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Integro-Differential Boundary Conditions to the Sequential ψ_1-Hilfer and ψ_2-Caputo Fractional Differential Equations

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Abstract: In this paper, we introduce and study a new class of boundary value problems, consisting of a mixed-type ψ_1-Hilfer and ψ_2-Caputo fractional order differential equation supplemented with integro-differential nonlocal boundary conditions. The uniqueness of solutions is achieved via the Banach contraction principle, while the existence of results is established by using the Leray–Schauder nonlinear alternative. Numerical examples are constructed illustrating the obtained results.

Keywords: ψ-Hilfer fractional derivative; Caputo fractional derivative; boundary value problems; nonlocal boundary conditions; existence; uniqueness; fixed point

MSC: 26A33; 34A08; 34B10

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1. Introduction

Fractional order differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of mathematics, physics, viscoelasticity, electrochemistry, engineering, control, porous media, electromagnetic, etc., see [1–5] and references cited therein. For a theoretical approach of fractional calculus, see the monographs [6–11]. Many processes in physics and engineering can be described accurately by using differential equations containing different types of fractional derivatives such as Riemann–Liouville, Caputo, Hadamard, Erdeyl–Kober, Hilfer, Caputo–Hadamard, etc. Hilfer proposed in [12] a fractional derivative operator generalizing both Riemann–Liouville and Caputo fractional derivative operators. For the advantages of the Hilfer derivative, see [13]. In [14], the ψ-Hilfer fractional derivative operator was introduced. Initial and boundary value problems including the ψ-Hilfer fractional derivative operator have been studied by many researchers, see [15–20] and references therein.

In the present paper, we investigate a new class of boundary value problems, consisting of mixed-type fractional differential equations including ψ_1-Hilfer and ψ_2-Caputo fractional derivative operators supplemented with nonlocal integro-differential boundary conditions. More precisely, we consider the following sequential ψ_1-Hilfer and ψ_2-Caputo fractional differential equation with nonlocal integro-differential boundary conditions

$$\begin{cases} {}^H\mathbb{D}^{\alpha,\beta;\psi_1}({}^C\mathbb{D}^{\gamma;\psi_2}\pi)(t) = \Pi(t,\pi(t)), \quad 0<\alpha,\beta,\gamma<1, \ t\in[0,x_1], \\ {}^C\mathbb{D}^{\gamma;\psi_2}\pi(0)=0, \quad \pi(T)=\sum_{i=1}^{m}\lambda_i\,{}^C\mathbb{D}^{\gamma;\psi_2}\pi(\eta_i)+\sum_{j=1}^{n}\delta_j\,\mathbb{I}^{\mu_j;\psi_2}\pi(\xi_j), \end{cases} \tag{1}$$

where ${}^H\mathbb{D}^{\alpha,\beta;\psi_1}$ and ${}^C\mathbb{D}^{\gamma;\psi_2}$ are the ψ_1-Hilfer and ψ_2-Caputo fractional derivatives with respect to functions ψ_1 and ψ_2, respectively, when $\psi_1'(t),\psi_2'(t)>0$ for all $t\in[0,x_1]$.

In addition, the given constants $\lambda_i, \delta_j \in \mathbb{R}$ and some points $\eta_i, \xi_j \in (0, x_1)$, $\mathbb{I}^{\mu_j;\psi_2}$ is the Riemann–Liouville fractional integral of order $\mu_j > 0$, with respect to a function ψ_2, for $i = 1, \cdots, m$, $j = 1, \cdots, n$ and $\Pi : [0, x_1] \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function. Existence and uniqueness are established via Banach's fixed point theorem and the Leray–Schauder nonlinear alternative.

The novelty of this study lies in the fact that we introduce a new class of nonlocal boundary value problems in which we combine ψ_1-Hilfer and ψ_2-Caputo fractional derivative operators and as far as we know, this is the only paper dealing with this combination. By fixing the parameters in the nonlocal integro-differential fractional boundary value problem (1), we obtain some new results as special cases. For example, we get to:

(i) Hilfer and Caputo fractional nonlocal integro-differential boundary value problem if $\psi_1(t) = \psi_2(t) = t$;
(ii) ψ_2-Hilfer and Caputo-type fractional nonlocal integro-differential boundary value problem if $\psi_1(t) = t$;
(iii) ψ_1-Hilfer and Caputo-type nonlocal integro-differential boundary value problem if $\psi_2(t) = t$.

The remaining part of this article is organized as follows: in Section 2, some preliminary definitions and results that will be applied in the next sections are recalled. In addition, an auxiliary result is proved to convert the problem (1) into a fixed point problem. In Section 3, the main results for the nonlocal integro-differential boundary value problem (1) are established, while in Section 4, these results are discussed for some special cases. Section 5 includes some numerical examples illustrating the main results.

2. Preliminaries

Now, some notations and definitions of fractional calculus are recalled. In the following, we assume that $\psi \in C^1([0, x_1], \mathbb{R})$ is an increasing function with $\psi'(t) > 0$ for all $t \in [0, x_1]$.

Definition 1 ([7])**.** *Given $\alpha > 0$ and $\hat{h} \in L^1([0, x_1], \mathbb{R})$, the ψ-Riemann–Liouville fractional integral of order α of a function $\hat{h}$ with respect to ψ is defined by*

$$\mathbb{I}_0^{\alpha;\psi}\hat{h}(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\hat{h}(s)ds.$$

To abbreviate, we use $\mathbb{I}_0^{\alpha;\psi}\hat{h}(t)$ as $\mathbb{I}^{\alpha;\psi}\hat{h}(t)$ throughout this paper.

Definition 2 ([14])**.** *Suppose that $n - 1 < \alpha < n$, $n \in \mathbb{N}$ and $\hat{h}, \psi \in C^n([0, x_1], \mathbb{R})$. The ψ-Hilfer fractional derivative ${}^H\mathbb{D}^{\alpha,\beta;\psi}(\cdot)$ of order α of a function $\hat{h}$ with a parameter $\beta \in [0, 1]$ is defined by*

$${}^H\mathbb{D}^{\alpha,\beta;\psi}\hat{h}(t) = \mathbb{I}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathbb{I}^{(1-\beta)(n-\alpha);\psi}\hat{h}(t),$$

provided that the right-hand side exists.

Definition 3 ([21])**.** *The ψ-Caputo fractional derivative ${}^C\mathbb{D}^{\alpha;\psi}(\cdot)$ of order α of a function $\hat{h}$ is expressed as*

$${}^C\mathbb{D}^{\alpha;\psi}\hat{h}(t) = \mathbb{I}^{n-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \hat{h}(t),$$

where $n - 1 < \alpha < n$, $n \in \mathbb{N}$ and $\hat{h}, \psi \in C^n([0, x_1], \mathbb{R})$.

Remark 1 ([22])**.** *The following relations hold:*

$$\rho = \alpha + \beta(n - \alpha), \quad n - 1 < \alpha, \quad \rho < n, \quad 0 \leq \beta \leq 1,$$

and

$$\rho \geq \alpha, \quad \rho > \beta, \quad n - \rho < n - \beta(n - \alpha).$$

Lemma 1 ([14]). *Let $\alpha, \mu > 0$ and $\delta > 1$ be constants. Then, we have:*

(i) $\mathbb{I}^{\alpha;\psi}\mathbb{I}^{\mu;\psi}\hat{h}(t) = \mathbb{I}^{\alpha+\mu;\psi}\hat{h}(t)$;

(ii) $\mathbb{I}^{\alpha;\psi}(\psi(t) - \psi(0))^{\delta-1} = \dfrac{\Gamma(\delta)}{\Gamma(\alpha+\delta)}(\psi(t) - \psi(0))^{\alpha+\delta-1}$.

Lemma 2. *Let $\hat{h} \in L(0, x_1)$, $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, $0 \leq \beta \leq 1$, $\rho = \alpha + n\beta - \alpha\beta$, $\left(\mathbb{I}^{(n-\alpha)(1-\beta);\psi}\hat{h}\right) \in AC^k[0, x_1]$. (Here, by $AC^k[0, x_1]$, we denote the space of k times absolutely continuous functions on $[0, x_1]$.) Then, we have*

$$\left(\mathbb{I}^{\alpha;\psi}\ {}^{H}\mathbb{D}^{\alpha,\beta;\psi}\hat{h}\right)(t) = \hat{h}(t) - \sum_{k=1}^{n} \frac{(\psi(t) - \psi(a))^{\rho-k}}{\Gamma(\rho - k + 1)}\, \hat{h}_{\psi}^{[n-k]}\left(\mathbb{I}^{(1-\beta)(n-\alpha);\psi}\hat{h}\right)(a),$$

where $\hat{h}_{\psi}^{[n-k]} = \left(\dfrac{1}{\psi'(t)}\dfrac{d}{dt}\right)^{n-k}$ *and*

$$\left(\mathbb{I}^{\alpha;\psi}\ {}^{C}\mathbb{D}^{\alpha;\psi}\hat{h}\right)(t) = \hat{h}(t) - \sum_{k=0}^{n-1} \frac{\hat{h}_{\psi}^{[k]}\hat{h}(a)}{k!}(\psi(t) - \psi(a))^{k}.$$

A linear variant of the sequential Hilfer–Caputo fractional integro-differential boundary value problem (1) is investigated in the next lemma.

Lemma 3. *Let $h \in C([0, x_1], \mathbb{R})$ be a given function and all constants are as in boundary value problem (1). Then, the sequential Hilfer–Caputo fractional integro-differential linear boundary value problem*

$$\begin{cases} {}^{H}\mathbb{D}^{\alpha,\beta;\psi_1}({}^{C}\mathbb{D}^{\gamma;\psi_2}\pi)(t) = h(t), \quad t \in [0, x_1], \\ {}^{C}\mathbb{D}^{\gamma;\psi_2}\pi(0) = 0, \quad \pi(x_1) = \displaystyle\sum_{i=1}^{m} \lambda_i\, {}^{C}\mathbb{D}^{\gamma;\psi_2}\pi(\eta_i) + \sum_{j=1}^{n} \delta_j\, \mathbb{I}^{\mu_j;\psi_2}\pi(\xi_j) \end{cases} \tag{2}$$

is equivalent to the integral equation

$$\begin{aligned} \pi(t) &= \frac{1}{A}\left(\sum_{i=1}^{m} \lambda_i \mathbb{I}^{\alpha;\psi_1}h(\eta_i) + \sum_{j=1}^{n} \delta_j \mathbb{I}^{\mu_j+\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}h(\xi_j) - \mathbb{I}^{\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}h(x_1)\right) \\ &\quad + \mathbb{I}^{\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}h(t), \end{aligned} \tag{3}$$

where it is assumed that

$$A := 1 - \sum_{j=1}^{n} \delta_j \frac{\left[\psi_2(\xi_j) - \psi_2(0)\right]^{\mu_j}}{\Gamma(\mu_j + 1)} \neq 0. \tag{4}$$

Proof. Operating the fractional integral $\mathbb{I}^{\alpha;\psi_1}$ to both sides of the first equation in (2) and applying Lemma 2, we obtain for $t \in [0, x_1]$, that

$${}^{C}\mathbb{D}^{\gamma;\psi_2}\pi(t) = \frac{c_0}{\Gamma(\rho_1)}(\psi_1(t) - \psi_1(0))^{\rho_1-1} + \mathbb{I}^{\alpha;\psi_1}h(t),$$

where $\rho_1 = \alpha + (1-\alpha)\beta$ and $c_0 \in \mathbb{R}$. Since $\rho_1 \in (\alpha, 1)$, by Remark 1, from ${}^{C}\mathbb{D}^{\gamma;\psi_2}\pi(0) = 0$, we have $c_0 = 0$. Therefore, we get

$${}^{C}\mathbb{D}^{\gamma;\psi_2}\pi(t) = \mathbb{I}^{\alpha;\psi_1}h(t), \tag{5}$$

which leads to

$$\sum_{i=1}^{m} \lambda_i \, {}^{C}\mathbb{D}^{\gamma;\psi_2} \pi(\eta_i) = \sum_{i=1}^{m} \lambda_i \mathbb{I}^{\alpha;\psi_1} h(\eta_i). \tag{6}$$

Acting $\mathbb{I}^{\gamma;\psi_2}$ in (5) yields

$$\pi(t) = c_1 + \mathbb{I}^{\gamma;\psi_2} \mathbb{I}^{\alpha;\psi_1} h(t). \tag{7}$$

In addition, we have

$$\sum_{j=1}^{n} \delta_j \mathbb{I}^{\mu_j;\psi_2} \pi(\xi_j) = c_1 \sum_{j=1}^{n} \delta_j \frac{[\psi_2(\xi_j) - \psi_2(0)]^{\mu_j}}{\Gamma(\mu_j + 1)} + \sum_{j=1}^{n} \delta_j \mathbb{I}^{\mu_j+\gamma;\psi_2} \mathbb{I}^{\alpha;\psi_1} h(\xi_j). \tag{8}$$

From the second boundary condition (2) with (6) and (8), we get

$$c_1 = \frac{1}{A}\left[\sum_{i=1}^{m} \lambda_i \mathbb{I}^{\alpha;\psi_1} h(\eta_i) + \sum_{j=1}^{n} \delta_j \mathbb{I}^{\mu_j+\gamma;\psi_2} \mathbb{I}^{\alpha;\psi_1} h(\xi_j) - \mathbb{I}^{\gamma;\psi_2} \mathbb{I}^{\alpha;\psi_1} h(x_1)\right], \tag{9}$$

where A is defined in (4). Substituting the value of c_1 in (7), we get the solution (3). On the other hand, by taking the fractional differential operator of ψ_2-Caputo and ψ_1-Hilfer of orders γ and α, respectively, we get the first equation in problem (2). By direct computation, it is easy to see that (3) satisfies the two boundary conditions in (2). Therefore, the proof is completed. $\square$

3. Main Results

In this section, we establish existence and uniqueness of solutions to the sequential Hilfer–Caputo fractional integro-differential boundary value problem (1) on an interval $J = [0, x_1]$. At first, we denote the Banach space of all continuous functions from J to $\mathbb{R}$ equipped with the norm $\|\pi\| = \sup\{|\pi(t)| : t \in J\}$ by $\mathcal{C} = C(J, \mathbb{R})$. Having in mind Lemma 3, we define an operator $\mathbb{W} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} (\mathbb{W}\pi)(t) \quad &= \quad \frac{1}{A}\Big[\sum_{i=1}^{m} \lambda_i \mathbb{I}^{\alpha;\psi_1} \Pi(\eta_i, \pi(\eta_i)) + \sum_{j=1}^{n} \delta_j \mathbb{I}^{\mu_j+\gamma;\psi_2} \mathbb{I}^{\alpha;\psi_1} \Pi(\xi_j, \pi(\xi_j)) \\ &\quad - \mathbb{I}^{\gamma;\psi_2} \mathbb{I}^{\alpha;\psi_1} \Pi(x_1, \pi(x_1))\Big] + \mathbb{I}^{\gamma;\psi_2} \mathbb{I}^{\alpha;\psi_1} \Pi(t, \pi(t)), \end{aligned} \tag{10}$$

where

$$\mathbb{I}^{\alpha;\psi_1} \Pi(\eta_i, \pi(\eta_i)) = \frac{1}{\Gamma(\alpha)} \int_0^{\eta_i} \psi_1'(s)(\psi_1(\eta_i) - \psi_1(s))^{\alpha-1} \Pi(s, \pi(s)) ds$$

and

$$\begin{aligned} &\mathbb{I}^{\phi;\psi_2} \mathbb{I}^{\alpha;\psi_1} \Pi(l, \pi(l)) \\ = \; &\frac{1}{\Gamma(\alpha)\Gamma(\phi)} \int_0^{l} \int_0^{u} \psi_1'(s) \psi_2'(u) (\psi_1(u) - \psi_1(s))^{\alpha-1} (\psi_2(l) - \psi_2(u))^{\phi-1} \Pi(s, \pi(s)) ds du, \end{aligned}$$

with $\phi \in \{\gamma, \mu_j + \gamma\}$ and $l \in \{t, x_1, \xi_j\}$. Note that if $\Pi(t, \pi) \equiv 1$, we have

$$\begin{aligned} \mathbb{I}^{\phi;\psi_2} \mathbb{I}^{\alpha;\psi_1}(1)(l) \quad &= \quad \frac{1}{\Gamma(\alpha+1)\Gamma(\phi)} \int_0^{l} \psi_2'(u)(\psi_1(u) - \psi_1(0))^{\alpha} (\psi_2(l) - \psi_2(u))^{\phi-1} du \\ &:= \quad A_{\psi_1,\psi_2}^{\alpha,\phi}(l). \end{aligned}$$

For convenience, we put

$$A_1 \quad = \quad \frac{1}{|A|}\left(\sum_{i=1}^{m} |\lambda_i| \frac{[\psi_1(\eta_i) - \psi_1(0)]^{\alpha}}{\Gamma(\alpha+1)} + \sum_{j=1}^{n} |\delta_j| A_{\psi_1,\psi_2}^{\alpha,\mu_j+\gamma}(\xi_j)\right)$$

$$+\left(\frac{|A|+1}{|A|}\right)A^{\alpha,\gamma}_{\psi_1,\psi_2}(x_1). \tag{11}$$

In the following theorem, we prove the existence and uniqueness of solutions of the fractional integro-differential boundary value problem of sequential Hilfer and Caputo fractional derivatives (1) by applying the Banach contraction mapping principle.

Theorem 1. *Let $\Pi : J \times \mathbb{R} \to \mathbb{R}$ such that:*
$(\mathbb{H}_1)$ *There exists $\mathbb{L} > 0$ such that*

$$|\Pi(t,\pi_1) - \Pi(t,\pi_2)| \leq \mathbb{L}|\pi_1 - \pi_2|, \tag{12}$$

$\forall t \in J$ and $\pi_1, \pi_2 \in \mathbb{R}$.
If

$$A_1\mathbb{L} < 1, \tag{13}$$

where A_1 is given by (11). Then, the fractional integro-differential boundary value problem of sequential Hilfer and Caputo fractional derivatives (1) *has a unique solution on J.*

Proof. Let $M = \sup\{|\Pi(t,0)| : t \in J\}$ and $B_r = \{\pi \in \mathcal{C} : \|\pi\| \leq r^*\}$ with

$$r^* \geq \frac{MA_1}{1 - A_1\mathbb{L}}. \tag{14}$$

Now, we will show that $\mathbb{W}B_{r^*} \subseteq B_{r^*}$. For any $\pi \in B_{r^*}$, we obtain

$$\begin{aligned}
|\mathbb{W}\pi(t)| &\leq \sup_{t\in J}|\mathbb{W}\pi(t)| \\
&\leq \frac{1}{|A|}\Bigg[\sum_{i=1}^{m}|\lambda_i|\mathbb{I}^{\alpha;\psi_1}\big(|\Pi(\eta_i,\pi(\eta_i)) - \Pi(\eta_i,0)| + |\Pi(\eta_i,0)|\big) \\
&\quad + \sum_{j=1}^{n}|\delta_j|\mathbb{I}^{\mu_j+\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}\big(|\Pi(\xi_j,\pi(\xi_j)) - \Pi(\xi_j,0)| + |\Pi(\xi_j,0)|\big) \\
&\quad + \mathbb{I}^{\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}\big(|\Pi(x_1,\pi(x_1)) - \Pi(x_1,0)| + |\Pi(x_1,0)|\big)\Bigg] \\
&\quad + \mathbb{I}^{\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}\big(|\Pi(x_1,\pi(x_1)) - \Pi(x_1,0)| + |\Pi(x_1,0)|\big) \\
&\leq \frac{1}{|A|}\Bigg[\sum_{i=1}^{m}|\lambda_i|(\mathbb{L}r^* + M)I^{\alpha;\psi_1}(1)(\eta_i) \\
&\quad + \sum_{j=1}^{n}|\delta_j|(\mathbb{L}r^* + M)I^{\mu_j+\gamma;\psi_2}I^{\alpha;\psi_1}(1)(\xi_j) \\
&\quad + (\mathbb{L}r^* + M)I^{\gamma;\psi_2}I^{\alpha;\psi_1}(1)(x_1)\Bigg] + (\mathbb{L}r^* + M)I^{\gamma;\psi_2}I^{\alpha;\psi_1}(1)(x_1) \\
&= \frac{1}{|A|}\Bigg[(\mathbb{L}r^* + M)\sum_{i=1}^{m}|\lambda_i|\frac{[\psi_1(\eta_i) - \psi_1(0)]^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + (\mathbb{L}r^* + M)\sum_{j=1}^{n}|\delta_j|A^{\alpha,\mu_j+\gamma}_{\psi_1,\psi_2}(\xi_j) + (\mathbb{L}r^* + M)A^{\alpha,\gamma}_{\psi_1,\psi_2}(x_1)\Bigg] \\
&\quad + (\mathbb{L}r^* + M)A^{\alpha,\gamma}_{\psi_1,\psi_2}(x_1) \\
&= (\mathbb{L}r^* + M)A_1 \leq r^*,
\end{aligned}$$

which holds from (14). This shows that $\mathbb{W}B_{r^*} \subseteq B_{r^*}$. Next, we let $\pi_1, \pi_2 \in B_{r^*}$, then we have

$$\begin{aligned}
|\mathbb{W}\pi_1(t) - \mathbb{W}\pi_2(t)| &\leq \sup_{t\in J} |\mathbb{W}\pi_1(t) - \mathbb{W}\pi_2(t)| \\
&\leq \frac{1}{|A|}\Bigg[\sum_{i=1}^{m} |\lambda_i| \mathbb{I}^{\alpha;\psi_1} |\Pi(\eta_i, \pi_1(\eta_i)) - \Pi(\eta_i, \pi_2(\eta_i))| \\
&\quad + \sum_{j=1}^{n} |\delta_j| \mathbb{I}^{\mu_j+\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1} |\Pi(\xi_j, \pi_1(\xi_j)) - \Pi(\xi_j, \pi_2(\xi_j))| \\
&\quad + \mathbb{I}^{\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1} |\Pi(x_1, \pi_1(x_1)) - \Pi(x_1, \pi_2(x_1))|\Bigg] \\
&\quad + \mathbb{I}^{\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1} |\Pi(x_1, \pi_1(x_1)) - \Pi(x_1, \pi_2(x_1))|, \\
&\leq \frac{\mathbb{L}}{|A|}\Bigg[\|\pi_1 - \pi_2\| \sum_{i=1}^{m} |\lambda_i| \mathbb{I}^{\alpha;\psi_1}(1)(\eta_i) \\
&\quad + \|\pi_1 - \pi_2\| \sum_{j=1}^{n} |\delta_j| \mathbb{I}^{\mu_j+\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}(1)(\xi_j) \\
&\quad + \|\pi_1 - \pi_2\| \mathbb{I}^{\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}(1)(x_1)\Bigg] + L\|\pi_1 - \pi_2\| \mathbb{I}^{\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}(1)(x_1) \\
&= A_1\mathbb{L}\|\pi_1 - \pi_2\|.
\end{aligned}$$

Therefore, the operator $\mathbb{W}$ satisfies the inequality $\|\mathbb{W}\pi_1 - \mathbb{W}\pi_2\| \leq A_1\mathbb{L}\|\pi_1 - \pi_2\|$. Since, $A_1\mathbb{L} < 1$, $\mathbb{W}$ is a contraction. Therefore, the operator $\mathbb{W}$ has a unique fixed point in the ball B_r, by Banach's contraction mapping. Consequently, the sequential Hilfer–Caputo fractional integro-differential boundary value problem (1) has a unique solution on J. □

Next, the nonlinear alternative of the Leray–Schauder-type [23] is used to prove the existence of at least one solution to the sequential Hilfer–Caputo fractional integro-differential boundary value problem (1).

Theorem 2. *Assume that $\Pi : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the conditions:*

($\mathbb{H}_2$) *There exists a continuous function $\Omega : [0, \infty) \to (0, \infty)$ which is nondecreasing and $u_1, u_2 : J \to \mathbb{R}^+$ two continuous functions such that*

$$|\Pi(t, \pi)| \leq u_1(t)\Omega(|\pi|) + u_2(t), \tag{15}$$

for all $t \in J$ and $\pi \in \mathbb{R}$;

($\mathbb{H}_3$) *There exists a positive constant K such that*

$$\frac{K}{(\|u_1\|\Omega(K) + \|u_2\|)A_1} > 1. \tag{16}$$

Then, the sequential Hilfer–Caputo fractional integro-differential boundary value problem (1) has at least one solution on J.

Proof. We show that the operator $\mathbb{W}$ defined by (10) is compact on a bounded ball B_ρ, when $B_\rho = \{\pi \in \mathcal{C} : \|\pi\| \leq \rho\}$. For any $\pi \in B_\rho$, we have

$$\begin{aligned}
&|\mathbb{W}\pi(t)| \leq \sup_{t\in J} |\mathbb{W}\pi(t)| \\
\leq\ & \frac{1}{|A|}\Bigg[\sum_{i=1}^{m} |\lambda_i| \mathbb{I}^{\alpha;\psi_1} |\Pi(\eta_i, \pi(\eta_i))| + \sum_{j=1}^{n} |\delta_j| \mathbb{I}^{\mu_j+\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1} |\Pi(\xi_j, \pi(\xi_j))|
\end{aligned}$$

$$
\begin{aligned}
& \left. +\mathbb{I}^{\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}|\Pi(x_1,\pi(x_1))| \right] + \mathbb{I}^{\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}|\Pi(x_1,\pi(x_1))|, \\
\leq\ & \frac{1}{|A|}\left[(\|u_1\|\Omega(\rho)+\|u_2\|)\sum_{i=1}^{m}|\lambda_i|\mathbb{I}^{\alpha;\psi_1}(\eta_i) \right. \\
& +(\|u_1\|\Omega(\rho)+\|u_2\|)\sum_{j=1}^{n}|\delta_j|\mathbb{I}^{\mu_j+\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}(\xi_j) \\
& \left. +(\|u_1\|\Omega(\rho)+\|u_2\|)\mathbb{I}^{\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}(x_1) \right] + (\|u_1\|\Omega(\rho)+\|u_2\|)\mathbb{I}^{\gamma;\psi_2}\mathbb{I}^{\alpha;\psi_1}(x_1) \\
=\ & (\|u_1\|\Omega(\rho)+\|u_2\|)A_1 := \Phi, \ \ \text{a constant,}
\end{aligned}
$$

which yields $\|\mathbb{W}\pi\| \leq \Phi$. Therefore, the set $\mathbb{W}(B_\rho)$ is uniformly bounded. To show that $\mathbb{W}(B_\rho)$ is an equicontinuous set, we let t_1 and t_2 be the two points in J such that $t_1 < t_2$. Thus, for any $\pi \in B_\rho$, we have

$$
\begin{aligned}
& |\mathbb{W}\pi(t_2) - \mathbb{W}\pi(t_1)| \\
=\ & \Big|\frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^{t_2}\int_0^{u}\psi_1'(s)\psi_2'(u)(\psi_1(u)-\psi_1(s))^{\alpha-1} \\
& \times(\psi_2(t_2)-\psi_2(u))^{\alpha-1}\Pi(s,\pi(s))dsdu \\
& -\frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^{t_1}\int_0^{u}\psi_1'(s)\psi_2'(u)(\psi_1(u)-\psi_1(s))^{\alpha-1} \\
& \times(\psi_2(t_1)-\psi_2(u))^{\alpha-1}\Pi(s,\pi(s))dsdu\Big| \\
=\ & \Big|\frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^{t_1}\int_0^{u}\psi_1'(s)\psi_2'(u)(\psi_1(u)-\psi_1(s))^{\alpha-1} \\
& \times(\psi_2(t_2)-\psi_2(u))^{\alpha-1}\Pi(s,\pi(s))dsdu \\
& +\frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_{t_1}^{t_2}\int_0^{u}\psi_1'(s)\psi_2'(u)(\psi_1(u)-\psi_1(s))^{\alpha-1} \\
& \times(\psi_2(t_2)-\psi_2(u))^{\alpha-1}\Pi(s,\pi(s))dsdu \\
& -\frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^{t_1}\int_0^{u}\psi_1'(s)\psi_2'(u)(\psi_1(u)-\psi_1(s))^{\alpha-1} \\
& \times(\psi_2(t_1)-\psi_2(u))^{\alpha-1}\Pi(s,\pi(s))dsdu\Big| \\
=\ & \Big|\frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^{t_1}\int_0^{u}\psi_1'(s)\psi_2'(u)(\psi_1(u)-\psi_1(s))^{\alpha-1}\Big\{(\psi_2(t_2)-\psi_2(u))^{\alpha-1} \\
& -(\psi_2(t_1)-\psi_2(u))^{\alpha-1}\Big\}\Pi(s,\pi(s))dsdu \\
& +\frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_{t_1}^{t_2}\int_0^{u}\psi_1'(s)\psi_2'(u)(\psi_1(u)-\psi_1(s))^{\alpha-1} \\
& \times(\psi_2(t_2)-\psi_2(u))^{\alpha-1}\Pi(s,\pi(s))dsdu\Big| \\
\leq\ & \frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^{t_1}\int_0^{u}\psi_1'(s)\psi_2'(u)(\psi_1(u)-\psi_1(s))^{\alpha-1} \\
& \times\Big|(\psi_2(t_2)-\psi_2(u))^{\alpha-1}-(\psi_2(t_1)-\psi_2(u))^{\alpha-1}\Big||\Pi(s,\pi(s))|dsdu \\
& +\frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_{t_1}^{t_2}\int_0^{u}\psi_1'(s)\psi_2'(u)(\psi_1(u)-\psi_1(s))^{\alpha-1} \\
& \times(\psi_2(t_2)-\psi_2(u))^{\alpha-1}|\Pi(s,\pi(s))|dsdu \\
\leq\ & \frac{(\|u_1\|\Omega(\rho)+\|u_2\|)}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^{t_1}\int_0^{u}\psi_1'(s)\psi_2'(u)(\psi_1(u)-\psi_1(s))^{\alpha-1}
\end{aligned}
$$

$$\times\Big|(\psi_2(t_2)-\psi_2(u))^{\alpha-1}-(\psi_2(t_1)-\psi_2(u))^{\alpha-1}\Big|dsdu$$
$$+\frac{(\|u_1\|\Omega(\rho)+\|u_2\|)}{\Gamma(\alpha)\Gamma(\gamma)}\int_{t_1}^{t_2}\int_0^u \psi_1'(s)\psi_2'(u)(\psi_1(u)-\psi_1(s))^{\alpha-1}$$
$$\times(\psi_2(t_2)-\psi_2(u))^{\alpha-1}dsdu.$$

Observe that if $t_1 \to t_2$, then we have $|\mathbb{W}\pi(t_2)-\mathbb{W}\pi(t_1)| \to 0$ independently of π. Therefore, the set $\mathbb{W}(B_\rho)$ is an equicontinuous set. Hence, the set $\mathbb{W}(B_\rho)$ is relatively compact. By applying the Arzelá–Ascoli theorem, the operator $\mathbb{W}$ is completely continuous.

Finally, we show that the set of all solutions to equations $\pi=\lambda\mathbb{W}\pi$ is bounded for $\lambda\in(0,1)$. Let $\pi\in\mathcal{C}$ and $\pi=\lambda\mathbb{W}\pi$ for some $\lambda\in(0,1)$. Then, for any $t\in J$, as in the first step, we obtain

$$\begin{aligned}|\pi(t)| &= \lambda|\mathbb{W}\pi(t)|\le\sup_{t\in J}|\mathbb{W}\pi(t)|\\ &\le (\|u_1\|\Omega(\|\pi\|)+\|u_2\|)A_1,\end{aligned}$$

and, consequently,

$$\frac{\|\pi\|}{(\|u_1\|\Omega(\|\pi\|)+\|u_2\|)A_1}\le 1.$$

From $(\mathbb{H}_3)$, $\|\pi\|\neq K$. After that, we define $U=\{\pi\in B_\rho:\|\pi\|<K\}$. Now, we can see that $\mathbb{W}:\overline{U}\to\mathcal{C}$ is continuous and completely continuous. Thus, there is no $\pi\in\partial U$ such that $\pi=\lambda\mathbb{W}\pi$ with $0<\lambda<1$. By the nonlinear alternative of the Leray–Schauder-type, we get that the operator $\mathbb{W}$ has a fixed point $\pi\in\overline{U}$, which is a solution of the nonlocal fractional integro-differential sequential Hilfer and Caputo boundary value problem (17) on J. The proof is completed. □

4. Some Special Cases

In this section, we present some special cases and some interesting behavior of solutions to the investigated problem (1).

Corollary 1. *Assume that $\Pi: J\times\mathbb{R}\to\mathbb{R}$ is a continuous function.*

(a) *If $|\Pi(t,\pi)|\le M$, where M is a positive constant, then the nonlocal fractional integro-differential sequential Hilfer and Caputo boundary value problem (17) has at least one solution J.*

(b) *If $u_1(t)=1$, $\Omega(u)=Bu+C$, $u_2(t)=D$, where $B\ge 0$, $C,D>0$, then the nonlocal fractional integro-differential sequential Hilfer and Caputo boundary value problem (17) has at least one solution J if $A_1B<1$.*

(c) *If $u_1(t)=1$, $\Omega(u)=Bu^2+C$, $u_2(t)=D$, where $B\ge 0$, $C,D>0$, then the nonlocal fractional integro-differential sequential Hilfer and Caputo boundary value problem (17) has at least one solution J, if $4A_1^2B(C+D)<1$.*

If we put $\psi_1(t)=\psi_2(t)=\psi(t)$, then the nonlocal fractional integro-differential sequential Hilfer and Caputo boundary value problem (17) is reduced to

$$\begin{cases}{}^H\mathbb{D}^{\alpha,\beta;\psi}({}^C\mathbb{D}^{\gamma;\psi}\pi)(t)=\Pi(t,\pi(t)), \quad 0<\alpha,\beta,\gamma<1,\ t\in[0,x_1],\\ {}^C\mathbb{D}^{\gamma;\psi}\pi(0)=0, \quad \pi(x_1)=\sum_{i=1}^m\lambda_i\,{}^C\mathbb{D}^{\gamma;\psi}\pi(\eta_i)+\sum_{j=1}^n\delta_j\,\mathbb{I}^{\mu_j;\psi}\pi(\xi_j).\end{cases} \tag{17}$$

The following constants are used in the next corollaries.

$$A^* = 1-\sum_{j=1}^n\delta_j\frac{[\psi(\xi_j)-\psi(0)]^{\mu_j}}{\Gamma(\mu_j+1)},$$

$$A_1^* = \frac{1}{|A^*|}\left(\sum_{i=1}^{m}|\lambda_i|\frac{[\psi(\eta_i)-\psi(0)]^{\alpha}}{\Gamma(\alpha+1)}+\sum_{j=1}^{n}|\delta_j|\frac{(\psi(\xi_j)-\psi(0))^{\alpha+\mu_j+\gamma}}{\Gamma(\alpha+\mu_j+\gamma+1)}\right)$$
$$+\left(\frac{|A^*|+1}{|A^*|}\right)\frac{(\psi(x_1)-\psi(0))^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}.$$

Corollary 2. *If f satisfies the Lipschitz condition in $(\mathbb{H}_1)$ and if $A_1^*L < 1$, then the nonlocal fractional integro-differential sequential Hilfer and Caputo boundary value problem (17) has a unique solution on J.*

Corollary 3. *If the continuous function f satisfies $(\mathbb{H}_2)$ in Theorem 2 and if there exists a positive constant M such that*
$$\frac{M}{(\|u_1\|\Omega(M)+\|u_2\|)A_1^*} > 1,$$
then the nonlocal fractional integro-differential sequential Hilfer and Caputo boundary value problem (17) has at least one solutions on J.

If $n = p + q$, and $\mu_w = 0$ for $w = 1, \dots, q$, then the problem (17) can be reduced to the following problem with integro-differential multi-point boundary conditions as

$$\begin{cases} {}^{H}\mathbb{D}^{\alpha,\beta;\psi}({}^{C}\mathbb{D}^{\gamma;\psi}\pi)(t) = \Pi(t,\pi(t)), \quad 0<\alpha,\beta,\gamma<1,\ t\in[0,x_1], \\ {}^{C}\mathbb{D}^{\gamma;\psi}\pi(0) = 0, \quad \pi(x_1) = \displaystyle\sum_{i=1}^{m}\lambda_i {}^{C}\mathbb{D}^{\gamma;\psi}\pi(\eta_i) + \sum_{j=1}^{p}\delta_j\,\mathbb{I}^{\mu_j;\psi}\pi(\xi_j) + \sum_{w=p+1}^{q}\zeta_w\,\pi(\theta_w). \end{cases} \tag{18}$$

In addition, we put

$$\hat{A} = 1 - \sum_{j=1}^{p}\delta_j\frac{[\psi(\xi_j)-\psi(0)]^{\mu_j}}{\Gamma(\mu_j+1)} - \sum_{w=p+1}^{q}\zeta_w,$$
$$\hat{A}_1 = \frac{1}{|\hat{A}|}\left\{\sum_{i=1}^{m}|\lambda_i|\frac{[\psi(\eta_i)-\psi(0)]^{\alpha}}{\Gamma(\alpha+1)}+\sum_{j=1}^{p}|\delta_j|\frac{(\psi(\xi_j)-\psi(0))^{\alpha+\mu_j+\gamma}}{\Gamma(\alpha+\mu_j+\gamma+1)}\right.$$
$$\left.+\sum_{w=p+1}^{q}|\zeta_w|\frac{(\psi(\theta_w)-\psi(0))^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}\right\}+\left(\frac{|\hat{A}|+1}{|\hat{A}|}\right)\frac{(\psi(x_1)-\psi(0))^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}.$$

The existence and uniqueness results for the integro-differential multi-point boundary value problem (18) are similar to the Corollaries 2 and 3 by replacing $\hat{A}_1$ with A_1^*.

5. Illustrative Examples

Example 1. *Let us consider the following integro-differential boundary conditions to the sequential ψ_1-Hilfer and ψ_2-Caputo fractional differential equation of the form*

$${}^{H}\mathbb{D}^{\frac{1}{4},\frac{3}{4};e^{\frac{1}{10}t}}\left({}^{C}\mathbb{D}^{\frac{1}{2};t^2+t}\pi\right)(t) = \Pi(t,\pi(t)), \quad 0<t<\frac{9}{8}, \tag{19}$$

$${}^{C}\mathbb{D}^{\frac{1}{2};t^2+t}\pi(0) = 0, \qquad \pi\left(\frac{9}{8}\right) = \frac{3}{67}{}^{C}\mathbb{D}^{\frac{1}{2};t^2+t}\pi\left(\frac{3}{8}\right)+\frac{5}{77}{}^{C}\mathbb{D}^{\frac{1}{2};t^2+t}\pi\left(\frac{5}{8}\right)$$
$$+\frac{7}{87}{}^{C}\mathbb{D}^{\frac{1}{2};t^2+t}\pi\left(\frac{7}{8}\right)+\frac{2}{39}\mathbb{I}^{\frac{4}{5};t^2+t}\pi\left(\frac{1}{2}\right)+\frac{4}{59}\mathbb{I}^{\frac{7}{5};t^2+t}\pi\left(\frac{3}{4}\right). \tag{20}$$

From the boundary value problem (19), we set constants as $\alpha = 1/4$, $\beta = 3/4$, $\gamma = 1/2$, $x_1 = 9/8$, $\lambda_1 = 3/67$, $\lambda_2 = 5/77$, $\lambda_3 = 7/87$, $\eta_1 = 3/8$, $\eta_2 = 5/8$, $\eta_3 = 7/8$, $\delta_1 = 2/39$, $\delta_2 = 4/59$, $\mu_1 = 4/5$, $\mu_2 = 7/5$, $\xi_1 = 1/2$, $\xi_2 = 3/4$ and functions $\psi_1(t) = e^{(1/10)t}$ and $\psi_2(t) = t^2 + t$. From above information, we can compute that $A \approx 0.8763925133$ and $A_1 \approx 2.374946616$. Observe that the two functions satisfy $\psi_1', \psi_2' > 0$.

(i) If the function Π is defined by

$$\Pi(t,\pi(t)) = \frac{1}{2t+5}\left(\frac{2|\pi|+\pi^2}{1+|\pi|}\right) + \frac{1}{7}t^2 + 8t + \frac{1}{9}. \tag{21}$$

From the given nonlinear unbounded Lipschitzian function in (21), we get $|\Pi(t,\pi) - \Pi(t,z)| \le (2/5)|\pi - z|$ for $t \in [0,9/8]$, $\pi, z \in \mathbb{R}$. Setting $L = 2/5$, we have $A_1 L \approx 0.9499786464 < 1$ which fulfills the condition in (13). The result in Theorem 1 can be used to conclude that the boundary value problem (19) and (20) with the given function in (21) has a unique solution on $[0,9/8]$

(ii) Let the function Π be defined as

$$\Pi(t,\pi(t)) = \frac{1}{t+4}\left(\frac{\pi^{2024}}{5(1+\pi^{2022})} + \frac{1}{3t+6}\right) + \frac{1}{2t+7}. \tag{22}$$

We have

$$|\Pi(t,\pi)| \le \frac{1}{t+4}\left(\frac{1}{5}\pi^2 + \frac{1}{6}\right) + \frac{1}{2t+7}.$$

Choosing $u_1(t) = 1/(t+4)$, $u_2(t) = 1/(2t+7)$ and $\Omega(\pi) = (1/5)\pi^2 + (1/6)$, we get $\|u_1\| = 1/4$, $\|u_2\| = 1/7$ and then, there exists a $K \in (0.463775263, 7.957466657)$ satisfying the inequality in (16). Therefore, all assumptions in Theorem 2 agree with function Π in (22). Then, using integro-differential boundary conditions to the sequential ψ_1-Hilfer and ψ_2-Caputo fractional differential Equations (19), (20) and (22) have at least one solution on $[0,9/8]$.

(iii) If $\psi_1(t) = \psi_2(t) = t^2 + t$, then (19) is expressed as

$${}^{H}\mathbb{D}^{\frac{1}{4},\frac{3}{4};t^2+t}\left({}^{C}\mathbb{D}^{\frac{1}{2};t^2+t}\pi\right)(t) = \Pi(t,\pi(t)), \quad 0 < t < \frac{9}{8}, \tag{23}$$

and we can find that $A^* \approx 0.8763925133$, $A_1^* \approx 4.810643110$. If

$$\Pi(t,pi) = \frac{1}{2t+10}\left(\frac{2|\pi|+\pi^2}{1+|x|}\right) + \frac{1}{7}t^2 + 8t + \frac{1}{9}. \tag{24}$$

Then, $L = 1/5$ and we have $A_1^* L \approx 0.9621286220$. This means that boundary value problem (23), (20) and (24) has a unique solution on $[0,9/8]$.

In addition, if function Π in (23) is given in (22), then there exists a constant $K \in (0.235228817, 3.922219479)$ which satisfies the Corollary 3. So, the boundary value problem (23), (20) and (22) has at least one solution on $[0,9/8]$.

(iv) If the boundary conditions in (20) is replaced by

$$\begin{aligned} {}^{C}\mathbb{D}^{\frac{1}{2};t^2+t}\pi(0) = 0, \qquad \pi\left(\frac{9}{8}\right) &= \frac{3}{67}{}^{C}\mathbb{D}^{\frac{1}{2};t^2+t}\pi\left(\frac{3}{8}\right) + \frac{5}{77}{}^{C}\mathbb{D}^{\frac{1}{2};t^2+t}\pi\left(\frac{5}{8}\right) \\ &+ \frac{7}{87}{}^{C}\mathbb{D}^{\frac{1}{2};t^2+t}\pi\left(\frac{7}{8}\right) + \frac{2}{39}\mathbb{I}^{\frac{4}{5};t^2+t}\pi\left(\frac{1}{2}\right) + \frac{4}{59}\pi\left(\frac{3}{4}\right). \end{aligned} \tag{25}$$

Then, we get $\hat{A} \approx 0.8884626894$, $\hat{A}_1 \approx 4.816166032$. If

$$\Pi(t,\pi) = W\pi^2 + Z, \tag{26}$$

where constants $W, Z > 0$ and $WZ < 1/(4\hat{A}_1^2) \approx 0.01077797341$. Then, there exists a positive constant M satisfying the Corollary 3 when replacing A_1^* by $\hat{A}_1$. Hence, the boundary value problem (23), (25) and (26) has at least one solution on $[0,9/8]$.

6. Conclusions

In this paper, we have studied a new kind of boundary value problem consisting of a combination of two fractional derivative operators, one ψ_1-Hilfer and one ψ_2-Caputo, supplemented with nonlocal integro-differential boundary conditions. This combination, as far as we know, is new in the literature. Our uniqueness result is derived via Banach's contraction principle, while the Leray–Schauder nonlinear alternative is used to derive the existence result. The main results are well illustrated by constructing numerical examples.

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