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Reductions and Exact Solutions of Nonlinear Wave-Type PDEs with Proportional and More Complex Delays

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Abstract: The study gives a brief overview of publications on exact solutions for functional PDEs with delays of various types and on methods for constructing such solutions. For the first time, second-order wave-type PDEs with a nonlinear source term containing the unknown function with proportional time delay, proportional space delay, or both time and space delays are considered. In addition to nonlinear wave-type PDEs with constant speed, equations with variable speed are also studied. New one-dimensional reductions and exact solutions of such PDEs with proportional delay are obtained using solutions of simpler PDEs without delay and methods of separation of variables for nonlinear PDEs. Self-similar solutions, additive and multiplicative separable solutions, generalized separable solutions, and some other solutions are presented. More complex nonlinear functional PDEs with a variable time or space delay of general form are also investigated. Overall, more than thirty wave-type equations with delays that admit exact solutions are described. The study results can be used to test numerical methods and investigate the properties of the considered and related PDEs with proportional or more complex variable delays.

Keywords: nonlinear wave-type equations; PDEs with proportional delay; delay Klein–Gordon equations; PDEs with variable delay; partial functional-differential equations; reductions and exact solutions; self-similar solutions; additive and multiplicative separable solutions; generalized separable solutions

MSC: 35L05; 35L70; 35R10; 35C05

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1. Introduction

1.1. Delay ODEs

Delay naturally arises in processes and systems that exhibit heredity, where the current state is influenced by the prehistory or, particularly, by a certain moment in the past. The simplest cases are described by delay ODEs of the form

$$u'_t = f(u, w), \quad w = u(t - \tau), \tag{1}$$

where $u = u(t)$ is the unknown function, $f(u, w)$ is a given function, and $\tau > 0$ is a constant delay.

Equations of the form (1) and more complex ODEs with constant delay are often used to model various processes in population theory, medicine, epidemiology, economics, and many other fields (e.g., see [1–21]).

Let us now consider a broader class of ODEs with a variable delay of general form:

$$u'_t = f(u, w), \quad w = u(\eta(t)), \quad t > 0, \tag{2}$$

where $\eta = \eta(t)$ is a given function that satisfies the conditions

$$\eta < t, \quad \eta'_t > 0 \quad \text{for all} \;\; t > 0. \tag{3}$$

The second inequality for the derivative in (3) is optional and is introduced here temporarily for a clearer interpretation of the models under consideration (note that the second condition is not used further in Section 4).

Inequalities (3) can be non-strict at one or more isolated points, including the initial point $t = 0$. The delay function $\eta(t)$ in (2) is often written in the alternative form $\eta(t) = t - \tau(t)$, where $\tau(t) > 0$. For example, a variable delay may occur when the transmission rate of the control signal from one object to another is finite and constant, and these objects are moving away from each other at a constant or variable speed.

Let us expand the function $\eta(t)$ in a Taylor series for small t. Neglecting the terms of the order of t^2 and higher, we obtain

$$\eta(t) = \alpha + qt, \tag{4}$$

where $\alpha = \eta(0)$ and $q = \eta'(0)$. From conditions (3) for small t, it follows that the coefficients in Formula (4) must satisfy the inequalities

$$\alpha \leq 0, \quad 0 < q \leq 1, \tag{5}$$

where two equal signs cannot be taken simultaneously.

When inequalities (5) hold, the linear function (4) satisfies condition (3) for all $t > 0$.

Note that in the neighborhood of any point t_0, one can obtain a local linear approximation of the delay function in the form $\eta = \alpha + q(t - t_0)$. In this case, the coefficients α and q will depend on the point t_0 and satisfy the inequalities $\alpha < t_0$ and $0 < q < 1$, which follow from conditions (3).

We show that the simple Formula (4) describes the two most important cases frequently arising in the mathematical modeling of processes with delay. Indeed, substituting (4) with $q = 1$ and $\alpha = -\tau$ ($\tau > 0$) into (2), we arrive at the ODE with constant delay (1). Next, substituting (4) with $\alpha = 0$ into (2), we come to the ODE with proportional delay

$$u'_t = f(u, w), \quad w = u(qt), \quad 0 < q < 1. \tag{6}$$

The function $w = u(qt)$ appearing on the right-hand side of ODE (6) differs from the function $u(t)$ by dilation of the argument t by a factor of $1/q$. Equations of the form (6) are sometimes called 'pantograph-type ODEs.'

Equations of the form (6) and more complex ODEs with proportional delay are used to describe various phenomena in electrodynamics [22], population theory [23], number theory [24], stochastic games [25], graph theory [26], risk and queueing theory [27], and theory of artificial neural networks [28–30]. Importantly, related ordinary functional differential equations with the function $w = u(qt)$, where $q > 1$, are used to describe various processes in biology [31–33] and astrophysics [34].

Approximate analytical and numerical methods for solving ODEs with proportional delay are relatively well developed (e.g., see [35–47] for analytical and [48–73] for numerical methods).

Currently, analytical and numerical methods for solving ODEs with constant and variable delays can be regarded as well elaborated.

Notably, the vast majority of analytical methods that allow one to successfully find exact solutions for nonlinear PDEs without delay either are inapplicable to constructing exact solutions for nonlinear PDEs with constant or variable delay or have a minimal area of applicability.

1.2. Delay First-Order PDEs

There are some studies concerned with first-order delay PDEs. A complete group classification of the first-order Hopf-type equation with constant delay

$$u_t + uu_x = f(u, w), \quad w = u(x, t - \tau),$$

is given in [74]. It obtained three exact solutions, other than traveling waves, with arbitrary functions and free parameters. The article [75] treats the first-order PDE with a variable speed, proportional space delay, and constant time delay

$$u_t + [g(x)u]_x = f(w), \quad w = u(px, t - \tau),$$

where $0 < p < 1$, as a model of proliferative stem and precursor cell production in the bone marrow. The case of several space variables is investigated in [76]. The existence, uniqueness, invariance, and asymptotic behavior of solutions of such PDEs are studied in [75,76] by using semigroup theory and characteristic theory, respectively. Another first-order PDE with constant time delay and variable space delay,

$$u_t + g(x)u_x = f(t, u, w), \quad w = u(\xi(x), t - \tau),$$

is used in [77] to describe the biological process of hematological cell development from a pluripotential stem cell population. Here, x is the so-called maturation variable, which could be associated with the intracellular hemoglobin concentration (for detailed biological description, see [77] Section 2.1). The variable space delay $\xi(x)$ is used to model the population behavior's dependence on the behavior at a previous maturation level.

The study [78] models the growth and division of cells structured by size with a linear first-order PDE with proportional delay. Its solution is shown to have the form of a series whose terms are determined from simpler equations without delay.

Numerical methods of integration of first-order delay PDEs are studied, for example, in [79–82].

1.3. Delay Reaction–Diffusion PDEs

1°. *Delay reaction–diffusion PDEs with constant delay*. More complex equations can take into account spatial heterogeneity, which leads to delay PDEs of the form

$$u_t = au_{xx} + f(u, w), \quad w = u(x, t - \tau), \tag{7}$$

which are often referred to as 'delay reaction–diffusion equations'. Such equations describe various processes in many areas, including population theory [83–98], medicine [99–107], epidemiology [108–118], biology [119–124], chemistry [125–129], control theory [130–133], theory of artificial neural networks [134–139] and many others (e.g., see [131,140]). Delay PDEs (7) admit traveling-wave solutions, $u = u(z)$, where $z = kx + \lambda t$ (e.g., see [131,141–143]), but do not have self-similar solutions, $u = t^{\beta}\varphi(xt^{\lambda})$, which non-delay PDEs often have. Reductions and exact solutions with additive, multiplicative, and generalized separation of variables and more complex solutions for delay PDEs are obtained in [144–156] (see also [157] for a brief review, which, in addition to exact solutions, describes the main numerical methods for solving such equations).

2°. *Delay reaction–diffusion PDEs with proportional delay*. Consider the following reaction–diffusion equations involving a delay with time variability:

$$u_t = au_{xx} + f(u, w), \quad w = u(x, \eta(t)).$$

For example, PDEs with variable time delay can be used to describe changes in population density when the maturation period (modeled by a variable time delay) depends on the ambient temperature that varies non-linearly. Choosing a more specific form of the delay, $\eta(t) = qt$, we arrive at PDEs with proportional delay that have the following general form:

$$u_t = au_{xx} + f(u, w), \quad w = u(x, qt), \tag{8}$$

where $0 < q < 1$. Referring to the example above, case (8) would correspond to a simpler model in which the ambient temperature would change linearly.

There are also PDEs with a proportional space argument,

$$u_t = au_{xx} + f(u, w), \quad w = u(px, t), \tag{9}$$

and proportional delays in both the time and space arguments,

$$u_t = au_{xx} + f(u, w), \quad w = u(px, qt), \tag{10}$$

where p and q are scaling parameters ($0 < p < 1$ and $0 < q < 1$). In what follows, PDEs containing a nonlinear function $f(u, w)$ with $w = u(px, qt)$ will often be called equations with two proportional arguments.

Publications on the properties and solutions of diffusion-type PDEs with proportional delay are much less common than those for PDEs with constant delay. In [158], an asymptotic solution of a linear reaction–diffusion equation with proportional delay is obtained. The study [159] constructs exact solutions of linear heat and wave equations with proportional delay using the method of separation of variables. Many nonlinear reaction–diffusion PDEs with proportional delay that admit reductions and exact solutions with additive, multiplicative, generalized, and functional separation of variables are described in [160–162]. Approximate analytical methods for solving some linear and nonlinear PDEs with proportional delays are considered in [163,164]. Numerical methods for solving PDEs with proportional delay are developed in [165,166], and those for PDEs with more complex varying delay are studied in [167].

1.4. Delay Wave-Type PDEs

Nonlinear wave-type equations with constant delay of the form

$$u_{tt} = au_{xx} + f(u, w), \quad w = u(x, t - \tau), \tag{11}$$

are more complex than Equation (7); here, a is a positive constant ($\sqrt{a}$ is the speed) and $f(u, w)$ is the source function. These equations are often referred to as 'delay Klein–Gordon equations'.

The studies [168–172] describe a number of exact solutions and reductions of nonlinear wave equations of the form (11) and related hyperbolic PDEs with constant delay.

The article [173] carries out a complete group analysis of a two-dimensional Klein–Gordon equation with a time-varying delay and describes its exact invariant solutions.

The present paper describes, for the first time, different classes of exact solutions to nonlinear wave-type functional PDEs:

$$u_{tt} = au_{xx} + f(u, w), \quad w = u(x, qt), \tag{12}$$

and

$$u_{tt} = [g(u)u_x]_x + f(u, w), \quad w = u(px, qt), \tag{13}$$

where $u = u(x, t)$ is the unknown function, $\sqrt{g(u)}$ is the variable speed, $f(u, w)$ is the source function, and p and q are scaling parameters.

In addition to Equations (12) and (13), more complex wave-type PDEs with an arbitrary delay function will also be considered.

Notably, reaction–diffusion equations with proportional delay (8) and wave type equations with proportional delay (12) do not have traveling-wave solutions, in contrast to related PDEs with constant delay (7) and (11).

2. Exact Solutions: Definition and Construction Methods

2.1. Reductions. Term 'Exact Solution' for Nonlinear PDEs with Proportional Delay

Reductions of a mathematical equation are generally understood as ways for constructing its solutions (perhaps, not all but some) using solutions of simpler equations. For nonlinear PDEs, one-dimensional reductions are the most important; these allow one to describe solutions for these PDEs in terms of solutions to ODEs, which are much simpler than PDEs. Moreover, reductions of various differential and functional differential equations are crucial in finding their exact solutions.

Below we will use the term 'exact solution' for nonlinear PDEs with constant or variable delay (including proportional delay) in cases where the solution is expressed in terms of:

(i) Elementary functions, functions appearing in the PDE (this is necessary when the PDE contains arbitrary functions), and indefinite or/and definite integrals;
(ii) Solutions of usual non-delay ODEs or systems of non-delay ODEs;
(iii) Solutions of ODEs with constant or variable delay (including proportional delay), or systems of such equations.

Solutions from Items (i)–(iii) can be combined. The above definition generalizes the term 'exact solution' used for nonlinear PDEs with constant delay in [146,148].

2.2. Construction of Exact Solutions to Nonlinear PDEs with Proportional Delay

In many cases, reductions and exact solutions of PDEs with proportional delay (12) and (13) are obtained using the principle of analogy of solutions, whose detailed description and application examples can be found in [160] (see also [161,162]). The method relies on the fact that a solution to a nonlinear partial differential equation with proportional delay can often (but not always) be sought in the same form as a solution to a simpler non-delay PDE obtained from (12) and (13) by setting $w = u$. The use of the principle of analogy of solutions has shown its high efficiency for nonlinear reaction–diffusion equations with proportional delay in [160].

Figure 1 displays a schematic of applying the principle of analogy to wave-type PDEs with proportional delays.

Below we illustrate the use of the principle of analogy of solutions with a simple specific example.

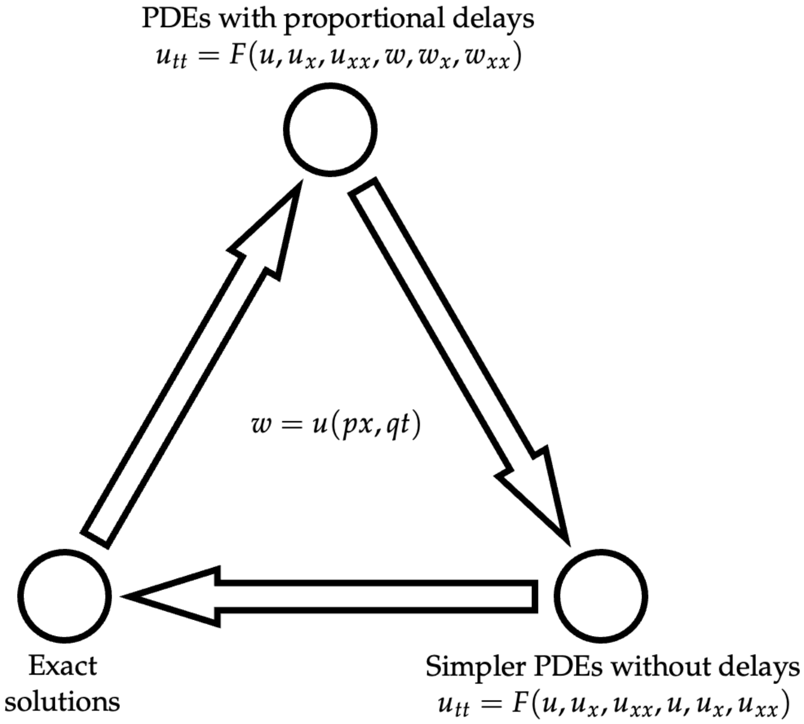

Figure 1. A schematic of using the principle of analogy to construct exact solutions for nonlinear wave-type PDEs with proportional delays.

Example 1. *Consider the wave-type PDE with proportional delay and power-law nonlinearity*

$$u_{tt} = au_{xx} + bw^k, \quad w = u(px, t). \tag{14}$$

Following the principle of analogy of solutions, we set $w = u$ in Equation (14) to obtain a simpler wave-type PDE without delay:

$$u_{tt} = au_{xx} + bu^k. \tag{15}$$

For $k \neq 1$, Equation (15) admits a self-similar solution of the form (e.g., see [174]):

$$u(x, t) = t^{\frac{2}{1-k}} U(z), \quad z = x/t. \tag{16}$$

Using the principle of analogy, we seek a solution of the wave-type PDE with proportional delay (14) also in the form (16). As a result, we get the following nonlinear second-order ODE with proportional delay for $U = U(z)$:

$$\frac{2(1+k)}{(1-k)^2}U - \frac{2(1+k)}{1-k}zU'_z + z^2U''_{zz} = aU''_{zz} + bW^k, \quad W = U(pz).$$

More complex reductions and exact solutions are constructed for nonlinear PDEs using the methods of separation of variables [174–176], modifications of the method of functional constraints [146,151], or by combining these methods with the principle of analogy.

As a result, we have found many nonlinear wave-type PDEs with proportional delay that admit self-similar solutions, solutions with additive, multiplicative, or generalized separation of variables, as well as some other exact solutions. Special attention has been paid to nonlinear wave-type PDEs with proportional delay that involve arbitrary functions. Such equations and their solutions are of utmost interest for testing numerical and approximate analytical methods of solving the corresponding nonlinear initial-boundary value problems.

In what follows, for the generality of the results, we will assume that in wave-type functional PDEs of the form (12) and (13), the scaling parameters p and q can be any positive constants, $0 < p < \infty$ and $0 < q < \infty$, that cannot simultaneously be equal to one. Such functional PDEs will be called 'PDEs with proportional arguments' (in the special case of $0 < p < 1$ and $0 < q < 1$, these equations are 'PDEs with proportional delays'). We also consider more complex PDEs with variable delay, which, in addition to the unknown function $u(x,t)$, contain the function $u(x,\eta)$, where $\eta = \eta(t)$ is any function that satisfies conditions (3) (one can take, for example, the linear function $\eta = \alpha + \beta t$).

The following sections arrange equations following the principle 'from simple to complex.' We only present the final results without intermediate calculations. The solutions below can be easily verified by directly substituting them into the equations considered.

Remark 1. *In the present paper, we dismiss trivial solutions of the form $u = const$ (which correspond to equilibria) for both the equations considered and the simpler equations obtained by reductions. Moreover, we also neglect degenerate solutions that depend on only one independent variable, t or x.*

3. Exact Solutions to Nonlinear Wave-Type PDEs with Proportional Arguments

3.1. Equations with Constant Speed

Equation 1. Consider the equation with proportional delay and quadratic nonlinearity

$$u_{tt} = au_{xx} + bw^2, \quad w = u(x, \tfrac{1}{2}t).$$

It has the following exact solutions.

1°. Multiplicative separable solution:

$$u = e^{-\lambda t}\varphi(x),$$

where λ is an arbitrary constant, and the function $\varphi = \varphi(x)$ is described by the second-order autonomous ODE $a\varphi''_{xx} + b\varphi^2 - \lambda^2\varphi = 0$.

2°. For a self-similar solution of this PDE with proportional delay, see Equation (17) below with $m = 0$, $k = 2$, $p = 1$, and $q = \frac{1}{2}$.

Equation 2.The equation with proportional time-argument and power-law nonlinearity

$$u_{tt} = au_{xx} + bw^{1/q}, \quad w = u(x, qt),$$

admits the following exact solutions.

1°. Multiplicative separable solution:

$$u = e^{-\lambda t}\varphi(x),$$

where λ is an arbitrary constant, and the function $\varphi = \varphi(x)$ is described by the second-order autonomous ODE $a\varphi''_{xx} + b\varphi^{1/q} - \lambda^2\varphi = 0$.

2°. For a self-similar solution of this PDE with proportional argument, see Equation (17) below with $m = 0$, $k = 1/q$, and $p = 1$.

Equation 3. The equation with two proportional arguments and power-law nonlinearity

$$u_{tt} = au_{xx} + bu^m w^k, \quad w = u(px, qt), \tag{17}$$

admits the following exact solutions.

1°. Self-similar solution for $k + m \neq 1$:

$$u(x,t) = t^{\frac{2}{1-k-m}}U(z), \quad z = x/t,$$

where the function $U = U(z)$ is described by the second-order ODE with proportional argument

$$\frac{2(1+k+m)}{(1-k-m)^2}U - \frac{2(1+k+m)}{1-k-m}zU'_z + z^2U''_{zz} = aU''_{zz} + bq^{\frac{2k}{1-k-m}}U^mW^k,$$
$$W = U(sz), \quad s = p/q.$$

2°. Traveling wave solution for $q = p$:

$$u(x,t) = U(z), \quad z = kx - \lambda t,$$

where k and λ are arbitrary constants, and the function $U = U(z)$ is described by the second-order ODE with proportional argument

$$(ak^2 - \lambda^2)U''_{zz} + bU^mW^k = 0, \quad W = U(pz).$$

3°. Multiplicative separable solution for $m = 1 - kq$:

$$u(x,t) = e^{-\lambda t}\varphi(x),$$

where λ is an arbitrary constant, and the function $\varphi = \varphi(x)$ is described by the second-order ODE with proportional argument

$$a\varphi''_{xx} - \lambda^2\varphi + b\varphi^{1-kq}\bar{\varphi}^k = 0, \quad \bar{\varphi} = \varphi(px).$$

4°. Multiplicative separable solution for $m = 1 - kp$:

$$u(x,t) = e^{\beta x}\psi(t),$$

where β is an arbitrary constant, and the function $\psi = \psi(t)$ is described by the second-order ODE with proportional argument

$$\psi''_{tt} = a\beta^2\psi + b\psi^{1-kp}\bar{\psi}^k, \quad \bar{\psi} = \psi(qt).$$

Equation 4. The equation with two proportional arguments and exponential nonlinearity

$$u_{tt} = au_{xx} + be^{\lambda w}, \quad w = u(px, qt),$$

has an exact solution of the form:

$$u(x,t) = U(z) - \frac{2}{\lambda}\ln t, \quad z = \frac{x}{t},$$

where the function $U = U(z)$ is described by the second-order ODE with proportional argument

$$(z^2U'_z)'_z + \frac{2}{\lambda} = aU''_{zz} + \frac{b}{q^2}e^{\lambda W}, \quad W = U(sz), \quad s = \frac{p}{q}.$$

For $p = q$, this equation simplifies to become a usual ODE.

Equation 5. The equation with two proportional arguments and more complex exponential nonlinearity

$$u_{tt} = au_{xx} + be^{\mu u + \lambda w}, \quad w = u(px, qt),$$

for $\mu + \lambda \neq 0$, admits an exact solution of the form

$$u(x,t) = U(z) - \frac{2}{\mu + \lambda}\ln t, \quad z = \frac{x}{t},$$

where the function $U = U(z)$ is described by the second-order ODE with proportional argument

$$(z^2 U'_z)'_z + \frac{2}{\mu+\lambda} = aU''_{zz} + bq^{-\frac{2\lambda}{\mu+\lambda}} e^{\mu U + \lambda W}, \quad W = U(sz), \quad s = \frac{p}{q}.$$

For $p = q$, this equation simplifies to become a usual ODE.

Equation 6. The equation with two proportional arguments and logarithmic nonlinearity

$$u_{tt} = au_{xx} + u(b\ln u + c\ln w), \quad w = u(px, qt),$$

admits a multiplicative separable solution

$$u(x,t) = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are described by the second-order ODEs with proportional argument

$$a\varphi''_{xx} + \varphi(b\ln\varphi + c\ln\bar{\varphi}) = 0, \quad \bar{\varphi} = \varphi(px);$$
$$\psi''_{tt} = \psi(b\ln\psi + c\ln\bar{\psi}), \quad \bar{\psi} = \psi(qt).$$

Equation 7. The equation with a proportional time argument and an arbitrary function f dependent on the difference $u - w$:

$$u_{tt} = au_{xx} + f(u - w), \quad w = u(x, qt),$$

has an additive separable solution

$$u(x,t) = C_1 x^2 + C_2 x + \psi(t),$$

where C_1 and C_2 are arbitrary constants, and the function $\psi = \psi(t)$ is described by the second-order ODE with proportional argument

$$\psi''_{tt} = 2aC_1 + f(\psi - \bar{\psi}), \quad \bar{\psi} = \psi(qt).$$

Equation 8. The equation with a proportional space argument and an arbitrary function f dependent on the difference $u - w$:

$$u_{tt} = au_{xx} + f(u - w), \quad w = u(px, t),$$

admits an additive separable solution

$$u(x,t) = C_1 t^2 + C_2 t + \varphi(x),$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(x)$ is described by the second-order ODE with proportional argument

$$a\varphi''_{xx} - 2C_1 + f(\varphi - \bar{\varphi}) = 0, \quad \bar{\varphi} = \varphi(px).$$

Equation 9. Consider the following equation with a proportional time argument and an arbitrary function f dependent on the difference $u - w$:

$$u_{tt} = au_{xx} + bu + f(u - w), \quad w = u(x, qt).$$

It has the following exact solutions.

1°. Additive separable solution for $ab < 0$:

$$u(x,t) = A\cosh(\lambda x) + B\sinh(\lambda x) + \psi(t), \quad \lambda = \sqrt{-b/a},$$

where A and B are arbitrary constants, and the function $\psi = \psi(t)$ is described by the second-order ODE with proportional argument

$$\psi''_{tt} = b\psi + f(\psi - \bar{\psi}), \quad \bar{\psi} = \psi(qt). \tag{18}$$

2°. Additive separable solution for $ab > 0$:

$$u(x,t) = A\cos(\lambda x) + B\sin(\lambda x) + \psi(t), \quad \lambda = \sqrt{b/a},$$

where A and B are arbitrary constants, and the function $\psi = \psi(t)$ is described by the second-order ODE with proportional argument (18).

Equation 10. The equation with two proportional arguments

$$u_{tt} = au_{xx} + e^{\lambda u} f(u - w), \quad w = u(px, qt),$$

where f is an arbitrary function dependent on the difference $u - w$, admits an exact solution of the form

$$u(x,t) = U(z) - \frac{2}{\lambda}\ln t, \quad z = \frac{x}{t},$$

where the function $U = U(z)$ is described by the second-order ODE with proportional argument

$$\begin{aligned} &(z^2 U'_z)'_z + \frac{2}{\lambda} = aU''_{zz} + e^{\lambda U} f\Big(U - W + \frac{2}{\lambda}\ln q\Big) = 0, \\ &W = U(sz), \quad s = p/q. \end{aligned}$$

For $p = q$, this equation simplifies to become a usual ODE.

Equation 11. Consider the following equation with a proportional space argument and an arbitrary function f dependent on the ratio w/u:

$$u_{tt} = au_{xx} + uf(w/u), \quad w = u(px, t).$$

It admits the following exact solutions.

1°. Multiplicative separable solution:

$$u(x,t) = (Ae^{-\lambda t} + Be^{\lambda t})\varphi(x),$$

where A, B, and λ are arbitrary constants; the function $\varphi = \varphi(x)$ is described by the second-order ODE with proportional argument

$$a\varphi''_{xx} + \varphi[f(\bar{\varphi}/\varphi) - \lambda^2] = 0, \quad \bar{\varphi} = \varphi(px).$$

2°. Multiplicative separable solution:

$$u(x,t) = [A\cos(\lambda t) + B\sin(\lambda t)]\varphi(x),$$

where A, B, and λ are arbitrary constants; the function $\varphi = \varphi(x)$ is described by the second-order ODE with proportional argument

$$a\varphi''_{xx} + \varphi[f(\bar{\varphi}/\varphi) + \lambda^2] = 0, \quad \bar{\varphi} = \varphi(px).$$

Equation 12. The equation with proportional time argument

$$u_{tt} = au_{xx} + uf(w/u^q), \quad w = u(x, qt),$$

where $f(z)$ is an arbitrary function, admits a multiplicative separable solution:

$$u = e^{-\lambda t}\varphi(x).$$

Here λ is an arbitrary constant, and the function $\varphi = \varphi(x)$ is described by the second-order autonomous ODE $a\varphi''_{xx} + \varphi f(\varphi^{1-q}) - \lambda^2\varphi = 0$.

3.2. Equations with Variable Speed

Equation 13. Consider the following equation with a proportional time argument and a nonlinear power-law speed:

$$u_{tt} = a(u^k u_x)_x + uf(w/u), \quad w = u(x, qt),$$

where f is an arbitrary function dependent on the ratio w/u. This equation admits a multiplicative separable solution:

$$u(x,t) = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are described by a non-delay ODE and an ODE with proportional argument:

$$\begin{aligned} &a(\varphi^k \varphi'_x)'_x = b\varphi, \\ &\psi''_{tt} = b\psi^{k+1} + \psi f(\bar{\psi}/\psi), \quad \bar{\psi} = \psi(qt), \end{aligned}$$

b is an arbitrary constant.

Equation 14. Consider the equation with a proportional space argument and a nonlinear power-law speed

$$u_{tt} = a(u^k u_x)_x + uf(w/u), \quad w = u(px, t),$$

where $f(z)$ is an arbitrary function. This equation admits the following solutions.

1°. Multiplicative separable solution:

$$u = x^{2/k}\varphi(t),$$

where the function $\varphi = \varphi(t)$ is described by the ODE

$$\varphi''_{tt} = \frac{2a(k+2)}{k^2}\varphi^{k+1} + f(p^{2/k})\varphi.$$

2°. This equation also has a solution of the more complex form

$$u(x,t) = e^{2\lambda t}U(z), \quad z = e^{-k\lambda t}x,$$

where λ is an arbitrary constant, and the function $U = U(z)$ is described by the ODE with proportional argument

$$4\lambda^2 U - 4k\lambda^2 zU'_z + k^2\lambda^2 z(zU'_z)'_z = a(U^k U'_z)'_z + Uf(W/U), \quad W = U(pz).$$

Equation 15. Consider the equation with a proportional time argument and a nonlinear power-law speed

$$u_{tt} = a(u^k u_x)_x + bu^{k+1} + uf(w/u), \quad w = u(x, qt),$$

where f is an arbitrary function dependent on the ratio w/u. This equation has the following exact solutions.

1°. Multiplicative separable solution with $b(k+1) > 0$:

$$u(x,t) = [C_1\cos(\beta x) + C_2\sin(\beta x)]^{1/(k+1)}\psi(t), \quad \beta = \sqrt{b(k+1)/a},$$

where C_1 and C_2 are arbitrary constants, and the function $\psi = \psi(t)$ is described by the second-order ODE with proportional argument

$$\psi''_{tt} = \psi f(\bar{\psi}/\psi), \quad \bar{\psi} = \psi(qt). \tag{19}$$

2°. Multiplicative separable solution with $b(k+1) < 0$:

$$u(x,t) = [C_1 \exp(-\beta x) + C_2 \exp(\beta x)]^{1/(k+1)}\psi(t), \quad \beta = \sqrt{-b(k+1)/a},$$

where C_1 and C_2 are arbitrary constants, and the function $\psi = \psi(t)$ is described by the second-order ODE with proportional argument (19).

3°. Multiplicative separable solution with $k = -1$:

$$u(x,t) = C_1 \exp\Big(-\frac{b}{2a}x^2 + C_2 x\Big)\psi(t),$$

where C_1 and C_2 are arbitrary constants, and the function $\psi = \psi(t)$ is described by the second-order ODE with proportional argument (19).

Equation 16. Consider the equation with a proportional time argument and a nonlinear power-law speed

$$u_{tt} = a(u^k u_x)_x + u^{k+1} f(w/u), \quad w = u(x,qt),$$

where f is an arbitrary function dependent on the ratio w/u. This equation admits an exact solution of the form

$$u(x,t) = t^{-2/k}\varphi(z), \quad z = x + \lambda \ln t,$$

where λ is an arbitrary constant, and the function $\varphi = \varphi(z)$ is described by the second-order ODE with proportional argument

$$\frac{2(k+2)}{k^2}\varphi - \lambda\frac{k+4}{k}\varphi'_z + \lambda^2\varphi''_{zz} = a(\varphi^k\varphi'_z)'_z + \varphi^{k+1} f(q^{-2/k}\bar{\varphi}/\varphi),$$
$$\bar{\varphi} = \varphi(z + \lambda \ln q).$$

For $\lambda = 0$, the last equation simplifies to become a usual ODE.

Equation 17. Consider the equation with two proportional arguments and a nonlinear power-law speed

$$u_{tt} = a(u^k u_x)_x + u^n f(w/u), \quad w = u(px,qt),$$

where f is an arbitrary function dependent on the ratio w/u. This equation admits the following exact solutions.

1°. Self-similar solution for $n \neq 1$:

$$u(x,t) = t^{\frac{2}{1-n}} U(z), \quad z = xt^{\frac{n-k-1}{1-n}},$$

where the function $U = U(z)$ is described by the second-order ODE with proportional argument

$$\frac{2(1+n)}{(1-n)^2}U + \frac{(n-k-1)(2n-k+2)}{(1-n)^2}zU'_z + \frac{(n-k-1)^2}{(1-n)^2}z^2U''_{zz}$$
$$= a(U^kU'_z)'_z + U^n f(q^{\frac{2}{1-n}}W/U), \quad W = U(sz), \quad s = pq^{\frac{n-k-1}{1-n}}.$$

For $p = q^{\frac{k-n+1}{1-n}}$, this equation simplifies to become a usual ODE.

2°. Traveling-wave solution for $q = p$:

$$u(x,t) = U(z), \quad z = kx - \lambda t,$$

where k and λ are arbitrary constants, and the function $U = U(z)$ is described by the second-order ODE with proportional argument

$$ak^2(U^k U'_z)'_z - \lambda^2 U''_{zz} + U^n f(W/U) = 0, \quad W = U(pz).$$

Equation 18. Consider the equation with a proportional time argument and a nonlinear exponential speed

$$u_{tt} = a(e^{\lambda u} u_x)_x + f(u - w), \quad w = u(x, qt),$$

where f is an arbitrary function dependent on the difference $u - w$. This equation has an additive separable solution:

$$u(x,t) = \frac{1}{\lambda} \ln(Ax^2 + Bx + C) + \psi(t),$$

where A, B, and C are arbitrary constants, and the function $\psi = \psi(t)$ is described by the second-order ODE with proportional argument

$$\psi''_{tt} = 2a(A/\lambda)e^{\lambda\psi} + f(\psi - \bar{\psi}), \quad \bar{\psi} = \psi(qt).$$

Equation 19. Consider the equation with two proportional arguments and a nonlinear exponential speed

$$u_{tt} = a(e^{\lambda u} u_x)_x + e^{\mu u} f(u - w), \quad w = u(px, qt),$$

where f is an arbitrary function dependent on the difference $u - w$. This equation admits an exact solution of the form

$$u(x,t) = U(z) - \frac{2}{\mu} \ln t, \quad z = xt^{\frac{\lambda-\mu}{\mu}},$$

where the function $U = U(z)$ is described by the second-order ODE with proportional argument

$$\begin{aligned} &\frac{2}{\mu} + \frac{\mu - \lambda}{\mu} zU'_z + \frac{(\lambda-\mu)^2}{\mu^2} z(zU'_z)'_z = a(e^{\lambda U} U'_z)'_z + e^{\mu U} f\Big(U - W + \frac{2}{\mu}\ln q\Big), \\ &W = U(sz), \quad s = pq^{\frac{\lambda-\mu}{\mu}}. \end{aligned}$$

For $p = q^{\frac{\mu-\lambda}{\mu}}$, this equation simplifies to become a usual ODE.

4. Exact Solutions to Nonlinear Wave-Type PDEs with Variable Delays of General Form

4.1. Equations with Constant Speed

Equation 20. The equation with variable time and space delays and a logarithmic nonlinearity

$$u_{tt} = au_{xx} + u(b \ln u + c \ln w + d), \quad w = u(\xi(x), \eta(t)).$$

admits a multiplicative separable solution:

$$u(x,t) = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are described by the ODEs with variable argument

$$\begin{aligned} &a\varphi''_{xx} + \varphi(b \ln \varphi + c \ln \bar{\varphi} - K) = 0, \quad \bar{\varphi} = \varphi(\xi(x)); \\ &\psi''_{tt} = \psi(b \ln \psi + c \ln \bar{\psi} + d + K), \quad \bar{\psi} = \psi(\eta(t)), \end{aligned}$$

K is an arbitrary constant.

Equation 21. The equation with a variable time delay and an arbitrary function f dependent on the ratio w/u:

$$u_{tt} = au_{xx} + uf(w/u), \quad w = u(x, \eta(t)),$$

has the following exact solutions.

1°. Multiplicative separable solution:

$$u(x,t) = [A\cosh(\lambda x) + B\sinh(\lambda x)]\psi(t),$$

where A, B, and λ are arbitrary constants, and the function $\psi = \psi(t)$ is described by the ODE with variable argument

$$\psi''_{tt} = a\lambda^2\psi + \psi f(\bar{\psi}/\psi), \quad \bar{\psi} = \psi(\eta(t)). \tag{20}$$

2°. Multiplicative separable solution:

$$u(x,t) = [A\cos(\lambda x) + B\sin(\lambda x)]\psi(t),$$

where A, B, and λ are arbitrary constants, and the function $\psi = \psi(t)$ is described by the ODE with variable argument

$$\psi''_{tt} = -a\lambda^2\psi + \psi f(\bar{\psi}/\psi), \quad \bar{\psi} = \psi(\eta(t)).$$

3°. Degenerate solution with multiplicative separation of variables:

$$u(x,t) = (Ax + B)\psi(t),$$

where A and B, and the function $\psi = \psi(t)$ is described by the ODE with variable argument (20) with $\lambda = 0$.

Equation 22. The equation with a variable space delay and an arbitrary function f dependent on the ratio w/u:

$$u_{tt} = au_{xx} + uf(w/u), \quad w = u(\xi(x), t),$$

admits the following exact solutions.

1°. Multiplicative separable solution:

$$u(x,t) = [A\cosh(\lambda t) + B\sinh(\lambda t)]\varphi(x),$$

where A, B, and λ are arbitrary constants, and the function $\varphi = \varphi(x)$ is described by the ODE with variable argument

$$a\varphi''_{xx} - \lambda^2\varphi + \varphi f(\bar{\varphi}/\varphi) = 0, \quad \bar{\varphi} = \varphi(\xi(x)). \tag{21}$$

2°. Multiplicative separable solution:

$$u(x,t) = [A\cos(\lambda t) + B\sin(\lambda t)]\varphi(x),$$

where A, B, and λ are arbitrary constants, and the function $\varphi = \varphi(x)$ is described by the ODE with variable argument

$$a\varphi''_{xx} + \lambda^2\varphi + \varphi f(\bar{\varphi}/\varphi) = 0, \quad \bar{\varphi} = \varphi(\xi(x)).$$

3°. Degenerate solution with multiplicative separation of variables:

$$u(x,t) = (At + B)\varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi = \varphi(x)$ is described by the ODE with variable argument (21) with $\lambda = 0$.

Equation 23. The equation with a variable time delay and an arbitrary function f dependent on the difference $u - w$:

$$u_{tt} = au_{xx} + f(u - w), \quad w = u(x, \eta(t)),$$

admits an additive separable solution:

$$u(x,t) = C_1 x^2 + C_2 x + \psi(t),$$

where C_1 and C_2 are arbitrary constants, and the function $\psi = \psi(t)$ is described by the ODE with variable argument

$$\psi''_{tt} = 2aC_1 + f(\psi - \bar{\psi}), \quad \bar{\psi} = \psi(\eta(t)).$$

Equation 24. The equation with a variable time delay and an arbitrary function f dependent on the ratio w/u:

$$u_{tt} = au_{xx} + bu \ln u + uf(w/u), \quad w = u(x, \eta(t)),$$

has a multiplicative separable solution:

$$u(x,t) = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are described by the ODE and the ODE with variable argument

$$\begin{aligned} &a\varphi''_{xx} = C_1\varphi - b\varphi \ln \varphi, \\ &\psi''_{tt} = C_1\psi + \psi f(\bar{\psi}/\psi) + b\psi \ln \psi, \quad \bar{\psi} = \psi(\eta(t)), \end{aligned}$$

C_1 is an arbitrary constant. A particular one-parameter solution of the first ODE can be represented in the explicit form

$$\varphi = \exp\Big[-\frac{b}{4a}(x + C_2)^2 + \frac{C_1}{b} + \frac{1}{2}\Big],$$

where C_2 is an arbitrary constant.

Equation 25. The equation with a variable time delay and an arbitrary function f dependent on the difference $u - w$:

$$u_{tt} = au_{xx} + bu + f(u - w), \quad w = u(x, \eta(t)),$$

admits the following exact solutions.

1°. Additive separable solution for $ab < 0$:

$$u(x,t) = A\cosh(\lambda x) + B\sinh(\lambda x) + \psi(t), \quad \lambda = \sqrt{-b/a},$$

where A and B are arbitrary constants, and the function $\psi = \psi(t)$ is described by the ODE with variable argument

$$\psi''_{tt} = b\psi + f(\psi - \bar{\psi}), \quad \bar{\psi} = \psi(\eta(t)).$$

2°. Additive separable solution for $ab > 0$:

$$u(x,t) = A\cos(\lambda x) + B\sin(\lambda x) + \psi(t), \quad \lambda = \sqrt{b/a},$$

where A and B are arbitrary constants, and the function $\psi = \psi(t)$ is described by the ODE with variable argument

$$\psi''_{tt} = b\psi + f(\psi - \bar{\psi}), \quad \bar{\psi} = \psi(\eta(t)).$$

Equation 26. The equation with a variable space delay and an arbitrary function f dependent on the difference $u - w$:

$$u_{tt} = au_{xx} + f(u - w), \quad w = u(\xi(x), t),$$

admits an additive separable solution:

$$u(x,t) = C_1 t^2 + C_2 t + \varphi(x),$$

where C is an arbitrary constant, and the function $\varphi = \varphi(x)$ is described by the ODE with variable argument

$$a\varphi''_{xx} - 2C_1 + f(\varphi - \bar{\varphi}) = 0, \quad \bar{\varphi} = \varphi(\xi(x)).$$

4.2. Equations with Variable Speed

Equation 27. Consider the equation with a variable time delay and a quadratic nonlinearity

$$u_{tt} = a(uu_x)_x + bw, \quad w = u(x, \eta(t)),$$

which admits the following exact solutions.

1°. Generalized separable solution quadratic in x:

$$u(x,t) = \varphi(t)x^2 + \psi(t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are described by the ODE system with variable delay

$$\begin{aligned} \varphi''_{tt} &= 6a\varphi^2 + b\bar{\varphi}, \quad \bar{\varphi} = \varphi(\eta(t)); \\ \psi''_{tt} &= 2a\varphi\psi + b\bar{\psi}, \quad \bar{\psi} = \psi(\eta(t)). \end{aligned}$$

2°. More complex generalized separable solution:

$$u(x,t) = \varphi(t)x^2 + \psi(t)\sqrt{x},$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are described by the ODE system with variable delay

$$\begin{aligned} \varphi''_{tt} &= 6a\varphi^2 + b\bar{\varphi}, \quad \bar{\varphi} = \varphi(\eta(t)); \\ \psi''_{tt} &= \tfrac{15}{4}a\varphi\psi + b\bar{\psi}, \quad \bar{\psi} = \psi(\eta(t)). \end{aligned}$$

Equation 28. Consider the equation with a variable time delay and a nonlinear power-law speed

$$u_{tt} = a(u^k u_x)_x + uf(w/u), \quad w = u(x, \eta(t)),$$

where f is an arbitrary function dependent on the ratio w/u. This equation has a multiplicative separable solution:

$$u(x,t) = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are described by the ODE and the ODE with variable argument

$$a(\varphi^k \varphi'_x)'_x = b\varphi,$$
$$\psi''_{tt} = b\psi^{k+1} + \psi f(\bar{\psi}/\psi), \quad \bar{\psi} = \psi(\eta(t));$$

b is an arbitrary constant.

Equation 29. Consider the equation with a variable time delay and a nonlinear power-law speed

$$u_{tt} = a(u^k u_x)_x + bu^{k+1} + uf(w/u), \quad w = u(x, \eta(t)),$$

where f is an arbitrary function dependent on the ratio w/u. This equation admits the following exact solutions.

1°. Multiplicative separable solution for $b(k+1) > 0$:

$$u(x,t) = [C_1 \cos(\beta x) + C_2 \sin(\beta x)]^{1/(k+1)} \psi(t), \quad \beta = \sqrt{b(k+1)/a},$$

where C_1 and C_2 are arbitrary constants, and the function $\psi = \psi(t)$ is described by the ODE with variable argument

$$\psi''_{tt} = \psi f(\bar{\psi}/\psi), \quad \bar{\psi} = \psi(\eta(t)). \tag{22}$$

2°. Multiplicative separable solution for $b(k+1) < 0$:

$$u(x,t) = [C_1 \exp(-\beta x) + C_2 \exp(\beta x)]^{1/(k+1)} \psi(t), \quad \beta = \sqrt{-b(k+1)/a},$$

where C_1 and C_2 are arbitrary constants, and the function $\psi = \psi(t)$ is described by the ODE with variable argument (22).

3°. Multiplicative separable solution for $k = -1$:

$$u(x,t) = C_1 \exp\Big(-\frac{b}{2a}x^2 + C_2 x\Big) \psi(t),$$

where C_1 and C_2 are arbitrary constants, and the function $\psi = \psi(t)$ is described by the ODE with variable argument (22).

Equation 30. Consider the equation with a variable space delay and a nonlinear power-law speed

$$u_{tt} = a(u^k u_x)_x + u^{k+1} f(w/u), \quad w = u(\xi(x), t),$$

where f is an arbitrary function dependent on the ratio w/u. This equation admits a multiplicative separable solution:

$$u = t^{-2/k} \varphi(x),$$

where the function $\varphi = \varphi(x)$ is described by the ODE with variable argument

$$a(\varphi^k \varphi'_x)'_x - \frac{2(k+2)}{k^2} \varphi + \varphi^{k+1} f(\bar{\varphi}/\varphi) = 0, \quad \bar{\varphi} = \varphi(\xi(x)).$$

Equation 31. Consider the equation with a variable time delay and a nonlinear speed of the exponential form

$$u_{tt} = a(e^{\lambda u} u_x)_x + f(u - w), \quad w = u(x, \eta(t)),$$

where f is an arbitrary function dependent on the difference $u - w$. This equation admits an additive separable solution:

$$u(x,t) = \frac{1}{\lambda}\ln(Ax^2 + Bx + C) + \psi(t),$$

where A, B, and C are arbitrary constants, and the function $\psi = \psi(t)$ is described by the ODE with variable argument

$$\psi''_{tt} = 2a(A/\lambda)e^{\lambda\psi} + f(\psi - \bar{\psi}), \quad \bar{\psi} = \psi(\eta(t)).$$

Equation 32. Consider the equation with a variable time delay and a nonlinear exponential speed

$$u_{tt} = a(e^{\lambda u}u_x)_x + be^{\lambda u} + f(u - w), \quad w = u(x, \eta(t)),$$

where f is an arbitrary function dependent on the difference $u - w$. This equation has the following exact solutions.

1°. Additive separable solution for $b\lambda > 0$:

$$u(x,t) = \frac{1}{\lambda}\ln[C_1\cos(\beta x) + C_2\sin(\beta x)] + \psi(t), \quad \beta = \sqrt{b\lambda/a},$$

where C_1 and C_2 are arbitrary constants, and the function $\psi = \psi(t)$ is described by the ODE with variable argument

$$\psi''_{tt} = f(\psi - \bar{\psi}), \quad \bar{\psi} = \psi(\eta(t)). \tag{23}$$

2°. Additive separable solution for $b\lambda < 0$:

$$u(x,t) = \frac{1}{\lambda}\ln[C_1\exp(-\beta x) + C_2\exp(\beta x)] + \psi(t), \quad \beta = \sqrt{-b\lambda/a},$$

where C_1 and C_2 are arbitrary constants, and the function $\psi = \psi(t)$ is described by the ODE with variable argument (23).

Equation 33. Consider the equation with a variable space delay and a nonlinear exponential speed

$$u_{tt} = a(e^{\lambda u}u_x)_x + e^{\lambda u}f(u - w), \quad w = u(\xi(x), t),$$

where f is an arbitrary function dependent on the difference $u - w$. This equation admits an additive separable solution:

$$u = -\frac{2}{\lambda}\ln t + \varphi(x),$$

where the function $\varphi = \varphi(x)$ is described by the ODE with variable argument

$$a(e^{\lambda\varphi}\varphi'_x)'_x - \frac{2}{\lambda} + e^{\lambda\varphi}f(\varphi - \bar{\varphi}) = 0, \quad \bar{\varphi} = \varphi(\xi(x)).$$

Equation 34. Consider the equation with a variable time delay and nonlinear logarithmic speed

$$u_{tt} = [(a\ln u + b)u_x]_x - cu\ln u + uf(w/u), \quad w = u(x, \eta(t)),$$

where f is an arbitrary function dependent on the ratio w/u. This equation has a multiplicative separable solution:

$$u(x,t) = \exp\left(\pm\sqrt{c/a}\,x\right)\psi(t),$$

where the function $\psi = \psi(t)$ is described by the ODE with variable argument

$$\psi''_{tt} = c(1 + b/a)\psi + \psi f(\bar{\psi}/\psi), \qquad \bar{\psi} = \psi(\eta(t)).$$

5. Brief Conclusions

We have considered many nonlinear wave-type PDEs that involve, in addition to the unknown function $u(x,t)$, also functions of the form $u(x,qt)$, or $u(px,t)$, or $u(px,qt)$, where p and q are free 'scaling' parameters (for PDEs with proportional delay, $0 < p < 1$ and $0 < q < 1$). A number of more complex nonlinear wave-type PDEs with variable delays of general form and involving functions $u(\xi(x),t)$ or $u(x,\eta(t))$ have also been investigated. We have described over thirty such equations that admit exact solutions. In addition, many self-similar, additive separable, multiplicative separable, generalized separable, and other solutions have been specified. The study results fit for testing numerical methods and investigating the properties of the considered and related equations with proportional delay.

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