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The Existence Problems of Solutions for a Class of Differential Variational–Hemivariational Inequality Problems

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Abstract: In this work, we used reflexive Banach spaces to study the differential variational–hemivariational inequality problems with constraints. We established a sequence of perturbed differential variational–hemivariational inequality problems with perturbed constraints and penalty coefficients. Then, for each perturbed inequality, we proved the unique solvability and convergence of the solutions to the problems. Following that, we proposed a mathematical model for a viscoelastic rod in unilateral contact equilibrium, where the unknowns were the displacement field and the history of the deformation. We used the abstract penalty method in the analysis of this inequality and provided the corresponding mechanical interpretations.

Keywords: differential variational inequality; unilateral constraints; penalty method; Mosco convergence; viscoelastic rod; inverse strongly monotonicity; Lipschitz continuity

MSC: 34G20; 47J20; 49J40; 49J53; 49N45; 35M86; 74M10; 90C26

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1. Introduction

Aubin and Cellina [1] were the first to present the concept of differential variational inequalities. A comprehensive study of differential variational inequalities in the environment of Euclidean spaces has been performed in [2–5].

Differential hemivariational inequalities, as well as differential variational–hemivariational inequalities, are important extensions of differential variational inequalities, even though they couple a differential or partial differential equation with a hemivariational inequality and a variational–hemivariational inequality, *respectively*, where the existence and uniqueness results for various classes of differential variational–hemivariational inequalities have been determined. The references in the field are [6–10].

Penalty techniques are a well-known mathematical tool for dealing with a wide range of problems with constraints. The constraints are alleviated in the traditional penalty technique by injecting an additional term defined by a penalty parameter. The unique solution of the original problem can be approached by the unique solution of the penalty problem as the penalty parameter approaches zero. Penalty methods can be used to verify the solvability of constrained problems and can also be used to solve the numerical solution of constrained problems, see [11–15].

In this work, we proposed a class of differential variational–hemivariational inequality problems with a set of constraints in abstract Banach spaces. We proceeded by introducing an approximating sequence of differential variational–hemivariational inequality problems with a set of constraints and a penalty parameter. Using the appropriate assumptions of data, we proved the existence and convergence solution to the differential variational–hemivariational inequality problems. Finally, we showed how to apply our result to analyse a viscoelastic rod in a unilateral contact problem, and the corresponding mechanical interpretations were discussed.

2. Preliminaries

Unless otherwise stated, everywhere in this paper, let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a real Banach space, while $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ a reflexible Banach space, $0_{\mathbb{E}}$ and $0_{\mathbb{F}}$ denote the zero elements of $\mathbb{E}$ and $\mathbb{F}$, *respectively*. $\mathbb{F}^*$ denotes the duality of $\mathbb{F}$ and $\langle\cdot,\cdot\rangle$ represents the duality pairing mapping. The $\mathfrak{L}(\mathbb{F},\mathbb{E})$ denotes the space of bounded linear continuous operators from $\mathbb{F}$ to $\mathbb{E}$ endowed with the norm $\|\cdot\|_{\mathfrak{L}(\mathbb{E})}$. Furthermore, we use $\mathbb{E}\times\mathbb{F}$ for the product of the spaces $\mathbb{E}$ and $\mathbb{F}$ endowed with the canonical product topology. In addition, let $T > 0$ and let I be the interval of time $I = [0, T]$. $\mathrm{C}(I,\mathbb{E})$ and $\mathrm{C}(I,\mathbb{F})$ be the space of continuous functions defined on I with values in $\mathbb{E}$ and $\mathbb{F}$, respectively, with the norm of the uniform convergence. Let $\mathcal{A} : \mathfrak{D}(\mathcal{A}) \subset \mathbb{E} \to \mathbb{E}$ be the infinitesimal generator of a ϑ_0-semigroup $\{T(\tau)\}_{\tau\geq 0}$ of linear continuous operators on $\mathbb{E}$. Moreover, suppose that $f : I\times\mathbb{E} \to \mathbb{E}$, $g : I\times\mathbb{E} \to \mathfrak{L}(\mathbb{F},\mathbb{E})$ and $x_0 \in \mathbb{E}$. We also consider a set $\Omega \subset \mathbb{F}$, the operators $\mathcal{B} : \mathbb{F}\times\mathbb{F} \to \mathbb{F}^*$ and $h : I\times\mathbb{E} \to \mathbb{F}^*$, and the functions $\varphi : \mathbb{F}\times\mathbb{F} \to \mathbb{R}$ and $\jmath : \mathbb{F} \to \mathbb{R}$. We assume that φ is convex with respect to the second argument, that the function $\jmath$ is locally Lipschitz, and $\jmath^0$ denotes its generalized (Clarke) directional derivative. From now on, we note that $g(\tau,\cdot) = g_\tau(\cdot)$, $f(\tau,\cdot) = f_\tau(\cdot)$ and $h(\tau,\cdot) = h_\tau(\cdot)$ unless otherwise specified.

With these notations, we offer the system of coupled differential equations with a variational–hemivariational inequality problem associated with initial conditions.

To find a pair of functions (x, u) with $x : I \to \mathbb{E}$ and $u : I \to \mathbb{F}$ such that $x(0) = x_0$ and for each $\tau \in I$, $u(\tau) \in \Omega$, the following hold:

$$\begin{cases} (a)\;\; x'(\tau) = \mathcal{A}x(\tau) + f_\tau(x(\tau)) + g_\tau(x(\tau))u(\tau), \\ (b)\;\; \langle \mathcal{B}(u(\tau),u(\tau)) - h_\tau(x(\tau)), v - u(\tau)\rangle + \varphi(u(\tau),v) - \varphi(u(\tau),u(\tau)) \\ \qquad + \jmath^0(u(\tau), v - u(\tau)) \geq 0, \;\; \forall v \in \Omega. \end{cases} \tag{1}$$

For solvability of (1), we consider the following assumptions on the data:

$$\begin{cases} \mathcal{A} : \mathfrak{D}(\mathcal{A}) \subset \mathbb{E} \to \mathbb{E} \text{ is the generator of a } \vartheta_0\text{-semigroup of linear} \\ \text{and continuous operators } \{T(\tau)\}_{\tau\geq 0} \text{ on the space } \mathbb{E}. \end{cases} \tag{2}$$

$$\begin{cases} f : I\times\mathbb{E} \to \mathbb{E} \text{ is such that:} \\ (a)\;\; f(\cdot,x) : I \to \mathbb{E} \text{ is measurable for all } x \in \mathbb{E}; \\ (b)\;\; \text{there exists a positive function } \mathcal{L}_f > 0 \text{ such that} \\ \quad \|f_\tau(x) - f_\tau(y)\|_{\mathbb{E}} \leq \mathcal{L}_f\|x-y\|_{\mathbb{E}},\;\; \forall x,y \in \mathbb{E} \text{ a.e. } \tau \in I; \\ (c)\;\; \|f_\tau(o_{\mathbb{E}})\|_{\mathbb{E}} \leq a(\tau) \text{ a.e. } \tau \in I \text{ with } a \in L^1(I,\mathbb{R}^+). \end{cases} \tag{3}$$

$$\begin{cases} g : I\times\mathbb{E} \to \mathfrak{L}(\mathbb{F},\mathbb{E}) \text{ is such that:} \\ (a)\;\; g(\cdot,x) : I \to \mathfrak{L}(\mathbb{F},\mathbb{E}) \text{ is continuous for all } x \in \mathbb{E}; \\ (b)\;\; \text{inverse strongly monotone with constant } \alpha_g > 0 \text{ such that} \\ \quad \langle g_\tau(x) - g_\tau(y), x-y\rangle \geq \alpha_g\|g_\tau(x) - g_\tau(y)\|^2 \text{ for a.e. } \tau \in I \text{ all } x,y \in \mathbb{E}; \\ (c)\;\; \text{there exists a constant } \mathcal{L}_g > 0 \text{ such that} \\ \quad \|g_\tau(x) - g_\tau(y)\|_{\mathfrak{L}(\mathbb{F},\mathbb{E})} \leq \mathcal{L}_g\|x-y\|_{\mathbb{E}} \;\forall\; \tau \in I,\;\; x,y \in \mathbb{E}. \\ (d)\;\; \|g_\tau(0_{\mathbb{E}})\|_{\mathfrak{L}(\mathbb{F},\mathbb{E})} \leq d(\tau) \text{ for } \tau \in I \text{ with } d \in \mathcal{C}(I,\mathbb{R}^+). \end{cases} \tag{4}$$

$$\begin{cases} \mathcal{B} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}^* \text{ is such that:} \\ (a) \quad \mathcal{B} \text{ is pseudomonotone;} \\ (b) \quad \mathcal{B} \text{ is inverse strongly monotone with constant } \alpha_{\mathcal{B}} > 0 \text{ such that} \\ \quad \langle \mathcal{B}(v_1, v_1) - \mathcal{B}(v_2, v_2), v_1 - v_2 \rangle \geq \alpha_{\mathcal{B}} \|\mathcal{B}(v_1, v_1) - \mathcal{B}(v_2, v_2)\|_{\mathbb{F}}^2 \text{ for any } v_1, v_2 \in \mathbb{F}; \\ (c) \quad \mathcal{B} \text{ is Lipschitz continuous with respect to the first argument with constant } \beta_{\mathcal{B}} > 0 \\ \quad \text{and the second argument with respect to the constant } \rho_{\mathcal{B}} > 0 \text{ such that} \\ \quad \|\mathcal{B}(v_1, v_1) - \mathcal{B}(v_2, v_2)\|_{\mathbb{F}} \leq \beta_{\mathcal{B}} \|v_1 - v_2\|_{\mathbb{F}} + \rho_{\mathcal{B}} \|v_1 - v_2\|_{\mathbb{F}} \text{ for any } v_1, v_2 \in \mathbb{F}. \end{cases} \tag{5}$$

$$\begin{cases} h : I \times \mathbb{E} \to \mathbb{F}^* \text{ is such that:} \\ (a) \quad h(\cdot, x) : I \to \mathbb{F}^* \text{ is continuous for all } x \in \mathbb{E}; \\ (b) \quad \|h_\tau(x)\|_{\mathbb{F}^*} \leq \ell \text{ for all } x \in \mathbb{E}; \\ (c) \quad \text{there exists a constant } \mathcal{L}_h > 0 \text{ such that} \\ \quad \|h_\tau(x) - h_\tau(y)\|_{\mathbb{F}^*} \leq \mathcal{L}_h \|x - y\|_{\mathbb{E}}, \ \forall \ \tau \in I, \ x, y \in \mathbb{E}. \end{cases} \tag{6}$$

$$\begin{cases} \varphi : \mathbb{F} \times \mathbb{F} \to \mathbb{R} \text{ is such that} \\ (a) \quad \varphi(\eta, \cdot) : \mathbb{F} \to \mathbb{R} \text{ is convex and lower semicontinuous function for all } \eta \in \mathbb{F}; \\ (b) \quad \varphi(u, \lambda v) = \lambda \varphi(u, v), \forall u, v \in \mathbb{F}, \lambda > 0; \\ (c) \quad \varphi(u, u) \geq 0, \ \forall \ u \in \mathbb{F}; \\ (d) \quad \text{there exists } \alpha_\varphi > 0 \text{ such that} \\ \quad \varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \leq \alpha_\varphi \|\eta_1 - \eta_2\|_{\mathbb{F}} \|v_1 - v_2\|_{\mathbb{F}}, \\ \quad \text{for all } \eta_1, \eta_2, v_1 . v_2 \in \mathbb{F}. \end{cases} \tag{7}$$

$$\begin{cases} \jmath : \mathbb{F} \to \mathbb{R} \text{ is a locally Lipschitz continuous function, such that} \\ (a) \quad \|\xi\|_{\mathbb{F}^*} \leq \varrho_0 + \varrho_1 \|v\|_{\mathbb{F}} \text{ for all } \xi \in \partial \jmath(v), v \in \mathbb{F} \text{ with } \varrho_0, \varrho_1 \geq 0; \\ (b) \quad \text{there exists } \alpha_\jmath > 0 \text{ such that} \\ \quad \jmath^0(v_1, v_2 - v_1) + \jmath^0(v_2, v_1 - v_2) \leq \alpha_\jmath \|v_1 - v_2\|_{\mathbb{F}}^2, \ \forall \ v_1, v_2 \in \mathbb{F}. \end{cases} \tag{8}$$

$$\emptyset \neq \Omega \text{ is a closed convex subset of } \mathbb{F} \text{ such that } 0_{\mathbb{F}} \in \Omega. \tag{9}$$

$$\alpha_\varphi + \alpha_\jmath < \alpha_{\mathcal{B}} (\beta_{\mathcal{B}} + \rho_{\mathcal{B}})^2. \tag{10}$$

$$x_0 \in \mathbb{E}. \tag{11}$$

Definition 1. *A pair of functions (x, u) is said to be a solution of system (1) if $x \in \mathcal{C}(I, \mathbb{E}), u \in \mathcal{C}(I, \mathbb{F})$, (1)(b) holds for all $\tau \in I$ and*

$$x(\tau) = T(\tau)x_0 + \int_0^\tau T(\tau - \sigma)[f_\sigma(x(\sigma)) + g_\sigma(x(\sigma))] \, ds \ \ \forall \tau \in I. \tag{12}$$

Definition 2 ([16,17])**.** *An operator $\mathcal{B} : \mathbb{F} \to \mathbb{F}^\star$ is said to be*

(i) *Monotone, if*

$$\langle \mathcal{B}(u) - \mathcal{B}(v), u - v \rangle \geq 0, \ \forall u, v \in \mathbb{F},$$

(ii) *Strongly monotone, if there exists $\alpha_{\mathcal{B}} > 0$, such that*

$$\langle \mathcal{B}(u) - \mathcal{B}(v), u - v \rangle \geq \alpha_{\mathcal{B}} \|u - v\|^2, \ \forall \ u, v \in \mathbb{F},$$

(iii) *Inverse strongly monotone, if there exists $\alpha_{\mathcal{B}} > 0$, such that*

$$\langle \mathcal{B}(u) - \mathcal{B}(v), u - v \rangle \geq \alpha_{\mathcal{B}} \|\mathcal{B}(u) - \mathcal{B}(v)\|^2, \ \forall \ u, v \in \mathbb{F},$$

(iv) *Lipschitz continuous, if there exists* $\beta_{\mathcal{B}} \geq 0$*, such that*

$$\|\mathcal{B}(u) - \mathcal{B}(v)\| \leq \beta_{\mathcal{B}}\|u - v\|, \ \forall\, u, v \in \mathbb{F},$$

(v) *Bounded, if it is maps bounded sets in* $\mathbb{F}$ *into bounded sets of* $\mathbb{F}^*$*,*

(vi) *Pseudomonotone, if* $\mathcal{B}$ *is bounded and for every sequence* $\{u_n\} \subseteq \mathbb{F}$ *converging weakly to* $u \in \mathbb{F}$*, such that*

$$\limsup_{n\to\infty} \langle \mathcal{B}(u_n), u_n - u\rangle \leq 0,$$

we have

$$\liminf_{n\to\infty} \langle \mathcal{B}(u_n), u_n - v\rangle \geq \langle \mathcal{B}(u), u - v\rangle, \ \forall\, v \in \mathbb{F},$$

(vii) *Hemicontinuous, if for all* $u, v, w \in \mathbb{F}$*, the function*

$$\lambda \rightarrowtail \langle \mathcal{B}(u + \lambda v), w\rangle$$

is continuous on $[0, 1]$*,*

(viii) *Demicontinuous, if* $u_n \to u \in \mathbb{F}$ *implies*

$$\mathcal{B}(u_n) \to \mathcal{B}(u) \quad \text{weakly in } \ \mathbb{F}^*.$$

Definition 3 ([18])**.** *An operator* $\mathcal{P} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}^*$ *is said to be a penalty operator of the set* $\Omega \subset \mathbb{F}$ *if* $\mathcal{P}$ *is bounded, demicontinuous, monotone and*

$$\Omega = \{u \in \mathbb{F} \mid \mathcal{P}(u, u) = 0_{\mathbb{F}^*}\}.$$

Definition 4 ([19])**.** *A function* $\varphi : \mathbb{F} \to \mathbb{R}$ *is said to be lower semicontinuous if*

$$\liminf_{n\to\infty} \varphi(u_n) \geq \varphi(u)$$

for any sequence $\{u_n\} \subset \mathbb{F}$ *with* $u_n \to u \in \mathbb{F}$*.*

Definition 5 ([19])**.** *Let* $\{\Omega_n\}$ *be a sequence of non-empty subsets of* $\mathbb{F}$ *and* $\tilde{\Omega}$ *a nonempty subset of* $\mathbb{F}$*. If the sequence*

$$\Omega_n \xrightarrow{Mosco} \tilde{\Omega},$$

then the following conditions hold:

(i) *For each* $v \in \tilde{\Omega}$*, there exists a sequence* $\{v_n\}$ *such that* $v_n \in \Omega_n$ *for each* $n \in \mathbb{N}$ *and* $v_n \to v \in \mathbb{F}$*.*

(ii) *For each sequence* $\{v_n\}$*, such that* $v_n \in \Omega_n$ *for each* $n \in \mathbb{N}$ *and* $v_n \to v$ *weakly in* $\mathbb{F}$*, we have* $v \in \tilde{\Omega}$*.*

We shall denote the convergence in the sense of Mosco by $\Omega_n \xrightarrow{M} \Omega$ proposed in [20].

Definition 6 ([21])**.** *The Clarke generalized directional derivative of a locally Lipschitz function* $\jmath : \mathbb{F} \to \mathbb{R}$ *at* x *in the direction* v*, denoted by* $\jmath^0(x; v)$*, is defined by*

$$\jmath^0(x; v) = \lim_{y\to x} \sup_{\lambda\to 0^+} \frac{\jmath(y + \lambda v) - \jmath(y)}{\lambda}, \ \ \forall x, v \in \mathbb{F}.$$

The generalized Clarke subdifferential of $\jmath$ *at* x *is a subset of* $\mathbb{F}^*$ *given by*

$$\partial\jmath(x) = \{x^* \in \mathbb{F}^* | \jmath^0(x, v) \geq (x^*, v), \ \ \forall\, v \in \mathbb{F}\}.$$

Lemma 1 ([22]). *If $\mathcal{A} : \mathbb{F} \to \mathbb{F}^*$ is a bounded, hemicontinuous and monotone operator, then it is pseudomonotone. Moreover, if $\mathcal{A}, \mathcal{B} : \mathbb{F} \to \mathbb{F}^*$ are pseudomonotone operators, then $\mathcal{A} + \mathcal{B} : \mathbb{F} \to \mathbb{F}^*$ is pseudomonotone, too.*

Lemma 2 ([21]). *Let $\jmath : \mathbb{F} \to \mathbb{R}$ be a locally Lipschitz function. Then, the following statements hold:*

(1) $\jmath^0(x, v) = \max\{(\xi, v) | \xi \in \partial\jmath(x)\}, \ \forall x, v \in \mathbb{F}.$

(2) *For each $x \in \mathbb{F}$, the function $\mathcal{U} \ni v \rightarrowtail \jmath^0(x, v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e.,*
$$\jmath^0(x, \lambda v) = \lambda\jmath^0(x, v), \ \forall \ \lambda \geq 0, v \in \mathcal{U}$$
and
$$\jmath^0(x, v_1 + v_2) \leq \jmath^0(x, v_1) + \jmath^0(x, v_2), \ \forall v_1, v_2 \in \mathbb{F}, \text{ respectively.}$$

Theorem 1 ([23]). *Assume that (2)–(11) hold. Then, there exists a unique solution $(x, u) \in \mathcal{C}(I, \mathbb{E}) \times \mathcal{C}(I, \mathbb{F})$ to problem (1).*

3. Main Results

In this section, we define a sequence of penalty problems (1) in order to prove their unique solvability and prove the convergence of the sequence of their solutions to the unique solution of (1). To this end, we examine an operator $\mathcal{P} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}^*$, two sequences $\{\Omega_n\} \subset \mathbb{F}, \{\gamma_n\} \subset \mathbb{R}$ and, for each $n \in \mathbb{N}$, the differential variational–hemivariational inequality problem for finding a pair of functions (x_n, u_n) with $x_n : I \to \mathbb{E}$ and $u_n : I \to \mathbb{F}$, such that
$$x_n(0) = x_0 \text{ and for each } \tau \in I, \ u_n(\tau) \in \Omega_n, \text{ it asserts that}$$
$$\begin{cases} (a) \quad x_n'(\tau) = \mathcal{A}x_n(\tau) + f_\tau(x_n(\tau)) + g_\tau(x_n(\tau))u_n(\tau), \\ (b) \quad \langle \mathcal{B}(u_n(\tau), u_n(\tau)) - h_\tau(x_n(\tau)), v - u_n(\tau)\rangle + \dfrac{1}{\gamma_n}\langle \mathcal{P}(u_n(\tau), u_n(\tau)), v - u_n(\tau)\rangle \\ \qquad + \varphi(u_n(\tau), v) - \varphi(u_n(\tau), u_n(\tau)) + \jmath^0(u_n(\tau), v - u_n(\tau)) \geq 0, \ \text{ for all } v \in \Omega_n. \end{cases} \tag{13}$$

The pair of functions (x_n, u_n) is said to be a solution to (13) if $x_n \in \mathcal{C}(I, \mathbb{E})$ and $u_n \in \mathcal{C}(I, \mathbb{F})$ and (13)(b) hold for all $\tau \in I$ and
$$x_n(\tau) = T(\tau)x_0 + \int_0^\tau T(\tau - \sigma)[f_\sigma(x_n(\sigma)) + g_\sigma(x_n(\sigma))u_n(\sigma)]ds, \ \ \tau \in I. \tag{14}$$

We evaluate the following hypotheses on the data in the research of (13).
$$\Big\{ \text{For every } n \in \mathbb{N}, \ \emptyset \neq \Omega_n \text{ is a closed convex subset of } \ \mathbb{F} \ \text{ and } \ \Omega_n \supset \Omega. \tag{15}$$
$$\text{For every } \ n \in \mathbb{N}, \ \gamma_n > 0. \tag{16}$$
$$\mathcal{P} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}^* \ \text{ is a bounded, demicontinuous and monotone operator.} \tag{17}$$
$$\begin{cases} \text{There exists a set } \tilde{\Omega}, \text{ such that} \\ (a) \quad \Omega_n \subset \tilde{\Omega} \subset \mathbb{F} \ \text{ for each } n \in \mathbb{N}. \\ (b) \quad \Omega_n \xrightarrow{M} \tilde{\Omega} \ \text{ as } \ n \to \infty. \\ (c) \quad \langle \mathcal{P}(u, u), v - u\rangle \leq 0, \ \forall u \in \tilde{\Omega} \text{ and } \ v \in \Omega. \\ (d) \quad \text{if } u \in \tilde{\Omega} \text{ and } \langle \mathcal{P}(u, u), v - u\rangle = 0, \ \forall \ v \in \Omega \ \text{ then } u \in \Omega. \end{cases} \tag{18}$$
$$\gamma_n \to 0 \ \text{ as } \ n \to \infty. \tag{19}$$

$$\begin{cases} \text{There exists a function } \chi_\varphi : \Omega \to \mathbb{R}^+, \text{ such that} \\ (a) \quad \varphi(u, v_1) - \varphi(u, v_2) \leq \chi_\varphi(u)\|v_1 - v_2\|_{\mathbb{F}}, \forall u, v_1, v_2 \in \mathbb{F}. \end{cases} \tag{20}$$

$$\begin{cases} \limsup_{n\to\infty} \jmath^0(u_n, v - u_n) \leq \jmath^0(u, v - u) \\ \text{as } u_n \xrightarrow{weakly} u \in \mathbb{F}, \ \forall u, v \in \mathbb{F}. \end{cases} \tag{21}$$

The main result of this paper is as follows.

Theorem 2. *Assume that (2)–(11), (15)–(21) hold. Then*

(1) For $n \in \mathbb{N}$, there exists a unique solution $(x_n, u_n) \in \mathcal{C}(I, \mathbb{E}) \times \mathcal{C}(I, \mathbb{F})$ to the problem (13).

(2) For $\tau \in I$, the solution (x_n, u_n) of the problem (13) converges to the solution (x, u) of the problem (1), i.e.,

$$(x_n(\tau), u_n(\tau)) \to (x(\tau), u(\tau)) \ \in \ \mathbb{E} \times \mathbb{F}, \ \text{as } \ n \to \infty. \tag{22}$$

Proof. (1) Let $n \in \mathbb{N}$ and consider the function $\mathcal{B}_n : \mathbb{F} \times \mathbb{F} \to \mathbb{F}^*$ defined by

$$\mathcal{B}_n(\cdot,\cdot) = \mathcal{B}(\cdot,\cdot) + \frac{1}{\gamma_n}\mathcal{P}(\cdot,\cdot).$$

Under the hypotheses (17), (19) and Lemma 1, it is simple to see that $\mathcal{B}_n$ is pseudomonotone, inversely strongly monotone and Lipschitz continuous with respect to both arguments with constants $\alpha_{\mathcal{B}}, \beta_{\mathcal{B}}$ and $\rho_{\mathbb{B}}$, *respectively*. Using Theorem 1 with Ω_n and $\mathcal{B}_n$ instead of Ω and $\mathcal{B}$, *respectively*, we determine that there exists a unique solution $(x_n, u_n) \in \mathcal{C}(I, \mathbb{E}) \times \mathcal{C}(I, \mathbb{F})$ to (13).

(2) Fixing $n \in \mathbb{N}$, we consider the auxiliary problem of finding a function $\tilde{u}_n \in C(I, \mathbb{F})$, such that

$$\begin{aligned} \tilde{u}_n(\tau) \in \Omega_n, \ \langle \mathcal{B}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)) - h_\tau(x(\tau)), v - \tilde{u}_n(\tau)\rangle + \frac{1}{\gamma_n}\langle \mathcal{P}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), v - \tilde{u}_n(\tau)\rangle \\ + \varphi(u(\tau), v) - \varphi(u(\tau), \tilde{u}_n(\tau)) + \jmath^0(\tilde{u}_n(\tau), v - \tilde{u}_n(\tau)) \geq 0, \ \forall v \in \Omega_n, \tau \in I. \end{aligned} \tag{23}$$

Utilizing a standard arguments, we see that Equation (23) has a unique solution $\tilde{u}_n \in \mathcal{C}(I, \mathbb{F})$.

The rest of the proof is now divided into five steps. Here, assume that $\Omega_n \neq \Omega$ and $\mathcal{P}$ satisfies (18)(c),(d).

Step (i) We assert that for any $\tau \in I$, there exists $\tilde{u}(\tau) \in \tilde{\Omega}$ and a subsequence of $\{\tilde{u}_n(\tau)\}$, again denoted by $\{\tilde{u}_n(\tau)\}$, such that

$$\tilde{u}_n(\tau) \xrightarrow{weakly} \tilde{u}(\tau) \ \in \mathbb{F} \ \text{ as } n \to \infty. \tag{24}$$

To fix $\tau \in I, n \in \mathbb{N}$ and $u_0 \in \Omega$. We put $v = u_0$ in (23) to obtain

$$\begin{aligned} &\langle \mathcal{B}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)) - h_\tau(x(\tau)), u_0 - \tilde{u}_n(\tau)\rangle + \frac{1}{\gamma_n}\langle \mathcal{P}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), u_0 - \tilde{u}_n(\tau)\rangle \\ &+ \varphi(u(\tau), u_0) - \varphi(u(\tau), \tilde{u}_n(\tau)) + \jmath^0(\tilde{u}_n(\tau), u_0 - \tilde{u}_n(\tau)) \geq 0. \end{aligned}$$

Using (18)(c) and (20) we have

$$\begin{aligned} \langle \mathcal{B}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)) - h_\tau(x(\tau)), \tilde{u}_n(\tau) - u_0\rangle \leq \chi_\varphi(u(\tau))\|u_0 - \tilde{u}_n(\tau)\|_{\mathbb{F}} \\ + \jmath^0(\tilde{u}_n(\tau), u_0 - \tilde{u}_n(\tau)). \end{aligned} \tag{25}$$

Next, from (8) and Lemma 2(1), we get

$$J^0(\tilde{u}_n(\tau), u_0 - \tilde{u}_n(\tau)) \le \alpha_J \|u_0 - \tilde{u}_n(\tau)\|_{\mathbb{F}}^2 + (\varrho_0 + \varrho_1 \|u_0\|_{\mathbb{F}}) \|u_0 - \tilde{u}_n(\tau)\|_{\mathbb{F}}. \quad (26)$$

Furthermore, using (5), (6)(b), (20), (25) and (26), we obtain

$$\begin{aligned} \alpha_{\mathcal{B}}(\beta_{\mathcal{B}} + \rho_{\mathbb{B}})^2 \|\tilde{u}_n(\tau) - u_0\|_{\mathbb{F}}^2 \le{}& \ell \|u_0 - \tilde{u}_n(\tau)\|_{\mathbb{F}} + \chi_\varphi(u(\tau)) \|u_0 - \tilde{u}_n(\tau)\|_{\mathbb{F}} \\ &+ \alpha_J \|u_0 - \tilde{u}_n(\tau)\|_{\mathbb{F}}^2 + (\varrho_0 + \varrho_1 \|u_0\|_{\mathbb{F}}) \|u_0 - \tilde{u}_n(\tau)\|_{\mathbb{F}} \\ &+ \|\mathcal{B}(u_0, u_0)\|_{\mathbb{F}^*} \|u_0 - \tilde{u}_n(\tau)\|_{\mathbb{F}}. \end{aligned} \quad (27)$$

Adding (27) together with (10) to get

$$\|\tilde{u}_n(\tau) - u_0\|_{\mathbb{F}} \le \frac{\Upsilon_0}{\alpha_{\mathcal{B}}(\beta_{\mathcal{B}} + \rho_{\mathbb{B}})^2 - \alpha_J}, \quad (28)$$

where

$$\Upsilon_0 = \ell + \chi_\varphi(u(\tau)) + \varrho_0 + \varrho_1 \|u_0\|_{\mathbb{F}} + \|\mathcal{B}(u_0, u_0)\|_{\mathbb{F}^*}.$$

Since Υ_0 depends on τ but does not depend on n, this implies that the sequence $\{\tilde{u}_n(\tau)\}$ is bounded in $\mathbb{F}$. Hence, the reflexivity of $\mathbb{F}$ implies that there exists an element $\tilde{u}(\tau) \in \mathbb{F}$ such that, passing to a subsequence if necessary, we find that

$$\tilde{u}_n(\tau) \xrightarrow{weakly} \tilde{u}(\tau) \in \mathbb{F} \text{ as } n \to \infty.$$

Since $\tilde{u}_n(\tau) \in \Omega_n$, therefore, the elimination of (18)(b) and Definition 5(ii) reveals that

$$\tilde{u}(\tau) \in \tilde{\Omega}.$$

Step (ii) We prove that $\tilde{u}(\tau) \in \Omega$ for all $\tau \in I$.
Let $n \in \mathbb{N}, \tau \in I$ and $v \in \Omega$. Then, Definition 5(i) assures us that there is a sequence $\{v_n\}$ such that $v_n \in \Omega_n$ for each $n \in \mathbb{N}$ and $v_n \to v \in \mathbb{F}$ as $n \to \infty$. We will utilize (23) and similar estimates from the previous step to get

$$\begin{aligned} \frac{1}{\gamma_n} \langle \mathcal{P}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - v_n \rangle &\le \alpha_J \|v_n - \tilde{u}_n(\tau)\|_{\mathbb{F}}^2 \\ &+ \left(\|\mathcal{B}(\tilde{u}_n(\tau), \tilde{u}_n(\tau))\|_{\mathbb{F}^*} + \ell + \chi_\varphi(u(\tau)) + \varrho_0 + \varrho_1 \|v_n\|_{\mathbb{F}} \right) \|v_n - \tilde{u}_n(\tau)\|_{\mathbb{F}}. \end{aligned}$$

Since $\{v_n\}$, $\{\tilde{u}_n(\tau)\}$ are bounded sequences and $\mathcal{B}$ is a bounded operator. Therefore, there exists a constant $\vartheta_0 > 0$ which does not depend on n, such that

$$\frac{1}{\gamma_n} \langle \mathcal{P}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - v_n \rangle \le \tilde{\vartheta}_0.$$

Hence,

$$\limsup_{n \to \infty} \langle \mathcal{P}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - v_n \rangle \le 0. \quad (29)$$

Again, since the sequence $\{\mathcal{P}(\tilde{u}_n(\tau), \tilde{u}_n(\tau))\}$ is bounded in $\mathbb{F}^*$ and $v_n \to v \in \mathbb{F}$, we have that

$$\begin{aligned} \limsup_{n \to \infty} \langle \mathcal{P}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - v \rangle \le{}& \limsup_{n \to \infty} \langle \mathcal{P}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - v_n \rangle \\ &+ \limsup_{n \to \infty} \langle \mathcal{P}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - v \rangle \\ ={}& \limsup_{n \to \infty} \langle \mathcal{P}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - v_n \rangle. \end{aligned}$$

Therefore, (29) yields

$$\limsup_{n\to\infty}\langle \mathcal{P}(\tilde{u}_n(\tau),\tilde{u}_n(\tau)),\tilde{u}_n(\tau)-v\rangle \le 0,\quad \forall v\in\Omega. \tag{30}$$

Moreover, the regularity of $\tilde{u}(\tau)\in\Omega$ allows us to take $v=\tilde{u}(\tau)$ in (30) to get

$$\limsup_{n\to\infty}\langle \mathcal{P}(\tilde{u}_n(\tau),\tilde{u}_n(\tau)),\tilde{u}_n(\tau)-\tilde{u}(\tau)\rangle \le 0. \tag{31}$$

However, the assumption (17) and Lemma 1 ensures that $\mathcal{P}$ is a pseudomonotone operator. From (31) and the pseudomonotonicity of $\mathcal{P}$, we have

$$\begin{aligned}\langle \mathcal{P}(\tilde{u}_n(\tau),\tilde{u}_n(\tau)),\tilde{u}_n(\tau)-v\rangle &\le \liminf_{n\to\infty}\langle \mathcal{P}(\tilde{u}_n(\tau),\tilde{u}_n(\tau)),\tilde{u}_n(\tau)-v\rangle\\ &\le \limsup_{n\to\infty}\langle \mathcal{P}(\tilde{u}_n(\tau),\tilde{u}_n(\tau)),\tilde{u}_n(\tau)-v\rangle,\forall v\in\mathbb{F}.\end{aligned}$$

Therefore, (30) yields

$$\langle \mathcal{P}(\tilde{u}(\tau),\tilde{u}(\tau)),\tilde{u}(\tau)-v\rangle \le 0,\forall v\in\tilde{\Omega}. \tag{32}$$

Since $\Omega\subset\tilde{\Omega}$, therefore, from (32), we derive that

$$\langle \mathcal{P}(\tilde{u}(\tau),\tilde{u}(\tau)),\tilde{u}(\tau)-v\rangle \le 0,\forall v\in\Omega. \tag{33}$$

Now, combining (33) with (18)(c) to get

$$\langle \mathcal{P}(\tilde{u}(\tau),\tilde{u}(\tau)),\tilde{u}(\tau)-v\rangle = 0,\forall v\in\Omega.$$

Hence, using (18)(d) to obtain the regularity

$$\tilde{u}(\tau)\in\Omega. \tag{34}$$

Step (iii) We now prove that $\tilde{u}_n(\tau)\rightharpoonup u(\tau)\in\mathbb{F}$, for all $\tau\in I$.
Let $n\in\mathbb{N},\tau\in I$ and $v\in\Omega$. We use Equation (23) and inclusion $\Omega\subset\Omega_n$ to see that

$$\begin{aligned}\langle \mathcal{B}(\tilde{u}_n(\tau),\tilde{u}_n(\tau)),\tilde{u}_n(\tau)-v\rangle \le &-\langle h_\tau(x(\tau)),v-\tilde{u}_n(\tau)\rangle\\ &+\frac{1}{\gamma_n}\langle \mathcal{P}(\tilde{u}_n(\tau),\tilde{u}_n(\tau)),v-\tilde{u}_n(\tau)\rangle+\varphi(u(\tau),v)\\ &-\varphi(u(\tau),\tilde{u}_n(\tau))+J^0(\tilde{u}_n(\tau),v-\tilde{u}_n(\tau)),\end{aligned}$$

and using (18)(c), we have

$$\begin{aligned}\langle \mathcal{B}(\tilde{u}_n(\tau),\tilde{u}_n(\tau)),\tilde{u}_n(\tau)-v\rangle \le &-\langle h_\tau(x(\tau)),v-\tilde{u}_n(\tau)\rangle+\varphi(u(\tau),v)\\ &-\varphi(u(\tau),\tilde{u}_n(\tau))+J^0(\tilde{u}_n(\tau),v-\tilde{u}_n(\tau)).\end{aligned} \tag{35}$$

Then, we use the lower semicontinuity of φ concerning the second argument and the hypothesis (21) to find that

$$\begin{aligned}\limsup_{n\to\infty}\langle \mathcal{B}(\tilde{u}_n(\tau),\tilde{u}_n(\tau)),\tilde{u}_n(\tau)-v\rangle \le &-\langle h_\tau(x(\tau)),v-\tilde{u}_n(\tau)\rangle+\varphi(u(\tau),v)\\ &-\varphi(u(\tau),\tilde{u}_n(\tau))+J^0(\tilde{u}_n(\tau),v-\tilde{u}_n(\tau)).\end{aligned} \tag{36}$$

Again, we put $v=\tilde{u}(\tau)\in\Omega$ in (36) to obtain that

$$\limsup_{n\to\infty}\langle \mathcal{B}(\tilde{u}_n(\tau),\tilde{u}_n(\tau)),\tilde{u}_n(\tau)-\tilde{u}(\tau)\rangle \le 0. \tag{37}$$

Together with the pseudomonotonicity of operator $\mathcal{B}$, this inequality implies that

$$\langle \mathcal{B}(\tilde{u}(\tau), \tilde{u}(\tau)), \tilde{u}(\tau) - v \rangle \leq \liminf_{n \to \infty} \langle \mathcal{B}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - v \rangle. \tag{38}$$

Now, adding (36) and (38) to get

$$\begin{aligned}\langle \mathcal{B}(\tilde{u}(\tau), \tilde{u}(\tau)), \tilde{u}(\tau) - v \rangle \leq -\langle h_\tau(x(\tau)), v - \tilde{u}(\tau) \rangle + \varphi(u(\tau), v) - \varphi(u(\tau), \tilde{u}(\tau)) \\ + \jmath^0(\tilde{u}(\tau), v - \tilde{u}(\tau)).\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \mathcal{B}(\tilde{u}(\tau), \tilde{u}(\tau)) - h_\tau(x(\tau)), v - \tilde{u}(\tau) \rangle + \varphi(u(\tau), v) - \varphi(u(\tau), \tilde{u}(\tau)) \\ + \jmath^0(\tilde{u}(\tau), v - \tilde{u}(\tau)) \geq 0.\end{aligned} \tag{39}$$

We take $v = \tilde{u}(\tau)$ in (1)(b) and $v = u(\tau)$ in (39), then we add the resulting inequalities to see that

$$\begin{aligned}\langle \mathcal{B}(u(\tau), u(\tau)) - \mathcal{B}(\tilde{u}(\tau), \tilde{u}(\tau)), \tilde{u}(\tau) - u(\tau) \rangle + \jmath^0(u(\tau), \tilde{u}(\tau) - u(\tau)) \\ + \jmath^0(\tilde{u}(\tau), u(\tau) - \tilde{u}(\tau)) \geq 0.\end{aligned}$$

Then, we use assumptions (5) and (8)(b) to find that

$$(\alpha_{\mathcal{B}}(\beta_{\mathcal{B}} + \rho_{\mathcal{B}})^2 - \alpha_\jmath)\|\tilde{u}(\tau) - u(\tau)\|_{\mathbb{F}} \leq 0.$$

This inequality, together with (10), implies that

$$\tilde{u}(\tau) = u(\tau).$$

Meanwhile, each weakly convergent subsequence of the sequence $\{\tilde{u}_n(\tau)\}$ converges weakly to $u(\tau)$ as $n \to \infty$. Furthermore, since the sequence $\{\tilde{u}_n(\tau)\}$ is bounded, it imply that the whole sequence $\{\tilde{u}_n(\tau)\}$ converges weakly to $u(\tau)$.

Step (iv) We now prove that $\tilde{u}_n(\tau) \to u(\tau) \in \mathbb{F}, \forall \tau \in I$.
Let $\tau \in I$. Since $\tilde{u}(\tau) = u(\tau)$, putting $v = u(\tau)$ in (38) and using (37), we get

$$\liminf_{n \to \infty} \langle \mathcal{B}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - u(\tau) \rangle \geq 0$$

and

$$\limsup_{n \to \infty} \langle \mathcal{B}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - u(\tau) \rangle \leq 0,$$

imply that

$$\langle \mathcal{B}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - u(\tau) \rangle \to 0.$$

Hence, from (5)(b),(c) and $\tilde{u}_n(\tau) \to u(\tau)$ weakly in $\mathbb{F}$, we have

$$\begin{aligned}\alpha_{\mathcal{B}}(\beta_{\mathcal{B}} + \rho_{\mathbb{B}})^2\|\tilde{u}_n(\tau) - u(\tau)\|_{\mathbb{F}}^2 &\leq \langle \mathcal{B}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)) - \mathcal{B}(u(\tau), u(\tau)), \tilde{u}_n(\tau) - u(\tau) \rangle \\ &= \langle \mathcal{B}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - u(\tau) \rangle \\ &\quad - \langle \mathcal{B}(u(\tau), u(\tau)), \tilde{u}_n(\tau) - u(\tau) \rangle \to 0.\end{aligned}$$

The proof of this step is completed.

Step (v) Finally, we prove that $(x_n(\tau), u_n(\tau)) \to (x(\tau), u(\tau)) \in \mathbb{E} \times \mathbb{F}, \forall \tau \in I$. Let $\tau \in I$ and $n \in \mathbb{N}$. We write (1)(b) with $v = u_n(\tau)$. Then, we take (13)(b) with $v = \tilde{u}_n(\tau)$ and add the resulting inequalities to see that

$$\begin{aligned}
&\langle \mathcal{B}(u_n(\tau), u_n(\tau)) - \mathcal{B}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - u_n(\tau) \rangle \\
&- \langle h_\tau(x_n(\tau)) - h_\tau(x(\tau)), \tilde{u}_n(\tau) - u_n(\tau) \rangle \\
&+ \frac{1}{\gamma_n} \langle \mathcal{P}(u_n(\tau), u_n(\tau)) - \mathcal{P}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)), \tilde{u}_n(\tau) - u_n(\tau) \rangle \\
&+ \varphi(u_n(\tau), \tilde{u}_n(\tau)) - \varphi(u_n(\tau), u_n(\tau)) + \varphi(u(\tau), u_n(\tau)) - \varphi(u(\tau), \tilde{u}_n(\tau)) \\
&+ \jmath^0(u_n(\tau), \tilde{u}_n(\tau) - u_n(\tau)) + \jmath^0(\tilde{u}_n(\tau), u_n(\tau) - \tilde{u}_n(\tau)) \geq 0.
\end{aligned}$$

Therefore, (6)–(8) and the monotonicity of the operator $\mathcal{P}$ yield

$$\begin{aligned}
\alpha_{\mathcal{B}}(\beta_{\mathcal{B}} + \rho_{\mathbb{B}})^2 \|\tilde{u}_n(\tau) - u_n(\tau)\|_{\mathbb{F}}^2 &\leq \langle \mathcal{B}(\tilde{u}_n(\tau), \tilde{u}_n(\tau)) - \mathcal{B}(u_n(\tau), u_n(\tau)), \tilde{u}_n(\tau) - u_n(\tau) \rangle \\
&\leq \mathcal{L}_h \|x_n(\tau) - x(\tau)\|_{\mathbb{E}} \|\tilde{u}_n(\tau) - u_n(\tau)\|_{\mathbb{F}} \\
&+ \alpha_\varphi \|u_n(\tau) - u(\tau)\|_{\mathbb{F}} \|\tilde{u}_n(\tau) - u_n(\tau)\|_{\mathbb{F}} \\
&+ \alpha_\jmath \|\tilde{u}_n(\tau) - u_n(\tau)\|_{\mathbb{F}}^2.
\end{aligned}$$

Thereby,

$$\begin{aligned}
\|\tilde{u}_n(\tau) - u_n(\tau)\|_{\mathbb{F}} \leq &\frac{\mathcal{L}_h}{\alpha_{\mathcal{B}}(\beta_{\mathcal{B}} + \rho_{\mathbb{B}})^2 - \alpha_\jmath} \|x_n(\tau) - x(\tau)\|_{\mathbb{E}} \\
&+ \frac{\alpha_\varphi}{\alpha_{\mathcal{B}}(\beta_{\mathcal{B}} + \rho_{\mathbb{B}})^2 - \alpha_\jmath} \|u_n(\tau) - u(\tau)\|_{\mathbb{F}}.
\end{aligned} \tag{40}$$

Hence,

$$\|u_n(\tau) - u(\tau)\|_{\mathbb{F}} \leq \|u_n(\tau) - \tilde{u}_n(\tau)\|_{\mathbb{F}} + \|\tilde{u}_n(\tau) - u(\tau)\|_{\mathbb{F}}.$$

Therefore, from (10) and (40), we derive that

$$\begin{aligned}
\left(1 - \frac{\alpha_\varphi}{\alpha_{\mathcal{B}}(\beta_{\mathcal{B}} + \rho_{\mathbb{B}})^2 - \alpha_\jmath}\right) \|u_n(\tau) - u(\tau)\|_{\mathbb{F}} \leq &\frac{\mathcal{L}_h}{\alpha_{\mathcal{B}}(\beta_{\mathcal{B}} + \rho_{\mathbb{B}})^2 - \alpha_\jmath} \|x_n(\tau) - x(\tau)\|_{\mathbb{E}} \\
&+ \|\tilde{u}_n(\tau) - u(\tau)\|_{\mathbb{F}},
\end{aligned}$$

which show that there exist two constants, $\zeta_0 > 0$ and $\zeta_1 > 0$, such that

$$\|u_n(\tau) - u(\tau)\|_{\mathbb{F}} \leq \zeta_0 \|x_n(\tau) - x(\tau)\|_{\mathbb{E}} + \zeta_1 \|\tilde{u}_n(\tau) - u(\tau)\|\mathbb{F}. \tag{41}$$

Meanwhile, using (3), (4), (12), (14), and (41), we find that there exist two constants, $\tilde{\zeta}_0 > 0$ and $\tilde{\zeta}_1 > 0$, such that

$$\|x_n(\tau) - x(\tau)\|_{\mathbb{E}} \leq \tilde{\zeta}_0 \int_0^\tau \|\tilde{u}_n(\sigma) - u(\sigma)\|_{\mathbb{F}}\, d\sigma + \tilde{\zeta}_1 \int_0^\tau \|x_n(\sigma) - x(\sigma)\|_{\mathbb{E}}\, d\sigma.$$

As a result of Gronwall inequality, it follows that there exists a constant $\zeta > 0$, such that

$$\|x(\tau) - x_n(\tau)\|_{\mathbb{E}} \leq \zeta \int_0^\tau \|\tilde{u}_n(\sigma) - u(\sigma)\|_{\mathbb{F}}\, d\sigma.$$

This inequality, the convergence $\tilde{u}_n(\sigma) \to u(\sigma) \in \mathbb{F}$, valid for each $\sigma \in [0, T]$, and the Lebesgue-dominated convergence theorem (see [13], Theorem 1.65) imply that

$$\lim \|x(\tau) - x_n(\tau)\|_{\mathbb{E}} \leq \zeta \int_0^\tau \lim \|\tilde{u}_n(\sigma) - u(\sigma)\|_{\mathbb{F}}\, d\sigma = 0.$$

Therefore, we conclude that

$$x_n(\tau) \to x(\tau) \in \mathbb{E}.$$

Using this convergence, we have

$$\tilde{u}_n(\tau) \to u \in \mathbb{F}$$

demonstrated in Step (iv), and from (41), we derive that

$$u_n(\tau) \to u(\tau) \in \mathbb{F}$$

and proof is completed.

□

4. A Mathematical Model for a Viscoelastic Rod in Unilateral Contact

In this section, we consider the viscoelastic rod defined on the interval $[0, L]$ on the Oz axis. The rod is fixed in $z = 0$ and is acted upon by body time-dependent forces of density f_b along Oz. Its extremity $z = L$ is in contact with an obstacle made of a rigid body covered by a rigid elastic layer of thickness $\omega > 0$. The time interval of interest is $I = [0, T]$ with $T > 0$. We denote by a prime the derivative with respect to the time variable $\tau \in I$ and by the subscript z the derivative with respect to the spatial variable $z \in [0, L]$, i.e., $x' = \frac{\partial x}{\partial \tau}$ and $u_z = \frac{\partial u}{\partial z}$.

Now, we depict the contact problem for finding a displacement field $u : [0, T] \times [0, L] \to \mathbb{R}$ and a stress field $\pi : [0, T] \times [0, L] \to \mathbb{R}$, such that

$$\pi(\tau, z) = \kappa u_z(\tau, z) + \tilde{h} \int_0^\tau \tilde{g}(\sigma) u_z(\sigma, z) d\sigma, \tag{42}$$

where the viscoelastic constitutive law in which $\kappa > 0$ is the Young modulus of the material and $\tilde{h}, \tilde{g}$ are constitutive functions. The equation

$$\pi_z(\tau, z) + f_b(\tau, z) = 0, \;\; \forall \tau \in I, z \in [0, L], \tag{43}$$

where f_b denotes the density of body forces acting on the rod, and

$$u(\tau, 0) = 0, \;\; \forall \tau \in I. \tag{44}$$

represents the displacement condition where the rod is assumed to be fixed at $z = 0$.

$$\begin{cases} u(\tau, L) \leq \omega, \;\; \forall \tau \in I \\ \pi(\tau, L) = 0 & \text{if} \;\; u(\tau, L) < 0, \\ -Q \leq \pi(\tau, L) \leq 0 & \text{if} \;\; u(\tau, L) = 0, \\ -\pi(\tau, L) = Q + p_e(u(\tau, L)) & \text{if} \;\; 0 < u(\tau, L) < \omega, \\ -\pi(\tau, L) \geq Q + p_e(u(\tau, L)) & \text{if} \;\; u(\tau, L) = \omega, \end{cases} \tag{45}$$

where the conditions of the contact of the point $z = L$ of the rod with a rigid body covered by a layer made of rigid elastic material, (*say*, a crust) and ω is the thickness of this layer, Q is its yield limit and p_e is a real-valued function that describes the elastic properties.

Using the notation $\varepsilon = u_z$, Equation (42) reads as

$$\pi(\tau) = \kappa \varepsilon(\tau) + \tilde{h}\left(\int_0^\tau \tilde{g}(\sigma) \varepsilon(\sigma) d\sigma \right), \tag{46}$$

where τ is the stress field and $\pi(\tau)$ can be split in two parts: an elastic part $\pi^{\mathbb{E}}(\tau) = \kappa \varepsilon(\tau)$ and an anelastic part $\pi^{\mathcal{A}N}(\tau) = \tilde{h}\left(\int_0^\tau \tilde{g}(\sigma) \varepsilon(\sigma) d\sigma \right)$.

We make the following assumptions on the data to investigate problem (42)–(45).

$$\begin{cases} \tilde{h} : \mathbb{R} \to \mathbb{R} \text{ is such that} \\ (a) \quad \text{there exists } \mathcal{L}_{\tilde{h}} > 0 \text{ such that} \\ \quad |\tilde{h}(\theta_1) - \tilde{h}(\theta_2)| \leq \mathcal{L}_{\tilde{h}} |\theta_1 - \theta_2|, \ \forall \theta_1, \theta_2 \in \mathbb{R}. \\ (b) \quad \text{there exists } I_{\tilde{h}} > 0 \text{ such that} \\ \quad |\tilde{h}(\theta)| \leq l_{\tilde{h}}, \ \forall \theta \in \mathbb{R}. \end{cases} \tag{47}$$

$$\tilde{g} : [0, T] \to \mathbb{R} \text{ is a continuous function.} \tag{48}$$

$$\begin{cases} (a) \quad p_e : \mathbb{R} \to \mathbb{R} \text{ is a continuous function.} \\ (b) \quad \text{There exist } \omega_0, \omega_1 \geq 0 \text{, such that} \\ \quad |p_e(\theta)| \leq \omega_0 + \omega_1 |\theta| \ \forall \theta \in \mathbb{R}. \\ (c) \quad \text{There exists } \alpha_e > 0 \text{, such that} \\ \quad \theta \rightarrowtail \alpha_e \theta + p_e(\theta) \text{ is nondecreasing.} \\ (d) \quad p_e(\theta) \geq 0 \text{ if } \theta > 0 \text{ and } p_e(\theta) = 0 \text{ if } \theta \leq 0. \end{cases} \tag{49}$$

$$f_b \in \mathcal{C}(I, L^2(0, L)), \omega > 0, \kappa > 0, Q \geq 0. \tag{50}$$

The real Hilbert spaces $\mathbb{E}$ and $\mathbb{F}$ are depicted as

$$\begin{cases} \mathbb{E} = L^2(0, L), \\ \mathbb{F} = \{v \in H^1(0, L) \mid v(0) = 0\} \end{cases} \tag{51}$$

with the inner products

$$(x, y)_{\mathbb{E}} = \int_0^L x(z) y(z) dz, \forall x, y \in \mathbb{E},$$

$$(u, v)_{\mathbb{F}} = \int_0^L u_z v_z dz, \forall u, v \in \mathbb{F}$$

and the associated norms $\| \cdot \|_{\mathbb{E}}$ and $\| \cdot \|_{\mathbb{F}}$, *respectively*. Moreover, based on the Sobolev trace theorem, it follows that

$$|v(L)| \leq \sqrt{L} \|v\|_{\mathbb{F}}, \ \forall v \in \mathbb{F}. \tag{52}$$

The duality of $\mathbb{F}$ is denoted by $\mathbb{F}^*$ and $\langle \cdot, \cdot \rangle$ by the duality pairing between $\mathbb{F}^*$ and $\mathbb{F}$, *respectively*, and the positive component of r is denoted by r^+.

Next, define the set Ω, the operators $\mathcal{A} : \mathbb{E} \to \mathbb{E}$, $\mathcal{B} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}^*$ and the functions $\varphi : \mathbb{F} \times \mathbb{F} \to \mathbb{R}$, $q : \mathbb{R} \to \mathbb{R}$, $\jmath : \mathbb{F} \to \mathbb{R}$, $f : I \times \mathbb{E} \to \mathbb{E}$, $g : I \times \mathbb{E} \to \mathfrak{L}(\mathbb{E}, \mathbb{F})$, $h : I \times \mathbb{E} \to \mathbb{F}^*$ by equalities

$$\mathcal{A} : \mathfrak{D}(\mathcal{A}) = \mathbb{E} \to \mathbb{E}, \quad \mathcal{A}x = x \forall x \in \mathbb{E}, \tag{53}$$

$$\Omega = \{u \in \mathbb{F} \mid u(L) \leq \omega\}, \tag{54}$$

$$\langle \mathcal{B}(u, u), v \rangle = \kappa \int_0^L u_z v_z dz, \ \forall u, v \in \mathbb{F}, \tag{55}$$

$$\varphi(u, v) = Q v^+(L) \ \forall u, v \in \mathbb{F}, \tag{56}$$

$$q(r) = \int_0^r p_e(\sigma) d\sigma \ \forall r \in \mathbb{R}, \tag{57}$$

$$\jmath(v) = q(v(L)) \;\; \forall v \in \mathbb{F}, \tag{58}$$

$$f_\tau(x)(z) = -x(z) \;\; \forall \tau \in I, x \in \mathbb{E}, \;\; \text{a.e. } z \in (0, L), \tag{59}$$

$$[g_\tau(x)u](z) = \tilde{g}_\tau u_z(z) \;\; \forall \tau \in I, x \in \mathbb{E}, u \in \mathbb{F}, \;\; \text{a.e. } z \in (0, L), \tag{60}$$

$$\langle h_\tau(x), v \rangle = \int_0^L (f_b(\tau, z) - \tilde{h}(x(z)))v(z)dz, \;\; \forall \tau \in I, x \in \mathbb{E}, v \in \mathbb{F}. \tag{61}$$

It is clear that this function belongs to $\mathbb{E}$, the operator $u \rightarrowtail g_\tau(x)u : \mathbb{F} \to \mathbb{E}$ is linear and continuous and that it belongs to $\mathfrak{L}(\mathbb{F}, \mathbb{E})$. Also note that Riesz's representation theorem is used to define the operator $\mathcal{B}$ and the function h. The function q is nonconvex and satisfies the equality

$$q^0(\sigma, \theta) = p_e(\sigma)\theta, \;\; \forall \sigma, \theta \in \mathbb{R}, \tag{62}$$

where $q^0(\sigma, \theta)$ denotes the generalized directional derivative of q at the point σ in the direction θ. Using a conventional argument (Lemma 8 (vi) in [22]), however, we obtain that

$$\jmath^0(u, v) = q^0(u(L), v(L)) \;\; \forall u, v \in \mathbb{F}, \tag{63}$$

where $\jmath^0(u, v)$ denotes the generalized directional derivative of $\jmath$ at the point u in the direction v.

Since (u, π) is a regular solution to (42)–(45), and considering the history of the deformation field $x : I \times [0, L] \to \mathbb{R}$ defined by

$$x(\tau, z) = \int_0^\tau \tilde{g}_\sigma u_z(\sigma, z)\, d\sigma, \;\; \forall \tau \in I, z \in [0, L]. \tag{64}$$

$$x'(\tau, z) = \tilde{g}_\tau u_z(\tau, z), \;\; \forall \tau \in I, z \in [0, L], \tag{65}$$

$$x(0, z) = 0, \;\; \forall z \in [0, L]. \tag{66}$$

Using (42), we derive that

$$\pi(\tau, z) = \kappa u_z(\tau, z) + \tilde{h}(x(\tau, z)), \;\; \forall \tau \in I, z \in [0, L]. \tag{67}$$

Furthermore, using (43)–(45) and performing integration by parts, it follows that

$$\begin{aligned} u(\tau) \in \Omega, \;\; &\int_0^L (\pi(\tau, z) - f_b(\tau, z))(v_z(z) - u_z)dz + Qv(\tau, L)^+ - Qu(\tau, z)^+ \\ &+ p_e(u(\tau, L))(v(L) - u(\tau, L)) \geq 0, \;\; \forall v \in \Omega, \tau \in I. \end{aligned}$$

Therefore, from (67), (62) and (63), we find $u(\tau) \in \Omega$, such that

$$\begin{aligned} &\int_0^L (\kappa u_z(\tau, z) + \tilde{h}(x(\tau, z)) - f_b(\tau, z))(v_z(z) - u_z(z))\, dz \\ &\quad + Qv(\tau, L)^+ - Qu(\tau, z)^+ + \jmath^0(u(\tau, L), v(L) - u(\tau, L)) \geq 0, \; \forall v \in \Omega, \tau \in I. \end{aligned} \tag{68}$$

Finally, from (53)–(61) and (65)–(68), we derive the following variational formulation of the contact problem to find a displacement field $u : I \to \mathbb{F}$ and a deformation field $x : I \to \mathbb{E}$ such that $x(0) = 0_{\mathbb{E}}$ and, for all $\tau \in I$, it holds that

$$\begin{cases} (a) \quad x'(\tau) = \mathcal{A}x(\tau) + f_\tau(x(\tau)) + g_\tau(x(\tau))u(\tau), \\ (b) \quad u(\tau) \in \Omega, \ \langle \mathcal{B}(u(\tau), u(\tau)) - h_\tau(x(\tau)), v - u(\tau) \rangle + \varphi(u(\tau), v) - \varphi(u(\tau), u(\tau)) \\ \quad + \jmath^0(u(\tau), v - u(\tau)) \geq 0, \ \forall v \in \Omega. \end{cases} \tag{69}$$

Next, we consider a function p, two sequences $\{\omega_n\}, \{\gamma_n\}$, and a positive number $\tilde{\omega}$, which satisfy the following properties:

$$\begin{cases} (a) \quad p : \mathbb{R} \to \mathbb{R} \text{ is nondecreasing.} \\ (b) \quad \text{There exists } \mathcal{L}_p > 0 \text{ such that} \\ \quad |p(\theta_1) - p(\theta_2)| \leq \mathcal{L}_p |\theta_1 - \theta_2|, \ \forall \theta_1, \theta_2 \in \mathbb{R}. \\ (c) \quad p(\theta) = 0 \text{ iff } \theta \leq 0. \end{cases} \tag{70}$$

$$\begin{cases} \text{For all } n \in \mathbb{N}, \ \tilde{\omega} \geq \omega_n \geq \omega, \\ \omega_n \to \tilde{\omega} \text{ as } n \to \infty. \end{cases} \tag{71}$$

$$\begin{cases} \text{For all } n \in \mathbb{N}, \ \gamma_n > 0, \\ \gamma_n \to 0 \text{ as } n \to \infty. \end{cases} \tag{72}$$

$$\begin{cases} \mathcal{P} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}^* \text{ is such that} \\ \langle \mathcal{P}(u, u), v \rangle = p(u(L) - \omega)v(L), \ \forall u, v \in \mathbb{F}, \end{cases} \tag{73}$$

$$\Omega_n = \{u \in \mathbb{F} \mid u(L) \leq \omega_n\}, \ \forall n \in \mathbb{N}. \tag{74}$$

We introduce the following perturbation problem to find a displacement field $u_n : I \to \mathbb{F}$ and a deformation field $x_n : I \to \mathbb{E}$, such that $x_n(0) = 0_{\mathbb{E}}$ and, for all $\tau \in I$, it holds that

$$\begin{cases} (a) \quad x_n'(\tau) = \mathcal{A}x_n(\tau) + f_\tau(x_n(\tau)) + g_\tau(x_n(\tau))u_n(\tau), \\ (b) \quad u_n(\tau) \in \Omega_n, \ \langle \mathcal{B}(u_n(\tau), u_n(\tau)) - h_\tau(x_n(\tau)), v - u_n(\tau) \rangle + \dfrac{1}{\gamma_n} \langle \mathcal{P}(u_n(\tau), u_n(\tau)), v - u_n(\tau) \rangle \\ \quad + \varphi(u_n(\tau), v) - \varphi(u_n(\tau), u_n(\tau)) + \jmath^0(u_n(\tau), v - u_n(\tau)) \geq 0, \ \forall v \in \Omega_n, \ n \in \mathbb{N}. \end{cases} \tag{75}$$

Theorem 3. *Assume (47)–(50), (70)–(72) and, in addition, assume that $\kappa > \alpha_e L$. Then, the following statements hold:*

(1) There exists a unique solution $(x, u) \in \mathcal{C}(I, \mathbb{E}) \times \mathcal{C}(I, \mathbb{F})$ to (69).
(2) For each $n \in \mathbb{N}$, there exists a unique solution $(x_n, u_n) \in \mathcal{C}(I, \mathbb{E}) \in \mathcal{C}(I, \mathbb{F})$ to (75).
(3) The solution (x_n, u_n) of (75) converges to (x, u) of (69), i.e.,

$$(x_n(\tau), u_n(\tau)) \to (x(\tau), u(\tau)) \in \mathbb{E} \times \mathbb{F}, \ \text{as } n \to \infty, \ \text{for all } \tau \in I. \tag{76}$$

Proof. Based on Theorems 1 and 2, we check the validity of the conditions of these theorems. First, note that the operator (53) is the generator of the semigroup $\{T(\tau)\}_{\tau \geq 0}$ defined by

$$T(\tau)x = e^\tau x \text{ for each } \tau \geq 0 \text{ and } x \in \mathbb{E}.$$

As a result, condition (2) is fulfilled. Furthermore, it is clear that the functions f and g, defined by (59) and (60), *respectively*, meet the conditions (3) and (4), *respectively*. In addition, the operator (55) satisfies condition (5) with $\alpha_{\mathcal{B}}(\beta_{\mathcal{B}} + \rho_{\mathcal{B}})^2 = \kappa$. Finally, assumptions (47) and (50) ensure that the function h defined by (61) fulfills condition (6). The function

φ defined by (56) satisfies condition (7) with $\alpha_\varphi = 0$ and, the function j defined by (58) satisfies the condition (8)(a). Using (63), (62) and (52), we have

$$\begin{aligned} j^0(u, v-u) + j^0(v, u-v) &= (p_e(u(L)) - p_e(v(L)))(v(L) - u(L)) \\ &\leq \alpha_e |u(L) - v(L)|^2 \\ &\leq \alpha_e L \|u(L) - v(L)\|_{\mathbb{F}}^2. \end{aligned}$$

It proves that given $\alpha_j = \alpha_e L$, condition (8)(b) holds. The inequality $\kappa > \alpha_e L$ also implies that (10) is satisfied. Finally, we can see that (9) and (11) are met. Thus, condition (15) is satisfied, and conditions (16) and (19) may now be recovered by assumption (72). Furthermore, using the properties (70) of the function p and the Inequality (52), it follows that the operator $\mathcal{P}$ defined by (73) is monotone and Lipschitz continuous, satisfying condition (17). Using the assumption (18), we consider the set

$$\tilde{\Omega} = \{u \in \mathbb{F} \mid u(L) \leq \tilde{\omega}\}. \tag{77}$$

Assumption (71) implies that $\tilde{\omega} \geq \omega$ and, therefore (18)(a) are satisfied. On the other hand, for each $n \in \mathbb{N}$, we have

$$\tilde{\Omega} = \frac{\tilde{\omega}}{\omega_n} \Omega_n$$

together with the assumption of compactness of the trace, implies that

$$\Omega_n \xrightarrow{Mosco} \tilde{\Omega} \in \mathbb{F}.$$

Hence, the condition (18)(b) is satisfied, too. Let $u \in \tilde{\Omega}$ and $v \in \Omega$. From (73), we have

$$\langle \mathcal{P}(u, u), v - u \rangle = p(u(L) - \omega)(v(L) - \omega) + p(u(L) - \omega)(\omega - u(L)). \tag{78}$$

Then, from the properties of the function p and inequality $\tilde{\omega} \geq \omega$ imply that each term in (78) is negative, i.e.,

$$\begin{cases} p(u(L) - \omega)(v(L) - \omega) \leq 0, \\ p(u(L) - \omega)(\omega - u(L)) \leq 0. \end{cases} \tag{79}$$

We observe from here that

$$\langle \mathcal{P}(u, u), v - u \rangle \leq 0.$$

Therefore, that condition (18)(c) holds. Assume now that

$$\langle \mathcal{P}(u, u), v - u \rangle = 0.$$

Then, (78) implies that

$$p(u(L) - \omega)(\omega - u(L)) = -p(u(L) - \omega)(v(L) - \omega).$$

Hence, (79) imply that $p(u(L) - \omega)(\omega - u(L))$ is both positive and negative. It follows from here that

$$p(u(L) - \omega)(\omega - u(L)) = 0.$$

This equality, combined with assumption (70)(c), shows that

$$u(L) \leq \omega.$$

We conclude that $u \in \Omega$ and, therefore, (18)(d) holds. Finally, using the compactness of the trace map, it follows that conditions (20) and (21) hold, too. The proof is based on standard arguments, and therefore we skip them. From above, we see that the assumptions of Theorems 1 and 2 are satisfied. Hence, we are in a position to conclude the proof is completed. □

5. Conclusions

The differential variational–hemivariational inequality problems can be viewed as a natural and innovative generalization of differential variational inclusion problems. Two of the most difficult and important problems related to these inequalities are the establishment of the sequences of the problem with a set of constraints and penalty parameters. In this work, we deal with the behaviour of the differential variational–hemivariational inequality problems and studied as the more general existing problem in the literature. The discussion of the differential variational–hemivariational inequality problem depends on the concepts of compactness, pseudo monotonicity, Mosco convergence, inverse strongly monotone and Lipschitz continuous mapping. Finally, we consider a mathematical model which describes the equilibrium of a viscoelastic rod in unilateral contact. The weak formulation of the model is in the form of a differential variational–hemivariational inequality in which the unknowns are the displacement field and the history of the deformation. Our mechanical interpretation is based on the penalty method in the analysis of said inequalities.

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References

1. Aubin, J.P.; Cellina, A. *Differential Inclusions*; Springer: New York, NY, USA, 1984.
2. Pang, J.S.; Stewart, D.E. Differential variational inequalities. *Math. Program.* **2008**, *113*, 345–424. [CrossRef]
3. Liang, L.U.; Lijie, L.I.; Sofonea, M. A generalized penalty method for differential variational-hemivariational inequalities. *Acta Math. Sci.* **2022**, *42*, 1–18.
4. Lu, L.; Liu, Z.H.; Obukhovskii, V. Second order differential variational inequalities involving anti-periodic boundary value conditions. *J. Math. Anal. Appl.* **2019**, *473*, 846–865. [CrossRef]
5. Chang, S.S.; Salahuddin; Wang, L.; Tang, J.; Zhao, L. The convergence results of differential variational inequality problems. *Symmetry* **2022**, *14*, 760. [CrossRef]
6. Liu, Z.H.; Zeng, S.D. Differential variational inequalities in infinite Banach spaces. *Acta Math. Sci.* **2017**, *37*, 26–52. [CrossRef]
7. Liu, Z.H.; Sofonea, M. Differential quasivariational inequalities in contact mechanics. *Math. Mech. Solids* **2019**, *24*, 845–661. [CrossRef]
8. Liu, Z.H.; Zeng, S.D.; Motreanu, D. Partial differential hemivariational inequalities. *Adv. Nonlinear Anal.* **2018**, *7*, 571–586.
9. Sitho, S.; Ntouyas, S.K.; Sudprasert, C.; Tariboon, J. Integrodifferential boundary conditions to the sequential ψ_1-Hilfer and ψ_2-caputo fractional differential equations. *Mathematics* **2023**, *11*, 867. [CrossRef]
10. Hakkar, N.; Dhayal, R.; Debbouche, A.; Torres, D.F.M. Approximate controllability of delayed fractional stochastic differential systems with mixed noise and impulsive effects. *Fractal Fract.* **2023**, *7*, 104. [CrossRef]
11. Sofonea, M.; Migóorski, S.; Han, W. A penalty method for history-dependent variational-hemivariational inequalities. *Comput. Math. Appl.* **2018**, *75*, 2561–2573. [CrossRef]
12. Chang, S.S.; Salahuddin; Wang, L.; Wang, G.; Zhao, Y.H. Existence and convergence results for generalized mixed quasi-variationa Hemivariational inequality problem. *Symmetry* **2021**, *13*, 1882. [CrossRef]
13. Migórski, S.; Ochal, A.; Sofonea, M. *Nonlinear Inclusions and Hemivariational Inequalities: Models and Analysis of Contact Problems*; Advances in Mechanics and Mathematics 26; Springer: New York, NY, USA, 2013.
14. Migórski, S.; Liu, Z.H.; Zeng, S.D. A class of history-dependent differential variational inequalities with application to contact problems. *Optimization* **2020**, *69*, 743–775. [CrossRef]
15. Liu, Z.H.; Zeng, S.D. Penalty method for a class of differential variational inequalities. *Appl. Anal.* **2021**, *100*, 1574–1589. [CrossRef]

16. Kim, J.K.; Salahuddin; Lim, W.H. General nonconvex split variational inequality problems. *Korean J. Math.* **2017**, *25*, 469–481.
17. Kim, J.K.; Dar, A.H.; Salahuddin. Existence solution for the generalized relaxed pseudomonotone variational inequalities. *Nonlinear Funct. Anal. Appl.* **2020**, *25*, 25–34.
18. Salahuddin. On penalty method for non-stationary general set valued equilibrium problems. *Commun. Appl. Nonlinear Anal.* **2016**, *23*, 82–92.
19. Chang, S.S.; Salahuddin; Ahmadini, A.A.H.; Wang, L.; Wang, G. The penalty method for generalized mixed variational-hemivariational inequality problems. *Carpathian J. Math.* **2022**, *38*, 357–381. [CrossRef]
20. Mosco, U. Convergence of convex sets and of solutions of variational inequalities. *Adv. Math.* **1969**, *3*, 510–585. [CrossRef]
21. Clarke, F.H. *Optimization and Nonsmooth Analysis*; Wiley: Hoboken, NJ, USA, 1983.
22. Sofonea, M.; Migórski, S. *Variational-Hemivariational Inequalities with Applications, Pure and Applied Mathematics*; Chapman & Hall: London, UK; CRC Press: Boca Raton, FL, USA, 2018.
23. Li, X.W.; Liu, Z.H.; Sofonea, M. Unique solvability and exponential stability of differential hemivariational inequalities. *Appl. Anal.* **2020**, *99*, 2489–2506. [CrossRef]