu\beta}\, U_{\alpha}(x). \tag{144}$$

Using Equation (116) with $(m) = (\alpha\beta)$ and $j = \gamma$, the angular-momentum tensor $M^{\mu}{}_{\alpha\beta} = \Theta^{\mu}_{(m)}(x)$ is

$$M^{\mu}{}_{\alpha\beta}(x) = -\sum_{\gamma=1}^{n}\sum_{q=1}^{2}(-1)^{q+1}\left(\frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q)}U_{\gamma}} + C^{\gamma\,\mu}_{q}\right)\widehat{\Im}_{(\mu,q)}\left(Y_{\gamma\,\alpha\beta} - \partial_{\nu}\,U_{\gamma}(x)\,X^{\mu}{}_{\alpha\beta}\right) -$$

$$\sum_{\gamma=1}^{n}\sum_{q=1}^{2}(-1)^{q+1}\,\widehat{\Im}^{\dagger}_{(\mu,q)}\left(\frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q+2)}U_{\gamma}} + C^{\gamma\,\mu}_{q+2}\right)\left(Y_{\gamma\,\alpha\beta} - \partial_{\nu}U_{\gamma}(x)\,X^{\nu}{}_{\alpha\beta}\right) - \Lambda\,X^{\mu}{}_{\alpha\beta}. \tag{145}$$

As a result, using (140) and (142), we obtain

$$M^{\mu}{}_{\alpha\beta}(x) = \sum_{\gamma=1}^{n}\sum_{q=1}^{2}(-1)^{q+1}\left(\frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q)}U_{\gamma}} + C^{\gamma\,\mu}_{q}\right)\widehat{\Im}_{(\mu,q)}\left(x_{\beta}\,\partial_{\alpha}\,U_{\gamma}(x) - x_{\alpha}\,\partial_{\beta}\,U_{\gamma}(x)\right) +$$

$$\sum_{\gamma=1}^{n}\sum_{q=1}^{2}(-1)^{q+1}\,\widehat{\Im}^{\dagger}_{(\mu,q)}\left(\frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q+2)}U_{\gamma}} + C^{\gamma\,\mu}_{q+2}\right)\left(x_{\beta}\,\partial_{\alpha}\,U_{\gamma}(x) - x_{\alpha}\,\partial_{\beta}\,U_{\gamma}(x)\right) - \Lambda\left(x_{\beta}\,\delta^{\mu}_{\alpha} - x_{\alpha}\,\delta^{\mu}_{\beta}\right) -$$

$$\sum_{\gamma=1}^{n}\sum_{q=1}^{2}(-1)^{q+1}\left(\frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q)}U_{\gamma}} + C^{\gamma\,\mu}_{q}\right)\widehat{\Im}_{(\mu,q)}\left(g_{\gamma\alpha}\,U_{\beta}(x) - g_{\gamma\beta}\,U_{\alpha}(x)\right) -$$

$$\sum_{\gamma=1}^{n}\sum_{q=1}^{2}(-1)^{q+1}\,\widehat{\Im}^{\dagger}_{(\mu,q)}\left(\frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q+2)}U_{\gamma}} + C^{\gamma\,\mu}_{q+2}\right)\left(g_{\gamma\alpha}\,U_{\beta}(x) - g_{\gamma\beta}\,U_{\alpha}(x)\right). \tag{146}$$

If the Lagrangian density Λ is independent of the GFDs of the RL types, i.e., $\widehat{\partial}_{(\mu,q)}U_{\gamma}(x)$ with $q = 1$, $q = 2$, and the non-holonomic functional does not depend on the variation of the GFDs of the RL types (i.e., $C^{\gamma\,\mu}_{q} = 0$ for with $q = 1, q = 2$), then

$$M^{\mu}{}_{\alpha\beta}(x) = \sum_{\gamma=1}^{n}\sum_{q=1}^{2}(-1)^{q+1}\,\widehat{\Im}^{\dagger}_{(\mu,q)}\left(\frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q+2)}U_{\gamma}} + C^{\gamma\,\mu}_{q+2}\right)\left(x_{\beta}\,\partial_{\alpha}\,U_{\gamma}(x) - x_{\alpha}\,\partial_{\beta}\,U_{\gamma}(x)\right) -$$

$$\sum_{\gamma=1}^{n}\sum_{q=1}^{2}(-1)^{q+1}\,\widehat{\Im}^{\dagger}_{(\mu,q)}\left(\frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q+2)}U_{\gamma}} + C^{\gamma\,\mu}_{q+2}\right)\left(g_{\gamma\alpha}\,U_{\beta}(x) - g_{\gamma\beta}\,U_{\alpha}(x)\right) - \Lambda\left(x_{\beta}\,\delta^{\mu}_{\alpha} - x_{\alpha}\,\delta^{\mu}_{\beta}\right). \tag{147}$$

Equation (147) can be represented in the form

$$M^{\mu}{}_{\alpha\beta}(x) = g_{\alpha\gamma}\,g_{\beta\delta}\,M^{\mu\gamma\delta}_{OR}(x) + S^{\mu}{}_{\alpha\beta}(x), \tag{148}$$

where $M^{\mu\gamma\delta}_{OR}(x)$ is the orbital angular-momentum tensor that is described as

$$M^{\mu\alpha\beta}_{OR}(x) = x^{\mu}\,T^{\alpha\beta}(x) - x^{\alpha}\,T^{\mu\beta}(x), \tag{149}$$

where $T^{\alpha\beta}(x)$ is the energy–momentum tensor defined by (135), and $S^{\mu}{}_{\alpha\beta}(x)$ is the spin angular-momentum tensor

$$S^{\mu}{}_{\alpha\beta}(x) = \sum_{\gamma=1}^{n}\sum_{q=1}^{2}(-1)^{q+1}\,\widehat{\Im}^{\dagger}_{(\mu,q)}\left(\frac{\partial\Lambda}{\partial\,\widehat{\partial}_{(\mu,q+2)}U_{\gamma}} + C^{\gamma\,\mu}_{q+2}\right)\left(g_{\gamma\alpha}\,U_{\beta}(x) - g_{\gamma\beta}\,U_{\alpha}(x)\right) \tag{150}$$

for vector fields. For scalar field, $S^{\mu}{}_{\alpha\beta}(x) = 0$ since $A^{\nu}_{\gamma\alpha\beta} = 0$, and the orbital angular-momentum is conserved $\partial_\beta M^{\mu\alpha\beta}_{OR}(x) = 0$.

The tensor $S^{\mu}{}_{\alpha\beta}(x)$ characterizes the polarization properties of the field and corresponds to the spin angular momentum of the particles described by the quantized fields $U^\mu(x)$.

The spatial density of the orbital and spin angular momenta is given by $M^{\mu\alpha n}_{OR}(x)$ and $S^{n}{}_{\alpha\beta}(x)$, respectively. Integrating these expressions over configuration space, one can obtain the orbital and spin angular momentum tensors in the form

$$M^{\mu\alpha}_{OR} = \int_{\partial R} M^{\mu\alpha n}_{OR}(x)\, d^{n-1}x, \quad S_{\alpha\beta} = \int_{\partial R} S^{n}{}_{\alpha\beta}(x)\, d^{n-1}x, \tag{151}$$

Contracting the space components of $S_{\alpha\beta}$ with the antisymmetric tensor of $\varepsilon_{\alpha\beta\gamma}$, one can obtain the components of the $(n-1)$-dimensional (pseudo) spin vector $S_\alpha = \varepsilon_{\alpha\beta\gamma}\, S_{\beta\gamma}$.

5.3. Example of Field Equations for Real Scalar Field

Let us consider example with the scalar field $U_j(x) = \varphi(x)$, and let us assume that the holonomic functional does not depend on the GFIs of the field and the non-holonomic functional does not depend on variations of these GFIs of the scalar field. Then, field Equation (71) for scalar field is

$$\left(\frac{\partial \Lambda}{\partial \varphi} + B\right) + \sum_{q=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial\, \widehat{\partial}_{(\mu,q)}\varphi} + C^{j\,\mu}_{q}\right) + \sum_{q,p=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)}\, \widehat{\partial}^{\dagger}_{(\nu,p)} \left(\frac{\partial \Lambda}{\partial\, \widehat{\partial}_{(\nu,p)}\, \widehat{\partial}_{(\mu,q)}\varphi} + C^{j\,\mu\nu}_{q,p}\right) = 0. \tag{152}$$

It should be emphasized that non-holonomic functional $\overline{\delta}\, W^*$ cannot be represented as a variation of holonomic functional. For the function $B = B(x,\, \varphi,\, \widehat{\partial}\varphi,\, \widehat{\partial}^2\varphi)$, this fact means that it cannot be represented as

$$B = \frac{\partial \mathcal{J}}{\partial \varphi} + \sum_{q=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} \frac{\partial \mathcal{J}}{\partial\, \widehat{\partial}_{(\mu,q)}\varphi} + \sum_{q,p=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)}\, \widehat{\partial}^{\dagger}_{(\nu,p)} \frac{\partial \mathcal{J}}{\partial\, \widehat{\partial}_{(\nu,p)}\, \widehat{\partial}_{(\mu,q)}\varphi}. \tag{153}$$

for a function $\mathcal{J} = \mathcal{J}(x,\, \varphi,\, \widehat{\partial}\varphi,\, \widehat{\partial}^2\varphi)$, i.e., there is no such function $\mathcal{J}$, for which Equation (153) is satisfied.

In standard field theory, field equations of scalar fields can be derived from holonomic variational equation only if the field equation is linear with respect to second-order derivatives [35] (p. 534).

Let us consider the function B as a bilinear combination of two GFDs $\widehat{\partial}_{(\nu,p)}\, \widehat{\partial}_{(\mu,q)}\varphi(x)$ such that

$$B = \sum_{q,p=1}^{4} \delta^{\mu\nu,\alpha\beta}_{qp}(x,\varphi)\, (\widehat{\partial}_{(\mu,p)}\, \widehat{\partial}_{(\alpha,q)}\varphi)(x)\, (\widehat{\partial}_{(\nu,p)}\, \widehat{\partial}_{(\beta,q)}\varphi)(x), \tag{154}$$

where

$$\delta^{\mu\nu,\alpha\beta}_{qp}(x,\varphi) = a_{qp}(x,\varphi)\, g^{\mu\nu}\, g^{\alpha\beta} + b_{qp}(x,\varphi)\, g^{\mu\alpha}\, g^{\nu\beta} + c_{qp}(x,\varphi)\, g^{\mu\beta}\, g^{\nu\alpha}. \tag{155}$$

In particular case a_{qp}, b_{qp}, c_{qp} can be arbitrary real numbers.

For Lagrangian density $\Lambda = \Lambda(x,\, \varphi,\, \widehat{\partial}\phi)$ and $C^{j\,\mu}_{q} = 0$ and $C^{j\,\mu\nu}_{q,p} = 0$, field Equation (152) is

$$\frac{\partial \Lambda}{\partial \varphi} + B(\varphi,\, \widehat{\partial}\varphi, \widehat{\partial}^2\varphi) + \sum_{q=1}^{4} \left(\widehat{\partial}^{\dagger}_{(\mu,q)} \frac{\partial \Lambda}{\partial\, \widehat{\partial}_{(\mu,q)}\varphi}\right) = 0, \tag{156}$$

where $B = B(\varphi,\, \widehat{\partial}\varphi, \widehat{\partial}^2\varphi)$ is defined by (154) and cannot be represented as (153).

For function (154) and the Lagrangian density

$$\Lambda(x,\, \varphi,\, \widehat{\partial}\phi) \;=\; -\frac{1}{2}\Big(\sum_{q,p=1}^{4} g^{\mu\nu}\, (\widehat{\partial}_{(\mu,q)}\varphi)(x)(\widehat{\partial}_{(\nu,p)}\varphi)(x) \;+\; m^2\, \varphi^2(x) \Big), \tag{157}$$

with $C^{j\,\mu}_{q} \;=\; 0$ and $C^{j\,\mu\nu}_{q,p} \;=\; 0$, field Equation (152) is

$$-\sum_{q=1}^{4} g^{\mu\nu}\, (\widehat{\partial}^{\dagger}_{(\mu,q)}\, g^{\mu\nu}\, \widehat{\partial}_{(\nu,p)}\,\varphi)(x) \;-\; m^2\, \varphi(x) +$$

$$\sum_{q,p=1}^{4} \delta^{\mu\nu,\alpha\beta}_{qp}(x,\varphi)\; (\widehat{\partial}_{(\mu,p)}\, \widehat{\partial}_{(\alpha,q)}\,\varphi)(x)\; (\widehat{\partial}_{(\nu,p)}\, \widehat{\partial}_{(\beta,q)}\,\varphi)(x) \;=\; 0, \tag{158}$$

where $B(\varphi,\, \widehat{\partial}\varphi, \widehat{\partial}^2\varphi)$ cannot be represented in the form (153) and $q \;\neq\; p$, in general.

Using (135) and field Equation (158), general energy–momentum tensor of the scalar field is

$$T^{\mu\nu}(x) \;=\; \sum_{q=1}^{2}(-1)^{q+1} \left(\frac{\partial\Lambda}{\partial\, \widehat{\partial}_{(\mu,q)}\,\varphi} \right) \widehat{\mathfrak{I}}_{(\mu,q)}\, g^{\nu\beta}\, \partial_{\beta}\, \varphi(x) +$$

$$\sum_{q=1}^{2}(-1)^{q+1}\, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial\Lambda}{\partial\, \widehat{\partial}_{(\mu,q+2)}\,\varphi} \right) g^{\nu\beta}\, \partial_{\beta}\varphi(x) \;-\; \Lambda\, g^{\mu\nu}. \tag{159}$$

Using Equation (145) with $Y_{\gamma\,\alpha\beta} \;=\; 0$, the angular-momentum tensor of the sclar field is

$$M^{\mu}_{\ \alpha\beta}(x) \;=\; \sum_{q=1}^{2}(-1)^{q+1} \left(\frac{\partial\Lambda}{\partial\, \widehat{\partial}_{(\mu,q)}\,\varphi} \right) \widehat{\mathfrak{I}}_{(\mu,q)} \left(\partial_{\nu}\, \varphi(x)\, X^{\mu}_{\ \alpha\beta} \right) +$$

$$\sum_{q=1}^{2}(-1)^{q+1}\, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial\Lambda}{\partial\, \widehat{\partial}_{(\mu,q+2)}\,\varphi} \right) \left(\partial_{\nu}\varphi(x)\, X^{\nu}_{\ \alpha\beta} \right) \;-\; \Lambda\, X^{\mu}_{\ \alpha\beta} \;= \tag{160}$$

$$\sum_{q=1}^{2}(-1)^{q+1} \left(\frac{\partial\Lambda}{\partial\, \widehat{\partial}_{(\mu,q)}\,\varphi} \right) \widehat{\mathfrak{I}}_{(\mu,q)}\, (x_{\beta}\, \partial_{\alpha}\, \varphi(x) \;-\; x_{\alpha}\, \partial_{\beta}\, \varphi(x)) +$$

$$\sum_{q=1}^{2}(-1)^{q+1}\, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial\Lambda}{\partial\, \widehat{\partial}_{(\mu,q+2)}\,\varphi} \right) (x_{\beta}\, \partial_{\alpha}\, \varphi(x) \;-\; x_{\alpha}\, \partial_{\beta}\, \varphi(x)) \;-\; \Lambda\, (x_{\beta}\, \delta^{\mu}_{\alpha} \;-\; x_{\alpha}\, \delta^{\mu}_{\beta}), \tag{161}$$

where we use that $A^{\nu}_{\mu\alpha\beta} \;=\; 0$ for scalar field.

If the Lagrangian density Λ is independent of the GFDs of the RL types that is $(\widehat{\partial}_{(\mu,q)} U_{\gamma}$ with $q \;=\; 1, q \;=\; 2)$, then

$$M^{\mu}_{\ \alpha\beta}(x) \;=\; \sum_{q=1}^{2}(-1)^{q+1}\, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial\Lambda}{\partial\, \widehat{\partial}_{(\mu,q+2)}\,\varphi} \right) (x_{\beta}\, \partial_{\alpha}\, \varphi(x) \;-\; x_{\alpha}\, \partial_{\beta}\, \varphi(x)). \tag{162}$$

and the orbital angular-momentum tensor

$$M^{\mu}_{\ \alpha\beta}(x) \;=\; g_{\alpha\gamma}\, g_{\beta\delta}\, M^{\mu\gamma\delta}_{OR}(x), \tag{163}$$

the spin angular-momentum tensor

$$S^{\mu}_{\ \alpha\beta}(x) \;=\; 0. \tag{164}$$

5.4. Example of Field Equations for Real Vector Field

Let us consider a vector field with components $U_j(x) = A_\mu(x)$. To simplify the consideration, we will assume that the Lagrangian density Λ is independent of the GFDs of the RL types, i.e., $\widehat{\partial}_{(\mu,q)} U_\gamma(x)$ with $q = 1$, $q = 2$, and the non-holonomic functional $\bar{\delta} W^*$ does not depend on the variation of the GFDs of the Riemann–Liouville types (i.e., $C_q^{\gamma\,\mu} = 0$ for with $q = 1$, $q = 2$). We will also assume that Λ does not depend on the GFIs $\widehat{\mathfrak{I}}_{(\mu,q)} U_\gamma(x)$ and non-holonomic functional does not depend on variation of the GFIs $\bar{\delta}\, \widehat{\mathfrak{I}}_{(\mu,q)} U_\gamma(x)$.

One can consider the Lagrangian density with bilinear combinations of the vector field functions A_μ and their GFDs of the Caputo type $\widehat{\partial}_{(\mu,q)} A_\nu$ with $q = 3$ and $q = 4$ in the form

$$\Lambda = -\frac{1}{2} \sum_{q,p=3}^{4} \Big(\delta_{q\,p}^{\mu\nu,\alpha\beta}\, \widehat{\partial}_{(\mu,q)}\, A_\alpha\, \widehat{\partial}_{(\nu,p)}\, A_\beta + m^2\, A_\mu A^\mu \Big) =$$

$$-\frac{1}{2} \sum_{q,p=3}^{4} \Big(a_{qp}\, \widehat{\partial}_{(\mu,q)}\, A_\alpha\, \widehat{\partial}^{(\mu,p)}\, A^\alpha + b_{qp}\, \widehat{\partial}_{(\mu,q)}\, A^\mu\, \widehat{\partial}_{(\alpha,p)}\, A^\alpha +$$

$$c_{qp}\, \widehat{\partial}_{(\mu,q)}\, A^\alpha\, \widehat{\partial}_{(\alpha,p)}\, A^\mu + m^2\, A_\mu A^\mu \Big), \tag{165}$$

where a_{pq}, b_{qp}, c_{qp} are arbitrary real constants, and $\delta_{q\,p}^{\mu\nu,\alpha\beta}$ is the bilinear combination of the metrics

$$\delta_{qp}^{\mu\nu,\alpha\beta} = a_{qp}\, g^{\mu\nu}\, g^{\alpha\beta} + b_{qp}\, g^{\mu\alpha}\, g^{\nu\beta} + c_{qp}\, g^{\mu\beta}\, g^{\nu\alpha}, \tag{166}$$

The non-holonomic functional (44) can be taken in the form

$$\bar{\delta}\, W^* = \lambda \int \Big(B^\mu(x, A(x), \widehat{\partial} A(x))\, \bar{\delta}\, A_\mu(x) + \sum_{q=3}^{4} C_q^{\nu\,\mu}(x, A(x), \widehat{\partial} A(x))\, \bar{\delta}\, \widehat{\partial}_{(\mu,q)} A_\nu(x) \Big)\, d^n x. \tag{167}$$

In this case, the field Equations (62) of the vector field are

$$\Big(\frac{\partial \Lambda}{\partial A_\alpha} + B^\alpha \Big) + \sum_{q=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} \Big(\frac{\partial \Lambda}{\partial\, \widehat{\partial}_{(\mu,q)} A_\alpha} + C_q^{\alpha\,\mu} \Big) = 0. \tag{168}$$

For the Lagrangian density (165), we obtain

$$\frac{\partial \Lambda}{\partial A_\alpha} = -\, m^2\, A^\alpha(x), \tag{169}$$

$$\frac{\partial \Lambda}{\partial (\widehat{\partial}_{(\mu,q)} A_\alpha)} = -\, \delta_{qp}^{\mu\beta,\nu\alpha}\, (\widehat{\partial}_{(\nu,p)}\, A_\beta)(x). \tag{170}$$

Using (169) and (170), field Equations (168) take the form

$$\Big(-\, m^2\, A^\alpha(x) + B^\alpha \Big) + \sum_{q=1}^{4} \widehat{\partial}^{\dagger}_{(\mu,q)} \Big(-\, \delta_{qp}^{\mu\beta,\nu\alpha}\, (\widehat{\partial}_{(\nu,p)}\, A_\beta)(x) + C_q^{\alpha\,\mu} \Big) = 0. \tag{171}$$

The energy–momentum tensor (135) is

$$T^{\mu\nu}(x) = \sum_{\alpha=1}^{n} \sum_{q=1}^{2} (-1)^{q+1}\, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \Big(\frac{\partial \Lambda}{\partial\, \widehat{\partial}_{(\mu,q+2)} A_\alpha} + C_{q+2}^{\alpha\,\mu} \Big)\, g^{\nu\beta}\, \partial_\beta A_\alpha(x) - \Lambda\, g^{\mu\nu} =$$

$$\sum_{\alpha=1}^{n} \sum_{q,p=3}^{4} (-1)^{q+1}\, \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \Big(-\, \delta_{qp}^{\mu\beta,\nu\alpha}\, (\widehat{\partial}_{(\nu,p)}\, A_\beta)(x) + C_q^{\alpha\,\mu} \Big)\, g^{\nu\beta}\, \partial_\beta A_\alpha(x) - \Lambda\, g^{\mu\nu}. \tag{172}$$

The angular-momentum tensor (147) is

$$M^{\mu}{}_{\alpha\beta}(x) = \sum_{\gamma=1}^{n} \sum_{q=1}^{2} (-1)^{q+1} \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q+2)} A_{\gamma}} + C^{\gamma\,\mu}_{q+2} \right) (x_{\beta}\, \partial_{\alpha} A_{\gamma}(x) - x_{\alpha}\, \partial_{\beta} A_{\gamma}(x)) -$$

$$\sum_{\gamma=1}^{n} \sum_{q=1}^{2} (-1)^{q+1} \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(\frac{\partial \Lambda}{\partial \widehat{\partial}_{(\mu,q+2)} A_{\gamma}} + C^{\gamma\,\mu}_{q+2} \right) (g_{\gamma\alpha}\, A_{\beta}(x) - g_{\gamma\beta}\, A_{\alpha}(x)) - \Lambda\, (x_{\beta}\, \delta^{\mu}_{\alpha} - x_{\alpha}\, \delta^{\mu}_{\beta}). \quad (173)$$

Substitution Equations (169) and (170) into Equation (173) gives

$$M^{\mu}{}_{\alpha\beta}(x) = \sum_{\gamma=1}^{n} \sum_{q,p=3}^{4} (-1)^{q+1} \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(-\, \delta^{\mu\beta,\nu\gamma}_{qp}\, (\widehat{\partial}_{(\nu,p)} A_{\beta})(x) + C^{\gamma\,\mu}_{q+2} \right) (x_{\beta}\, \partial_{\alpha} A_{\gamma}(x) - x_{\alpha}\, \partial_{\beta} A_{\gamma}(x)) -$$

$$\sum_{\gamma=1}^{n} \sum_{q,p=3}^{4} (-1)^{q+1} \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(-\, \delta^{\mu\beta,\nu\gamma}_{qp}\, (\widehat{\partial}_{(\nu,p)} A_{\beta})(x) + C^{\gamma\,\mu}_{q} \right) (g_{\gamma\alpha}\, A_{\beta}(x) - g_{\gamma\beta}\, A_{\alpha}(x)) - \Lambda\, (x_{\beta}\, \delta^{\mu}_{\alpha} - x_{\alpha}\, \delta^{\mu}_{\beta}). \quad (174)$$

Equation (174) can be represented in the form

$$M^{\mu}{}_{\alpha\beta}(x) = g_{\alpha\gamma}\, g_{\beta\delta}\, M^{\mu\gamma\delta}_{OR}(x) + S^{\mu}{}_{\alpha\beta}(x), \quad (175)$$

where $M^{\mu\gamma\delta}_{OR}(x)$ is the orbital angular-momentum tensor that is described as

$$M^{\mu\alpha\beta}_{OR}(x) = x^{\mu}\, T^{\alpha\beta}(x) - x^{\alpha}\, T^{\mu\beta}(x), \quad (176)$$

where $T^{\alpha\beta}(x)$ is the energy–momentum tensor defined by (172), and $S^{\mu}{}_{\alpha\beta}(x)$ is the spin angular-momentum tensor

$$S^{\mu}{}_{\alpha\beta}(x) = \sum_{\gamma=1}^{n} \sum_{q,p=3}^{4} (-1)^{q+1} \widehat{\mathfrak{I}}^{\dagger}_{(\mu,q)} \left(-\, \delta^{\mu\beta,\nu\gamma}_{qp}\, (\widehat{\partial}_{(\nu,p)} A_{\beta})(x) + C^{\gamma\,\mu}_{q} \right) (g_{\gamma\alpha}\, A_{\beta}(x) - g_{\gamma\beta}\, A_{\alpha}(x)). \quad (177)$$

Note that the sum over q and p is realized from 3 to 4.

6. Conclusions

In this paper, the following generalizations have been proposed.

1. A generalization of the standard action principle and the usual fractional action principle by using the Luchko general fractional calculus and non-holonomic variational equations of Sedov type. The proposed general fractional action principle is formulated as a non-holonomic variational equation that depends on variations of coordinates, GF derivatives and GF integrals. The GF integrals are considered in the left-sided and right sided. The GF derivatives of Riemann–Liouville and Caputo types are considered in both the left-sided and right-sided versions.

2. A generalization of the standard Noether theorem and the usual fractional Noether theorem by using the Luchko general fractional calculus and non-holonomic variational equations of Sedov type. The proposed general fractional Noether theorem is formulated by using the non-holonomic variational equations and general fractional action principle for equations with the GF derivatives and GF integrals.

3. Examples of field equations with GFDs and GFIs are suggested. The energy–momentum tensor, orbital angular-momentum tensor and spin angular-momentum tensor are given for general fractional non-Lagrangian field theories. Examples of the application of generalized first Noether's theorem are suggested for scalar end vector fields of non-Lagrangian field theory.

The proposed GF action principle and GF Noether theorem allow us to take into account a wide range of space–time nonlocalities and a wide class of irreversible processes, dissipative and open systems, and non-Lagrangian and non-Hamiltonian field theories and systems.

Let us note new directions for the development and application of the suggested approach to the description of nonlocal processes and systems.

(1) The GF action principle and the GF Noether theorem can be used for modern continuum mechanics including electrodynamics, hydrodynamics, and elasticity theory, where properties of spatial nonlocality and memory with dissipation and irreversibility should be taken into account [147,148].

(2) The GF action principle and the GF Noether theorem can be used in nonlocal quantum mechanics. The standard Schwinger action principle (see in [2] (Section 2.1, pp. 60–78), and [174]) can be generalized for spatial nonlocality and memory. It should be emphasized that to take into account dissipative and irreversible processes, quantum mechanics should be described by Lindblad equations and its generalizations instead of the Schrodinger equations for the wave function.

(3) The basic postulate of quantization of classical fields use the Noether theorem (see classical book of Bogoliubov and Shirkov [1] (p. 89)), This postulate and the GF Noether theorem can be used to formulate the quantization of non-Lagrangian field theories with space–time nonlocality. This approach can be used to formulate quantum field theories similar to quantum mechanics of open, non-Hamiltonian and dissipative systems [170–173].

(4) The GF action principle and the GF Noether theorem can be formulated for spinor fields, gauge symmetries, and corresponding conservation laws to generalize ths standard gauge field theories [7,11,18,111]. It should be emphasized that the general fractional second Noether theorem should be described with the generalization for spinor fields that are considered in the standard gauge theories [7,11,18,111].

(5) The GF action principle and the GF Noether theorem can also be generalized by using the multi-kernel GFC of arbitrary order [144], the GFC in the Riesz form for multi-dimensional space [142], the scale-Invariant GFC [143].

There are many other directions of development and applications of the proposed GF nonholonomic variational equations and the GF Noether theorem.

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