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# Infinitesimal Transformations of Riemannian Manifolds—The Geometric Dynamics Point of View

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**Abstract:** In the present paper, we study the geometry of infinitesimal conformal, affine, projective, and harmonic transformations of complete Riemannian manifolds using the concepts of geometric dynamics and the methods of the modern version of the Bochner technique.

**Keywords:** complete Riemannian manifold; geometric dynamical systems; Liouville-type theorems; infinitesimal transformations

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## 1. Introduction

A smooth vector field $\xi$ on an $n$-dimensional differentiable (that is, of class $C^\infty$) manifold $M$ may be interpreted alternatively as the right-hand side of an autonomous system of first-order ordinary differential equations, i.e., a flow (see [1], pp. 12–13). Along with this, a *dynamical system* on a differentiable manifold $M$ is a smooth vector field $\xi$ that generates a flow on this manifold (see, for example, Introduction in [2]). Side by side, C. Udrişte showed that any flow on a differentiable manifold $M$ could be developed by conservative dynamics using a pseudo-Riemannian metric $g$ on $M$. He called this kind of dynamics *geometric dynamics* on a pseudo-Riemannian manifold $(M, g)$ (see [3]). The concept of geometric dynamics has many applications in mathematics and physics (see, for example, [2,4,5]).

In the present paper, we study the geometry of well-known infinitesimal conformal, affine, projective, and harmonic transformations of complete Riemannian manifolds using the concepts of geometric dynamics and the methods of the modern version of the *Bochner technique* for such manifolds (see, for example, [6–8]). The result of our study will be a series of Liouville-type theorems for such transformations for complete Riemannian manifolds. At the same time, we note that Liouville-type theorems of subharmonic and superharmonic functions on complete manifolds have been known for a long time (see, for example, [6,9]). In particular, our theorems generalize several results that have already become classical in the global theory of infinitesimal transformations of compact Riemannian manifolds.

In conclusion, the study methods explain the extent to which our research area will be explored. We use the generalized Bochner technique (see, for example, [6–8]), which is intended for complete Riemannian manifolds, unlike the classical Bochner technique (see [10], pp. 333–364; [11]). Therefore, our research area is an infinitesimal transformation of complete Riemannian manifolds.

## 2. Infinitesimal Transformations of Complete Riemannian Manifolds and Dynamical Systems on Them

We recall here some facts from the theory of groups of infinitesimal transformations. Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $\xi$ be a differentiable vector field on $(M, g)$. In terms of a local coordinate system $x^1, \ldots, x^n$ of the coordinate neighborhood $U$ of an arbitrary point $x \in M$, a vector field $\xi$ may be expressed by $\xi = \xi^k \partial_k$, where $\partial_k = \partial / \partial x^k$ and $\xi^1, \ldots, \xi^n$ are differentiable functions defined in the coordinate neighborhood $U$, called the components of $\xi$ with respect to a local coordinate system $x^1, \ldots, x^n$.

It is well-known (see [1], pp. 12–14) that in a neighborhood $U$ of a point $x$ of the manifold $(M, g)$ the field $\xi$ generates a local flow, which is a local one-parameter group of infinitesimal diffeomorphisms or, in other words, transformations $\varphi_t \colon U \to M$ for an arbitrary $t \in (-\varepsilon, +\varepsilon) \subset \mathbb{R}$. The converse assertion is also valid (see [1], pp. 21–22), namely, a local flow or, in other words, a local one-parameter group of infinitesimal transformations of the manifold $(M, g)$ consisting of diffeomorphisms $\varphi_t \colon U \to M$, for some open set $U \subset M$, an arbitrary $t \in (-\varepsilon, +\varepsilon) \subset \mathbb{R}$ and any $x \in U$, induces a vector field $\xi$ on $U$ as follows. At each point $x \in U$, we define a vector $\xi_x$ tangent to the curve $x(t) = \varphi_t$ and such that $\xi^k = dx^k/dt$ for $k = 1, \ldots, n$ in a local coordinate system $x^1, \ldots, x^n$ in $U$. The vector field $\xi$ is called an (autonomous) *dynamical system* on $M$ and the curve $x(t) = \varphi_t$ is also called the *trajectory* of the flow (see, for example, [5]). In this case, $\xi$ is called the *velocity vector* or the *infinitesimal generator* of the flow (see [12], p. 274). If there exists a global 1-parameter group of transformations of $M$ which induces a vector field $\xi$, then $\xi$ is called *complete* (see [1], p. 13). In this case, any trajectory of the flow is a curve defined on all $t$ of $\mathbb{R}$ (see [12], p. 273). Moreover, there is a one-to-one correspondence between global flows and complete vector fields on a manifold (see [12], p. 276). Accordance to [5,13], we formulate here our definition for the case of complete Riemannian manifolds.

**Definition 1.** *A dynamical system on a complete Riemannian manifold $(M, g)$ is a vector field $\xi$ that generates a global flow on $(M, g)$.*

In particular, a vector field $\xi$ will be called a *parallel dynamical system* (compare this with the definition given in [13]) if $\xi$ is parallel with respect to the Levi–Civita connection $\nabla$ of $(M, g)$, i.e., $\nabla \xi = 0$.

**Remark 1.** *According to the well-known de Rham theorem (see [1], pp. 179, 192), if a vector field $\xi$ is parallel on a simply connected and complete manifold $(M, g)$, then $(M, g)$ is reducible and isometric to the Riemannian product of some one-dimensional manifold tangent to the field $\xi$ and some $(n-1)$-dimensional integral manifold of its orthogonal integrable complement $\xi^\perp$. We note that the manifold $(M, g)$ is irreducible if it is not reducible.*

By the above definition, we can conclude that a complete vector field $\xi$ defined on a complete Riemannian manifold $(M, g)$ is a dynamical system on it. In particular, any smooth vector field on a compact manifold $(M, g)$ is complete (see [1], p. 14; [12], p. 273) and, therefore, it is a dynamical system on compact $(M, g)$.

We recall that the (local) *volume element* or, in other words, *volume form* of $(M, g)$ is defined by the equation $\omega_g(\partial_1, \ldots, \partial_n) = \sqrt{\det g}$ with respect to a local coordinate system $x^1, \ldots, x^n$. We remark here that a Riemannian manifold $(M, g)$ has a (global) volume element if and only if $(M, g)$ is orientable (see [14], p. 195). We also recall here that a volume form on a connected manifold $(M, g)$ has a single global invariant, namely the (overall) volume, denoted by $\mathrm{Vol}_g(M)$ which is invariant under volume-form preserving transformations. In symbols, $\mathrm{Vol}_g(M) = \int_M \omega_g$. The volume $\mathrm{Vol}_g(M)$ can be infinite or finite. For example, $\mathrm{Vol}_g(M) < \infty$ for a compact manifold $(M, g)$. On the other hand, a complete non-compact Riemannian manifold with non-negative Ricci curvature has infinite volume (see [9]).

For the volume form $\omega_g$ of $(M, g)$ one can consider its *Lie derivative* along trajectories of the flow with the velocity vector $\xi$. Namely, we have (see [1], p. 281; [14], p. 195)

$$L_\xi \omega_g = (\operatorname{div} \xi)\, \omega_g.$$

According to the definition of the Lie derivative, $L_\xi \omega_g$ measures the rate of the change of the volume form $\omega_g$ under deformations determined by a one-parameter group of differentiable transformations $\varphi_t$ (or a flow) generated by the vector field $\xi$ (see, for example, [4], p. 39). On the other hand, in the well-known monograph [14] (p. 195) the function $\operatorname{div} \xi$ was called the *logarithmic rate of change of volume* (or, in other words, *rate of volume expansion*) under the flow $\varphi_t$ generated by the vector field $\xi$.

For a vector field $\xi$ on a compact oriented Riemannian manifold $(M, g)$, *Green's theorem* is valid (see [1], p. 259):

$$\int_M \operatorname{div} \xi \; d\mathrm{vol}_g = 0, \tag{1}$$

where we denoted a selected volume element of $(M, g)$ in classical style by $d\mathrm{vol}_g$. Obviously, the conditions $\operatorname{div} \xi > 0$ or $\operatorname{div} \xi < 0$ for the logarithmic rate of change of volume $\operatorname{div} \xi$ contradict (1). On the other hand, if $\operatorname{div} \xi \geq 0$ or $\operatorname{div} \xi \leq 0$, then (1) implies that $\operatorname{div} \xi = 0$ (see [15], p. 39). This means that $L_\xi \omega_g = 0$, i.e., the one-parameter group of differentiable transformations $\varphi_t \colon M \to M$ for all $t \in \mathbb{R}$ leaves $\omega_g$ invariant and the vector field $\xi$ is an infinitesimal automorphism of the volume structure (see [15], p. 6). In dynamical systems, such a vector field $\xi$ is said to be *divergence-free* and the flow generated by it is said to be *incompressible* (see [10], p. 125). Moreover, the geometric dynamics of divergence-free vector fields were studied in detail in the monograph [4].

We can formulate a similar assertion on conditions for the non-divergence of vector fields on a complete Riemannian manifold. To do this, we use the proposition from [16], which we formulate in terms of geometric dynamics.

**Proposition 1.** *Let $\xi$ be a dynamical system on a complete non-compact oriented Riemannian manifold $(M, g)$ such that its length is integrable. If, moreover, the logarithmic rate of its volumetric expansion does not change the sign on $(M, g)$, then the flow generated by $\xi$ is incompressible.*

**Remark 2.** *We will insist on the function $f$ being in $L^p(M)$, if the* $p$th *power of the absolute value of $f$ is integrable on $(M, g)$ (see also [17]). For example, the integrability condition for the length of a vector field $\xi$ means that its length satisfies the condition $\|\xi\| \in L^1(M)$, where $\|\xi\| = \sqrt{g(\xi, \xi)}$.*

From Proposition 1 we can conclude the following corollary.

**Corollary 1.** *Let a dynamical system on a complete noncompact oriented Riemannian manifold $(M, g)$ have a velocity vector $\xi$ of constant length. Furthermore, if the logarithmic rate of its volumetric expansion* $\operatorname{div} \xi \in L^1(M)$ *and it does not decrease under the flow it creates, then the flow is incompressible.*

**Proof.** Let a dynamical system on a complete noncompact oriented Riemannian manifold $(M, g)$ have a velocity vector $\xi$ of constant length. Elementary calculations allow us to conclude that the following equality holds

$$\operatorname{div}((\operatorname{div} \xi)\xi) = L_\xi(\operatorname{div} \xi) + (\operatorname{div} \xi)^2, \tag{2}$$

where $L_\xi(\operatorname{div} \xi) = \xi(\operatorname{div} \xi)$ is the Lie derivative of $\operatorname{div} \xi$ with respect $\xi$.

Then from (2), it is easy to see that the inequality $L_\xi(\operatorname{div} \xi) \geq 0$, which is valid everywhere on $(M, g)$, implies the inequality $\operatorname{div}((\operatorname{div} \xi)\xi) \geq 0$. To complete the proof, it suffices to refer to Proposition 1. Namely, if the conditions $\operatorname{div}((\operatorname{div} \xi)\xi) \geq 0$ and $\int_M \|(\operatorname{div} \xi)\xi\| \, d\mathrm{vol}_g = \|\xi\| \int_M |\operatorname{div} \xi| \, d\mathrm{vol}_g < \infty$ hold on an oriented and complete manifold $(M, g)$, then $\operatorname{div}((\operatorname{div} \xi)\xi) \geq 0$. It follows from (2) that $\operatorname{div} \xi = 0$. In this case, $L_\xi \omega_g$ is equal to zero due to (1). $\square$

**Remark 3.** *We considered in [18] a kinematic world model as a four-dimensional space-time $(M, g)$ which admits fluid flows of matter with a time-like velocity vector of unit length. In particular, we presented interesting applications of the generalized Landau–Raichaudhuri equation to the theory of* *the logarithmic velocity of the volume expansion of space-time. At the same time, Killing vector fields of constant length and the corresponding flows on complete Riemannian manifolds were studied in [19].*

The *energy density* $e(\xi)$ of the flow generated by the vector field $\xi$ is the scalar function defined by (see, for example, [17], p. 434)

$$e(\xi) = 1/2\ \|\xi\|^2 := 1/2\ g(\xi, \xi). \tag{3}$$

The energy density $e(\xi)$ has interesting properties imposed either by the behavior of the gradient, Hessian or Laplacian of $e(\xi)$, or by the behavior of $e(\xi)$ along field lines (see [3] where $e(\xi)$ is called the energy of the flow generated by $\xi$). In turn, the *kinetic energy* of the flow generated by $\xi$ is defined by the integral formula (see [4], pp. 2, 19, 37; [6], p. 437)

$$E(\xi) = \int_M e(\xi)\, d\mathrm{vol}_g.$$

The kinetic energy $E(\xi)$ can be infinite or finite. For example, $E(\xi) < \infty$ for a smooth vector field $\xi$ on a compact manifold $(M, g)$. Kinetic energy plays an important role in Hamilton dynamics (see [4]).

Let us now consider an example in which the concepts of geometric dynamics defined above are applied.

**Theorem 1.** *Let $(M, g)$ be a complete non-compact Riemannian manifold $(M, g)$ with non-negative Ricci curvature. There does not exist a non-zero dynamical system $\xi$ on $(M, g)$ such that*

*(i) $\xi$ is closed;*
*(ii) the logarithmic rate* $\mathrm{div}\,\xi$ *of volumetric expansion is a non-decreasing function under the flow of $\xi$;*
*(iii) the kinetic energy of the flow $E(\xi)$ is finite.*

**Proof.** Let $\xi$ be a dynamical system on a complete manifold $(M, g)$ such that the 1-form $\theta$ dual to $\xi$ with respect to $g$ is closed. In turn, the vector field $\xi$ is also called *closed*. This means that $\nabla\theta$ is symmetric with respect to the Levi–Civita connection $\nabla$ of the metric $g$. In this case, the following formula holds (see [10], p. 337)

$$\Delta e(\xi) = \|\nabla\xi\|^2 + L_\xi(\mathrm{div}\,\xi) + Ric(\xi, \xi) \tag{4}$$

where $\Delta := \mathrm{trace}_g\nabla^2$, $\|\nabla\xi\|^2 = g(\nabla\xi, \nabla\xi)$ and $Ric$ is the Ricci tensor of $\nabla$. We will assume that $Ric \geq 0$ and the logarithmic rate $\mathrm{div}\,\xi$ of the volumetric expansion does not decrease along the flow generated by $\xi$, i.e., $L_\xi(\mathrm{div}\,\xi) \geq 0$ everywhere on $(M, g)$. In this case, the inequality $\Delta e(\xi) \geq 0$ follows from (4). Then the energy density $e(\xi)$ is a non-negative *subharmonic function* (see [9]). On the other hand, if $f$ is a non-negative $L^p$ subharmonic function at least for one $0 < p < \infty$ defined on a complete manifold $(M, g)$ with non-negative Ricci curvature, then $f$ must be identically constant (see [20]). Furthermore, this constant must be zero if $(M, g)$ has infinite volume (see, for example, [17]). We recall here that every complete non-compact Riemannian manifold with non-negative Ricci curvature has infinite volume (see [9]). In conclusion, we note that the finiteness condition for $E(\xi)$ means that $e(\xi) \in L^2(M)$. Summarizing the above, we can formulate the above theorem as a conclusion. □

We recall that $\theta$ is a *harmonic form* if $\theta$ is both closed and co-closed, i.e., $\nabla\theta$ is symmetric with under the Levi–Civita connection $\nabla$ of the metric $g$ and $\nabla^*\theta = 0$, where $\nabla^*$ is the formal-adjoint of the differential operator $\nabla$ defined by the formula $\nabla^*\theta = -\mathrm{trace}_g(\nabla\theta)$ (see [11], p. 378). In this case $\mathrm{div}\,\xi = -\nabla^*\theta = 0$ for the 1-form $\theta$ dual to $\xi$ with respect to $g$. Green and Wu proved in [12] the following theorem: If $(M, g)$ is a complete noncompact manifold with non-negative Ricci curvature, then no nonzero harmonic 1-form is in $L^p(M)$

for any $1 < p < \infty$. If, moreover, the sectional curvature of $(M, g)$ is non-negative outside of some compact set, then no nonzero harmonic 1-form is in $L^1(M)$. This theorem generalized Yau result from [9]. In turn, the following corollary of our Theorem 1 is a generalization of the theorem proved by Green and Wu in [21].

**Corollary 2.** *There does not exist a nonzero harmonic $L^p$ one-form for any $0 < p < \infty$ on a complete noncompact Riemannian manifold with non-negative Ricci curvature.*

## 3. Conformal Dynamical Systems on Complete Riemannian Manifolds

We recall that a diffeomorphism $f$ of $(M, g)$ onto itself is called a *conformal mapping* if $f^*g = e^{2\sigma}g$ for some scalar function $\sigma$ on $M$. If $\sigma$ is constant, $f$ is called a *homothety*. For $\sigma = 0$ a homothety is an *isometry* (see [22], p. 269).

A vector field $\xi$ on $(M, g)$ is called an *infinitesimal conformal transformation* or, in other words, *conformal Killing vector field* if this field generates a local one-parameter group of conformal transformations $\varphi_t \colon U \to U$ in a neighborhood $U$ of any point $x \in M$. The vector field $\xi$ is a conformal Killing vector field on $(M, g)$ if and only if $L_\xi g = 2\sigma g$ for $\sigma = 1/n \operatorname{div} \xi$ (see [5]; [22], p. 282; [15], p. 50). Particular cases of a conformal Killing vector field is a homothety infinitesimal conformal transformation if $\sigma = \text{const}$ and a *Killing vector field* (or *an infinitesimal isometry*) if $\sigma = 0$ (see [1], p. 237). The local geometry of Riemannian manifolds and the global geometry of compact and complete Riemannian manifolds of infinitesimal conformal transformations are studied in detail. Information about these results can be found in numerous articles and well-known monographs [1,15,22] and others.

In turn, a complete vector field $\xi$ on a complete Riemannian manifold $(M, g)$ will be called a *conformal dynamical system* if it generates a global one-parameter group $\varphi_t \colon M \to M$ for all $t \in \mathbb{R}$ of infinitesimal conformal transformations on $(M, g)$. These transformations preserve the angles defined by the Riemannian metric of $(M, g)$. One can formulate a vanishing theorem for conformally Killing vector fields on a complete Riemannian manifold. To do this, we use the proposition from [16], which we formulate in terms of geometric dynamics.

**Theorem 2** (see [16])**.** *Let $(M, g)$ be a complete noncompact Riemannian manifold with non-positive Ricci curvature and $\xi$ be a conformal dynamical system with finite kinetic energy. Then $\xi$ is a parallel dynamical system and consequently generates an incompressible flow.*

We prove here a theorem generalizing the above result. Let $\xi$ be a conformal dynamical system with $L_\xi g = 2\sigma g$ defined on a complete manifold $(M, g)$. Then for the 1-form $\theta$ dual to $\xi$ with respect to $g$ we have (see [22], p. 285; [23])

$$\bar{\Delta}\,\theta = \frac{n-2}{n}\,\nabla(\operatorname{div}\xi) + Ric(\xi, \cdot) \tag{5}$$

where $\bar{\Delta} = \nabla^*\nabla$ is the *rough* (or *Bochner*) *Laplacian* defined by $\bar{\Delta}\,\theta(X) = -(\operatorname{trace}_g \nabla^2\theta)(X)$ for an arbitrary vector field $X$ on $(M, g)$ (see [10]; [11], p. 377). In this case, the well-known second *Kato inequality* (see [11], p. 380)

$$\|\xi\|\;\Delta\|\xi\| \geq -g(\bar{\Delta}\,\xi, \xi)$$

can be rewritten in the form

$$\|\xi\|\;\Delta\|\xi\| \geq -\frac{n-2}{n}\,L_\xi(\operatorname{div}\xi) - Ric(\xi, \xi). \tag{6}$$

The assumptions $Ric \leq 0$ and $L_\xi(\operatorname{div}\xi) \leq 0$ imply that $\|\xi\|\;\Delta\|\xi\| \geq 0$. By the oldest theorem of geometric analysis (see [9]) we deduce that either $\int_M \|\xi\|^p = \infty$ for a positive number $p > 1$ or $\|\xi\|$ = const. Therefore, if $\|\xi\| \in L^p(M)$ at least for one $p > 1$, then

$p > 1$ then $\|\xi\|$ = const. At the same time, this constant must be zero if $(M, g)$ has an infinite volume. On the other hand, from (6) we obtain $Ric(\xi, \xi) = 0$ and $L_\xi(\operatorname{div}\xi) = 0$ if the following three conditions: $\|\xi\|$ = const, $Ric \leq 0$ and $L_\xi(\operatorname{div}\xi) \leq 0$ hold. In this case, using (5) we have

$$0 = \frac{1}{2}\,\Delta\, g(\xi, \xi) = -g(\bar{\Delta}\theta, \theta) + \|\nabla\xi\|^2 = \frac{n-2}{n}\, L_\xi(\operatorname{div}\xi) - Ric(\xi, \xi) + \|\nabla\xi\|^2 = \|\nabla\xi\|^2.$$

Then $\xi$ is a parallel vector field. In this case, $(M, g)$ is reducible (see [1], p. 179 and also our remark in the second paragraph). On the other hand, if $(M, g)$ is irreducible, then there are no parallel vector fields on $(M, g)$. Therefore, the following theorem holds.

**Theorem 3.** *Let $(M, g)$ be a complete noncompact Riemannian manifold with non-positive Ricci curvature and $\xi$ be a conformal dynamical system such that $\|\xi\| \in L^p(M)$ at least for one $p > 1$. If the logarithmic rate $\operatorname{div}\xi$ is a non-increasing function under the flow of $\xi$, then $\xi$ is a parallel dynamical system and consequently generates an incompressible flow. Furthermore, if the volume of $(M, g)$ is infinite or $(M, g)$ is irreducible, then $\xi$ is identically equal to zero everywhere on $(M, g)$.*

Recall that a Riemannian manifold $(M, g)$ is called a *Yamabe soliton* (see [24]) if there is a smooth vector filed $\xi$ and constant $\rho$ such that $L_\xi g = 2(s - \rho)g$, where $s = \operatorname{trace}_g Ric$ is the scalar curvature of $(M, g)$. Therefore, the vector field $\xi$ of the Yamabe soliton $(M, g, \xi, \rho)$ is an example of a conformal Killing vector field. Based on Theorem 2 and the theorem from [23], we can formulate an assertion about the Yamabe soliton for which, as it is easy to prove, $L_\xi(\operatorname{div}\xi) = nL_\xi s$.

**Corollary 3.** *Let $(M, g, \xi, \rho)$ be a complete noncompact Yamabe soliton with non-positive Ricci curvature. If either of the following conditions holds:*

*(i)* $\|\xi\| \in L^2(M)$;

*(ii)* $\|\xi\| \in L^p(M)$ *at least for one* $p > 1$ *and the scalar curvature $s$ of $g$ is a non-increasing function under the flow of $\xi$,*

*then $\xi$ is a parallel vector field. Furthermore, if the volume of $(M, g)$ is infinite or $(M, g)$ is irreducible, then $\xi$ is identically equal to zero on $(M, g)$.*

**Remark 4.** *To illustrate the above statements, recall that Hadamard manifold is a simply connected complete Riemannian manifold of non-positive sectional curvature. It has an infinite volume, which follows from the Cartan–Hadamard theorem (see [10], p. 241). Furthermore, $Ric(X, X) = \sum_{i=1,\dots,n} g(R(X, e_i)e_i, X) \leq 0$ for the curvature tensor $R$ of a Hadamard manifold $(M, g)$ and for an orthonormal basis $e_1, \dots, e_n$ of $T_xM$ at an arbitrary $x \in M$.*

If $(M, g)$ is complete, then every infinitesimal isometry is a complete vector field (see [25], p. 46). Therefore, we can consider an arbitrary infinitesimal isometry as an isometric dynamical system on a complete manifold $(M, g)$. Let $\xi$ be an isometric dynamical system with $L_\xi g = 0$ defined on a complete manifold $(M, g)$. In this case, $\operatorname{div}\xi = 0$ and, therefore, repeating the arguments from the proof of Theorem 2, we arrive at the following corollary.

**Corollary 4.** *Let $(M, g)$ be a complete noncompact Riemannian manifold with non-positive Ricci curvature and $\xi$ be an isometric dynamical system such that $\|\xi\| \in L^p(M)$ at least for one $p > 1$. Then $\xi$ is a parallel dynamical system and consequently generates an incompressible flow. Furthermore, if the volume of $(M, g)$ is infinite or $(M, g)$ is irreducible, then $\xi$ is identically equal to zero everywhere on $(M, g)$.*

**Remark 5.** *Our theorem generalizes the following theorem from [23]: Suppose that a complete non-compact Riemannian manifold $(M, g)$ has non-positive Ricci curvature, then every Killing vector field $\xi$ is a parallel vector field on $(M, g)$ if $\|\xi\| \in L^2(M)$.*

**Proposition 2.** *Let $(M, g)$ be a Hadamard manifold and $\xi$ be an isometric dynamical system such that the energy density $e(\xi)$ of the flow generated by $\xi$ is bounded on $(M, g)$, then $(M, g)$ is reducible and isometric to a Riemannian product of some trajectory of the flow and its some orthogonal $(n-1)$-dimensional complement.*

**Proof.** The proposition is a corollary from the two theorems. First, this is a theorem from [19] with the following content: If the length of a Killing vector field $\xi$ is bounded on a Riemannian manifold $(M, g)$ with non-positive sectional curvature $(M, g)$ then $\xi$ is parallel on $(M, g)$. Second, it is the de Rham decomposition theorem on a simply connected and complete Riemannian manifold $(M, g)$ (see [1], p. 192). □

### 4. Affine Dynamical Systems on Complete Riemannian Manifolds

Let $(M, g)$ and $(M', g')$ be Riemannian manifolds with the Levi–Civita connections $\nabla$ and $\nabla'$, respectively. A differentiable mapping $f\colon (M, g) \to (M', g')$ is called an *affine mapping* if it maps every parallel vector field along any curve $\gamma$ in $(M, g)$ into a parallel vector field along the curve $f(\gamma)$ in $(M', g')$ (see [11]). Clearly, $f$ maps every geodesic in $(M, g)$ into a geodesic in $(M', g')$. An affine mapping $f$ of a manifold $(M, g)$ onto itself is called an *affine transformation* of $(M, g)$. We recall the well-known theorem (see [25], p. 126): If $(M, g)$ is an irreducible and complete Riemannian manifold, then the group of all affine transformations of $(M, g)$ is equal to the group of all isometric transformations of $(M, g)$, except the case when $(M, g)$ is the 1-dimensional Euclidean space.

In turn, vector field $\xi$ on $(M, g)$ is called an *infinitesimal affine transformation* or *affine Killing vector field* if the local 1-parameter group of local transformations $\varphi_t$ for $t \in (-\varepsilon, +\varepsilon) \subset \mathbb{R}$ generated by this field in a neighborhood $U$ of a point $x \in M$ preserves the connection $\nabla$, i.e., if $\varphi_t\colon U \to M$ is an affine transformation (see [10], p. 230). Therefore, if $\xi$ is an infinitesimal affine transformation, then (see [22], p. 224; [5,23])

$$L_\xi \nabla = 0. \tag{7}$$

Local and global geometries of infinitesimal affine transformations are studied in detail. Information about the obtained results can be found in numerous articles and well-known monographs [1,15,22] and others. In turn, we will consider the theory of these transformations from the point of view of dynamical systems and apply the generalized Bochner technique to their study.

Let now $(M, g)$ be a complete manifold, then every infinitesimal affine transformation is a complete vector field on $(M, g)$ (see Theorem 2.5 in [25], p. 46). Therefore, we can consider an infinitesimal affine transformation as a dynamical system on a complete Riemannian manifold $(M, g)$.

It is well-known that $\operatorname{div} \xi = 0$ for an infinitesimal affine transformation $\xi$ defined on a compact Riemannian manifold $(M, g)$ (see [25], p. 45). On the other hand, it is directly verified that Equation (7) is equivalent to the condition (see also [26])

$$\nabla(L_\xi g) = 0. \tag{8}$$

It is well-known that if a simply connected Riemannian manifold $(M, g)$ is irreducible, then any field of parallel symmetric 2-tensors $G$ is defined by the condition $G = \rho\, g$ for some constant $\rho$. Therefore, from (8) we can deduce the condition $L_\xi g = \rho\, g$ for some constant $\rho$. Hence $\xi$ is an infinitesimal homothetic transformation. On the other hand, if a simply connected complete Riemannian manifold admits a one-parameter group of non-isometric homothetic transformations, then it is isometric to a Euclidean space of the same dimension (see [26]). As a result, the following proposition holds.

**Proposition 3.** *If a simply connected complete and irreducible Riemannian manifold admits an affine dynamical system that is not an isometric dynamical system, then it is isometric to a Euclidean space of the same dimension.*

In turn, from (8) it follows that $\nabla \mathrm{trace}_g(L_\xi g) = 2\nabla(\mathrm{div}\,\xi) = 0$. In this case, the identity $\mathrm{div}\,\xi = \mathrm{constant}$ holds (see also [25], p. 45). Consider now an affine dynamical system $\xi$ on a complete Riemannian manifold $(M, g)$. If $\|\xi\| \in L^1(M)$, then by Proposition 1 we conclude that $\mathrm{div}\,\xi = 0$. As a result, we have the following statement.

**Proposition 4.** *Let $(M, g)$ be a complete Riemannian manifold and $\xi$ be an affine dynamical system such that its length is integrable, then the flow generated by $\xi$ is incompressible.*

Since Equation (7) follows from the condition $L_\xi g = 0$, we can conclude that the Killing vector field is an example of an infinitesimal affine transformation. Moreover, we can formulate a condition for the coincidence of an infinitesimal affine transformation and a Killing vector field on a complete Riemannian manifold. To do this, we use the proposition from [23], which we formulate in terms of geometric dynamics.

**Theorem 4** (see [23])**.** *Let $\xi$ be an affine dynamical system on a complete Riemannian manifold $(M, g)$. If the kinetic energy of $\xi$ is finite, then it is an isometric dynamical system.*

For an affine dynamical system $\xi$ on a complete manifold Riemannian manifold $(M, g)$ we have the following Equation (see [25], p. 56)

$$(\mathrm{Hess}_g e(\xi))(X, X) = \|\nabla_X \xi\| - g(R(\xi, X)X, \xi), \tag{9}$$

where $\mathrm{Hess}_g e(\xi) = \nabla d e(\xi)$ for the energy density function $e(\xi)$ of the flow generated by the vector field $\xi$ and for an arbitrary smooth vector field $X$ on $(M, g)$. If the section curvature of $(M, g)$ is non-positive, then the inequality $g(R(\xi, X)X, \xi) \leq 0$ holds. In this case, from (9) we obtain the inequality $\mathrm{Hess}_g e(\xi) \geq 0$. Therefore, $e(\xi)$ is a non-negative smooth *convex function* (see also [27]). In [27], was proved that an arbitrary convex function on a complete Riemannian manifold $(M, g)$ is constant on each closed geodesic in $(M, g)$ and, moreover, the critical points of a convex function are its absolute minimum points in $(M, g)$. Using this statement we can formulate a proposition.

**Proposition 5.** *Let $\xi$ be an affine dynamical system on a complete Riemannian manifold with non-positive sectional curvature, then its energy density function $e(\xi)$ is constant on each closed geodesic in $(M, g)$ and an arbitrary critical point of $e(\xi)$ is its absolute minimum point in $(M, g)$.*

A convex function is an example of a subharmonic function. At the same time, well-known from [20] that if a Riemannian manifold $(M, g)$ is complete, simply connected and has non-positive sectional curvature or, in other words, Hadamard manifold, then for each $p \in (0, +\infty)$ every nonnegative $L^p$ subharmonic function on $(M, g)$ is constant. Therefore, the energy density $L^p$ function $e(\xi)$ on a complete Riemannian manifold with non-positive sectional curvature must be a constant. On the other hand, a Hadamard manifold has an infinite volume, which follows the Cartan–Hadamard theorem. This forces the constant function $e(\xi)$ to be zero. As a result, the following corollary holds.

**Corollary 5.** *The Hadamard manifold $(M, g)$ does not admit a non-zero affine dynamical system $\xi$ such that $e(\xi) \in L^p(M)$ at least for one $p \in (0, +\infty)$.*

From (9) we obtain $\nabla^*(L_\xi g) = 0$, which is equivalent to Equation (see also [23])

$$\bar{\Delta}\,\theta = -Ric(\xi, \cdot\,). \tag{10}$$

In this case, the second Kato inequality (6) can be rewritten in the form

$$\|\xi\|\;\Delta\|\xi\| \geq -Ric(\xi, \xi).$$

Next, repeating our arguments of the third paragraph, we arrive at the following conclusion.

**Theorem 5.** *Let $(M,g)$ be a complete noncompact Riemannian manifold with non-positive Ricci curvature and $\xi$ be an affine dynamical system such that $\|\xi\| \in L^p(M)$ at least for one $p>1$. Then $\xi$ is a parallel dynamical system and consequently generates an incompressible flow. Furthermore, if the volume of $(M,g)$ is infinite or $(M,g)$ is irreducible, then $\xi$ is identically equal to zero everywhere on $(M,g)$.*

## 5. Projective Dynamical Systems on Complete Riemannian Manifolds

The classic geometrical problem of determining Riemannian metrics $g$ and $g'$ that have corresponding geodesics arose in connection with the dynamic problem on transformations of the equations of motion of mechanical systems in such a way that the trajectories are preserved. Various ways of identifying the geodesics of a pseudo-Riemannian manifold $(M,g)$ with the trajectories of conservative and nonconservative dynamical systems give the possibility of widely applying the results of the theory of projective transformations to physics and mechanics (see [4,5]).

A transformation of a Riemannian manifold $(M,g)$ which maps geodesics into geodesics is called *projective*. The main results of the local theory of projective transformations are presented in our monograph [22]. In the case of compact Riemannian manifolds, the results of the global theory of projective transformations can be found in the well-known monograph [15].

In turn, an infinitesimal transformation $\xi$ on $(M,g)$ is said to be *projective* if an arbitrary transformation $\varphi_t$ from the flow of the vector field $\xi$ preserves the geodesic curves of $(M,g)$, i.e., if any geodesic is invariant under the action of the (local) one-parameter group of (local) transformations that are generated by the vector field $\xi$. If $(M,g)$ is complete and $\xi$ generates a global one-parameter group of infinitesimal projective transformations on $(M,g)$, then we call $\xi$ a *projective dynamical system*. Therefore, $\xi$ is an infinitesimal projective transformation if and only if (see [5])

$$(L_\xi \nabla)(X,Y) = Y(\phi)X + X(\phi)Y$$

for $\phi = \frac{1}{n+1}\operatorname{div}\xi$ and any smooth vector field $X$ and $Y$ on $(M,g)$. Side by side, Equation (7) is equivalent to the condition (see [28])

$$\nabla_Z(L_\xi \nabla)(X,Y) = 2g(X,Y)Z(\phi) + g(X,Z)Y(\phi) + g(Y,Z)X(\phi) \tag{11}$$

for any smooth vector field $X$, $Y$ and $Z$ on $(M,g)$. From (11) we obtain Equation (see also [5,23])

$$\bar{\Delta}\,\theta = Ric(\xi,\cdot\,) - \frac{2}{n+1}\,\nabla(\operatorname{div}\xi).$$

In this case, the second Kato inequality (6) can be rewritten in the form

$$\|\xi\|\,\Delta\|\xi\| \geq -Ric(\xi,\xi) + \frac{2}{n+1}\,L_\xi(\operatorname{div}\xi). \tag{12}$$

Using (12), we can prove the projective dynamical system theorem. Moreover, the proof of the following theorem is no different from the proof of our Theorem 3.

**Theorem 6.** *Let $(M,g)$ be a complete noncompact Riemannian manifold with non-positive Ricci curvature and $\xi$ be a projective dynamical system such that $\|\xi\| \in L^p(M)$ at least for one $p>1$. If the logarithmic rate* div $\xi$ *is a non-decreasing function under the flow of $\xi$, then $\xi$ is a parallel dynamical system and consequently generates an incompressible flow. Furthermore, if the volume of $(M,g)$ is infinite or $(M,g)$ is irreducible, then $\xi$ is identically equal to zero everywhere on $(M,g)$.*

**Remark 6.** *Our theorem complements the following theorem from [23]: Let $(M,g)$ be a complete non-compact Riemannian manifold with non-positive Ricci curvature, then every projective infinitesimal transformation $\xi$ on $(M,g)$ is a parallel vector field if $\|\xi\| \in L^2(M)$ or, in other words, every projective dynamical system with finite kinetic energy is a parallel vector field.*

Let $(M, g, J)$ be a $2m$-dimensional *Kählerian manifold* where $(M, g)$ is $2m$-dimensional Riemannian manifold and $J$ is a tensor field such that $J \in C^\infty(T^*M \otimes TM)$, $J^2 = -\mathrm{id}$, $g(J, J) = g$ and $\nabla J = 0$ for the Levi–Civita connection $\nabla$ (see, for example, [14], p. 160). We shall say a vector field $\xi$ is an *infinitesimal holomorphically projective transformation* if it satisfies (see [22], p. 499; [28])

$$(L_\xi \nabla)(X, Y) = Y(\phi)X + X(\phi)Y - g(\xi, JX)JY - g(\xi, JY)JX \tag{13}$$

for the Levi–Civita connection $\nabla$ and smooth vector fields $X, Y$ on $M$. In this case, $\phi$ is called the associated one-form of the transformation. If $\phi$ vanishes, then the transformation reduces to an affine one. From (13) follows Equation (10). Hence the following corollary is true. Moreover, this statement is an analog of Corollary 4 and Theorem 5.

**Corollary 6.** *Let $(M, g, J)$ be a complete Kählerian manifold with non-positive Ricci curvature. If $\|\xi\| \in L^p(M)$ at least for one $p > 1$, then $\xi$ is a parallel vector field. Furthermore, if the volume of $(M, g, J)$ is infinite or $(M, g, J)$ is irreducible, then $\xi$ is identically equal to zero everywhere on $(M, g, J)$.*

**Remark 7.** *This corollary generalizes a similar assertion from [28] which was proved for a compact Kählerian manifold.*

### 6. Harmonic Dynamical Systems on Complete Riemannian Manifolds

Suppose a map $f\colon (M, g) \to (M', g')$ of Riemannian manifolds $(M, g)$ and $(M', g')$. Its differential $df\colon\ TM \ \to\ TM'$ determines the *energy density* by the formula $e(f) = 1/2\,\|df\|^2 := 1/2\,\mathrm{trace}_g f^* g'$ (see [6], p. 436). This map is said to be *harmonic* if it determines an extremum of the *energy functional* $E_U(f) = \int_M e(f)\,d\mathrm{vol}_g$ for any open set $U$ in $M$ relatively compact with respect to the variations of $f$ compactly supported on $U$ (see [6], p. 438). It is well-known that $f\colon (M, g) \to (M', g')$ is a harmonic mapping if and only if it satisfies the *Euler–Lagrange equation* $\mathrm{trace}_g(\bar{\nabla} df) = 0$ for the canonical connection $\bar{\nabla} = \nabla \oplus \nabla'$ in the vector bundle $T^*M \otimes f^*TM'$ (see also [6], p. 435). A harmonic mapping of $(M, g)$ onto itself is called a *harmonic transformation*.

A vector field $\xi$ on $(M, g)$ is called an infinitesimal harmonic transformation, if this field generates a local one-parameter group of harmonic transformations $\varphi_t\colon U \to U$ in a neighborhood $U$ of any point $x \in M$ (see [22], p. 262; [29]). In this case, the Euler–Lagrange equation can be rewritten in the form $\mathrm{trace}_g(L_\xi \nabla) = 0$. Therefore, an infinitesimal affine transformation is an example of an infinitesimal harmonic transformation. The geometry of infinitesimal harmonic transformations is studied in detail in [22,29]. In turn, we will consider the theory of these transformations from the point of view of dynamical systems and apply the generalized Bochner technique to their study.

In turn, a complete vector field $\xi$ is called a *harmonic dynamical system* on a complete Riemannian manifold $(M, g)$ if it generates a flow, which is a globally defined on $(M, g)$ one-parameter group of infinitesimal transformations $\varphi_t\colon M \to M$ for all $t \in \mathbb{R}$.

We proved in [29] that a vector field $\xi$ is an infinitesimal harmonic transformation if and only if

$$\bar{\Delta}\,\theta = Ric(\xi, \cdot\,) \tag{14}$$

the 1-form $\theta$ dual to $\xi$ with respect to $g$. In this case, the well-known second *Kato inequality* (see [11])

$$\|\xi\|\ \Delta\|\xi\| \geq -g(\bar{\Delta}\,\xi, \xi)$$

can be rewritten in the form

$$\|\xi\|\ \Delta\|\xi\| \geq -Ric(\xi, \xi). \tag{15}$$

Then the assumption $Ric \leq 0$ implies that $\|\xi\|\Delta\|\xi\| \geq 0$. By the Yau's theorem (see [6]), if $\|\xi\| \in L^p(M)$ at least for one $p > 1$, then $\|\xi\|$ = const. At the same time, this constant must be zero if $(M, g)$ has an infinite volume (see [17]). On the other hand, from (15) we obtain $Ric(\xi, \xi) = 0$ if the following conditions $\|\xi\| =$ constant and $Ric \leq 0$ are satisfied. In this case, using (14) we have

$$0 = 1/2\, \Delta g(\xi, \xi) = -g(\bar{\Delta}\,\theta, \theta) + \|\nabla\xi\|^2 = -Ric(\xi, \xi) + \|\nabla\xi\|^2 = \|\nabla\xi\|^2.$$

In accordance with the theory of harmonic mappings, we can define the energy density of the flow on $(M, g)$ generated by the infinitesimal harmonic system $\xi$ by equality (3), and the energy of the flow, by equality (4). Using these definitions, we can formulate the following theorem.

**Theorem 7.** *Let $(M, g)$ a complete noncompact Riemannian manifold with non-positive Ricci curvature and $\xi$ be a harmonic dynamical system on $(M, g)$. If $\|\xi\| \in L^p(M)$ at least for one $p > 1$ or, in particular, $\xi$ has finite energy on $(M, g)$, then $\xi$ is a parallel dynamical system and consequently generates an incompressible flow. Furthermore, if the volume of $(M, g)$ is infinite or $(M, g)$ is irreducible, then $\xi$ is identically equal to zero everywhere on $(M, g)$.*

Hamilton introduced the concept of Ricci solitons in mid 80 s. They are natural generalizations of Einstein manifolds. Suppose that $(M, g)$ is a complete Riemannian manifold such that the equation

$$-2Ric = 2\lambda\, g + L_V g$$

holds for some constant $\lambda$ and some complete vector field $V$ on $M$. In this case, we say $g$ is a *Ricci soliton* (see [30], pp. 37–38). The Ricci soliton is usually denoted as $(M, g, \xi, \lambda)$. In this case, $trace_g(L_\xi \nabla) = 0$ (see [22], p. 264). Then a vector field $\xi$ that makes a Riemannian metric $g$ a Ricci soliton metric is necessarily a harmonic dynamical system on a complete Riemannian manifold $(M, g)$. Therefore, the following corollary holds.

**Corollary 7.** *Let $(M, g, \xi, \lambda)$ be a Ricci soliton with a complete Riemannian metric $(M, g)$ and non-positive Ricci curvature. If $\|\xi\| \in L^p(M)$ at least for one $p > 1$, then $\xi$ is a parallel vector field. Furthermore, if the volume of $(M, g)$ is infinite or $(M, g)$ is irreducible, then $\xi$ is identically equal to zero everywhere on $(M, g)$.*

**Remark 8.** *This corollary generalizes our similar assertion from [22] (p. 265) which was proved for a compact Ricci soliton manifold.*

Let $(M, g, J)$ be an *almost Kählerian manifold* where $(M, g)$ is $2m$-dimensional Riemannian manifold and $J$ is a tensor field such that $J \in C^\infty(T^*M \otimes TM)$, $J^2 = -\,id$, $g(J, J) = g$ and $(\nabla_X J)Y + (\nabla_Y J)X = 0$ for the Levi–Civita connection $\nabla$ and smooth vector fields on $M$ (see, for example, [22], p. 263).

A *holomorphic vector field* $\xi$ on $(M, g, J)$ is defined by the condition $L_\xi J = 0$. In this case, Equation (13) holds (see also [22], p. 263). Therefore, a holomorphic vector field on an almost Kählerian manifold is an example of an infinitesimal harmonic transformation. Moreover, a complete holomorphic vector field $\xi$ is necessarily a harmonic dynamical system on a complete Riemannian manifold $(M, g)$. Hence the following corollary is true.

**Corollary 8.** *Let $(M, g, J)$ be a complete almost Kählerian manifold with non-positive Ricci curvature and $\xi$ be a holomorphic vector field on $(M, g, J)$. If $\|\xi\| \in L^p(M)$ at least for one $p > 1$, then $\xi$ is a parallel vector field. Furthermore, if the volume of $(M, g, J)$ is infinite or $(M, g, J)$ is irreducible, then $\xi$ is identically equal to zero everywhere on $(M, g, J)$.*

**Remark 9.** *This corollary is a new statement despite a large number of articles on this topic.*

## 7. Conclusions

The study of the geometry of infinitesimal transformations of Riemannian manifolds using the concepts of geometric dynamics and the methods of the modern version of the Bochner technique distinguishes our work from other similar articles. Therefore, our paper has the potential to become a good research article.

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## References

1. Kobayashi, S.; Nomizu, K. *Foundations of Differential Geometry*; Inters. Publ. John Wiley & Sons, Inc.: New York, NY, USA, 1963; Volume I.
2. Ginoux, J.M. *Differential Geometry Applied to Dynamical Systems*; World Scientific Publishing: Singapore, 2009.
3. Udrişte, C. *Geometric Dynamics*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2000.
4. Arnold, V.I.; Klesin, B.A. *Topological Methods in Hydrodynamics*; Springer: New York, NY, USA, 1998.
5. Aminova, A.V. Projective transformations of pseudo-Riemannian manifolds. *J. Math. Sci.* **2003**, *113*, 367–470. [CrossRef]
6. Wu, H.-H. The Bochner technique in differential geometry. *Math. Rep.* **1988**, *3*, 289–541. [CrossRef]
7. Mikeš, J.; Stepanov, S.E. What is the Bochner technique and where is it applied. *Lobachevskii J. Math.* **2022**, *43*, 709–719.
8. Stepanov, S.E.; Mikeš, J. Liouville-type theorems for some classes of Riemannian almost product manifolds and for special mappings of Riemannian manifolds. *Differ. Geom. Appl.* **2017**, *54 Pt A*, 111–121. [CrossRef]
9. Yau, S.-T. Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. *Indiana Univ. Math. J.* **1976**, *25*, 659–670. [CrossRef]
10. Petersen, P. *Riemannian Geometry*, 3rd ed.; Graduate Texts in Math., 171; Springer: Cham, Switzerland, 2016.
11. Berard, P.H. From vanishing theorems to estimating theorems: The Bochner technique revisited. *Bull. Am. Math. Soc. (N.S.)* **1988**, *19*, 371–406. [CrossRef]
12. Hilgert, J.; Neeb, K.-H. *Structure and Geometry of Lie Groups*; Springer: Berlin, Germany, 2012.
13. Markus, L. Parallel dynamical systems. *Topology* **1969**, *8*, 47–57. [CrossRef]
14. O'Neil, B. *Semi-Riemannian Geometry with Applications to Relativity*; Academic Press: London, UK, 1983.
15. Yano, K. *Integral Formulas in Riemannian Geometry*; Pure and Applied Math., No. 1; Marcel Dekker, Inc.: New York, NY, USA, 1970.
16. Caminha, A. The geometry of closed conformal Killing vector fields on Riemannian spaces. *Bull. Braz. Math. Soc.* **2011**, *41*, 277–300. [CrossRef]
17. Li, P. Function theory on complete Riemannian manifolds. *Astérisque* **1985**, *132*, 277–284.
18. Stepanov, S.E.; Mikeš, J. The generalized Landau-Raychaudhuri equation and its applications. *Int. J. Geom. Methods Mod. Phys.* **2015**, *12*, 1560026. [CrossRef]
19. Berestovskii, V.N.; Nikonorov, Y.G. Killing vector fields of constant length on Riemannian manifolds. *Sib. Math. J.* **2008**, *49*, 395–407. [CrossRef]
20. Li, P.; Schoen, R. $L^p$ and mean value properties of subharmonic functions on Riemannian manifolds. *Acta Math.* **1984**, *153*, 279–301. [CrossRef]
21. Greene, R.E.; Wu, H. Harmonic forms on noncompact Riemannian and Kähler manifolds. *Mich. Math. J.* **1981**, *28*, 63–81. [CrossRef]
22. Mikeš, J.; Stepanova, E.; Vanžurová, A.; Bácsó, S.; Berezovski, V.E.; Chepurna, O.; Chodorová, M.; Chudá, H.; Gavrilchenko, M.L.; Haddad, M.; et al. *Differential Geometry of Special Mappings*, 2nd ed.; Palacky University Press: Olomouc, Czech Republic, 2019.
23. Yoroza, S. Affine and projective vector fields on complete non-compact Riemannian manifolds. *Yokohama Math. J.* **1983**, *31*, 41–46.
24. Ma, L.; Cheng, L. Properties of complete non-compact Yamabe solitons. *Ann. Global. Ann. Geom.* **2011**, *40*, 379–387. [CrossRef]
25. Kobayashi, S. *Transformations Groups in Differential Geometry*; Springer: Heidelberg, Germany, 1995.
26. Yano, K.; Nagano, T. Einstein spaces admitting a one-parameter group of conformal transformations. *Ann. Math.* **1959**, *69*, 451–461. [CrossRef]
27. Bishop, R.L.; O'Neill, B. Manifolds of negative curvature. *Trans. Am. Math. Soc.* **1969**, *145*, 1–49. [CrossRef]

28. Tachibana, S.; Ishihara, S. On infinitesimal holomorphically projective transformations in Kählerian manifolds. *Tohoku Math. J.* **1960**, *12*, 77–101. [CrossRef]
29. Stepanov, S.E.; Shandra, I.G. Geometry of infinitesimal harmonic transformations. *Ann. Glob. Anal. Geom.* **2003**, *24*, 291–299. [CrossRef]
30. Morgan, J.; Tian, G. *Ricci Flow and Poincare Conjecture*; Clay Math. Monographs, 3; AMS: Providence, RI, USA; Clay Mathematics Institute: Cambridge, MA, USA, 2007.