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Common Fixed Point Results in Bicomplex Valued Metric Spaces with Application

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Abstract: The purpose of this paper is to establish common fixed points of six mappings in the context of bicomplex valued metric spaces. In this way, we generalize some previous well-known results from the literature. Moreover, we provide a non-trivial example to demonstrate the authenticity of established outcomes. As an application, we investigate the solution of an Urysohn integral equation by applying our results.

Keywords: common fixed point; generalized contractions; bicomplex valued metric space; Urysohn integral equation

MSC: 46S40; 47H10; 54H25

1. Introduction

The disclosure of complex numbers was established in the 17th century by Sir Carl Fredrich Gauss, but his work was not on the record. Then, in the year 1840, Augustin Louis Cauchy started analyzing complex numbers. Cauchy is known to be an effective founder of complex analysis. The theory of complex numbers has its source in the solution of $ax^2 + bx + c = 0$, which was not worthwhile for $b^2 - 4ac < 0$, in the set of real numbers. Based on this background, Euler was the first mathematician to present the symbol i, for $\sqrt{-1}$ with the property $i^2 = -1$.

The starting point for bicomplex numbers was provided by Segre [1], supporting a commutative substitute for the skew field of quaternions. These numbers generalize and extend the complex numbers more firmly and specifically to quaternions. For an excellent investigation of the study of bicomplex numbers, we refer the reader to [2]. In 2011, Azam et al. [3] proposed the theory of complex valued metric space (CVMS) as a generalization and extension of cone metric space and classical metric space. In 2017, Choi et al. [4] linked the concepts of bicomplex numbers and complex valued metric spaces and introduced the notion of bicomplex valued metric spaces (bi-CVMS). They established common fixed point results for weakly compatible mappings. Subsequently, Jebril et al. [5], utilized this notion of newly introduced space and proved common fixed point results under rational contractions for a pair of mappings in the background of bi-CVMS. Later on, Beg et al. [6] strengthened the notion of bi-CVMS and obtained generalized fixed point theorems. Recently, Gnanaprakasam et al. [7] established some common fixed point results for rational contraction in bi-CVMSs and solved a system of linear equations as an application of their main result. For more characteristics in the direction of CVMS and bi-CVMS, we refer the researchers to [8–27].

In this article, we establish common fixed points of six self-mappings in the context of bicomplex valued metric spaces. Some previous well-known results of literature are generalized in this way. Moreover, we provide a non-trivial example to show the authenticity of

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established outcomes. As an application, we investigate the solution of an Urysohn integral equation by applying our results.

2. Preliminaries

We describe $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$ as the set of real, complex and bicomplex numbers, correspondingly. Segre [1] set out the notion of bicomplex numbers in this manner.

$$\hbar = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$, and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We represent the set of bicomplex numbers by $\mathbb{C}_2$ and it is defined as

$$\mathbb{C}_2 = \{\hbar : \hbar = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 : a_1, a_2, a_3, a_4 \in \mathbb{C}_0\},$$

that is,

$$\mathbb{C}_2 = \{\hbar : \hbar = z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$. If $\hbar = z_1 + i_2 z_2$ and $\wp = \omega_1 + i_2 \omega_2$ are any two bicomplex numbers, then the sum is

$$\hbar \pm \wp = (z_1 + i_2 z_2) \pm (\omega_1 + i_2 \omega_2) = (z_1 \pm \omega_1) + i_2 (z_2 \pm \omega_2),$$

and the product is

$$\hbar \cdot \wp = (z_1 + i_2 z_2) \cdot (\omega_1 + i_2 \omega_2) = (z_1 \omega_1 - z_2 \omega_2) + i_2 (z_1 \omega_2 + z_2 \omega_1).$$

There are four idempotent elements in $\mathbb{C}_2$, which are $0, 1, e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$, of which e_1 and e_2 are non-trivial, such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Every bicomplex number $z_1 + i_2 z_2$ can uniquely be given as the combination of $e_1$and e_2, namely

$$\hbar = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2.$$

This characterization of $\hbar$ is studied as the idempotent characterization of $\mathbb{C}_2$ and the complex coefficients $\hbar_1 = (z_1 - i_1 z_2)$ and $\hbar_2 = (z_1 + i_1 z_2)$ are known as idempotent components of $\hbar$.

A member $\hbar = z_1 + i_2 z_2 \in \mathbb{C}_2$ is called invertible if there is one more member $\wp \in \mathbb{C}_2$, such that $\hbar \wp = 1$ and $\wp$ is called the multiplicative inverse of $\hbar$. Accordingly $\hbar$ is called the multiplicative inverse of $\wp$. A member which has an inverse in $\mathbb{C}_2$ is called a non-singular element of $\mathbb{C}_2$ and a member which does not have an inverse in $\mathbb{C}_2$ is called a singular element of $\mathbb{C}_2$.

A member $\hbar = z_1 + i_2 z_2 \in \mathbb{C}_2$ is non-singular if and only if $\left|z_1^2 + z_2^2\right| \neq 0$ and singular if and only if $\left|z_1^2 + z_2^2\right| = 0$. The inverse of $\hbar$ is defined as

$$\hbar^{-1} = \wp = \frac{z_1 - i_2 z_2}{z_1^2 + z_2^2}.$$

0 in $\mathbb{C}_0$ and $0 = 0 + i0$ in $\mathbb{C}_1$ are the only members which do not have multiplicative inverses. We represent the set of singular elements of $\mathbb{C}_0$ and $\mathbb{C}_1$ by $\aleph_0$ and $\aleph_1$, respectively. However, in $\mathbb{C}_2$, more than one member does not have a multiplicative inverse. We represent the set of a singular member of $\mathbb{C}_2$ by $\aleph_2$. Evidently, $\aleph_0 = \aleph_1 \subset \aleph_2$.

A bicomplex number $\hbar = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_{2\times 2}$$

is degenerated. In that case, $\hbar^{-1}$ exists, and it is also degenerated.

The norm $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined by

$$\begin{aligned} \|\hbar\| &= \|z_1 + i_2 z_2\| = \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} \\ &= \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}}, \end{aligned}$$

where $\hbar = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with reference to the defined norm is a normed linear space; also, $\mathbb{C}_2$ is complete, hence $\mathbb{C}_2$ is a Banach space. If $\hbar, \wp \in \mathbb{C}_2$, then

$$\|\hbar\wp\| \leq \sqrt{2}\|\hbar\|\|\wp\|,$$

holds instead of

$$\|\hbar\wp\| \leq \|\hbar\|\|\wp\|,$$

therefore $\mathbb{C}_2$ is not a Banach algebra. The partial-order relation $\preceq_{i_2}$ on $\mathbb{C}_2$ is defined as:

$$\hbar \preceq_{i_2} \wp \Leftrightarrow \text{Re}(z_1) \preceq \text{Re}(\omega_1) \text{ and } \text{Im}(z_2) \preceq \text{Im}(\omega_2),$$

where $\hbar = z_1 + i_2 z_2$, $\wp = \omega_1 + i_2 \omega_2 \in \mathbb{C}_2$.

It follows that

$$\hbar \preceq_{i_2} \wp,$$

if one of these assertions is satisfied:

$$\begin{aligned} \text{(i) } z_1 &= \omega_1,\ z_2 \prec \omega_2, \\ \text{(ii) } z_1 &\prec \omega_1,\ z_2 = \omega_2, \\ \text{(iii) } z_1 &\prec \omega_1,\ z_2 \prec \omega_2, \\ \text{(iv) } z_1 &= \omega_1,\ z_2 = \omega_2. \end{aligned}$$

Specifically, we can write $\hbar \precnsim_{i_2} \wp$ if $\hbar \prec_{i_2} \wp$ and $\hbar \neq \wp$; that is, (i), (ii) or (iii) is satisfied, and we will write $\hbar = \wp$ if only (iv) is satisfied. For $\hbar, \wp \in \mathbb{C}_2$, we have

(i) $\hbar \preceq_{i_2} \wp \Longrightarrow \|\hbar\| \leq \|\wp\|$,
(ii) $\|\hbar + \wp\| \leq \|\hbar\| + \|\wp\|$,
(iii) $\|a\hbar\| \leq a\|\wp\|$, where a is a non-negative real number,
(iv) $\|\hbar\wp\| \leq \sqrt{2}\|\hbar\|\|\wp\|$,
(v) $\left\|\hbar^{-1}\right\| = \|\hbar\|^{-1}$,
(vi) $\left\|\frac{\hbar}{\wp}\right\| = \frac{\|\hbar\|}{\|\wp\|}$, if $\wp$ is a degenerated bicomplex number.

Choi et al. [4] defined the bicomplex valued metric space (bi-CVMS) as follows:

Definition 1 ([4]). *Let $\mathcal{Z} \neq \emptyset$ and $\sigma : \mathcal{Z} \times \mathcal{Z} \to \mathbb{C}_2$ be a mapping satisfying*

(i) $0 \preceq_{i_2} \sigma(\hbar, \wp)$ *and* $\sigma(\hbar, \wp) = 0$ *if and only if* $\hbar = \wp$,
(ii) $\sigma(\hbar, \wp) = \sigma(\wp, \hbar)$,
(iii) $\sigma(\hbar, \wp) \preceq_{i_2} \sigma(\hbar, \nu) + \sigma(\nu, \wp)$,

for all $\hbar, \wp, \nu \in \mathcal{Z}$, then $(\mathcal{Z}, \sigma)$ is a bi-CVMS.

Example 1 ([6]). *Let $\mathcal{Z} = \mathbb{C}_2$ and $\hbar, \wp \in \mathcal{Z}$. Define $\sigma : \mathcal{Z} \times \mathcal{Z} \to \mathbb{C}_2$ by*

$$\sigma(\hbar, \wp) = |z_1 - \omega_1| + i_2|z_2 - \omega_2|,$$

where $\hbar = z_1 + i_2 z_2$, $\wp = \omega_1 + i_2 \omega_2 \in \mathbb{C}_2$. Then, $(\mathcal{Z}, \sigma)$ is a bi-CVMS.

Definition 2 ([4]). *Let $(\mathcal{Z}, \sigma)$ be a bi-CVMS and let $\aleph, \mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ be self-mappings. The mappings $\aleph$ and $\mathcal{Q}$ are said to be commuting if $\aleph\mathcal{Q}\hbar = \mathcal{Q}\aleph\hbar$, for all $\hbar \in \mathcal{Z}$. The mappings $\aleph$ and $\mathcal{Q}$ are said to be compatible if*

$$\lim_{\kappa\to\infty} \sigma(\aleph\mathcal{Q}\hbar_\kappa, \mathcal{Q}\aleph\hbar_\kappa) = 0,$$

whenever $\{\hbar_\kappa\}$ is a sequence in $\mathcal{Z}$, such that $\lim_{\kappa\to\infty} \aleph\hbar_\kappa = \mathcal{Q}\hbar_\kappa = t$ for some $t \in \mathcal{Z}$. The mappings $\aleph$ and $\mathcal{Q}$ are said to be weakly compatible if $\aleph\mathcal{Q}\hbar = \mathcal{Q}\aleph\hbar$, whenever $\mathcal{Q}\hbar = \aleph\hbar$.

Lemma 1 ([4]). *Let $(\mathcal{Z}, \sigma)$ be a bi-CVMS and $\mathfrak{L}, \mathcal{T} : \mathcal{Z} \to \mathcal{Z}$. If the pair $(\mathfrak{L}, \mathcal{T})$ on $(\mathcal{Z}, \sigma)$ is said to be compatible, then the pair $(\mathfrak{L}, \mathcal{T})$ is weakly compatible, but the converse is not true in general.*

Lemma 2 ([6]). *Let $(\mathcal{Z}, \sigma)$ be a bi-CVMS and let $\{\hbar_\kappa\} \subseteq \mathcal{Z}$. Then, $\{\hbar_\kappa\}$ converges to $\hbar$ if and only if $\|\sigma(\hbar_\kappa, \hbar)\| \to 0$ as $\kappa \to \infty$.*

Lemma 3 ([6]). *Let $(\mathcal{Z}, \sigma)$ be a bi-CVMS and let $\{\hbar_\kappa\} \subseteq \mathcal{Z}$. Then, $\{\hbar_\kappa\}$ is a Cauchy sequence if and only if $\|\sigma(\hbar_\kappa, \hbar_{\kappa+m})\| \to 0$ as $\kappa \to \infty$, where $m \in \mathbb{N}$.*

3. Main Results

We state our main result in this way.

Theorem 1. *Let $(\mathcal{Z}, \sigma)$ be a complete bi-CVMS and let $\Im, \Re, \mathfrak{L}, \mathcal{T}, \aleph, \mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ be self-mappings. If there exists some $\theta \in [0, 1)$ such that the following conditions hold:*

(i)

$$\sigma(\aleph\hbar, \mathcal{Q}\wp) \preceq_{i_2} \theta\mathfrak{Q}(\hbar, \wp) \tag{1}$$

for all $\hbar, \wp \in \mathcal{Z}$, where

$$\mathfrak{Q}(\hbar, \wp) \in \left\{ \begin{array}{c} \sigma(\Im\Re\hbar, \mathfrak{L}\mathcal{T}\wp), \sigma(\Im\Re\hbar, \aleph\hbar), \sigma(\mathfrak{L}\mathcal{T}\wp, \mathcal{Q}\wp), \\ \frac{1}{2}(\sigma(\mathfrak{L}\mathcal{T}\wp, \aleph\hbar) + \sigma(\Im\Re\hbar, \mathcal{Q}\wp)) \end{array} \right\};$$

(ii) $\aleph(\mathcal{Z}) \subset \mathfrak{L}\mathcal{T}(\mathcal{Z}), \mathcal{Q}(\mathcal{Z}) \subset \Im\Re(\mathcal{Z})$;
(iii) $\Im\Re = \Re\Im$, $\mathfrak{L}\mathcal{T} = \mathcal{T}\mathfrak{L}, \aleph\Re = \Re\aleph$, $\mathcal{Q}\mathcal{T} = \mathcal{T}\mathcal{Q}$;
(iv) the pair $(\aleph, \Im\Re)$ is compatible and the pair $(\mathcal{Q}, \mathfrak{L}\mathcal{T})$ is weakly compatible;
(v) either the mapping $\Im\Re$ or the mapping $\aleph$ is continuous.

Then, there exists a unique point $\hbar' \in \mathcal{Z}$, such that $\Im\hbar' = \Re\hbar' = \mathfrak{L}\hbar' = \mathcal{T}\hbar' = \aleph\hbar' = \mathcal{Q}\hbar' = \hbar'$.

Proof. Let $\hbar_0$ be an arbitrary point in $\mathcal{Z}$. According to hypothesis (ii), there exist $\hbar_1, \hbar_2 \in \mathcal{Z}$, such that $\aleph\hbar_0 = \mathfrak{L}\mathcal{T}\hbar_1 = \wp_0$ and $\mathcal{Q}\hbar_1 = \Im\Re\hbar_2 = \wp_1$. We can generate two sequences $\{\hbar_\kappa\}$ and $\{\wp_\kappa\}$ in $\mathcal{Z}$ successively in this way

$$\wp_{2\kappa} = \mathfrak{L}\mathcal{T}\hbar_{2\kappa+1} = \aleph\hbar_{2\kappa} \text{ and } \wp_{2\kappa+1} = \Im\Re\hbar_{2\kappa+2} = \mathcal{Q}\hbar_{2\kappa+1}, \tag{2}$$

for $\kappa = 0, 1, 2, \cdots$. Now, according to (i), we have

$$\sigma(\wp_{2\kappa}, \wp_{2\kappa+1}) = \sigma(\aleph\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\mathfrak{Q}(\hbar_{2\kappa}, \hbar_{2\kappa+1}), \tag{3}$$

where

$$\begin{aligned} \mathfrak{Q}(\hbar_{2\kappa}, \hbar_{2\kappa+1}) &\in \left\{ \begin{array}{c} \sigma(\Im\Re\hbar_{2\kappa}, \mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}), \sigma(\Im\Re\hbar_{2\kappa}, \aleph\hbar_{2\kappa}), \sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \mathcal{Q}\hbar_{2\kappa+1}), \\ \frac{1}{2}(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \aleph\hbar_{2\kappa}) + \sigma(\Im\Re\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1})), \end{array} \right\} \\ &= \left\{ \begin{array}{c} \sigma(\wp_{2\kappa-1}, \wp_{2\kappa}), \sigma(\wp_{2\kappa-1}, \wp_{2\kappa}), \sigma(\wp_{2\kappa}, \wp_{2\kappa+1}), \\ \frac{1}{2}((\sigma(\wp_{2\kappa}, \wp_{2\kappa}) + \sigma(\wp_{2\kappa-1}, \wp_{2\kappa+1})), \end{array} \right\} \\ &= \left\{ \sigma(\wp_{2\kappa-1}, \wp_{2\kappa}), \sigma(\wp_{2\kappa}, \wp_{2\kappa+1}), \frac{1}{2}\sigma(\wp_{2\kappa-1}, \wp_{2\kappa+1}) \right\}. \end{aligned} \tag{4}$$

Using the inequality (3) and expression (4), we discuss the following three cases as follows:

If $\mathfrak{Q}(\hbar_{2\kappa}, \hbar_{2\kappa+1}) = \sigma(\wp_{2\kappa-1}, \wp_{2\kappa})$, then we have

$$\sigma(\wp_{2\kappa}, \wp_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\wp_{2\kappa-1}, \wp_{2\kappa}),$$

which implies that

$$\|\sigma(\wp_{2\kappa}, \wp_{2\kappa+1})\| \leq \theta\|\sigma(\wp_{2\kappa-1}, \wp_{2\kappa})\|. \tag{5}$$

If $\mathfrak{Q}(\hbar_{2\kappa}, \hbar_{2\kappa+1}) = \sigma(\wp_{2\kappa}, \wp_{2\kappa+1})$, then we have

$$\sigma(\wp_{2\kappa}, \wp_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\wp_{2\kappa}, \wp_{2\kappa+1}),$$

which implies that

$$\|\sigma(\wp_{2\kappa}, \wp_{2\kappa+1})\| \leq \theta\|\sigma(\wp_{2\kappa}, \wp_{2\kappa+1})\|$$

which is a contradiction because $\theta < 1$.

If $\mathfrak{Q}(\hbar_{2\kappa}, \hbar_{2\kappa+1}) = \frac{1}{2}\sigma(\wp_{2\kappa-1}, \wp_{2\kappa+1})$, then we have

$$\sigma(\wp_{2\kappa}, \wp_{2\kappa+1}) \preceq_{i_2} \frac{\theta}{2}\sigma(\wp_{2\kappa-1}, \wp_{2\kappa+1}) \preceq_{i_2} \frac{\theta}{2}\sigma(\wp_{2\kappa-1}, \wp_{2\kappa}) + \frac{\theta}{2}\sigma(\wp_{2\kappa}, \wp_{2\kappa+1}),$$

that is,

$$\sigma(\wp_{2\kappa}, \wp_{2\kappa+1}) \preceq_{i_2} \frac{\theta}{2}\sigma(\wp_{2\kappa-1}, \wp_{2\kappa}) + \frac{\theta}{2}\sigma(\wp_{2\kappa}, \wp_{2\kappa+1}).$$

Taking the norm on both sides of above inequality, we have

$$\begin{aligned} \|\sigma(\wp_{2\kappa}, \wp_{2\kappa+1})\| &\leq \left\| \frac{\theta}{2}\sigma(\wp_{2\kappa-1}, \wp_{2\kappa}) + \frac{\theta}{2}\sigma(\wp_{2\kappa}, \wp_{2\kappa+1}) \right\| \\ &\leq \left|\frac{\theta}{2}\right| \|\sigma(\wp_{2\kappa-1}, \wp_{2\kappa})\| + \left|\frac{\theta}{2}\right| \|\sigma(\wp_{2\kappa}, \wp_{2\kappa+1})\| \\ &= \frac{\theta}{2}\|\sigma(\wp_{2\kappa-1}, \wp_{2\kappa})\| + \frac{\theta}{2}\|\sigma(\wp_{2\kappa}, \wp_{2\kappa+1})\|. \end{aligned}$$

Now, using the fact that $\theta < 1$, we have

$$\begin{aligned} \|\sigma(\wp_{2\kappa}, \wp_{2\kappa+1})\| &\leq \frac{\theta}{2}\|\sigma(\wp_{2\kappa-1}, \wp_{2\kappa})\| + \frac{\theta}{2}\|\sigma(\wp_{2\kappa}, \wp_{2\kappa+1})\| \\ &\leq \frac{\theta}{2}\|\sigma(\wp_{2\kappa-1}, \wp_{2\kappa})\| + \frac{1}{2}\|\sigma(\wp_{2\kappa}, \wp_{2\kappa+1})\|, \end{aligned}$$

which implies that

$$\|\sigma(\wp_{2\kappa}, \wp_{2\kappa+1})\| \leq \theta\|\sigma(\wp_{2\kappa-1}, \wp_{2\kappa})\|.$$

Thus, in all cases, we have

$$\|\sigma(\wp_{2\kappa}, \wp_{2\kappa+1})\| \leq \theta\|\sigma(\wp_{2\kappa-1}, \wp_{2\kappa})\|. \tag{6}$$

Similarly, using (*i*), we have

$$\sigma(\wp_{2\kappa+1}, \wp_{2\kappa+2}) = \sigma(\mathcal{Q}\hbar_{2\kappa+1}, \aleph\hbar_{2\kappa+2}) = \sigma(\aleph\hbar_{2\kappa+2}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\mathfrak{Q}(\hbar_{2\kappa+2}, \hbar_{2\kappa+1}), \tag{7}$$

where

$$\begin{aligned}\mathfrak{Q}(\hbar_{2\kappa+2},\hbar_{2\kappa+1}) &\in \left\{\begin{array}{c}\sigma(\Im\Re\hbar_{2\kappa+2},\mathfrak{LT}\hbar_{2\kappa+1}),\sigma(\Im\Re\hbar_{2\kappa+2},\aleph\hbar_{2\kappa+2}),\sigma(\mathfrak{LT}\hbar_{2\kappa+1},\mathcal{Q}\hbar_{2\kappa+1}),\\ \frac{1}{2}(\sigma(\mathfrak{LT}\hbar_{2\kappa+1},\aleph\hbar_{2\kappa+2})+\sigma(\Im\Re\hbar_{2\kappa+2},\mathcal{Q}\hbar_{2\kappa+1})),\end{array}\right\}\\ &= \left\{\begin{array}{c}\sigma(\wp_{2\kappa+1},\wp_{2\kappa}),\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2}),\sigma(\wp_{2\kappa},\wp_{2\kappa+1}),\\ \frac{1}{2}(\sigma(\wp_{2\kappa},\wp_{2\kappa+2})+\sigma(\wp_{2\kappa+1},\wp_{2\kappa+1})),\end{array}\right\}\\ &= \left\{\sigma(\wp_{2\kappa},\wp_{2\kappa+1}),\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2}),\frac{1}{2}\sigma(\wp_{2\kappa},\wp_{2\kappa+2})\right\}.\end{aligned} \tag{8}$$

Using the inequality (7) and expression (9), we discuss the following three cases as follows:

If $\mathfrak{Q}(\hbar_{2\kappa+2},\hbar_{2\kappa+1})=\sigma(\wp_{2\kappa},\wp_{2\kappa+1})$, then we have

$$\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2}) \preceq_{i_2} \theta\sigma(\wp_{2\kappa},\wp_{2\kappa+1}),$$

which implies that

$$\|\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})\| \leq \theta\|\sigma(\wp_{2\kappa},\wp_{2\kappa+1})\|. \tag{9}$$

If $\mathfrak{Q}(\hbar_{2\kappa+2},\hbar_{2\kappa+1})=\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})$, then we have

$$\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2}) \preceq_{i_2} \theta\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2}),$$

which implies that

$$\|\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})\| \leq \theta\|\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})\|,$$

which is a contradiction, because $\theta<1$.

If $\mathfrak{Q}(\hbar_{2\kappa+2},\hbar_{2\kappa+1})=\frac{1}{2}\sigma(\wp_{2\kappa},\wp_{2\kappa+2})$, then we have

$$\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2}) \preceq_{i_2} \frac{\theta}{2}\sigma(\wp_{2\kappa},\wp_{2\kappa+2}) \preceq_{i_2} \frac{\theta}{2}\sigma(\wp_{2\kappa},\wp_{2\kappa+1})+\frac{\theta}{2}\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2});$$

that is,

$$\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2}) \preceq_{i_2} \frac{\theta}{2}\sigma(\wp_{2\kappa},\wp_{2\kappa+1})+\frac{\theta}{2}\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2}).$$

Taking the norm on both sides of above inequality, we have

$$\begin{aligned}\|\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})\| &\leq \left\|\frac{\theta}{2}\sigma(\wp_{2\kappa},\wp_{2\kappa+1})+\frac{\theta}{2}\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})\right\|\\ &\leq \left|\frac{\theta}{2}\right|\|\sigma(\wp_{2\kappa},\wp_{2\kappa+1})\|+\left|\frac{\theta}{2}\right|\|\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})\|\\ &= \frac{\theta}{2}\|\sigma(\wp_{2\kappa},\wp_{2\kappa+1})\|+\frac{\theta}{2}\|\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})\|.\end{aligned}$$

Now, using the fact that $\theta<1$, we have

$$\begin{aligned}\|\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})\| &\leq \frac{\theta}{2}\|\sigma(\wp_{2\kappa},\wp_{2\kappa+1})\|+\frac{\theta}{2}\|\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})\|\\ &\leq \frac{\theta}{2}\|\sigma(\wp_{2\kappa},\wp_{2\kappa+1})\|+\frac{1}{2}\|\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})\|,\end{aligned}$$

which implies that

$$\|\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})\| \leq \theta\|\sigma(\wp_{2\kappa},\wp_{2\kappa+1})\|.$$

Thus, in all cases, we have

$$\|\sigma(\wp_{2\kappa+1},\wp_{2\kappa+2})\| \leq \theta\|\sigma(\wp_{2\kappa},\wp_{2\kappa+1})\|. \tag{10}$$

Thus, using (6) and (10), we have

$$\|\sigma(\wp_\kappa, \wp_{\kappa+1})\| \leq \theta\|\sigma(\wp_{\kappa-1}, \wp_\kappa)\|. \tag{11}$$

for all $\kappa \in \mathbb{N}$. It yields that

$$\|\sigma(\wp_\kappa, \wp_{\kappa+1})\|| \leq \theta\|\sigma(\wp_{\kappa-1}, \wp_\kappa)\| \leq \cdots \leq \theta^\kappa\|\sigma(\wp_0, \wp_1)\|. \tag{12}$$

Using the inequality (12) and the triangle inequality, for all $m > \kappa$, we have

$$\begin{aligned}\|\sigma(\wp_\kappa, \wp_m)\| &\leq \|\sigma(\wp_\kappa, \wp_{\kappa+1})\| + \|\sigma(\wp_{\kappa+1}, \wp_{\kappa+2})\| + \cdots + \|\sigma(\wp_{m-1}, \wp_m)\| \\ &\leq [\theta^\kappa + \theta^{\kappa+1} + \cdots + \theta^{m-1}]\|\sigma(\wp_0, \wp_1)\| \\ &\leq [\frac{\theta^\kappa}{1-\theta}]\|\sigma(\wp_0, \wp_1)\| \to 0 \text{ as } \kappa \to \infty.\end{aligned}$$

It follows that $\{\wp_\kappa\}$ is a Cauchy sequence in bi-CVMS $(\mathcal{Z}, \sigma)$. As $(\mathcal{Z}, \sigma)$ is complete, there exists some $\hbar' \in \mathcal{Z}$, such that $\wp_\kappa \to \hbar'$ as $\kappa \to \infty$. For its sub sequences, we also have $\mathcal{Q}\hbar_{2\kappa+1} \to \hbar'$, $\mathfrak{LT}\hbar_{2\kappa+1} \to \hbar'$, $\aleph\hbar_{2\kappa} \to \hbar'$ and $\Im\Re\hbar_{2\kappa} \to \hbar'$. Now, according to hypothesis (v), we will have the following two cases:

Case 1. *If $\Im\Re$ is continuous.*

Then, $\Im\Re\Im\Re\hbar_{2\kappa} \to \Im\Re\hbar'$ and $\Im\Re\aleph\hbar_{2\kappa} \to \Im\Re\hbar'$, as $\kappa \to \infty$. Additionally, since the pair $(\aleph, \Im\Re)$ is compatible, it follows that $\aleph\Im\Re\hbar_{2\kappa} \to \Im\Re\hbar'$. Because, using triangle inequality, we have

$$\sigma(\aleph\Im\Re\hbar_{2\kappa}, \Im\Re\hbar') \preceq_{i_2} \sigma(\aleph\Im\Re\hbar_{2\kappa}, \Im\Re\aleph\hbar_{2\kappa}) + \sigma(\Im\Re\aleph\hbar_{2\kappa}, \Im\Re\hbar'),$$

which implies

$$\left\|\sigma(\aleph\Im\Re\hbar_{2\kappa}, \Im\Re\hbar')\right\| \leq \|\sigma(\aleph\Im\Re\hbar_{2\kappa}, \Im\Re\aleph\hbar_{2\kappa})\| + \left\|\sigma(\Im\Re\aleph\hbar_{2\kappa}, \Im\Re\hbar')\right\| \to 0$$

as $\kappa \to \infty$.

(a) First, we prove that $\Im\Re\hbar' = \hbar'$. We assume, on the contrary, that $\Im\Re\hbar' \neq \hbar'$. Then, $0 \prec_{i_2} \sigma(\Im\Re\hbar', \hbar')$. Now, by using triangle inequality two times, we have

$$\sigma(\Im\Re\hbar', \hbar') \preceq_{i_2} \sigma(\Im\Re\hbar', \aleph\Im\Re\hbar_{2\kappa}) + \sigma(\aleph\Im\Re\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) + \sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar'). \tag{13}$$

Now, using hypothesis (i), we have

$$\sigma(\aleph\Im\Re\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\mathfrak{Q}(\Im\Re\hbar_{2\kappa}, \hbar_{2\kappa+1}), \tag{14}$$

where

$$\mathfrak{Q}(\Im\Re\hbar_{2\kappa}, \hbar_{2\kappa+1}) \in \left\{ \begin{array}{c} \sigma(\Im\Re\Im\Re\hbar_{2\kappa}, \mathfrak{LT}\hbar_{2\kappa+1}), \sigma(\Im\Re\Im\Re\hbar_{2\kappa}, \aleph\Im\Re\hbar_{2\kappa}), \\ \sigma(\mathfrak{LT}\hbar_{2\kappa+1}, \mathcal{Q}\hbar_{2\kappa+1}), \\ \frac{1}{2}(\sigma(\mathfrak{LT}\hbar_{2\kappa+1}, \aleph\Im\Re\hbar_{2\kappa}) + \sigma(\Im\Re\Im\Re\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1})) \end{array} \right\}.$$

Now, we have the following four cases:
If $\mathfrak{Q}(\Im\Re\hbar_{2\kappa}, \hbar_{2\kappa+1}) = \sigma(\Im\Re\Im\Re\hbar_{2\kappa}, \mathfrak{LT}\hbar_{2\kappa+1})$, then by (14), we have

$$\sigma(\aleph\Im\Re\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\Im\Re\Im\Re\hbar_{2\kappa}, \mathfrak{LT}\hbar_{2\kappa+1}).$$

Using the triangle inequality, we have

$$\sigma(\aleph\Im\Re\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\Im\Re\Im\Re\hbar_{2\kappa}, \Im\Re\hbar') + \theta\sigma(\Im\Re\hbar', \hbar') + \theta\sigma(\hbar', \mathfrak{LT}\hbar_{2\kappa+1}). \tag{15}$$

Now, using (15) in (13), we have

$$\begin{aligned}\left\|\sigma(\Im\Re\hbar^{/},\hbar^{/})\right\| &\leq \frac{1}{1-\theta}\left\|\sigma(\Im\Re\hbar^{/},\aleph\Im\Re\hbar_{2\kappa})\right\|+\frac{\theta}{1-\theta}\left\|\sigma(\Im\Re\Im\Re\hbar_{2\kappa},\Im\Re\hbar^{/})\right\|\\&\quad+\frac{\theta}{1-\theta}\left\|\sigma(\hbar^{/},\mathfrak{LT}\hbar_{2\kappa+1})\right\|+\frac{1}{1-\theta}\left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1},\hbar^{/})\right\|.\end{aligned}$$

Taking the limit as $\kappa\to\infty$, we get

$$\left\|\sigma(\Im\Re\hbar^{/},\hbar^{/})\right\|\leq 0,$$

that is, $\left\|\sigma(\Im\Re\hbar^{/},\hbar^{/})\right\|=0$, a contradiction. Thus, $\Im\Re\hbar^{/}=\hbar^{/}$.

If $\mathfrak{Q}(\Im\Re\hbar_{2\kappa},\hbar_{2\kappa+1})=\sigma(\Im\Re\Im\Re\hbar_{2\kappa},\aleph\Im\Re\hbar_{2\kappa})$, then, according to (14), we have

$$\sigma(\aleph\Im\Re\hbar_{2\kappa},\mathcal{Q}\hbar_{2\kappa+1})\preceq_{i_2}\theta\sigma(\Im\Re\Im\Re\hbar_{2\kappa},\aleph\Im\Re\hbar_{2\kappa}).$$

Using the triangle inequality, we have

$$\sigma(\aleph\Im\Re\hbar_{2\kappa},\mathcal{Q}\hbar_{2\kappa+1})\preceq_{i_2}\theta\sigma(\Im\Re\Im\Re\hbar_{2\kappa},\Im\Re\hbar^{/})+\theta\sigma(\Im\Re\hbar^{/},\aleph\Im\Re\hbar_{2\kappa}).\tag{16}$$

Now, using (16) in (13), we have

$$\begin{aligned}\left\|\sigma(\Im\Re\hbar^{/},\hbar^{/})\right\| &\leq (1+\theta)\left\|\sigma(\Im\Re\hbar^{/},\aleph\Im\Re\hbar_{2\kappa})\right\|+\theta\left\|\sigma(\Im\Re\Im\Re\hbar_{2\kappa},\Im\Re\hbar^{/})\right\|\\&\quad+\left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1},\hbar^{/})\right\|.\end{aligned}$$

Now, taking the limit as $\kappa\to\infty$, we get

$$\left\|\sigma(\Im\Re\hbar^{/},\hbar^{/})\right\|\leq 0,$$

that is, $\left\|\sigma(\Im\Re\hbar^{/},\hbar^{/})\right\|=0$, a contradiction. Thus, $\Im\Re\hbar^{/}=\hbar^{/}$.

If $\mathfrak{Q}(\Im\Re\hbar_{2\kappa},\hbar_{2\kappa+1})=\sigma(\mathfrak{LT}\hbar_{2\kappa+1},\mathcal{Q}\hbar_{2\kappa+1})$, then according to (14), we have

$$\sigma(\aleph\Im\Re\hbar_{2\kappa},\mathcal{Q}\hbar_{2\kappa+1})\preceq_{i_2}\theta\sigma(\mathfrak{LT}\hbar_{2\kappa+1},\mathcal{Q}\hbar_{2\kappa+1}).$$

Using the triangle inequality, we have

$$\sigma(\aleph\Im\Re\hbar_{2\kappa},\mathcal{Q}\hbar_{2\kappa+1})\preceq_{i_2}\theta\sigma(\mathfrak{LT}\hbar_{2\kappa+1},\hbar^{/})+\theta\sigma(\hbar^{/},\mathcal{Q}\hbar_{2\kappa+1}).\tag{17}$$

Now, using (17) in (13), we have

$$\begin{aligned}\left\|\sigma(\Im\Re\hbar^{/},\hbar^{/})\right\| &\leq \left\|\sigma(\Im\Re\hbar^{/},\aleph\Im\Re\hbar_{2\kappa})\right\|+(1+\theta)\left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1},\hbar^{/})\right\|\\&\quad+\theta\left\|\sigma(\mathfrak{LT}\hbar_{2\kappa+1},\hbar^{/})\right\|.\end{aligned}$$

Letting $\kappa\to\infty$ in the above inequality and using the fact that $(\aleph,\Im\Re)$ is compatible, so we have

$$\left\|\sigma(\Im\Re\hbar^{/},\hbar^{/})\right\|\leq 0,$$

that is, $\left\|\sigma(\Im\Re\hbar^{/},\hbar^{/})\right\|=0$, a contradiction. Thus, $\Im\Re\hbar^{/}=\hbar^{/}$.

If $\mathfrak{Q}(\Im\Re\hbar_{2\kappa},\hbar_{2\kappa+1})=\frac{1}{2}(\sigma(\mathfrak{LT}\hbar_{2\kappa+1},\aleph\Im\Re\hbar_{2\kappa})+\sigma(\Im\Re\Im\Re\hbar_{2\kappa},\mathcal{Q}\hbar_{2\kappa+1}))$, then according to (14), we have

$$\sigma(\aleph\Im\Re\hbar_{2\kappa},\mathcal{Q}\hbar_{2\kappa+1})\preceq_{i_2}\frac{\theta}{2}(\sigma(\mathfrak{LT}\hbar_{2\kappa+1},\aleph\Im\Re\hbar_{2\kappa})+\sigma(\Im\Re\Im\Re\hbar_{2\kappa},\mathcal{Q}\hbar_{2\kappa+1})).$$

Using the triangle inequality, we have

$$\begin{aligned}\sigma(\aleph\Im\Re\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) &\preceq_{i_2} \frac{\theta}{2}\Big(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar^{/}) + \sigma(\hbar^{/}, \Im\Re\hbar^{/}) + \sigma(\Im\Re\hbar^{/}, \aleph\Im\Re\hbar_{2\kappa})\Big)\\ &+ \frac{\theta}{2}\Big(\sigma(\Im\Re\Im\Re\hbar_{2\kappa}, \Im\Re\hbar^{/}) + \sigma(\Im\Re\hbar^{/}, \hbar^{/}) + \sigma(\hbar^{/}, \mathcal{Q}\hbar_{2\kappa+1})\Big)\\ &\preceq_{i_2} \frac{\theta}{2}\Big(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar^{/}) + \sigma(\hbar^{/}, \mathcal{Q}\hbar_{2\kappa+1})\Big)\\ &+ \frac{\theta}{2}\Big(\sigma(\Im\Re\Im\Re\hbar_{2\kappa}, \Im\Re\hbar^{/}) + \sigma(\Im\Re\hbar^{/}, \aleph\Im\Re\hbar_{2\kappa})\Big) + \theta\sigma(\Im\Re\hbar^{/}, \hbar^{/}). \end{aligned} \tag{18}$$

Now, using (18) in (13), we have

$$\begin{aligned}\Big\|\sigma(\Im\Re\hbar^{/}, \hbar^{/})\Big\| \leq\ & \frac{1}{1-\theta}\Big\|\sigma(\Im\Re\hbar^{/}, \aleph\Im\Re\hbar_{2\kappa})\Big\|\\ &+\frac{\theta}{2(1-\theta)}\Big(\Big\|\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar^{/})\Big\| + \Big\|\sigma(\hbar^{/}, \mathcal{Q}\hbar_{2\kappa+1})\Big\|\Big) +\\ &+\frac{\theta}{2(1-\theta)}\Big(\Big\|\sigma(\Im\Re\Im\Re\hbar_{2\kappa}, \Im\Re\hbar^{/})\Big\| + \Big\|\sigma(\Im\Re\hbar^{/}, \aleph\Im\Re\hbar_{2\kappa})\Big\|\Big)\\ &+\frac{1}{1-\theta}\Big\|\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar^{/})\Big\|.\end{aligned}$$

Now, taking the limit as $\kappa \to \infty$, we have

$$\Big\|\sigma(\Im\Re\hbar^{/}, \hbar^{/})\Big\| \leq 0,$$

that is, $\Big\|\sigma(\Im\Re\hbar^{/}, \hbar^{/})\Big\| = 0$, a contradiction. Hence, $\Im\Re\hbar^{/} = \hbar^{/}$. Thus, in all cases $\Im\Re\hbar^{/} = \hbar^{/}$.

(b) Now, we show that $\aleph\hbar^{/} = \hbar^{/}$. We assume, on the contrary, that $\aleph\hbar^{/} \neq \hbar^{/}$. Then, $0 \prec_{i_2} \sigma(\aleph\hbar^{/}, \hbar^{/})$. Based on the triangle inequality, we have

$$\sigma(\aleph\hbar^{/}, \hbar^{/}) \preceq_{i_2} \sigma(\aleph\hbar^{/}, \mathcal{Q}\hbar_{2\kappa+1}) + \sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar^{/}). \tag{19}$$

Using (i), with $\hbar = \hbar^{/}, \wp = \hbar_{2\kappa+1}$, we have

$$\sigma(\aleph\hbar^{/}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\mathfrak{Q}(\hbar^{/}, \hbar_{2\kappa+1}), \tag{20}$$

where

$$\mathfrak{Q}(\hbar^{/}, \hbar_{2\kappa+1}) \in \left\{ \begin{array}{c} \sigma(\Im\Re\hbar^{/}, \mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}), \sigma(\Im\Re\hbar^{/}, \aleph\hbar^{/}), \sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \mathcal{Q}\hbar_{2\kappa+1}), \\ \frac{1}{2}(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \aleph\hbar^{/}) + \sigma(\Im\Re\hbar^{/}, \mathcal{Q}\hbar_{2\kappa+1})) \end{array} \right\}.$$

Now, we have the following four cases:
If $\mathfrak{Q}(\hbar^{/}, \hbar_{2\kappa+1}) = \sigma(\Im\Re\hbar^{/}, \mathfrak{L}\mathcal{T}\hbar_{2\kappa+1})$, then according to (20), we have

$$\sigma(\aleph\hbar^{/}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\Im\Re\hbar^{/}, \mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}).$$

Since $\Im\Re\hbar^{/} = \hbar^{/}$, we have

$$\sigma(\aleph\hbar^{/}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\hbar^{/}, \mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}). \tag{21}$$

Now, using (21) in (19), we have

$$\Big\|\sigma(\aleph\hbar^{/}, \hbar^{/})\Big\| \leq \theta\Big\|\sigma(\hbar^{/}, \mathfrak{L}\mathcal{T}\hbar_{2\kappa+1})\Big\| + \Big\|\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar^{/})\Big\|.$$

Now, letting $\kappa \to \infty$ in the above inequality, we have

$$\left\|\sigma(\aleph\hbar', \hbar')\right\| \leq 0,$$

that is, $\left\|\sigma(\aleph\hbar', \hbar')\right\| = 0$, a contradiction. Hence, $\aleph\hbar' = \hbar'$.

If $\mathfrak{Q}(\hbar', \hbar_{2\kappa+1}) = \sigma(\Im\Re\hbar', \aleph\hbar')$, then, according to (20), we have

$$\sigma(\aleph\hbar', \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\Im\Re\hbar', \aleph\hbar').$$

Since $\Im\Re\hbar' = \hbar'$, we have

$$\sigma(\aleph\hbar', \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\hbar', \aleph\hbar'). \tag{22}$$

Now, using (22) in (19), we have

$$\begin{aligned}\left\|\sigma(\aleph\hbar', \hbar')\right\| &\leq \theta\left\|\sigma(\hbar', \aleph\hbar')\right\| + \left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar')\right\| \\ &\leq \frac{1}{1-\theta}\left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar')\right\|.\end{aligned}$$

Now, letting $\kappa \to \infty$ in the above inequality, we have

$$\left\|\sigma(\aleph\hbar', \hbar')\right\| \leq 0,$$

that is, $\left\|\sigma(\aleph\hbar', \hbar')\right\| = 0$, a contradiction. Hence, $\aleph\hbar' = \hbar'$.

If $\mathfrak{Q}(\hbar', \hbar_{2\kappa+1}) = \sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \mathcal{Q}\hbar_{2\kappa+1})$, then according to (20), we have

$$\sigma(\aleph\hbar', \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \mathcal{Q}\hbar_{2\kappa+1}).$$

Using triangle inequality, we have

$$\sigma(\aleph\hbar', \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar') + \theta\sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1}). \tag{23}$$

Now, using (23) in (19), we have

$$\left\|\sigma(\aleph\hbar', \hbar')\right\| \leq (1+\theta)\left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar')\right\| + \theta\left\|\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar')\right\|.$$

Now, letting $\kappa \to \infty$ in the above inequality, we have

$$\left\|\sigma(\aleph\hbar', \hbar')\right\| \leq 0,$$

that is, $\left\|\sigma(\aleph\hbar', \hbar')\right\| = 0$, a contradiction. Hence, $\aleph\hbar' = \hbar'$.

If $\mathfrak{Q}(\hbar', \hbar_{2\kappa+1}) = \frac{1}{2}(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \aleph\hbar') + \sigma(\Im\Re\hbar', \mathcal{Q}\hbar_{2\kappa+1}))$, then according to (20), we have

$$\sigma(\aleph\hbar', \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \frac{\theta}{2}\Big(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \aleph\hbar') + \sigma(\Im\Re\hbar', \mathcal{Q}\hbar_{2\kappa+1})\Big).$$

Since $\Im\Re\hbar' = \hbar'$, we have

$$\sigma(\aleph\hbar', \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \frac{\theta}{2}\Big(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \aleph\hbar') + \sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1})\Big).$$

According to the triangle inequality, we have

$$\sigma(\aleph\hbar', \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \frac{\theta}{2}\Big(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar') + \sigma(\hbar', \aleph\hbar')\Big) + \frac{\theta}{2}\sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1}). \tag{24}$$

Now, using (24) in (19), we have

$$\sigma(\aleph\hbar^{/},\hbar^{/}) \preceq_{i_2} \frac{\theta}{2}\Big(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1},\hbar^{/}) + \sigma(\hbar^{/},\aleph\hbar^{/})\Big) + (\frac{\theta}{2}+1)\sigma(\hbar^{/},\mathcal{Q}\hbar_{2\kappa+1}).$$

This implies that

$$\Big\|\sigma(\aleph\hbar^{/},\hbar^{/})\Big\| \leq \frac{\theta}{2}\Big(\Big\|\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1},\hbar^{/})\Big\| + \Big\|\sigma(\hbar^{/},\aleph\hbar^{/})\Big\|\Big) + (\frac{\theta}{2}+1)\Big\|\sigma(\hbar^{/},\mathcal{Q}\hbar_{2\kappa+1})\Big\|.$$

Now, letting $\kappa \to \infty$ in the above inequality, we have

$$\Big\|\sigma(\aleph\hbar^{/},\hbar^{/})\Big\| \leq 0,$$

that is, $\Big\|\sigma(\aleph\hbar^{/},\hbar^{/})\Big\| = 0$, a contradiction. Thus, $\aleph\hbar^{/} = \hbar^{/}$.

(c) Now, we show that $\Re\hbar^{/} = \hbar^{/}$. We assume, on the contrary, that $\Re\hbar^{/} \neq \hbar^{/}$. Then, $0 \prec_{i_2} \sigma(\Re\hbar^{/},\hbar^{/})$. Now, using the triangle inequality, we get

$$\sigma(\Re\hbar^{/},\hbar^{/}) = \sigma(\Re\aleph\hbar^{/},\hbar^{/}) = \sigma(\aleph\Re\hbar^{/},\hbar^{/}) \preceq_{i_2} \sigma(\aleph\Re\hbar^{/},\mathcal{Q}\hbar_{2\kappa+1}) + \sigma(\mathcal{Q}\hbar_{2\kappa+1},\hbar^{/}). \quad (25)$$

According to (1) with $\hbar = \Re\hbar^{/}$ and $\wp = \hbar_{2\kappa+1}$, we have

$$\sigma(\aleph\Re\hbar^{/},\mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\mathfrak{Q}(\Re\hbar^{/},\hbar_{2\kappa+1}), \quad (26)$$

where

$$\begin{aligned}
\mathfrak{Q}(\Re\hbar^{/},\hbar_{2\kappa+1}) &\in \left\{ \begin{array}{c} \sigma(\Im\Re\Re\hbar^{/},\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}), \sigma(\Im\Re\Re\hbar^{/},\aleph\Re\hbar^{/}), \sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1},\mathcal{Q}\hbar_{2\kappa+1}), \\ \frac{1}{2}(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1},\aleph\hbar^{/}) + \sigma(\Im\Re\hbar^{/},\mathcal{Q}\hbar_{2\kappa+1})) \end{array} \right\} \\
&= \left\{ \begin{array}{c} \sigma(\Re\Im\Re\hbar^{/},\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}), \sigma(\Re\Im\Re\hbar^{/},\Re\aleph\hbar^{/}), \sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1},\mathcal{Q}\hbar_{2\kappa+1}), \\ \frac{1}{2}(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1},\aleph\hbar^{/}) + \sigma(\Im\Re\hbar^{/},\mathcal{Q}\hbar_{2\kappa+1})) \end{array} \right\} \\
&= \left\{ \begin{array}{c} \sigma(\Re\hbar^{/},\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}), \sigma(\Re\hbar^{/},\Re\hbar^{/}), \sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1},\mathcal{Q}\hbar_{2\kappa+1}), \\ \frac{1}{2}(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1},\hbar^{/}) + \sigma(\hbar^{/},\mathcal{Q}\hbar_{2\kappa+1})) \end{array} \right\}.
\end{aligned}$$

Now, we have the following four sub cases:

If $\mathfrak{Q}(\Re\hbar^{/},\hbar_{2\kappa+1}) = \sigma(\Im\Re\Re\hbar^{/},\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1})$, then according to (26) and $\Im\Re\hbar^{/} = \hbar^{/}$, we have

$$\sigma(\aleph\Re\hbar^{/},\mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\Re\hbar^{/},\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\Re\hbar^{/},\hbar^{/}) + \theta\sigma(\hbar^{/},\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}). \quad (27)$$

Now, using (27) in (25), we have

$$\Big\|\sigma(\Re\hbar^{/},\hbar^{/})\Big\| \leq \theta\Big\|\sigma(\Re\hbar^{/},\hbar^{/})\Big\| + \theta\Big\|\sigma(\hbar^{/},\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1})\Big\| + \Big\|\sigma(\mathcal{Q}\hbar_{2\kappa+1},\hbar^{/})\Big\|,$$

which implies that

$$\Big\|\sigma(\Re\hbar^{/},\hbar^{/})\Big\| \leq \frac{\theta}{1-\theta}\Big\|\sigma(\hbar^{/},\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1})\Big\| + \frac{1}{1-\theta}\Big\|\sigma(\mathcal{Q}\hbar_{2\kappa+1},\hbar^{/})\Big\|.$$

Now, letting $\kappa \to \infty$ in the above inequality, we have

$$\Big\|\sigma(\Re\hbar^{/},\hbar^{/})\Big\| \leq 0,$$

that is, $\Big\|\sigma(\Re\hbar^{/},\hbar^{/})\Big\| = 0$ a contradiction. Hence, $\Re\hbar^{/} = \hbar^{/}$.

If $\mathfrak{Q}(\Re\hbar', \hbar_{2\kappa+1}) = \sigma(\Im\Re\Re\hbar', \aleph\Re\hbar')$, then according to (26) and $\Im\Re\hbar' = \hbar'$, we have

$$\sigma(\aleph\Re\hbar', \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\Re\hbar', \Re\hbar'). \tag{28}$$

Now, using (28) in (25), we have

$$\left\|\sigma(\Re\hbar', \hbar')\right\| \leq \left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar')\right\|.$$

Now, letting $\kappa \to \infty$ in the above inequality, we get $\left\|\sigma(\Re\hbar', \hbar')\right\| = 0$, a contradiction. Hence, $\Re\hbar' = \hbar'$.

If $\mathfrak{Q}(\Re\hbar', \hbar_{2\kappa+1}) = \sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \mathcal{Q}\hbar_{2\kappa+1})$, then according to (26), we have

$$\sigma(\aleph\Re\hbar', \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar') + \theta\sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1}). \tag{29}$$

Now, using (29) in (25), we have

$$\sigma(\Re\hbar', \hbar') \preceq_{i_2} \theta\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar') + \theta\sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1}) + \sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar'),$$

which implies that

$$\left\|\sigma(\Re\hbar', \hbar')\right\| \leq \theta\left\|\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar')\right\| + \theta\left\|\sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1})\right\| + \left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar')\right\|.$$

Now, letting $\kappa \to \infty$ in the above inequality, we get $\left\|\sigma(\Re\hbar', \hbar')\right\| = 0$, a contradiction. Hence, $\Re\hbar' = \hbar'$.

If $\mathfrak{Q}(\Re\hbar', \hbar_{2\kappa+1}) = \frac{1}{2}\Big(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar') + \sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1})\Big)$, then by (26), we have

$$\sigma(\aleph\Re\hbar', \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \frac{\theta}{2}\Big(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar') + \sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1})\Big). \tag{30}$$

Now, using (30) in (25), we have

$$\sigma(\Re\hbar', \hbar') \preceq_{i_2} \frac{\theta}{2}\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar') + (\frac{\theta}{2} + 1)\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar'),$$

which implies that

$$\left\|\sigma(\aleph\Re\hbar', \hbar')\right\| \leq \frac{\theta}{2}\left\|\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar')\right\| + (\frac{\theta}{2} + 1)\left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar')\right\|.$$

Now, letting $\kappa \to \infty$ in the above inequality, we get $\left\|\sigma(\Re\hbar', \hbar')\right\| = 0$, a contradiction. Hence, $\Re\hbar' = \hbar'$.

Thus, in all cases, we get $\Re\hbar' = \hbar'$.

(d) As $\aleph(\mathcal{Z}) \subset \mathfrak{L}\mathcal{T}(\mathcal{Z})$, there exists $v \in \mathcal{Z}$, such that $\hbar' = \aleph\hbar' = \mathfrak{L}\mathcal{T}v$. First, we shall show that $\mathfrak{L}\mathcal{T}v = \mathcal{Q}v$. According to (1) with $\hbar = \hbar'$ and $\wp = v$, we have

$$\sigma(\mathfrak{L}\mathcal{T}v, \mathcal{Q}v) = \sigma(\aleph\hbar', \mathcal{Q}v) \preceq_{i_2} \theta\mathfrak{Q}(\hbar', v), \tag{31}$$

where

$$\mathfrak{Q}(\hbar', v) \in \left\{ \begin{array}{c} \sigma(\Im\Re\hbar', \mathfrak{L}\mathcal{T}v), \sigma(\Im\Re\hbar', \aleph\hbar'), \sigma(\mathfrak{L}\mathcal{T}v, \mathcal{Q}v), \\ \frac{1}{2}(\sigma(\mathfrak{L}\mathcal{T}v, \aleph\hbar') + \sigma(\Im\Re\hbar', \mathcal{Q}v)) \end{array} \right\}.$$

Since $\Im\Re\hbar' = \hbar'$ and $\hbar' = \aleph\hbar' = \mathfrak{L}\mathcal{T}v$, so we have

$$\mathfrak{Q}(\hbar', v) \in \left\{\sigma(\hbar', \hbar'), \sigma(\hbar', \hbar'), \sigma(\mathfrak{L}\mathcal{T}v, \mathcal{Q}v), \frac{1}{2}(\sigma(\hbar', \hbar') + \sigma(\hbar', \mathcal{Q}v))\right\}.$$

This implies that

$$\mathfrak{Q}(\hbar^{/}, v) \in \{0, \sigma(\mathfrak{L}\mathcal{T}v, \mathcal{Q}v), \frac{1}{2}\sigma(\mathfrak{L}\mathcal{T}v, \mathcal{Q}v)\}. \tag{32}$$

From (31) and (32), it follows that

$$\|\sigma(\mathfrak{L}\mathcal{T}v, \mathcal{Q}v)\| = 0,$$

that is, $\mathfrak{L}\mathcal{T}v = \mathcal{Q}v = \hbar^{/}$. As the pair $(\mathcal{Q}, \mathfrak{L}\mathcal{T})$ is weakly compatible, we have $\mathfrak{L}\mathcal{T}\mathcal{Q}v = \mathcal{Q}\mathfrak{L}\mathcal{T}v$. Thus,

$$\mathfrak{L}\mathcal{T}\hbar^{/} = \mathcal{Q}\hbar^{/}.$$

(e) Now, we prove that $\mathcal{Q}\hbar^{/} = \hbar^{/}$. According to (1), we have

$$\sigma(\hbar^{/}, \mathcal{Q}\hbar^{/}) = \sigma(\aleph\hbar^{/}, \mathcal{Q}\hbar^{/}) \preceq_{i_2} \theta\mathfrak{Q}(\hbar^{/}, \hbar^{/}), \tag{33}$$

where

$$\begin{aligned} \mathfrak{Q}(\hbar^{/}, \hbar^{/}) &\in \left\{ \begin{array}{c} \sigma(\Im\Re\hbar^{/}, \mathfrak{L}\mathcal{T}\hbar^{/}), \sigma(\Im\Re\hbar^{/}, \aleph\hbar^{/}), \sigma(\mathfrak{L}\mathcal{T}\hbar^{/}, \mathcal{Q}\hbar^{/}), \\ \frac{1}{2}(\sigma(\mathfrak{L}\mathcal{T}\hbar^{/}, \aleph\hbar^{/}) + \sigma(\Im\Re\hbar^{/}, \mathcal{Q}\hbar^{/})) \end{array} \right\} \\ &= \left\{ \begin{array}{c} \sigma(\hbar^{/}, \mathcal{Q}\hbar^{/}), \sigma(\hbar^{/}, \hbar^{/}), \sigma(\mathcal{Q}\hbar^{/}, \mathcal{Q}\hbar^{/}), \\ \frac{1}{2}(\sigma(\mathcal{Q}\hbar^{/}, \hbar^{/}) + \sigma(\hbar^{/}, \mathcal{Q}\hbar^{/})) \end{array} \right\} \\ &= \left\{0, \sigma(\hbar^{/}, \mathcal{Q}\hbar^{/})\right\}. \end{aligned} \tag{34}$$

From (33) and (35), we get

$$\left\|\sigma(\hbar^{/}, \mathcal{Q}\hbar^{/})\right\| = 0,$$

that is,

$$\mathcal{Q}\hbar^{/} = \hbar^{/}.$$

(f) Now, we show that $\mathcal{T}\hbar^{/} = \hbar^{/}$. According to (1), we have

$$\sigma(\hbar^{/}, \mathcal{T}\hbar^{/}) = \sigma(\aleph\hbar^{/}, \mathcal{T}\mathcal{Q}\hbar^{/}) = \sigma(\aleph\hbar^{/}, \mathcal{Q}\mathcal{T}\hbar^{/}) \preceq_{i_2} \theta\mathfrak{Q}(\hbar^{/}, \mathcal{T}\hbar^{/}), \tag{35}$$

where

$$\begin{aligned} \mathfrak{Q}(\hbar^{/}, \mathcal{T}\hbar^{/}) &\in \left\{ \begin{array}{c} \sigma(\Im\Re\hbar^{/}, \mathfrak{L}\mathcal{T}\mathcal{T}\hbar^{/}), \sigma(\Im\Re\hbar^{/}, \aleph\hbar^{/}), \sigma(\mathfrak{L}\mathcal{T}\mathcal{T}\hbar^{/}, \mathcal{Q}\mathcal{T}\hbar^{/}), \\ \frac{1}{2}(\sigma(\mathfrak{L}\mathcal{T}\mathcal{T}\hbar^{/}, \aleph\hbar^{/}) + \sigma(\Im\Re\hbar^{/}, \mathcal{Q}\mathcal{T}\hbar^{/})) \end{array} \right\} \\ &= \left\{ \begin{array}{c} \sigma(\hbar^{/}, \mathcal{T}\mathfrak{L}\mathcal{T}\hbar^{/}), \sigma(\hbar^{/}, \hbar^{/}), \sigma(\mathcal{T}\mathfrak{L}\mathcal{T}\hbar^{/}, \mathcal{T}\mathcal{Q}\hbar^{/}), \\ \frac{1}{2}(\sigma(\mathcal{T}\mathfrak{L}\mathcal{T}\hbar^{/}, \aleph\hbar^{/}) + \sigma(\Im\Re\hbar^{/}, \mathcal{T}\mathcal{Q}\hbar^{/})) \end{array} \right\} \\ &= \left\{ \begin{array}{c} \sigma(\hbar^{/}, \mathcal{T}\hbar^{/}), \sigma(\hbar^{/}, \hbar^{/}), \sigma(\mathcal{T}\hbar^{/}, \mathcal{T}\hbar^{/}), \\ \frac{1}{2}(\sigma(\mathcal{T}\hbar^{/}, \hbar^{/}) + \sigma(\hbar^{/}, \mathcal{T}\hbar^{/})) \end{array} \right\} \\ &= \left\{0, \sigma(\hbar^{/}, \mathcal{T}\hbar^{/})\right\}. \end{aligned} \tag{36}$$

From (35) and (37), we get

$$\left\|\sigma(\hbar^{/}, \mathcal{T}\hbar^{/})\right\| = 0,$$

that is, $\mathcal{T}\hbar^{/} = \hbar^{/}$. Since $\mathfrak{L}\mathcal{T}\hbar^{/} = \hbar^{/}$, it follows that $\mathfrak{L}\hbar^{/} = \hbar^{/}$. Hence, if $\Im\Re$ is continuous, then we show that

$$\Im\hbar^{/} = \Re\hbar^{/} = \mathfrak{L}\hbar^{/} = \mathcal{T}\hbar^{/} = \aleph\hbar^{/} = \mathcal{Q}\hbar^{/} = \hbar^{/}.$$

Case 2. *if* $\aleph$ *is continuous.*

As $\aleph$ is continuous, then $\aleph^2\hbar_{2\kappa} \to \aleph\hbar'$ and $\aleph\Im\Re\hbar_{2\kappa} \to \aleph\hbar'$, as $\kappa \to \infty$. As the pair $(\aleph, \Im\Re)$ is compatible, we have $\Im\Re\aleph\hbar_{2\kappa} \to \aleph\hbar'$, as $\kappa \to \infty$, because according to triangle inequality, we have

$$\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar') \preceq_{i_2} \sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\Im\Re\hbar_{2\kappa}) + \sigma(\aleph\Im\Re\hbar_{2\kappa}, \aleph\hbar'),$$

which implies that

$$\left\|\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar')\right\| \leq \|\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\Im\Re\hbar_{2\kappa})\| + \left\|\sigma(\aleph\Im\Re\hbar_{2\kappa}, \aleph\hbar')\right\| \to 0, \text{ as } \kappa \to \infty.$$

(a) We show that $\aleph\hbar' = \hbar'$.
According to the triangle inequality, we have

$$\sigma(\aleph\hbar', \hbar') \preceq_{i_2} \sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa}) + \sigma(\aleph^2\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) + \sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar'). \tag{37}$$

According to (1) with $\hbar = \aleph\hbar_{2\kappa}$ and $\wp = \hbar_{2\kappa+1}$, we have

$$\sigma(\aleph^2\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\mathfrak{Q}(\aleph\hbar_{2\kappa}, \hbar_{2\kappa+1}), \tag{38}$$

where

$$\mathfrak{Q}(\aleph\hbar_{2\kappa}, \hbar_{2\kappa+1}) \in \left\{ \begin{array}{c} \sigma(\Im\Re\aleph\hbar_{2\kappa}, \mathfrak{LT}\hbar_{2\kappa+1}), \sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph^2\hbar_{2\kappa}), \sigma(\mathfrak{LT}\hbar_{2\kappa+1}, \mathcal{Q}\hbar_{2\kappa+1}), \\ \frac{1}{2}(\sigma(\mathfrak{LT}\hbar_{2\kappa+1}, \aleph^2\hbar_{2\kappa}) + \sigma(\Im\Re\aleph\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1})), \end{array} \right\}.$$

Now, we have the following four sub cases:
If $\mathfrak{Q}(\aleph\hbar_{2\kappa}, \hbar_{2\kappa+1}) = \sigma(\Im\Re\aleph\hbar_{2\kappa}, \mathfrak{LT}\hbar_{2\kappa+1})$, then according to (38), we have

$$\sigma(\aleph^2\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\Im\Re\aleph\hbar_{2\kappa}, \mathfrak{LT}\hbar_{2\kappa+1}).$$

According to triangle inequality, we have

$$\sigma(\aleph^2\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar') + \theta\sigma(\aleph\hbar', \hbar') + \theta\sigma(\hbar', \mathfrak{LT}\hbar_{2\kappa+1}). \tag{39}$$

Now, using (39) in (37), we have

$$\begin{aligned} \sigma(\aleph\hbar', \hbar') \preceq_{i_2} \sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa}) &+ \theta\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar') \\ &+ \theta\sigma(\aleph\hbar', \hbar') + \theta\sigma(\hbar', \mathfrak{LT}\hbar_{2\kappa+1}) + \sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar'), \end{aligned}$$

which implies that

$$\begin{aligned} \left\|\sigma(\aleph\hbar', \hbar')\right\| \leq\ & \left\|\sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa})\right\| + \theta\left\|\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar')\right\| \\ & + \theta\left\|\sigma(\aleph\hbar', \hbar')\right\| + \theta\left\|\sigma(\hbar', \mathfrak{LT}\hbar_{2\kappa+1})\right\| + \left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar')\right\|. \end{aligned}$$

lThis yields

$$\begin{aligned} \left\|\sigma(\aleph\hbar', \hbar')\right\| \leq\ & \frac{1}{1-\theta}\left\|\sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa})\right\| + \frac{\theta}{1-\theta}\left\|\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar')\right\| \\ & + \frac{\theta}{1-\theta}\left\|\sigma(\hbar', \mathfrak{LT}\hbar_{2\kappa+1})\right\| + \frac{1}{1-\theta}\left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar')\right\|. \end{aligned}$$

Letting $\kappa \to \infty$ in the above inequality, we have

$$\left\|\sigma(\aleph\hbar', \hbar')\right\| = 0.$$

Hence, $\aleph\hbar' = \hbar'$.
If $\mathfrak{Q}(\aleph\hbar_{2\kappa}, \hbar_{2\kappa+1}) = \sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph^2\hbar_{2\kappa})$, then according to (38), we have

$$\sigma(\aleph^2\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph^2\hbar_{2\kappa}).$$

According to the triangle inequality, we have

$$\sigma(\aleph^2\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar') + \theta\sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa}). \tag{40}$$

Now, using (40) in (37), we have

$$\sigma(\aleph\hbar', \hbar') \preceq_{i_2} \sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa}) + \theta\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar') + \theta\sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa}) + \sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar'),$$

which implies

$$\begin{aligned}\left\|\sigma(\aleph\hbar', \hbar')\right\| \quad \leq \quad & \left\|\sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa})\right\| + \theta\left\|\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar')\right\| \\ & + \theta\left\|\sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa})\right\| + \left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar')\right\|.\end{aligned}$$

Letting $\kappa \to \infty$ in the above inequality, we have

$$\left\|\sigma(\aleph\hbar', \hbar')\right\| = 0.$$

Hence, $\aleph\hbar' = \hbar'$.
If $\mathfrak{Q}(\aleph\hbar_{2\kappa}, \hbar_{2\kappa+1}) = \sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \mathcal{Q}\hbar_{2\kappa+1})$, then according to (38), we have

$$\sigma(\aleph^2\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \mathcal{Q}\hbar_{2\kappa+1}) \preceq_{i_2} \theta\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar') + \theta\sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1}). \tag{41}$$

Now, using (41) in (37), we have

$$\sigma(\aleph\hbar', \hbar') \preceq_{i_2} \sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa}) + \theta\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar') + \theta\sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1}) + \sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar'),$$

which implies that

$$\sigma(\aleph\hbar', \hbar') \preceq_{i_2} \sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa}) + \theta\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar') + (1+\theta)\sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1}).$$

It yields

$$\left\|\sigma(\aleph\hbar', \hbar')\right\| \leq \left\|\sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa})\right\| + \theta\left\|\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar')\right\| + (1+\theta)\left\|\sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1})\right\|.$$

Letting $\kappa \to \infty$ in the above inequality, we have

$$\left\|\sigma(\aleph\hbar', \hbar')\right\| = 0.$$

Hence $\aleph\hbar' = \hbar'$.
If $\mathfrak{Q}(\aleph\hbar_{2\kappa}, \hbar_{2\kappa+1}) = \frac{1}{2}(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \aleph^2\hbar_{2\kappa}) + \sigma(\Im\Re\aleph\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}))$, then according to (38), we have

$$\begin{aligned}\sigma(\aleph^2\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1}) &\preceq_{i_2} \frac{\theta}{2}(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \aleph^2\hbar_{2\kappa}) + \sigma(\Im\Re\aleph\hbar_{2\kappa}, \mathcal{Q}\hbar_{2\kappa+1})) \\ &\preceq_{i_2} \frac{\theta}{2}(\sigma(\mathfrak{L}\mathcal{T}\hbar_{2\kappa+1}, \hbar') + \sigma(\hbar', \aleph\hbar') + \sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa})) \\ &\quad + \frac{\theta}{2}(\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar') + \sigma(\aleph\hbar', \hbar') + \sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1})).\end{aligned} \tag{42}$$

Now, using (42) in (37), we have

$$\sigma(\aleph\hbar', \hbar') \preceq_{i_2} \sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa}) + \frac{\theta}{2}(\sigma(\mathfrak{LT}\hbar_{2\kappa+1}, \hbar') + \sigma(\hbar', \aleph\hbar') + \sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa}))$$
$$+ \frac{\theta}{2}\Big(\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar') + \sigma(\aleph\hbar', \hbar') + \sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1})\Big) + \sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar').$$

It implies that

$$\sigma(\aleph\hbar', \hbar') \preceq_{i_2} \frac{1}{1-\theta}\sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa}) + \frac{\theta}{2(1-\theta)}\sigma(\mathfrak{LT}\hbar_{2\kappa+1}, \hbar') + \frac{\theta}{2(1-\theta)}\sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa})$$
$$+ \frac{\theta}{2(1-\theta)}\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar') + \frac{\theta}{2(1-\theta)}\sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1}) + \frac{1}{1-\theta}\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar').$$

It yields

$$\begin{aligned}\left\|\sigma(\aleph\hbar', \hbar')\right\| \leq\ & \frac{1}{1-\theta}\left\|\sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa})\right\| + \\ & \frac{\theta}{2(1-\theta)}\left\|\sigma(\mathfrak{LT}\hbar_{2\kappa+1}, \hbar')\right\| + \frac{\theta}{2(1-\theta)}\left\|\sigma(\aleph\hbar', \aleph^2\hbar_{2\kappa})\right\| \\ & + \frac{\theta}{2(1-\theta)}\left\|\sigma(\Im\Re\aleph\hbar_{2\kappa}, \aleph\hbar')\right\| \\ & + \frac{\theta}{2(1-\theta)}\left\|\sigma(\hbar', \mathcal{Q}\hbar_{2\kappa+1})\right\| + \frac{1}{1-\theta}\left\|\sigma(\mathcal{Q}\hbar_{2\kappa+1}, \hbar')\right\|.\end{aligned}$$

Letting $\kappa \to \infty$ in the above inequality, we get

$$\left\|\sigma(\aleph\hbar', \hbar')\right\| \leq 0;$$

that is, $\left\|\sigma(\aleph\hbar', \hbar')\right\| = 0$, a contradiction. Hence, $\aleph\hbar' = \hbar'$. Thus, in all sub cases, $\aleph\hbar' = \hbar'$.

Now, utilizing steps (d), (e) and (f), and continuing the step (f) gives us

$$\mathcal{Q}\hbar' = \mathfrak{L}\hbar' = \mathcal{T}\hbar' = \hbar'.$$

Now, as $\mathcal{Q}(\mathcal{Z}) \subset \Im\Re(\mathcal{Z})$, so there exists $w \in \mathcal{Z}$, such that

$$\hbar' = \mathcal{Q}\hbar' = \Im\Re w.$$

Now, we show that

$$\aleph w = \Im\Re w = \hbar'.$$

By (1), we have

$$\sigma(\aleph w, \Im\Re w) = \sigma(\aleph w, \mathcal{Q}\hbar') \preceq_{i_2} \theta\mathfrak{Q}(w, \hbar'),$$

where

$$\begin{aligned}\mathfrak{Q}(w, \hbar') \in & \left\{ \begin{array}{c} \sigma(\Im\Re w, \mathfrak{LT}\hbar'), \sigma(\Im\Re w, \aleph w), \sigma(\mathfrak{LT}\hbar', \mathcal{Q}\hbar'), \\ \frac{1}{2}(\sigma(\mathfrak{LT}\hbar', \aleph w) + \sigma(\Im\Re w, \mathcal{Q}\hbar')) \end{array} \right\} \\ = & \{0, \frac{1}{2}(\sigma(\hbar', \aleph w)\}.\end{aligned}$$

This implies that

$$\sigma(\aleph w, \Im\Re w) = \sigma(\aleph w, \mathcal{Q}\hbar') = \sigma(\aleph w, \hbar') \preceq_{i_2} \frac{\theta}{2}(\sigma(\hbar', \aleph w),$$

i.e., $\aleph w = \Im\Re w = \hbar^{/}$.

Since $(\aleph, \Im\Re)$ is compatible, so it must be weakly compatible, and so we have

$$\aleph\hbar^{/} = \Im\Re\hbar^{/}.$$

Moreover, $\Re\hbar^{/} = \hbar^{/}$ according to step (c). Hence, $\Im\hbar^{/} = \Re\hbar^{/} = \aleph\hbar^{/} = \hbar^{/}$ and we establish that $\hbar^{/}$ is the common fixed point of $\Im, \Re, \mathfrak{L}, \mathcal{T}, \aleph$ and $\mathcal{Q}$ in this case, too.

Now, we prove that this common fixed point is unique. Let $\hbar^{*}$ be another common fixed point of $\Im, \Re, \mathfrak{L}, \mathcal{T}, \aleph$ and $\mathcal{Q}$; then

$$\Im\hbar^{*} = \Re\hbar^{*} = \mathfrak{L}\hbar^{*} = \mathcal{T}\hbar^{*} = \aleph\hbar^{*} = \mathcal{Q}\hbar^{*} = \hbar^{*}.$$

Using (1) with $\hbar = \hbar^{/}, \wp = \hbar^{*}$, we have

$$\sigma(\hbar^{/}, \hbar^{*}) = \sigma(\aleph\hbar^{/}, \mathcal{Q}\hbar^{*})) \preceq_{i_2} \theta\mathfrak{Q}(\hbar^{/}, \hbar^{*}), \tag{43}$$

where

$$\begin{aligned} \mathfrak{Q}(\hbar^{/}, \hbar^{*}) &\in \left\{ \begin{array}{c} \sigma(\Im\Re\hbar^{/}, \mathfrak{L}\mathcal{T}\hbar^{*}), \sigma(\Im\Re\hbar^{/}, \aleph\hbar^{/}), \sigma(\mathfrak{L}\mathcal{T}\hbar^{*}, \mathcal{Q}\hbar^{*}), \\ \frac{1}{2}(\sigma(\mathfrak{L}\mathcal{T}\hbar^{*}, \aleph\hbar^{/}) + \sigma(\Im\Re\hbar^{/}, \mathcal{Q}\hbar^{*})) \end{array} \right\} \\ &= \{0, \sigma(\hbar^{/}, \hbar^{*})\}. \end{aligned} \tag{44}$$

Using (44) in (43), we have

$$\left\| \sigma(\hbar^{/}, \hbar^{*}) \right\| = 0,$$

which implies that $\hbar^{/} = \hbar^{*}$. Thus, $\hbar^{/}$ is the unique common fixed point of $\Im, \Re, \mathfrak{L}, \mathcal{T}, \aleph$ and $\mathcal{Q}$. □

Corollary 1. *Let $(\mathcal{Z}, \sigma)$ be a complete bi-CVMS and let $\Im, \mathfrak{L}, \aleph, \mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ be a self-mapping. If there exists some $\theta \in [0, 1)$ such that the following conditions hold:*

(i)

$$\sigma(\aleph\hbar, \mathcal{Q}\wp) \preceq_{i_2} \theta\mathfrak{Q}(\hbar, \wp) \tag{45}$$

for all $\hbar, \wp \in \mathcal{Z}$, where

$$\mathfrak{Q}(\hbar, \wp) \in \left\{ \begin{array}{c} \sigma(\Im\hbar, \mathfrak{L}\wp), \sigma(\Im\hbar, \aleph\hbar), \sigma(\mathfrak{L}\wp, \mathcal{Q}\wp), \\ \frac{1}{2}(\sigma(\mathfrak{L}\wp, \aleph\hbar) + \sigma(\Im\hbar, \mathcal{Q}\wp)) \end{array} \right\};$$

(ii) $\aleph(\mathcal{Z}) \subset \mathfrak{L}(\mathcal{Z}), \mathcal{Q}(\mathcal{Z}) \subset \Im(\mathcal{Z})$;
(iii) *the pair $(\aleph, \Im)$ is compatible and the pair $(\mathcal{Q}, \mathfrak{L})$ is weakly compatible;*
(iv) *either the mapping $\Im$ or the mapping $\aleph$ is continuous.*
Then, there exists a unique point $\hbar^{/} \in \mathcal{Z}$, such that $\Im\hbar^{/} = \mathfrak{L}\hbar^{/} = \aleph\hbar^{/} = \mathcal{Q}\hbar^{/} = \hbar^{/}$.

Proof. Take $\Re = \mathcal{T} = I_{\mathcal{Z}}$, the identity mapping on $\mathcal{Z}$ in Theorem 1. □

Corollary 2. *Let $(\mathcal{Z}, \sigma)$ be a complete bi-CVMS and let $\aleph, \mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ be a self-mapping. If there exists some $\theta \in [0, 1)$, such that*

$$\sigma(\aleph\hbar, \mathcal{Q}\wp) \preceq_{i_2} \theta\mathfrak{Q}(\hbar, \wp), \tag{46}$$

for all $\hbar, \wp \in \mathcal{Z}$, where

$$\mathfrak{Q}(\hbar, \wp) \in \left\{ \begin{array}{c} \sigma(\hbar, \wp), \sigma(\hbar, \aleph\hbar), \sigma(\wp, \mathcal{Q}\wp), \\ \frac{1}{2}(\sigma(\wp, \aleph\hbar) + \sigma(\hbar, \mathcal{Q}\wp)) \end{array} \right\}.$$

Then, there exists a unique point $\hbar^{/} \in \mathcal{Z}$, such that $\aleph\hbar^{/} = \mathcal{Q}\hbar^{/} = \hbar^{/}$.

Proof. Take $\Im = \Re = \mathfrak{L} = \mathcal{T} = I_{\mathcal{Z}}$, the identity mapping on $\mathcal{Z}$ in Theorem 1. □

Corollary 3. *Let $(\mathcal{Z}, \sigma)$ be a complete bi-CVMS and let $F : \mathcal{Z} \to \mathcal{Z}$ be a self-mapping. If there exists some $\theta \in [0, 1)$ such that*

$$\sigma(F\hbar, F\wp)) \preceq \theta\mathfrak{Q}(\hbar, \wp), \tag{47}$$

for all $\hbar, \wp \in \mathcal{Z}$, where

$$\mathfrak{Q}(\hbar, \wp) \in \left\{\sigma(\hbar, \wp), \sigma(\hbar, F\hbar), \sigma(\wp, F\wp), \frac{1}{2}\{\sigma(\hbar, F\wp) + \sigma(\wp, F\hbar)\}\right\}.$$

Then, there exists a unique point $\hbar' \in \mathcal{Z}$ such that $F\hbar' = \hbar'$.

Proof. Take $\aleph = \mathcal{Q} = F$ and $\Im = \Re = \mathfrak{L} = \mathcal{T} = I_{\mathcal{Z}}$, the identity mapping on $\mathcal{Z}$ in Theorem 1. □

Example 2. *Let $\mathcal{Z} = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$ and define a mapping $\sigma : \mathcal{Z} \times \mathcal{Z} \to \mathbb{C}_2$ as*

$$\sigma(\hbar, \wp) = |\hbar_1 - \wp_1| + i|\hbar_2 - \wp_2|,$$

where $\hbar = \hbar_1 + i\hbar_2, \wp = \wp_1 + i\wp_2$, then $(\mathcal{Z}, \sigma)$ is a complete bi-CVMS. Define $F : \mathcal{Z} \to \mathcal{Z}$ by

$$F(\hbar_1 + i\hbar_2) = 2|\hbar_1 - \hbar_2| + 3i|\hbar_1 - \hbar_2|,$$

for all $\hbar = \hbar_1 + i\hbar_2 \in \mathcal{Z}$. Then, there exists $\theta = \frac{1}{2} \in [0, 1)$, such that the mapping $F : \mathcal{Z} \to \mathcal{Z}$ satisfies all the assertions of Corollary 3, and there exists a unique point $(2, 3) \in \mathcal{Z}$ such that $F(2, 3) = (2, 3)$.

4. Applications

Let $\mathcal{Z} = C([a, b], \mathbb{R}), a > 0$ where $C[a, b]$ denotes the set of all real continuous functions defined on the closed interval $[a, b]$ and $\sigma : \mathcal{Z} \times \mathcal{Z} \to \mathbb{C}_2$ be defined in this way

$$\sigma(\hbar, \wp) = (1 + i)(|\hbar(t) - \wp(t)|),$$

for all $\hbar, \wp \in \mathcal{Z}$ and $t \in [a, b]$, where $|\cdot|$ is the usual real modulus. Then, $(\mathcal{Z}, \sigma)$ is a complete bi-CVMS. Consider the Urysohn integral equations

$$\hbar(t) = \frac{1}{b - a}\int_a^b K_1(t, s, \hbar(s))\sigma s + g(t), \tag{48}$$

$$\hbar(t) = \frac{1}{b - a}\int_a^b K_2(t, s, \hbar(s))\sigma s + l(t), \tag{49}$$

where $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $g, l : [a, b] \to \mathbb{R}$ are continuous and $t \in [a, b]$. We define partial order $\preceq_{i_2}$ in $\mathbb{C}_2$ as follows $\hbar(t) \preceq_{i_2} \wp(t)$ if and only if $\hbar \leq \wp$.

Theorem 2. *Let $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$, such that $\mathcal{S}_\hbar(t), G_\hbar(t) \in \mathcal{Z}$ for each $\hbar \in \mathcal{Z}$, where*

$$\mathcal{S}_\hbar(t) = \frac{1}{b - a}\int_a^b K_1(t, s, \hbar(s))\sigma s, \quad G_\hbar(t) = \frac{1}{b - a}\int_a^b K_2(t, s, \hbar(s))\sigma s$$

for all $t \in [a, b]$. Suppose the following inequality

$$(1 + i_2)(|\mathcal{S}_\hbar(t) - G_\wp(t) + g(t) - h(t)|) \preceq_{i_2} \theta\mathfrak{Q}(\hbar, \wp),$$

holds, for all $\hbar, \wp \in \mathcal{Z}$ *with* $\hbar \neq \wp$ *and* $\theta<1$, *where*

$$\mathfrak{Q}(\hbar, \wp)(t) \in \{A(\hbar, \wp)(t), B(\hbar, \wp)(t), C(\hbar, \wp)(t), D(\hbar, \wp)(t)\},$$

$$\begin{aligned} A(\hbar, \wp)(t) &= (1+i_2)(|\hbar(t) - \wp(t)|), \\ B(\hbar, \wp)(t) &= (1+i_2)(|\mathcal{S}_\hbar(t) + g(t) - \hbar(t)|), \\ C(\hbar, \wp)(t) &= (1+i_2)(|G_\wp(t) + l(t) - \wp(t)|), \\ D(\hbar, \wp)(t) &= (1+i_2)\frac{(|G_\wp(t) + l(t) - \hbar(t)| + |\mathcal{S}_\hbar(t) + g(t) - \wp(t)|)}{2}. \end{aligned}$$

Then, the integral operators defined by (48) and (49) have a unique common solution.

Proof. Define continuous mappings $\aleph, \mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ by

$$\aleph\hbar(t) = \mathcal{S}_\hbar(t) + g(t),$$

$$\mathcal{Q}\hbar(t) = G_\hbar(t) + g(t),$$

for all $t \in [a, b]$. Then

$$\sigma(\aleph\hbar, \mathcal{Q}\wp) = (1+i_2)(|\mathcal{S}_\hbar(t) - G_\wp(t) + g(t) - l(t)|),$$

$$\sigma(\hbar, \wp) = A(\hbar, \wp)(t),$$

$$\sigma(\hbar, \aleph\hbar) = B(\hbar, \wp)(t),$$

$$\sigma(\wp, \mathcal{Q}\wp) = C(\hbar, \wp)(t),$$

$$\frac{1}{2}\{\sigma(\hbar, \mathcal{Q}\wp) + \sigma(\wp, \aleph\hbar)\} = D(\hbar, \wp)(t).$$

It is very simple to show that $\sigma(\aleph\hbar, \mathcal{Q}\wp) \preceq_{i_2} \theta\mathfrak{Q}(\hbar, \wp)$, where

$$\mathfrak{Q}(\hbar, \wp) \in \left\{\sigma(\hbar, \wp), \sigma(\hbar, \aleph\hbar), \sigma(\wp, \mathcal{Q}\wp), \frac{1}{2}\{\sigma(\hbar, \mathcal{Q}\wp) + \sigma(\wp, \aleph\hbar)\}\right\}.$$

Hence, all the assumptions of Corollary (2) are satisfied and the integral Equations (48) and (49) have a unique common solution. □

5. Conclusions

This article expands on the concept of bicomplex valued metric space in order to establish common fixed points of six mappings for generalized contractions. A non-trivial example is also provided to show the validity of the obtained results. At the end of this paper, we applied our result to discuss the solution of Urysohn integral equation. We believe that the established results in this paper will establish a contemporary link for investigators working in bicomplex valued metric space.

Common fixed points of multivalued mappings and fuzzy mappings in the context of bicomplex valued metric space can be an interesting outline for future work in this direction. Differential and integral inclusions can be investigated as applications of these results.

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References

1. Segre, C. Le Rappresentazioni Reali delle Forme Complesse a Gli Enti Iperalgebrici. *Math. Ann*. **1892**, *40*, 413–467. [CrossRef]
2. Price, G.B. *An Introduction to Multicomplex Spaces and Functions*; Marcel Dekker: New York, NY, USA, 1991.
3. Azam, A.; Fisher, B.; Khan, M. Common fixed point theorems in complex valued metric spaces. *Num. Funct. Anal. Optim.* **2011**, *32*, 243–253. [CrossRef]
4. Choi, J.; Datta, S.K.; Biswas, T.; Islam, N. Some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. *Honam. Math. J.* **2017**, *39*, 115–126. [CrossRef]
5. Jebril, I.H.; Datta, S.K.; Sarkar, R.; Biswas, N. Common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces. *J. Interdiscip. Math*. **2020**, *22*, 1071–1082. [CrossRef]
6. Beg, I.; Datta, S.K.; Pal, D. Fixed point in bicomplex valued metric spaces. *Int. J. Nonlinear Anal. Appl.* **2021**, *12*, 717–727.
7. Gnanaprakasam, A.J.; Boulaaras, S.M.; Mani, G.; Cherif, B.; Idris, S.A. Solving system of linear equations via bicomplex valued metric space. *Demonstr. Math*. **2021**, *54*, 474–487. [CrossRef]
8. Gu, Z.; Mani, G.; Gnanaprakasam, A.J.; Li, Y. Solving a system of nonlinear integral equations via common fixed point theorems on bicomplex partial metric space. *Mathematics* **2021**, *9*, 1584. [CrossRef]
9. Rouzkard. F.; Imdad, M. Some common fixed point theorems on complex valued metric spaces. *Comp. Math. Appl.* **2012**, *64*, 1866–1874. [CrossRef]
10. Sintunavarat, W.; Kumam, P. Generalized common fixed point theorems in complex valued metric spaces and applications. *J. Inequal Appl.* **2012**, *84*, 1–12. [CrossRef]
11. Sitthikul, K.; Saejung, S. Some fixed point theorems in complex valued metric spaces. *Fixed Point Theory Appl.* **2012**, *189*, 1–11. [CrossRef]
12. Ahmad, A.; Klin-eam, C.; Azam, A. Common fixed points for multivalued mappings in complex valued metric spaces with applications. *Abstr. Appl. Anal.* **2013**, *2013*, 1–12. [CrossRef]
13. Azam, A.; Ahmad, J.; Kumam, P. Common fixed point theorems for multi-valued mappings in complex-valued metric spaces. *J. Inequal Appl.* **2013**, *578*, 1–12. [CrossRef]
14. Albargi, A.H.; Ahmad, J. Common α-fuzzy fixed point results for Kannan type contractions with application. *J. Funct. Spaces* **2022**, *2022*, 1–9. [CrossRef]
15. Carmel Pushpa Raj, J.; Arul Xavier, A.; Maria Joseph, J.; Marudai, M. Common fixed point theorems under rational contractions in complex valued extended b-metric spaces. *Int. J. Nonlinear Anal. Appl.* **2022**, *13*, 3479–3490.
16. Humaira, M.; Sarwar, G.; Kishore, N.V. Fuzzy fixed point results for φ contractive mapping with applications. *Complexity* **2018**, *2018*, 1–12. [CrossRef]
17. Humaira, M.; Sarwar, G.; Kumam, P. Common fixed point results for fuzzy mappings on complex-valued metric spaces with Homotopy results. *Symmetry* **2019**, *11*, 61. [CrossRef]
18. Klin-eam, C.; Suanoom, C. Some common fixed point theorems for generalized contractive type mappings on complex valued metric spaces. *Abstr. Appl. Anal*. **2013**, *2013*, 1–6. [CrossRef]
19. Kutbi, M.K.; Ahmad, J.; Azam, A.; Hussain, N. On fuzzy fixed points for fuzzy maps with generalized weak property. *J. Appl. Math*. **2014**, *2014*, 1–13. [CrossRef]
20. Kumar, J. Common Fixed point theorem for generalized contractive type paps on complex valued b-metric spaces. *Int. J. Math. Anal*. **2015**, *9*, 2327–2334. [CrossRef]
21. Mukheimer, A.A. Some common fixed point theorems in complex valued b-metric spaces. *Sci. World J*. **2014**, *2014*, 1–6. [CrossRef]
22. Mustafa, Z.; Parvaneh, V.; Roshan, J.R.; Kadelburg, Z. b_2-Metric spaces and some fixed point theorems. *Fixed Point Theory Appl*. **2014**, *1*, 1–23.
23. Rao, P.; Swamy, R.; Prasad, J.R. A common fixed point theorem in complex valued b-metric spaces. *Bull. Math. Stat. Res.* **2013**, *1*, 1–8.
24. Ullah, N.; Shagari, M.S.; Azam, A. Fixed point theorems in complex valued extended b-metric spaces. *Moroc. J. Pure Appl. Anal*. **2019**, *5*, 140–163. [CrossRef]
25. Ullah, N.; Shagari, M.S.; Khan, T.A.; Khan, A.U.; Khan, M.A.U. Common fixed point theorems in complex valued non-negative extended b -metric space. *e-J. Anal. Appl. Math.* **2021**, *2012*, 35–47. [CrossRef]
26. Ullah, N.; Shagari, M.S. Fixed point results in complex valued extended b-metric spaces and related applications. *Annals. Math. Comp. Sci.* **2021**, *1*, 1–11.
27. Shammaky, A.E.; Ahmad, J.; Sayed, A.F. On fuzzy fixed point results in complex valued extended b-metric spaces with application. *J. Math*. **2021**, *2021*, 1–9. [CrossRef]